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PARTIAL DYNAMICAL SYSTEMS AND C^*-ALGEBRAS GENERATED BY PARTIAL ISOMETRIES

RUY EXEL, MARCELO LACA, and JOHN QUIGG

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Abstract. A collection of partial isometries whose range and initial projections satisfy a specified set of conditions often gives rise to a partial representation of a group. The corresponding C^*-algebra is thus a quotient of the universal C^*-algebra for partial representations of the group, from which it inherits a crossed product structure, of an abelian C^*-algebra by a partial action of the group. This allows us to characterize faithful representations and simplicity, and to study the ideal structure of these C^*-algebras in terms of amenability and topological freeness of the associated partial action. We also consider three specific applications: to partial representations of groups, to Toeplitz algebras of quasi-lattice ordered groups, and to Cuntz-Krieger algebras.

Keywords: *Partial representation, partial action, partial group algebra, Cuntz-Krieger algebra, crossed product, faithful representation.*

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INTRODUCTION

In this paper we develop tools to analyze partial dynamical systems and use them in our general approach to C^*-algebras generated by partial isometries. Our work builds upon [8], where a certain crossed product is shown to be universal for the partial representations of a group. Here we realize the universal C^*-algebra for partial representations of a group subject to relations as the crossed product of a partial action of the group on a commutative C^*-algebra. The key feature of our method is a machine that, given a partial representation with relations, produces an explicit description of the spectrum of this commutative C^*-algebra and of the partial action of the group. This machine, which is the main contribution of this work, has already been applied elsewhere ([9]) to define and study Cuntz-Krieger algebras for arbitrary infinite matrices, a generalization that had previously eluded other approaches.

We begin by reviewing the definition and basic construction of crossed products by partial actions in Section 1. In Section 2 we adapt the notion of topological freeness for group actions ([1]) to the context of partial actions on abelian C^*-algebras. The main technical result is Theorem 2.6, where we show that for a locally compact Hausdorff space X the ideals in the reduced crossed product of $C_0(X)$ by a topologically free partial action of a discrete group G necessarily intersect $C_0(X)$ nontrivially; hence a representation of the reduced crossed product is faithful if and only if it is faithful on $C_0(X)$. This leads to a sufficient condition for simplicity of the reduced crossed product in Corollary 2.9.

In Section 3 we consider invariant ideals of a partial action and the ideals they generate in the crossed product. In Proposition 3.1 we give a general short exact sequence relating an invariant ideal of a partial action, the corresponding ideal of the crossed product, and the crossed product by the quotient partial action. After discussing the approximation property introduced in [6], which implies amenability of a partial action and hence equality of the full and reduced crossed products, we prove Theorem 3.5, the main result of the section. Specifically, the result is that if a partial action has the approximation property and is topologically free on closed invariant subsets, then the ideals of the crossed product are in one-to-one correspondence with the invariant ideals, and hence with invariant open subsets under the partial action.

We begin Section 4 by introducing a class of partial dynamical systems arising from partial representations whose range projections satisfy a given set of relations. In Proposition 4.1 we describe the spectrum associated to the relations and give a canonical partial action of the group on this spectrum. The resulting crossed product has a universal property with respect to partial representations of the group satisfying the relations; this is proved in Theorem 4.4, which is our main result. It is through this that the results of the first three sections become available to study the C^*-algebras generated by partial representations subject to relations. The partial crossed product realization entails, as one of its useful features, that these C^*-algebras have canonical gradings over the group (corresponding to the dual coaction of the partial action).

In the final sections we apply the main results to some concrete situations. In Section 5 we show that the partial dynamical system canonically associated to a discrete group in [8] is topologically free if and only if the group is infinite. Since the reduced partial C^*-algebra of such a group is a crossed product by a partial action, we are able to characterize its faithful representations. In Section 6 we realize the Toeplitz C^*-algebras associated by Nica in [16] to quasi-lattice ordered groups as crossed products by partial actions. We show that the corresponding partial dynamical systems are always topologically free. This allows us to prove a stronger version of a result from [14] about faithful representations of Toeplitz algebras. Indeed, we show in Theorem 6.7 that a representation of the Toeplitz algebra of a quasi-lattice ordered group is faithful if and only if it is faithful on the diagonal, without any amenability assumptions. Finally, in Section 7 we briefly indicate how to realize the Cuntz-Krieger algebra $\mathcal{O}_{\mathcal{A}}$ associated to a $\{0,1\}$-valued $n \times n$ matrix A as a crossed product by a partial action of the free group on n generators. As with the Toeplitz algebras, our description of the crossed product realization of Cuntz-Krieger algebras is explicit, as opposed to the indirect method of [21]. We also indicate that the Cuntz-Krieger uniqueness theorem can be obtained from the results of Section 2 and an amenability result from [6].

1. CROSSED PRODUCTS BY PARTIAL ACTIONS

Let α be a partial action of the discrete group G on the C^*-algebra A in the sense of [5], [15], [8], [21]. That is, for each $s \in G$ we have an isomorphism $\alpha_s : D_{s^{-1}} \to D_s$ between closed ideals of A, such that

$$a_e = \mathrm{id}_A \quad \text{and} \alpha_{st} \text{ extends } \alpha_s \alpha_t \text{ for } s, t \in G.$$

We say that the triple (A, G, α) is a partial dynamical system. There are two C^*-algebras associated with a partial dynamical system: the full crossed product $A \rtimes_\alpha G$ and the reduced crossed product $A \rtimes_{\alpha,r} G$, cf. [15]. These are defined, in analogy with the crossed products of group actions, as certain C^*-completions of the convolution $*$-algebra of finite sums $\big\{ \sum_t a_t \delta_t : a_t \in D_t \big\}$. It is also possible to view the full crossed product as a universal C^*-algebra for covariant representations as in [21]. Since we will exploit this point of view, we briefly review some definitions and basic facts. In [8], Definition 6.2, a *partial representation* of G on a Hilbert space H is defined as a map $u : G \to B(H)$ such that

$$u_e = 1, \quad u_{t^{-1}} = u_t^*, \quad \text{and} \quad u_s u_t u_{t^{-1}} = u_{st} u_{t^{-1}}, \quad \text{for } s, t \in G.$$

Note that these conditions imply that the u_t are partial isometries on H. An equivalent definition which is sometimes easier to verify is given in [21], Definition 1.7, where one only requires that the u_t be partial isometries with commuting range projections, satisfying $u(e)u(e)^* = 1$, $u(s)^* u(s) = u(s^{-1}) u(s^{-1})^*$, and that $u(st)$ extends $u(s)u(t)$ in the sense of [21], Lemma 1.6. The equivalence is proved in [21], Lemma 1.8.

A *covariant representation* of the partial dynamical system (A, G, α) on a Hilbert space H is a pair (π, u) in which π is a nondegenerate representation of A on H and u is a partial representation of G on H such that for each $t \in G$ we have that $u_t u_t^*$ is the projection onto the subspace $\overline{\operatorname{span}}\, \pi(D_t) H$ and

$$\pi(\alpha_t(a)) = u_t \pi(a) u_{t^{-1}}, \quad \text{for } a \in D_{t^{-1}}.$$

As in the case of actions of groups, covariant representations of a partial action correspond to representations of the associated crossed product. This correspondence was first proved in [5], Propositions 5.5 and 5.6, in the case of a single partial automorphism, and it was generalized to partial actions of discrete groups in [15], Propositions 2.7 and 2.8. Our definition of covariant representations is slightly different from those of [5] and [15], but this poses no problem because the various definitions have been shown to be equivalent ([21], Remark 1.12; see also [21], Section 3). There is a one-to-one correspondence $(\pi, u) \mapsto \pi \times u$ between covariant representations of (A, G, α) on H and non-degenerate representations of $A \rtimes_\alpha G$ on H, determined by

$$(\pi \times u)(a\delta_t) = \pi(a) u_t \quad \text{for } t \in G,\ a \in D_t.$$

2. TOPOLOGICALLY FREE PARTIAL ACTIONS

We will mostly be concerned with partial actions arising from partial homeomorphisms of a locally compact space X, so that for every $t \in G$ there is an open subset U_t of X and a homeomorphism $\theta_t : U_{t^{-1}} \to U_t$ such that θ_{st} extends $\theta_s\theta_t$ ([8], [15], [7]). The partial action α of G on $C_0(X)$ corresponding to θ is given by

$$\alpha_t(f)(x) := f(\theta_{t^{-1}}(x)), \quad f \in C_0(U_{t^{-1}}).$$

So, here the ideals are $D_t = C_0(U_t)$. We will talk about the partial action at either the topological or the C^*-algebraic level, according to convenience.

