TYPE III ACTIONS ON BOUNDARIES OF $\widetilde{A}_n$ BUILDINGS

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ABSTRACT. Let Γ be a group of type rotating automorphisms of a building $\mathfrak{X}$ of type $\widetilde{A}_n$ and order q. Suppose that Γ acts freely and transitively on the vertex set of $\mathfrak{X}$. Then the action of Γ on the boundary of $\mathfrak{X}$ is ergodic, of type $\mathrm{III}_{1/q}$ or type III_{1/q^2} depending on whether n is odd or even.

KEYWORDS: *Affine building, boundary action, type III factor.*

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0. INTRODUCTION

Let M be a compact Riemannian manifold of negative sectional curvature, and let $\Gamma = \pi_1(M)$. Then Γ acts on the sphere at infinity S of the universal cover $\widetilde{M}$ of M. The main result of [14] is that the action of Γ on S is ergodic, amenable and type III_1. This applies in particular to a cocompact Fuchsian group in $G = \mathrm{PGL}(2, \mathbb{R})$ acting on the circle.

A discrete analogue of this result was proved in [11]. Namely, let Γ be a free group acting simply transitively on the vertices of a locally finite homogeneous tree $\mathcal{T}$ of degree $q + 1$. Then $\mathcal{T}$ is the universal covering space of a graph with fundamental group Γ. It was shown in [11] that the action of Γ on the boundary of the tree is ergodic, amenable and of type $\mathrm{III}_{1/q}$.

Turning to higher rank spaces of nonpositive curvature, it is known that if Γ is a lattice in $G = \mathrm{PGL}(n+1, \mathbb{R})$ with $n \geqslant 1$, and if $\Omega = G/B$ where B is the Borel subgroup of upper triangular matrices in G, then the action of Γ on Ω is ergodic of type III_1. Here Ω is the maximal boundary of Furstenberg ([10], VI.1.7). A similar result holds more generally for a lattice Γ in any semisimple noncompact Lie group G ([18], 4.3.15).

The discrete analogue of this construction is obtained by replacing $\mathbb{R}$ by a nonarchimedean local field $\mathbb{F}$ with residue field of order q. The affine Bruhat-Tits building $\mathfrak{X}$ of $G = \mathrm{PGL}(n + 1, \mathbb{F})$ is a building of type $\widetilde{A}_n$ [15]. The vertex set of $\mathfrak{X}$ may be identified with the homogeneous space G/K, where K is a maximal

compact subgroup of G, and G acts on the boundary $\Omega = G/B$, where B is a Borel subgroup of G.

The precise higher rank analogue of the setup in [11] is as follows. Let Γ be a group of type rotating automorphisms of a building $\mathfrak{X}$ of type $\widetilde{A}_n$, and suppose that Γ acts simply transitively on the vertices of $\mathfrak{X}$. In view of the fact that $\widetilde{A}_1$ buildings are trees, such groups Γ should be regarded as higher rank analogues of free groups. Note however that not every $\widetilde{A}_2$ building $\mathfrak{X}$ is the Bruhat-Tits building of $\mathrm{PGL}\,(3, \mathbb{K})$ where $\mathbb{K}$ is a local field ([6], II Section 8). Geometrically, an $\widetilde{A}_n$ building $\mathfrak{X}$ is an n-dimensional contractible simplicial complex in which each codimension one simplex lies on $q+1$ maximal simplices (*chambers*). If $n \geqslant 2$ then the number q is necessarily a prime power and is referred to as the *order* of the building. The boundary Ω of $\mathfrak{X}$ is a totally disconnected compact Hausdorff space and is endowed with a natural family of mutually absolutely continuous Borel probability measures. In [12] it was proved that if $n = 2$ and $q \geqslant 3$, then the action of Γ on Ω is ergodic and of type III_{1/q^2}. The purpose of the present article is to remove both these hypotheses and prove the following general result.

0.1. THEOREM. *Let $n \geqslant 2$ and let $\mathfrak{X}$ be a locally finite thick $\widetilde{A}_n$ building of order q. Let Γ be a group of type rotating automorphisms of $\mathfrak{X}$ which acts simply and transitively on the vertices of $\mathfrak{X}$. Then the action of Γ on the boundary Ω of $\mathfrak{X}$ is amenable, ergodic and of type III_λ, where*

$$\lambda = \begin{cases} \frac{1}{q} & \text{if } n \text{ is odd,} \\ \frac{1}{q^2} & \text{if } n \text{ is even.} \end{cases}$$

The proof of this result will be completed in Section 3. In Section 4 we deal with freeness of the action, which is required in order to prove that the associated von Neumann algebra is a factor. In particular, Section 4 removes a gap in the proof of freeness in [12]. We therefore obtain the following consequence.