DEFINITION 2.1. (cf. [1]) The partial action θ is *topologically free* if for every $t \in G \setminus \{e\}$ the set $F_t := \{x \in U_{t^{-1}} : \theta_t(x) = x\}$ has empty interior.

We point out that although the set F_t need not be closed in X, it is relatively closed in the domain $U_{t^{-1}}$ of θ_t. A standard argument gives the following equivalent version of topological freeness which is more appropriate for our purposes.

LEMMA 2.2. *The partial action θ on X is topologically free if and only if for every finite subset $\{t_1, t_2, \dots, t_n\}$ of $G \setminus \{e\}$, the set $\bigcup_{i=1}^{n} F_{t_i}$ has empty interior.*

Proof. It suffices to show that for every $t \in G \setminus \{e\}$, the fixed point set F_t is nowhere dense (i.e., its closure in X has empty interior), and then use the fact that a finite union of nowhere dense sets is nowhere dense.

Since F_t is closed relative to U_t we can write $F_t = C \cap U_t$ with C closed in X. Suppose $V \subset \overline{F}_t$ is open. Since the set $V \cap U_t$ is contained in $C \cap U_t = F_t$, it must be empty, by the assumption of topological freeness. Thus V and U_t are disjoint open sets, so each one is disjoint from the other's closure. But $V \subset \overline{C \cap U_t} \subset C \cap \overline{U}_t \subset \overline{U}_t$, so V itself is empty and hence F_t is nowhere dense. ∎

LEMMA 2.3. *Let $t \in G \setminus \{e\}$, $f \in D_t$, and $x_0 \notin F_t$. For every $\varepsilon > 0$ there exists $h \in C_0(X)$ such that:*
(i) $h(x_0) = 1$,
(ii) $\|h(f\delta_t)h\| \leqslant \varepsilon$, *and*
(iii) $0 \leqslant h \leqslant 1$.

Proof. We separate the proof into two cases according to x_0 being in the domain U_t of $\theta_{t^{-1}}$ or not. If $x_0 \notin U_t$, let $K := \{x \in U_t : |f(x)| \geqslant \varepsilon\}$. Then K is a compact subset of D_t and $x_0 \notin K$, so there is $h \in C_0(X)$ such that $0 \leqslant h \leqslant 1$, $h(x_0) = 1$ and $h(K) = 0$. Since f is bounded by ε off K, it follows that $\|hf\| \leqslant \varepsilon$, so (ii) holds too.

If $x_0 \in U_t$ then $\theta_{t^{-1}}(x_0)$ is defined and not equal to x_0. Take disjoint open sets V_1 and V_2 such that $x_0 \in V_1$ and $\theta_{t^{-1}}(x_0) \in V_2$. We may assume that $V_1 \subset U_t$ and $V_2 \subset U_{t^{-1}}$.

Letting $V := V_1 \cap \theta_t(V_2)$, we have that $x_0 \in V \subset V_1$ and $\theta_{t^{-1}}(V) \subset V_2$, from which it follows that $\theta_{t^{-1}}(V) \cap V = \emptyset$. Take now $h \in C_0(X)$ such that $0 \leqslant h \leqslant 1$, $h(x_0) = 1$ and $h(X \setminus V) = 0$. It remains to prove that h satisfies (ii). In fact, the product $hf\delta_t h = \alpha_t(\alpha_{t^{-1}}(hf)h)\delta_t$ vanishes because the support of $\alpha_{t^{-1}}(hf)$ is contained in $\theta_{t^{-1}}(V)$ and the support of h is in V. ∎

The reduced crossed product associated to a partial dynamical system in [15], Section 3 can also be obtained as the reduced cross-sectional algebra of the Fell bundle determined by the partial action ([6], Definition 2.3).

This reduced crossed product is a topologically graded algebra and the conditional expectation, denoted by E_r, from $C_0(X) \rtimes_r G$ onto $C_0(X)$ is a faithful positive map ([6], Proposition 2.12; see also [19], Corollary 3.9 and Lemma 1.4 and [21], Corollary 3.8).

Proposition 2.4. *If $(C_0(X), G, \alpha)$ is a topologically free partial action then for every $c \in C_0(X) \rtimes_r G$ and every $\varepsilon > 0$ there $h \in C_0(X)$ such that:*
(i) $\|hE_r(c)h\| \geqslant \|E_r(c)\| - \varepsilon$,
(ii) $\|hE_r(c)h - hch\| \leqslant \varepsilon$, *and*
(iii) $0 \leqslant h \leqslant 1$.

Proof. Assume first c is a finite linear combination of the form $\sum_{t\in T} a_t\delta_t$, where T denotes a finite subset of G, in which case $E_r(c) = a_e$ (where we put $a_e = 0$ if $e \notin T$). Let $V = \{x \in X : |a_e(x)| > \|a_e\| - \varepsilon\}$, which is clearly open and nonempty. By Lemma 2.2 there exists $x_0 \in V$ such that $x_0 \notin F_t$ for every $t \in T \setminus \{e\}$, and by Lemma 2.3 there exist functions h_t satisfying

$$h_t(x_0) = 1, \quad \|h_t(a_t\delta_t)h_t\| \leqslant \frac{\varepsilon}{|T|}, \quad \text{and} \quad 0 \leqslant h_t \leqslant 1.$$

Let $h := \prod_{t\in T\setminus\{e\}} h_t$. Then (iii) is immediate, and (i) also holds because $x_0 \in V$ so $\|ha_eh\| \geqslant |a_e(x_0)| > \|a_e\| - \varepsilon$. For (ii), we have

$$\|ha_eh - hah\| = \Big\| \sum_{t\in T\setminus\{e\}} ha_t\delta_t h\Big\| \leqslant \sum_{t\in T\setminus\{e\}} \|ha_t\delta_t h\| \leqslant \sum_{t\in T\setminus\{e\}} \|h_t a_t\delta_t h_t\| < \varepsilon.$$

Since the elements of the form $\sum_{t\in T} a_t\delta_t$ are dense in the crossed product and the conditional expectation E_r is contractive, a standard approximation argument gives the general case. ∎

Remark 2.5. It is easy to see that the Proposition 2.4 also holds with the full crossed product replacing the reduced one.

Theorem 2.6. *Suppose $(C_0(X), G, \alpha)$ is a topologically free partial action. If I is an ideal in $C_0(X) \rtimes_r G$ with $I \cap C_0(X) = \{0\}$, then $I = \{0\}$. A representation of the reduced crossed product $C_0(X) \rtimes_r G$ is faithful if and only if it is faithful on $C_0(X)$.*

Proof. Denote by $\pi : C_0(X) \rtimes_r G \to (C_0(X) \rtimes_r G)/I$ the quotient map, and let $a \in I$ with $a \geqslant 0$, so that $\pi(a) = 0$. Given $\varepsilon > 0$ take $h \in C_0(X)$ satisfying conditions (i), (ii) and (iii) of Proposition 2.4. Then

$$\|\pi(hE_r(a)h)\| = \|\pi(h(E_r(a) - a)h)\| \leqslant \varepsilon,$$

because $\pi(a) = 0$. Since π is isometric on $C_0(X)$, because $I \cap C_0(X) = \{0\}$, it follows that $\|hE_r(a)h\| \leqslant \varepsilon$. By Proposition 2.4 (i), $\|E_r(a)\| - \varepsilon \leqslant \|hE_r(a)h\|$, so $\|E_r(a)\| \leqslant 2\varepsilon$, and $E_r(a)$ has to vanish. Since the conditional expectation E_r is faithful on the reduced crossed product this implies that $a = 0$ and hence that $I = \{0\}$. This proves the first assertion, which, applied to the kernel of a representation, gives the second one. ∎

DEFINITION 2.7. A subset V of X is *invariant* under the partial action θ on X if $\theta_s(V \cap U_{s^{-1}}) \subset V$ for every $s \in G$.

An ideal J in $C_0(X)$ is *invariant* under the corresponding partial action α on $C_0(X)$ if $\alpha_t(J \cap D_{t^{-1}}) \subset J$ for every $t \in G$.

It is easy to see that if U is an invariant open set then the associated ideal $C_0(U)$ is invariant and, conversely, every invariant ideal corresponds to an invariant open set.

DEFINITION 2.8. The partial action θ on X is *minimal* if there are no θ-invariant open subsets of X other than $\emptyset$ and X or, equivalently, if the partial action α on $C_0(X)$ has no nontrivial proper invariant ideals.

The complement of an invariant set is invariant too, so the partial action is minimal if and only if it has no nontrivial proper closed invariant subsets.

COROLLARY 2.9. *If a partial action is topologically free and minimal then the associated reduced crossed product is simple.*

Proof. Suppose J is the kernel of a representation $\rho = \pi \times v$ of $C_0(X) \rtimes_r G$. Then $J \cap C_0(X)$ is an ideal in $C_0(X)$ which is invariant under α because for every $f \in J \cap D_{t^{-1}}$, we have, by covariance, that $\pi(\alpha_t(f)) = v_t \pi(f) v_t^* = 0$, and hence that $\alpha_t(f) \in J$.

By assumption α is minimal, so either $J \cap C_0(X) = C_0(X)$, in which case $\pi = 0$, hence $\rho = 0$, or else $J \cap C_0(X) = \{0\}$, in which case the representation π is faithful by Theorem 2.6. This proves that the crossed product is simple. ∎

3. INVARIANT IDEALS AND THE APPROXIMATION PROPERTY

Let α be a partial action on the C^*-algebra A. For each invariant ideal I of A there is a restriction of α to a partial action on I, with ideals $D_t \cap I$ as ranges of the restricted partial automorphisms, and there is also a quotient partial action $\dot{\alpha}_t$ of G on A/I, defined by passage to the quotient: the domain of $\dot{\alpha}_t$ is the ideal $\dot{D}_{t^{-1}} := \{a + I \in A/I : a \in D_{t^{-1}}\}$ and $\dot{\alpha}_t(a + I) = \alpha_t(a) + I$.

We will show that the quotient of the crossed product $A \rtimes G$ modulo the ideal generated by I is isomorphic to the crossed product of the quotient partial action modulo I. This result generalizes [20], Proposition 3.4, which proves the case $G = \mathbb{Z}$, and extends part of [15], Proposition 5.1, which only concerns ideals. The original argument, for group actions, is from [10], Lemma 1. We will denote by $\langle S \rangle$ the ideal generated by a subset S of a C^*-algebra B.

PROPOSITION 3.1. *Suppose α is a partial action on A and assume I is an α-invariant ideal of A. Then the map $a\delta_t \in I \rtimes G \mapsto a\delta_t \in A \rtimes G$ extends to an injection of $I \rtimes G$ onto the ideal $\langle I \rangle$ generated by I in $A \rtimes G$, and $\langle I \rangle \cap A = I$.*

The map $a\delta_t \in A \rtimes G \mapsto (a + I)\delta_t \in (A/I) \rtimes G$ extends to a surjective homomorphism, giving the short exact sequence

$$0 \to I \rtimes G \to A \rtimes G \to (A/I) \rtimes G \to 0.$$

Proof. The assertion that $I \rtimes G$ injects as an ideal in $A \rtimes G$ is proved in [15], Proposition 5.1 and Corollary 5.2 and, as done there, we identify $I \rtimes G$ with $\overline{\operatorname{span}}\{a\delta_t : a \in D_t \cap I,\, t \in G\}$; we also identify I with its canonical image $I\delta_e$ in $A \rtimes G$. It is clear that $\langle I \rangle$ is contained in $I \rtimes G$. To prove the reverse inclusion it suffices to show that $a\delta_t \in \langle I \rangle$ for every $a \in D_t \cap I$ and $t \in G$. Assume $a \in D_t \cap I$ and let b_λ be an approximate unit for the ideal D_t. Then $ab_\lambda\delta_t = (a\delta_e)(b_\lambda\delta_t) \in \langle I \rangle$ so $a\delta_t = \lim_\lambda ab_\lambda\delta_t \in \langle I \rangle$. This proves that $I \rtimes G = \langle I \rangle$, from which it is obvious that $I = \langle I \rangle \cap A$.

The map $a\delta_t \mapsto (a + I)\delta_t$ induces a $*$-homomorphism from $\ell^1(G, A)$ onto $\ell^1(G, A/I)$. Since $A \rtimes G$ is the enveloping C^*-algebra of $\ell^1(G, A)$, there is C^*-homomorphism φ of $A \rtimes G$ onto $(A/I) \rtimes G$ which sends $a\delta_t$ to $(a + I)\delta_t$ for every $a \in D_t$ and every $t \in G$. To finish the proof we need to show that $\ker \varphi = \langle I \rangle$.

It is clear that $\ker \varphi$ contains the ideal $\langle I \rangle$ generated by I in $A \rtimes G$. It remains to prove that $\ker \varphi \subset \langle I \rangle$. Let $\pi \times u$ be a representation of $A \rtimes G$ with kernel $\langle I \rangle$. Since the kernel of π contains I, π factors through the quotient map $A \to A/I$; denote by $\widetilde{\pi}$ the corresponding representation of A/I. The pair $(\widetilde{\pi}, u)$ is covariant and determines a representation $\widetilde{\pi} \times u$ of $(A/I) \rtimes G$. Then $\pi \times u = (\widetilde{\pi} \times u) \circ \varphi$, so $\ker \varphi \subset \ker(\pi \times u)$. ∎

REMARK 3.2. We point out that, at the level of *reduced* crossed products, it is always true that $I \rtimes_r G$ injects as an ideal in $A \rtimes_r G$ ([15], Proposition 5.1), but whether the quotient is the reduced crossed product $(A/I) \rtimes_r G$ is a subtler question. We refer to the discussion at the end of [6], Section 4 for related considerations.

When $A = C_0(X)$ the α-invariant ideals are in one to one correspondence with invariant open sets; the corresponding quotients are naturally identified with the continuous functions on the complements of these invariant open sets. Specifically, if α is a partial action on $C_0(X)$ and U is an invariant open subset of X then $C_0(U)$ is an invariant ideal in $C_0(X)$, and every invariant ideal is of this form. Moreover, $C_0(X)/C_0(U) \cong C_0(\Omega)$ with $\Omega = X \setminus U$, the quotient map being simply restriction. The quotient partial action $\dot{\alpha}$ of G on $C_0(\Omega)$ is given by $\dot{\alpha}_t(f|\Omega) = \alpha_t(f)|\Omega$ for $f \in D_{t^{-1}}$ (the domain of $\dot{\alpha}_t$ consists of the restrictions to Ω of functions in $D_{t^{-1}}$).