0.2. COROLLARY. *Let Γ and Ω be as above. Then the crossed product von Neumann algebra $L^\infty(\Omega) \rtimes \Gamma$ is the AFD factor of type III_λ, where $\lambda = 1/q$ if n is odd, and $\lambda = 1/q^2$ if n is even.*

A simple variation on the arguments leading to Theorem 0.1 proves the following result: see Subsection 3.2.

0.3. THEOREM. *Let $p \geqslant 2$ be a prime number, let $n \geqslant 1$, and let Ω be the boundary of the affine building of $\mathrm{PGL}(n+1, \mathbb{Q}_p)$. That is $\Omega = \mathrm{PGL}(n+1, \mathbb{Q}_p)/B$, where B is the Borel subgroup of upper triangular matrices. Then the action of $\mathrm{PGL}(n+1, \mathbb{Q})$ on Ω is ergodic and of type III_λ, where*

$$\lambda = \begin{cases} \frac{1}{p} & \text{if } n \text{ is odd,} \\ \frac{1}{p^2} & \text{if } n \text{ is even.} \end{cases}$$

Similar results can be stated for linear groups over other local fields, but this is perhaps the most striking case. Note that, in contrast to Theorem 0.1, $\mathrm{PGL}(n+1, \mathbb{Q})$ is not a lattice in $\mathrm{PGL}(n+1, \mathbb{Q}_p)$, and its action on the boundary is not amenable.

Given an $\widetilde{A}_n$ building $\mathfrak{X}$, there is a type map τ defined on the vertices of $\mathfrak{X}$ such that $\tau(v) \in \mathbb{Z}/(n+1)\mathbb{Z}$ for each vertex $v \in \mathfrak{X}$. Every chamber of $\mathfrak{X}$ has precisely one vertex of each type. An automorphism α of $\mathfrak{X}$ is said to be *type-rotating* if there exists $i \in \{0, 1, \ldots, n\}$ such that $\tau(\alpha v) = \tau(v) + i$ for all vertices $v \in \mathfrak{X}$. An $\widetilde{A}_1$ building is a tree with two types of vertices, and every automorphism of the tree is type rotating. We shall refer to a group Γ satisfying the hypotheses of Theorem 0.1 as an $\widetilde{A}_n$ group. In [2] it was shown that there is a 1-1 correspondence between $\widetilde{A}_n$ groups and "triangle presentations". The right Cayley graph of an $\widetilde{A}_n$ group Γ relative to a natural set of generators is the 1-skeleton of the $\widetilde{A}_n$ building $\mathfrak{X}$. We shall frequently refer to [4], which lays much of the groundwork for dealing with the higher rank $\widetilde{A}_n$ buildings.

Throughout the paper $\mathfrak{X}$ will denote a thick, locally finite $\widetilde{A}_n$ building, and the vertices of the building will be denoted by $\mathfrak{X}^0$. If $\mathfrak{X}$ is associated with the $\widetilde{A}_n$ group Γ then the underlying set of the group Γ will be identified with $\mathfrak{X}^0$, and the action of Γ on the building will be by left multiplication. The identity of Γ will be denoted by 1 throughout. For $x, y \in \mathfrak{X}^0$, $d(x, y)$ will denote the graph distance between those vertices in the 1-skeleton of $\mathfrak{X}$, and $|x| = d(x, 1)$.

Further information on buildings can be found in [3], [15], [1] and [13]. The first two of these references are introductory, while the last two provide a fuller account of the theory of buildings.

1. PRELIMINARIES

This section mainly recalls material from [4], to which we refer for a more complete discussion. An $\widetilde{A}_n$ building is a union of *apartments*. An apartment is isomorphic to a Coxeter complex of type $\widetilde{A}_n$. Let Σ denote the Coxeter complex of type $\widetilde{A}_n$. The n-simplices of Σ are referred to as *chambers* and can be regarded as forming a tessellation of $\mathbb{R}^n$. The vertex set of Σ can be identified with $\mathbb{Z}^{n+1}/\mathbb{Z}(1, 1, \ldots, 1)$. Two vertices $[a], [b] \in \Sigma$, $[a] = a + \mathbb{Z}(1, 1, \ldots, 1)$ and $[b] = b + \mathbb{Z}(1, 1, \ldots, 1)$, are adjacent if there exist representative vectors $(a_1, a_2, \ldots, a_{n+1}) \in [a]$ and $(b_1, b_2, \ldots, b_{n+1}) \in [b]$ such that $a_i \leqslant b_i \leqslant a_i + 1$ for all $1 \leqslant i \leqslant n$. The type $\tau(x) \in \mathbb{Z}/(n+1)\mathbb{Z}$ of a vertex $[x] = [(x_1, x_2, \ldots, x_{n+1})] \in \Sigma$ is given by

$$\tau(x) = \left(\sum_i x_i \right) \pmod{n+1}.$$

Each chamber of Σ has precisely one vertex of each type.