In general there may be more ideals in a crossed product than those of the form $\langle I \rangle$ with I an invariant ideal in A. Easy examples abound even for full actions; for instance write $C^*(G) = \mathbb{C} \rtimes G$ (with the trivial action). If G has more than one element, then the kernel of the trivial homomorphism $s \mapsto 1$ from $C^*(G)$ to $\mathbb{C}$ is a proper nontrivial ideal which is not generated by an ideal in $\mathbb{C}$.

The quotient system $(C_0(\Omega), G, \alpha)$, obtained by restricting the partial action α to a closed invariant subset Ω of X, need not be topologically free even if $(C_0(X), G, \alpha)$ is. An easy example of this phenomenon is obtained by restricting the action of $\mathbb{Z}$ by translation on $C(\mathbb{Z}\cup\{\pm\infty\})$ to the subset $\{\pm\infty\}$. However, we will see that if topological freeness holds on quotients of a partial action having the approximation property introduced in [], then all the ideals of the crossed product are obtained from their intersections with $C_0(X)$, via the map $I \mapsto \langle I\rangle$. Before we prove this we briefly review some basic facts about amenability and the approximation property.

A partial dynamical system $(C_0(X), G, \alpha)$ is *amenable* if the canonical homomorphism from the full crossed product to the reduced one is faithful. Amenability is equivalent to faithfulness (as a positive map) of the conditional expectation from the full crossed product $C_0(X) \rtimes G$ onto $C_0(X)$ ([6], Proposition 4.2); it is also equivalent to *normality* of the dual coaction in the sense of [18] and to amenability of the associated semi-direct product Fell bundle ([7], Definition 2.8).

Definition 3.3. The partial dynamical system $(C_0(X), G, \alpha)$ has the *approximation property* if the semi-direct product bundle has the approximation property of [6], Definition 4.4, that is, if there exists a net (a_i) of finitely supported functions $a_i : G \to C_0(X)$ such that

$$\sup_i \Big\| \sum_{t\in G} a_i(t)^* a_i(t) \Big\| < \infty$$

and

$$\lim_i \sum_{t\in G} a_i(st)^* f\delta_s a_i(t) = f\delta_s \quad s \in G,\ f \in D_s.$$

This approximation property implies amenability ([6], Theorem 4.6), over which it has the advantage of being inherited by graded quotients of Fell bundles in the sense specified in the next proposition. We do not know at present whether the approximation property is actually equivalent to amenability.

Although we will only need the following result in the special situation of crossed products by partial actions, it is more convenient to formulate it for the topologically graded algebras studied in [6].

Proposition 3.4. *Suppose the C^*-algebra B is topologically graded over G, and assume the associated Fell bundle has the approximation property. Let $J \subset B$ be such that $J = \langle J \cap B_e\rangle$. Then the quotient B/J is topologically graded over G and its associated Fell bundle also has the approximation property.*

Proof. Let $\pi : B \to B/J$ be the quotient map. That B/J is topologically graded is proved in [6], Proposition 3.1. To prove the second claim, suppose that the a_i are the approximating functions for B. Then the collection of functions $t \mapsto \pi(a_i(t))$ can be used to show that the approximation property holds for B/J. ∎

Theorem 3.5. *Let $(C_0(X), G, \alpha)$ be a partial dynamical system which is topologically free on every closed invariant subset of X and which satisfies the approximation property. Then the map*

$$U \mapsto \langle C_0(U) \rangle$$

is a lattice isomorphism of the invariant open subsets of X onto the ideals of $C_0(X) \rtimes G$.

Proof. It is clear that the map $U \mapsto \langle C_0(U) \rangle$ maps invariant open subsets of X to ideals in $C_0(X) \rtimes G$, and that if $U_1 \subset U_2$ then $\langle C_0(U_1) \rangle \subset \langle C_0(U_2) \rangle$. Since $C_0(U) = \langle C_0(U) \rangle \cap C_0(X)$ by Proposition 3.1, the map is one-to-one. Next we show that every ideal in $C_0(X) \rtimes G$ is of this form; this will prove that $U \mapsto \langle C_0(U) \rangle$ is an order preserving bijection, hence a lattice isomorphism.

Suppose J is an ideal of $C_0(X) \rtimes G$, and let $I := J \cap C_0(X)$. Then $I = C_0(U)$ for an open invariant subset $U \subset X$, and it is clear that $\langle C_0(U) \rangle \subset J$; we will show that in fact $\langle C_0(U) \rangle = J$.

The set $\Omega := X \setminus U$ is closed and invariant, so $\langle C_0(U) \rangle$ is the kernel of the homomorphism

$$\varphi : C_0(X) \rtimes G \to C_0(\Omega) \rtimes G$$

by the preceding proposition. Let $b \in \varphi(J) \cap C_0(\Omega)$, so that $b = \varphi(a)$ for some $a \in J$ and $b = \varphi(a_1)$ for some $a_1 \in C_0(X)$. Thus $a - a_1 \in \ker \varphi$, and since $\ker \varphi = \langle C_0(U) \rangle \subset J$ it follows that a_1 itself is in J. But then $a_1 \in J \cap C_0(X) = C_0(U)$, so $b = \varphi(a_1) = 0$. This shows that the ideal $\varphi(J)$ of $C_0(\Omega) \rtimes G$ has trivial intersection with $C_0(\Omega)$.

By Proposition 3.4 the partial action on the quotient $C_0(\Omega)$ satisfies the approximation property, so it is amenable and the reduced and full crossed products coincide, by [6], Proposition 4.2.

Since by assumption α is topologically free on Ω, $\varphi(J)$ is trivial by Proposition 2.6, and thus $J \subset \ker \varphi = \langle C_0(U) \rangle$ as required. ∎

4. PARTIAL REPRESENTATIONS SUBJECT TO CONDITIONS

The C^*-algebra of an inverse semigroup of isometries was realized as the crossed product of a partial action in [8]. In this section we show how to generalize and extend that construction to the C^*-algebra generated by a partial representation subject to relations. In order to set up the notation we briefly review the main ideas from [8]. Consider the compact Hausdorff space $\{0,1\}^G$, and let e denote the identity element in G. The subset

$$X_G := \{\omega \in \{0,1\}^G : e \in \omega\}$$

is a compact Hausdorff space with the relative topology inherited from $\{0,1\}^G$.

The sets $X_t := \{\omega \in X_G : t \in \omega\}$ are clopen, and we define a partial homeomorphism θ_t on $X_{t^{-1}}$ by $\theta_t(\omega) = t\omega$, where $t\omega = \{tx : x \in \omega\}$. This gives a partial action $(\{X_t\}, \{\theta_t\})_{t \in G}$ canonically associated to the group G.

At the algebra level, denote by 1_t the characteristic function of X_t; then $C(X_G)$ is the C^*-algebra generated by the projections $\{1_s : s \in G\}$. The domain of the partial automorphism α_t is $C_0(X_{t^{-1}}) = \overline{\operatorname{span}}\{1_s 1_{t^{-1}} : s \in G\}$, and α_t is determined by

$$\alpha_t(1_s 1_{t^{-1}}) = 1_{ts} 1_t.$$

The crossed product $C(X_G) \rtimes_\alpha G$ has the following universal property:

(U1) For every partial representation u of G there is a unique representation ρ_u of $C(X_G)$ satisfying $\rho_u(1_t) = u_t u_t^*$, and (ρ_u, u) is a covariant representation of $(C(X_G), G, \alpha)$;

(U2) every representation of $C(X_G) \rtimes_\alpha G$ is of the form $\rho_u \times u$ with (ρ_u, u) as above.

Since $C^*(\{u_t : t \in G\})$ coincides with the C^*-algebra generated by the range of $\rho \times u$, this justifies referring to the crossed product $C(X_G) \rtimes G$ as the universal C^*-algebra for partial representations of G or, simply, as the *partial group algebra* of G ([8], Definition 6.4).

Notice that since X_t is clopen the partial isometries themselves belong to the crossed product and, indeed, they generate it. We will denote by $[t]$ the partial isometry corresponding to the group element t in the universal partial representation of G, and, by abuse of notation, we will also write $1_t = [t][t]^*$. When u is a partial representation of G we will denote the range projections $u_t u_{t^{-1}} = u_t u_t^*$ by $e^u(t)$ or simply by $e(t)$. Notice that the initial projection of u_t is the range projection of $u_{t^{-1}}$, so we only need to mention range projections.

We are interested here in partial representations u whose range projections satisfy a set of relations of the form

$$\sum_i \prod_j \lambda_{ij} e(t_{ij}) = 0,$$

where the λ_{ij} are scalars and the sums and products are over finite sets. Observe that if u is such a representation then, since $\rho_u(1_t) = u_t u_t^* = e(t)$, we have

$$\rho_u\Bigl(\sum_i \prod_j \lambda_{ij} 1_{t_{ij}}\Bigr) = \sum_i \prod_j \lambda_{ij} e(t_{ij}) = 0.$$

It follows that ρ_u vanishes on $\sum_i \prod_j \lambda_{ij} 1_{t_{ij}}$. More generally, given a collection of functions $\mathcal{R}$ in $C(X_G)$ we will say that the partial representation u of G *satisfies the relations* $\mathcal{R}$ if ρ_u vanishes on every $f \in \mathcal{R}$. When the relations in $\mathcal{R}$ are of the form specified above, this amounts to saying that the generating partial isometries satisfy $\sum_i \prod_j \lambda_{ij} u_{t_{ij}} u_{t_{ij}}^* = 0$.

PROPOSITION 4.1. *Let $\mathcal{R}$ be a collection of functions in $C(X_G)$. Then the smallest α-invariant (closed, two-sided) ideal of $C(X_G)$ containing $\mathcal{R}$ is the ideal, denoted I, generated by the set $\{\alpha_t(f 1_{t^{-1}}) : t \in G,\ f \in \mathcal{R}\}$. Moreover the zero set of I coincides with*

$$\Omega_{\mathcal{R}} := \{\omega \in X_G : f(t^{-1}\omega) = 0 \text{ for all } t \in \omega,\ f \in \mathcal{R}\}, \tag{4.1}$$

from which it follows that $\Omega_{\mathcal{R}}$ is a compact invariant subset of X_G such that $I = C_0(X_G \setminus \Omega_R)$, and the quotient $C(X_G)/I$ is canonically isomorphic to $C(\Omega_{\mathcal{R}})$.

Proof. Notice first that for every $f \in C(X_G)$ the function $f1_{t^{-1}}$ is in $D_{t^{-1}}$ so that it makes sense to talk about $\alpha_t(f1_{t^{-1}})$. Moreover, identifying $C(X_G)$ with its image in the crossed product and using covariance, we have $\alpha_t(f1_{t^{-1}}) = [t]f[t^{-1}]$. Let I be the ideal generated by $\{\alpha_t(f1_{t^{-1}}) : t \in G,\ f \in \mathcal{R}\}$. Since any invariant ideal which contains f must contain $\alpha_t(f1_{t^{-1}})$, the smallest invariant ideal containing $\mathcal{R}$ must contain I. The reverse inclusion will follow once we show that I is invariant, i.e., that $\alpha_s(I \cap D_{s^{-1}}) \subset I$ for every $s \in G$. Since $I \cap D_{s^{-1}} = 1_{s^{-1}}I$, we need to show that $\alpha_s(1_{s^{-1}}I) \subset I$ for every $s \in G$.

For $g \in C(X_G)$, $f \in \mathcal{R}$ and $s,t \in G$, we have

$$\begin{aligned}\alpha_s(\alpha_t(f1_{t^{-1}})g1_{s^{-1}}) &= [s]([t]f[t^{-1}]g)[s^{-1}] = [st][t^{-1}][t]f[t^{-1}]g[s^{-1}][s][s^{-1}]\\ &= [st]f[t^{-1}][s^{-1}][s]g[s^{-1}] = [st]f[(st)^{-1}]\alpha_s(g1_{s^{-1}})\\ &= \alpha_{st}(f1_{(st)^{-1}})\alpha_s(g1_{s^{-1}}).\end{aligned}$$

Since the linear span of the elements $\alpha_t(f1_{t^{-1}})g$ is dense in I, it follows that $\alpha_s(1_{s^{-1}}I) \subset I$ for every $s \in G$.

Since $\alpha_t(f1_{t^{-1}})(\omega)$ is equal to $f(t^{-1}\omega)$ when $t \in \omega$, and 0 otherwise, the characterization of I given in the first part implies that $f(t^{-1}\omega) = 0$ for every $t \in \omega$ and $f \in \mathcal{R}$ if and only if $F(\omega) = 0$ for every $F \in I$. This proves that $\Omega_{\mathcal{R}}$ is indeed the zero set of I, finishing the proof. ∎

DEFINITION 4.2. The set $\Omega_{\mathcal{R}}$ is called the *spectrum of the relations* $\mathcal{R}$.

The spectrum of a set of relations is invariant under the partial action α on X_G so there is a partial action (also denoted α) on $\Omega_{\mathcal{R}}$ obtained by restricting the partial homeomorphisms to $\Omega_{\mathcal{R}}$. The restricted partial homeomorphisms have compact open (relative to $\Omega_{\mathcal{R}}$) sets as domains and ranges so for each group element t the partial isometry $v_t = (v_t v_t^*)v_t$ belongs to the crossed product. We will show that the crossed product $C(\Omega_{\mathcal{R}}) \rtimes G$ has a universal property with respect to partial representations of G subject to the relations $\mathcal{R}$.

DEFINITION 4.3. Suppose G is a group and let $\mathcal{R} \subset C_0(X_G)$ be a set of relations. A partial representation v of G is *universal for the relations* $\mathcal{R}$ if

(i) v satisfies $\mathcal{R}$, i.e., $\rho_v(\mathcal{R}) = \{0\}$, and

(ii) for every partial representation V of G satisfying $\mathcal{R}$ the map $v_t \mapsto V_t$ extends to a C^*-algebra homomorphism from $C^*(\{v_t : t \in G\})$ onto $C^*(\{V_t : t \in G\})$.

The C^*-algebra generated by a universal partial representation for $\mathcal{R}$ (which is clearly unique up to canonical isomorphism) will be called the *universal C^*-algebra for partial representations of G subject to the relations* $\mathcal{R}$ and denoted $C^*_{\mathrm{p}}(G;\mathcal{R})$.

THEOREM 4.4. *Suppose $\mathcal{R}$ is a collection of relations in $C(X_G)$ with spectrum*

$$\Omega_{\mathcal{R}} = \{\omega \in X_G : f(t^{-1}\omega) = 0 \text{ for all } t \in \omega,\ f \in \mathcal{R}\}.$$

(i) *If $\rho \times V$ is a representation of $C(\Omega_{\mathcal{R}}) \rtimes G$ then V is a partial representation of G satisfying the relations $\mathcal{R}$.*

(ii) *Conversely, if V is a partial representation of G satisfying the relations $\mathcal{R}$, then $1_t \mapsto V(t)V(t)^*$ extends uniquely to a representation ρ_V of $C(\Omega_{\mathcal{R}})$, and the pair (ρ_V, V) is covariant.*

(iii) $C(\Omega_{\mathcal{R}}) \rtimes G$ *is isomorphic to the universal* C^**-algebra* $C^*_{\mathrm{p}}(G;\mathcal{R})$ *for partial representations of* G *subject to the relations* $\mathcal{R}$.

Proof. (i) holds because the range projections of the partial isometries v_t are in $C(\Omega_{\mathcal{R}})$, which was defined precisely so that the relations $\mathcal{R}$ be satisfied.

Next we prove (ii). If V is a partial representation satisfying the relations, then the representation of $C(X_G)$ determined by the range projections $1_t \mapsto V_tV_t^*$ factors through $C(\Omega_{\mathcal{R}})$, and hence gives a covariant representation of $(C(\Omega_{\mathcal{R}}), G, \alpha)$.

By (i) and (ii) there is a bijection between partial representations satisfying the relations and covariant representations of $(C(\Omega_{\mathcal{R}}), G, \alpha)$. Furthermore, the range of a partial representation generates the same C^*-algebra as the range of the corresponding covariant representation. Since $C^*_{\mathrm{p}}(G;\mathcal{R})$ and $C(\Omega_{\mathcal{R}}) \rtimes G$ are both generated by the ranges of universal representations, $C(\Omega_{\mathcal{R}}) \rtimes G$ is a realization of $C^*_{\mathrm{p}}(G;\mathcal{R})$. ∎

In the remaining sections we consider several situations that fall naturally into the framework of partial representations with relations.

5. NO RELATIONS: THE PARTIAL GROUP ALGEBRA $C^*_{\mathrm{p}}(G)$

Let $\mathcal{R}$ be the empty set of relations and consider *all* partial representations of a group G, subject to no restrictions. This is the situation from [8] mentioned at the beginning of Section 4. In this section we will characterize topological freeness and the approximation property for the corresponding partial action. The spectrum $\Omega_\emptyset$ is $X_G := \{\omega \subset 2^G : e \in \omega\}$ and the canonical partial action θ is given by $\theta_t(\omega) = t\omega$ for $\omega \ni t^{-1}$. By Theorem 4.4, the crossed product $C(X_G) \rtimes_\alpha G$ is the universal C^*-algebra $C^*_{\mathrm{p}}(G) := C^*_{\mathrm{p}}(G;\emptyset)$ for partial representations of G.

PROPOSITION 5.1. *The canonical partial action of a nontrivial group* G *on* X_G *is topologically free if and only if* G *is infinite.*

Proof. When the group G is finite, the spectrum X_G has the discrete topology. Since the point $G \in X_G$ is fixed by every group element, the partial action associated to partial representations of G is never topologically free.

Assume now that G is infinite and let

$$U := \{\chi \in X_G : a_i \in \chi \text{ and } b_j \notin \chi \text{ for } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n\}$$

be a typical basic (nonempty) open set in X_G where $a_i, b_j \in G$. It suffices to show that for every element $t \in G \setminus \{e\}$ there is some $\omega_0 \in U$ which is not fixed by t. We may restrict our attention to the intersection of U with the domain of θ_t, by assuming that one of the a_i's (and none of the b_j's) is equal to t^{-1}.

Since G is infinite there exists an element $c \in G$ different from the a_i and the b_j and such that tc is different from the a_i. Then $\omega_0 := \{e, a_1, a_2, \ldots, a_m, c\}$ is in U and $\theta_t(\omega_0) = \{t, ta_1, ta_2, \ldots, ta_m, tc\}$ is different from ω_0 because tc is not in ω_0. Thus ω_0 is not fixed by t, finishing the proof. ∎

COROLLARY 5.2. *For infinite* G, *a representation of* $C(X_G) \rtimes_r G$ *is faithful if and only if its restriction to* $C(X_G)$ *is faithful.*

Proof. Direct application of Theorem 2.6. ∎

REMARK 5.3. The singleton $\{G\} \subset X_G$ is always closed and invariant under the partial action, so topological freeness fails at least for the restriction to $\{G\}$. Because of this the situation of Theorem 3.5 never arises for the empty set of relations, and a characterization of the ideals in $C^*_{\mathrm{p}}(G)$ lies beyond the present techniques.

THEOREM 5.4. *The canonical partial action of G on $C(X_G)$ has the approximation property if and only if G is amenable.*

Proof. That the partial action of an amenable group G on X_G satisfies the approximation property is an easy consequence of [7], Theorem 4.7.

To prove the converse suppose the action of G on X_G satisfies the approximation property. Then by Proposition 3.4 the (trivial) action on the closed invariant singleton $\{G\}$ satisfies the approximation property. Hence this trivial action is amenable and the reduced and full crossed products coincide. Since they correspond to the reduced and full group C^*-algebras of G, G itself must be an amenable group. ∎

REMARK 5.5. Since we do not know whether amenability itself is inherited by quotients, we do not know whether amenability of the partial action of G on $C(X_G)$ entails amenability of G.

6. NICA COVARIANCE: THE TOEPLITZ ALGEBRAS OF QUASI-LATTICE GROUPS

Let (G, P) be a quasi-lattice ordered group, as defined by Nica in [16], Section 2. The semigroup P induces a partial order in G via $x \leqslant y$ if and only if $x^{-1}y \in P$. The quasi-lattice condition says that if for $x, y \in G$ the set $\{z \in P : x \leqslant z,\ y \leqslant z\}$ is nonempty, then it has a smallest element, denoted $x \vee y$, and referred to as the least common upper bound in P of x and y, (if there is no common upper bound, we write $x \vee y = \infty$). It is easy to see that x has an upper bound in P if and only if $x \in PP^{-1}$.

An *isometric representation* of P on H is a map $V : P \to B(H)$ such that $V_x^* V_x = 1$ and $V_x V_y = V_{xy}$. The isometric representation V is *covariant* if it satisfies

$$V_x V_x^* V_y V_y^* = V_{x\vee y} V_{x\vee y}^*, \quad x, y \in P,$$

here we use the convention that $V_\infty = 0$, so that if x and y do not have a common upper bound in P then the corresponding isometries have orthogonal ranges.

The Toeplitz (or Wiener-Hopf) algebra $\mathcal{T}(G, P)$ is the C^*-algebra generated by the left regular representation T of P on $\ell^2(P)$ ([16]), which is easily seen to be covariant. The universal C^*-algebra $C^*(G, P)$ is the C^*-algebra generated by a universal covariant semigroup of isometries. When (G, P) is amenable, the canonical homomorphism $C^*(G, P) \mapsto \mathcal{T}(G, P)$ is an isomorphism ([16], [14]).

Every $x \in PP^{-1}$ can be written in a "most efficient way" as $x = \sigma(x)\tau(x)^{-1}$, where $\sigma(x) := x \vee e$ is the smallest upper bound of x in P and $\tau(x) := \sigma(x^{-1}) = x^{-1}\sigma(x)$. Using this factorization Raeburn and the third author have shown in [21], Theorem 6.6 that $\mathcal{T}(G, P)$ is a crossed product by a partial action on its diagonal subalgebra. Their proof involves extending isometric covariant representations of P to partial representations of G, and can be pushed further to describe the class of such extensions in terms of relations satisfied by the range projections, which we do next.

PROPOSITION 6.1. *Let (G, P) be a quasi-lattice ordered group.*

(1) *If V is a covariant isometric representation of P then*

$$u_x = \begin{cases} V_{\sigma(x)}V^*_{\tau(x)} & \text{if } x \in PP^{-1}, \\ 0 & \text{if } x \notin PP^{-1}, \end{cases} \tag{6.1}$$

is a partial representation of G satisfying the relations:

($\mathcal{N}_1$) *$u_t^*u_t = 1$ for $t \in P$, and*

($\mathcal{N}_2$) *$u_x u_x^* u_y u_y^* = u_{x\vee y}u^*_{x\vee y}$ for $x, y \in G$,*

which we denote collectively by ($\mathcal{N}$).

(2) *Conversely, every partial representation u_t of G satisfying the relations ($\mathcal{N}$) arises this way from a covariant isometric representation of P.*

Proof. (1) That u_x is a partial representation is proved in [21], Theorem 6.6 and that it satisfies ($\mathcal{N}_1$) is obvious. We prove ($\mathcal{N}_2$) next. Let $x, y \in G$ and assume both are in PP^{-1}, for otherwise both sides are zero and there is nothing to prove. Notice first that $u_x u_x^* = V_{\sigma(x)}V^*_{\tau(x)}V_{\tau(x)}V^*_{\sigma(x)} = V_{\sigma(x)}V^*_{\sigma(x)}$, so

$$u_x u_x^* u_y u_y^* = V_{\sigma(x)}V^*_{\sigma(x)}V_{\sigma(y)}V^*_{\sigma(y)} = V_{\sigma(x)\vee\sigma(y)}V^*_{\sigma(x)\vee\sigma(y)}.$$

Since $x \vee y = x \vee e \vee y = \sigma(x) \vee \sigma(y)$ this proves ($\mathcal{N}_2$).

(2) Assume now that u_x is a partial representation of G satisfying ($\mathcal{N}$). Then u_t is an isometry for every $t \in P$, and $u_s u_t = u_s u_t u_t^* u_t = u_{st}u_t^* u_t = u_{st}$. Thus the restriction of u to P is an isometric representation, which is covariant by ($\mathcal{N}_2$).

It only remains to check that u arises from its restriction V to P as in (6.1). Let $x \in G$. Then $u_x u_x^* = u_x u_x^* u_e u_e^* = u_{\sigma(x)}u^*_{\sigma(x)}$ by ($\mathcal{N}_2$). If $x \notin PP^{-1}$, then $\sigma(x) = \infty$ and $u_x u_x^*$ vanishes. If $x \in PP^{-1}$, then

$$u_x = u_x u_x^* u_x = u_{\sigma(x)}u^*_{\sigma(x)}u_x.$$

The last two factors can be combined because u is a partial representation, and since $\sigma(x)^{-1}x = \tau(x)^{-1}$ we conclude that $u_x = u_{\sigma(x)}u^*_{\tau(x)}$. ∎

DEFINITION 6.2. A subset ω of G is *hereditary* if $xP^{-1} \subset \omega$ for every $x \in \omega$. It is *directed* if for every $x, y \in \omega$ there exists $z \in \omega \cap P$ with $x \leqslant z$ and $y \leqslant z$.

Notice that a hereditary subset ω is directed if and only if the least upper bound of any two of its elements exists and is in ω; in particular, hereditary, directed subsets are contained in the set PP^{-1}.

LEMMA 6.3. *The set of hereditary, directed subsets of G containing e is invariant under the partial action θ on X_G.*

Proof. Suppose $\omega \in X_G$ is hereditary and directed and let $z^{-1} \in \omega$. In order to see that $z\omega$ is hereditary, suppose $zx \in z\omega$ with $x \in \omega$ and let $t \in P$. Then $xt^{-1} \in \omega$ and $zxt^{-1} \in z\omega$.

Next we show that $z\omega$ is directed. Assume zx and zy are elements of $z\omega$. Since ω is directed and contains x, y, and z^{-1}, it follows that $(x \vee y \vee z^{-1}) \in P \cap \omega$. It is easy to see using the definition that $z(x \vee y \vee z^{-1}) \in P \cap z\omega$ is a common upper bound for zx and zy. Thus $zx \vee zy \leqslant z(x \vee y \vee z^{-1})$ and, since $z\omega$ is hereditary, $zx \vee zy \in z\omega$. ∎

Theorem 6.4. *The spectrum $\Omega_{\mathcal{N}}$ of the relations $(\mathcal{N})$ is the set of hereditary, directed subsets of G which contain the identity element.*

The crossed product $C(\Omega_{\mathcal{N}}) \rtimes_\alpha G$ is canonically isomorphic to the universal C^*-algebra $C^*(G,P)$ for covariant isometric representations of P.

Proof. Let H be the set of hereditary, directed subsets of G containing the identity element. Then clearly $H \subset X_G$.

First we show that every $\omega \in \Omega_{\mathcal{N}}$ is hereditary and directed; that $e \in \omega$ is obvious because $\Omega_{\mathcal{N}} \subset X_G$. If $x \in \omega$ then ω is in the domain of the partial homeomorphism $\theta_{x^{-1}}$, and since $\Omega_{\mathcal{N}}$ is invariant we have $x^{-1}\omega = \theta_{x^{-1}}(\omega) \in \Omega_{\mathcal{N}}$. By the relation $(\mathcal{N}_1)$, for $t \in P$ we obtain $[t]^*[t](x^{-1}\omega) = 1$, which means that $t^{-1} \in x^{-1}\omega$. Since $xt^{-1} \in \omega$ for every $t \in P$ and every $x \in \omega$, ω is hereditary.

If x and y are elements of ω, then $1 = [x][x]^*[y][y]^*(\omega) = [x \vee y][x \vee y]^*(\omega)$ by $(\mathcal{N}_2)$. Thus $x \vee y \in \omega$ and ω is directed.

Conversely, by the preceding lemma, if $\omega \in X_G$ is hereditary and directed, and if $z^{-1} \in \omega$, then $z\omega$ is also hereditary and directed, so it suffices to show that the relations $(\mathcal{N})$ hold at every hereditary, directed $\omega \in X_G$.

It is trivial to verify $(\mathcal{N}_1)$, since $e \in \omega$ implies $et^{-1} \in \omega$ for every $t \in P$ by hereditarity of ω. For $(\mathcal{N}_2)$ we need to show that $[x][x]^*[y][y]^*(\omega) = [x \vee y][x \vee y]^*(\omega)$, or, equivalently, that x and y are in ω if and only if $x \vee y \in \omega$. The "only if" holds because ω is directed, and the "if" holds because it is hereditary, since $(x \vee y)^{-1}x \in P$ and $x = (x \vee y)\,(x \vee y)^{-1}x$.

The crossed product is isomorphic to $C^*(G,P)$ because of Proposition 6.1. ∎

Remark 6.5. Hereditary directed subsets of the semigroup P were introduced by Nica in [16], Section 6.2, where he showed that the spectrum of the diagonal subalgebra in the Toeplitz algebra is (homeomorphic to) the space of hereditary, directed, and nonempty subsets of P. The homeomorphism of our spectrum $\Omega_{\mathcal{N}}$ to the space considered by Nica is obtained simply by sending an element ω of $\Omega_{\mathcal{N}}$ to its intersection with P.

Proposition 6.6. *The canonical partial action θ on $\Omega_{\mathcal{N}}$ is topologically free.*

Proof. For each $t \in P$ the set $tP^{-1} = \{x \in G : x \leqslant t\}$ is hereditary and directed; moreover, $t \neq t'$ implies $tP^{-1} \neq t'P^{-1}$. This gives a copy of P inside $\Omega_{\mathcal{N}}$ which is in fact dense ([16], Section 6.2).

Suppose $x \in G$; it is easy to see that the point tP^{-1} is in the domain of the partial homeomorphism θ_x if and only if $xt \in P$. In this case $\theta_x(tP^{-1}) = (xt)P^{-1} \neq tP^{-1}$. Since no point in this dense subset is fixed by θ_x for $x \neq e$, the proof is finished. ∎

As an application we obtain a characterization of faithful representations of the reduced crossed product which is more general than [14], Theorem 3.7 because it does away with the amenability hypothesis by focusing on the reduced crossed product. From this point of view, it becomes apparent that the faithfulness theorem for representations is really a theorem about *reduced* crossed products and that it is a manifestation of topological freeness.

THEOREM 6.7. *Suppose (G,P) is a quasi-lattice ordered group. A representation of the reduced crossed product $C(\Omega_{\mathcal{N}})\rtimes_{\alpha,r}G$ is faithful if and only if it is faithful on the diagonal $C(\Omega_{\mathcal{N}})$.*

Proof. Since α is topologically free, the result follows from Theorem 2.6. ∎

Of course we may use [14], Proposition 2.3 (3) to decide when the restriction to $C(\Omega_{\mathcal{N}})$ of a representation $\pi \times v$ of $C(\Omega_{\mathcal{N}})\rtimes_{\alpha,r}G$ is faithful in terms of the generating partial isometries: the condition is that

$$\pi\bigg(\prod_{t\in F}(1-u_tu_t^*)\bigg) \neq 0$$

for every finite subset F of $P \setminus \{e\}$.

Since the diagonal algebra in Nica's Wiener-Hopf C^*-algebra $T(G,P)$ is a faithful copy of $C(\Omega_{\mathcal{N}})$ we can use the faithfulness theorem to express $T(G,P)$ as the reduced crossed product by a partial action.

COROLLARY 6.8. *If (G,P) is a quasi-lattice ordered group, then*

$$C(\Omega_{\mathcal{N}})\rtimes_{\alpha,r}G \cong \mathcal{T}(\mathcal{G},\mathcal{P}).$$

This isomorphism is essentially ([21], Theorem 6.6); although the partial action there is not given explicitly, it is not hard to see that it is the one above.

7. CUNTZ-KRIEGER RELATIONS: THE UNIVERSAL $\mathcal{O}_A$

Let $A=[a_{ij}]$ be a $\{0,1\}$-valued n by n matrix with no zero rows. A *Cuntz-Krieger A-family* is a collection of partial isometries $\{s_i\}_{i=1}^n$ such that

$$(\mathcal{CK}) \qquad \sum_j s_js_j^* = 1, \quad \text{and} \quad \sum_j a_{ij}s_js_j^* = s_i^*s_i \quad \text{for } i=1,2,\dots,n.$$

A Cuntz-Krieger A-family determines a semisaturated partial representation of the free group $\mathbb{F}_n$ on n generators ([6], Theorem 5.2). This partial representation satisfies the $\mathcal{CK}$ relations above. Conversely, any such partial representation comes from a Cuntz-Krieger A-family, and thus the C^*-algebra $\mathcal{O}_{\mathcal{A}}$ (universal for Cuntz-Krieger A-families) is universal for semisaturated partial representations of $\mathbb{F}_n$ satisfying the $\mathcal{CK}$ relations.

The machinery developed in Section 4 then gives a natural isomorphism of $\mathcal{O}_{\mathcal{A}}$ to a crossed product of the form $C(\Omega_{\mathcal{CK}})\rtimes_\alpha \mathbb{F}_n$. The spectrum $\Omega_{\mathcal{CK}}$ given by Proposition 4.1 is covariantly homeomorphic to the infinite path space associated to A on which the action of a generator is defined by multiplication on the left; this action is partial because one needs to require that the resulting path be admissible.

One can then verify that Cuntz and Krieger's condition (I) amounts to topological freeness of the partial action of $\mathbb{F}_n$ on infinite path space, so that their uniqueness theorem ([4], Theorem 2.13) can be obtained from our Theorem 2.6 and the isomorphism of reduced and full crossed products provided by [8], Theorem 6.6.

It is also not hard to see that if A is irreducible, and not a permutation matrix, then the partial action on P_A^∞ is minimal and topologically free, so the simplicity result for $\mathcal{O}_{\mathcal{A}}$ ([4], Theorem 2.14) follows from our Corollary 2.9.

In addition to the original direct approach of [4] and [3], Cuntz-Krieger algebras have since been realized and studied using various other methods. We refer to [12] and [13] for a groupoid approach, to [17] for a bimodule algebra realization, and to [21] for considerations involving reduced partial crossed products.

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REFERENCES

1. R.J. Archbold, J. Spielberg, Topologically free actions and ideals in discrete dynamical systems, *Proc. Edinburgh Math. Soc. (2)* **37**(1993), 119–124.
2. J. Cuntz, Simple C^*-algebras generated by isometries, *Comm. Math. Phys.* **57** (1977), 173–185.
3. J. Cuntz, A class of C^*-algebras and topological Markov chains. II: reducible chains and the Ext-functor for C^*-algebras, *Invent. Math.* **63**(1981), 25–40.
4. J. Cuntz, W. Krieger, A class of C^*-algebras and topological Markov chains, *Invent. Math.* **56**(1980), 251–268.
5. R. Exel, Circle actions on C^*-algebras, partial automorphisms and a generalized Pimsner–Voiculescu exact sequence, *J. Funct. Anal.* **122**(1994), 361–401.
6. R. Exel, Amenability for Fell bundles, *J. Reine Angew. Math.* **492**(1997), 41-73.
7. R. Exel, Twisted partial actions: A classification of regular C^*-algebraic bundles, *Proc. London Math. Soc.* **74**(1997), 417–443.
8. R. Exel, Partial actions of groups and actions of inverse semigroups, *Proc. Amer. Math. Soc.* **126**(1998), 3481–3494.
9. R. Exel, M. Laca, Cuntz-Krieger algebras for infinite matrices, *J. Reine Angew. Math.* **512**(1999), 119–172.
10. P. Green, C^*-algebras of transformation groups with smooth orbit spaces, *Pacific J. Math.* **72**(1977), 71–97.
11. A. an Huef, I. Raeburn, The ideal structure of Cuntz–Krieger algebras, *Ergodic Theory Dynam. Systems* **17**(1997), 611–624.
12. A. Kumjian, D. Pask, I. Raeburn, Cuntz-Krieger algebras of directed graphs, *Pacific J. Math.* **184**(1998), 161–174.
13. A. Kumjian, D. Pask, I. Raeburn, J. Renault, Graphs, groupoids and Cuntz Krieger algebras, *J. Funct. Anal.* **144**(1997), 505–541.
14. M. Laca, I. Raeburn, Semigroup crossed products and Toeplitz algebras of nonabelian groups, *J. Funct. Anal.* **139**(1996), 415–440.
15. K. McClanahan, K-theory for partial crossed products by discrete groups, *J. Funct. Anal.* **130**(1995), 77–117.
16. A. Nica, C^*-algebras generated by isometries and Wiener-Hopf operators, *J. Operator Theory* **27**(1992), 17–52.

17. M.V. PIMSNER, A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$, in *Free Probability Theory*, Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 189–212.
18. J.C. QUIGG, Full and reduced C^*-coactions, *Math. Proc. Cambridge Philos. Soc.* **116**(1994), 435–450.
19. J.C. QUIGG, Discrete C^*-coactions and C^*-algebraic bundles, *J. Austral. Math. Soc. Ser. A* **60**(1996), 204–221.
20. J.C. QUIGG, Crossed product duality for partial C^*-automorphisms, *Rocky Mountain J. Math.* **28**(1998), 1067–1088.
21. J.C. QUIGG, I. RAEBURN, Characterisations of crossed products by partial actions, *J. Operator Theory* **37**(1997), 311–340.

RUY EXEL
Departamento de Matemática
Universidade Federal de Santa Catarina
88010–970 Florianópolis SC
BRASIL
E-mail: exel@mtm.ufsc.br

MARCELO LACA
Department of Mathematics
University of Newcastle
NSW 2308
AUSTRALIA
Current address:
Institute of Mathematics
University of Muenster
Einsteinstr, 63
D-48149 Münster
GERMANY
E-mail: laca@math.uni-muenster.de

JOHN QUIGG
Department of Mathematics
Arizona State University
Tempe, Arizona 85287
USA
E-mail: quigg@math.la.asu.edu

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