Let $\mathbf{b}_i = (0, \ldots, 0, 1, \ldots, 1)$, where precisely i entries equal 1. Note that each $x \in \mathbb{Z}^{n+1}$ can be written as

$$x = x_1(1, 1, \ldots, 1) + \sum(x_{i+1} - x_i)\mathbf{b}_i.$$

Hence, there is a mapping $\mathbb{Z}^{n+1}/\mathbb{Z}(1, 1, \ldots, 1) \to \mathbb{Z}^n$ defined by

$$[x] \mapsto (x_2 - x_1, x_3 - x_2, \ldots, x_{n+1} - x_n).$$

This mapping is a canonical group homomorphism between $\mathbb{Z}^{n+1}/\mathbb{Z}(1, 1, \ldots, 1)$ and $\mathbb{Z}^n$, and by means of it the vertices of Σ can be coordinatized by $\mathbb{Z}^n$. Throughout this paper, if $k \in \mathbb{Z}^n$, then k_i denotes the ith entry of k.

1.1. THE BOUNDARY OF AN $\widetilde{A}_n$ BUILDING. Given an $\widetilde{A}_n$ building $\mathfrak{X}$, one can define the boundary of $\mathfrak{X}$ by means of equivalence classes of sectors. The central concern of this paper is the boundary regarded as a measure space. For a discussion of the geometric structure of the boundary, the reader is referred to Chapters 9 and 10 of [13].

Let $\mathcal{S}_0$ be the simplicial cone in the $\widetilde{A}_n$ Coxeter complex Σ with vertex set coordinatized by $\mathbb{Z}_+^n$. A subcomplex S of $\mathfrak{X}$ is called a *sector* if there is an apartment A containing S and a type-rotating isomorphism $\phi : A \to \Sigma$ such that $\phi(S) \mapsto \mathcal{S}_0$. (Recall that the isometry ϕ is said to be *type rotating* if there exists $j \in \mathbb{Z}/(n+1)\mathbb{Z}$ such that, for each vertex v of S, $\tau(\phi(v)) = \tau(v) + j \pmod{n+1}$. Note that if a, b are vertices in a sector S of $\mathfrak{X}$, and $\phi : S \to \mathcal{S}_0$ is a type preserving isomorphism such that $\phi(a) = (0, 0, \ldots, 0)$ and $\phi(b) = (k_1, k_2, \ldots, k_n)$, where $k_i \in \mathbb{Z}_+$, then the k_i do not depend on the particular apartment A containing S ([4], Lemma 2.3). Thus, for $x \in \mathfrak{X}^0$ and $k \in \mathbb{Z}_+^n$, one can define a set $S_k(x)$ consisting of those elements $y \in \mathfrak{X}^0$ such that there exists a sector S containing x and y, and a type rotating isomorphism $\phi : S \to \mathcal{S}_0$ such that $\phi(x) = (0, 0, \ldots, 0)$ and $\phi(y) = k$. Given a sector S and a type rotating isomorphism with $\phi(S) = \mathcal{S}_0$, the basepoint of S is $v = \phi^{-1}(0, 0, \ldots, 0)$. If $x, y \in \mathfrak{X}$ and $y \in S_k(x)$, with $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n$, then the graph distance between x and y is given by $d(x, y) = \sum_i k_i$.

Two sectors S_1, S_2 are said to be *equivalent* if $S_1 \cap S_2$ contains a subsector. Let Ω be the set of all such equivalence classes of sectors. Then Ω is called the *boundary* of $\mathfrak{X}$. Given $\omega \in \Omega$ and $x \in \mathfrak{X}^0$, there exists an unique sector with basepoint x which is contained in the equivalence class ω ([13], Lemma 9.7). Denote this sector by $[x, \omega)$. Also, for $m \in \mathbb{Z}_+^n$, let $s_m^x(\omega)$ be the unique element in the intersection $S_m(x) \cap [x, \omega)$. What follows is based on [4].

1.1.1. LEMMA. *Let* $\omega \in \Omega$, *and let* $x, y \in \mathfrak{X}^0$. *Then there exists* $m(x, y; \omega) \in \mathbb{Z}^n$ *such that*

$$s_k^x(\omega) = s_{k'}^y(\omega) \quad \text{where} \quad k' = k + m(x, y; \omega),$$

for all $k \in \mathbb{Z}_+$ *such that* $k_i + m_i(x, y; \omega) \geqslant 0$ *for* $1 \leqslant i \leqslant n$.

Proof. (c.f. [5], Lemma 2.1.) Since $[x, \omega)$ is in the same equivalence class as $[y, \omega)$, $[x, \omega) \cap [y, \omega)$ contains a subsector. Choose

$$u = s_k^x(\omega) = s_{k'}^y(\omega) \in [x, \omega) \cap [y, \omega).$$

Let $T = \{s_{k+l}^x(\omega) : l \in \mathbb{Z}_+^n\}$, and $T' = \{s_{k'+l}^y(\omega) : l \in \mathbb{Z}_+^n\}$. Then T, T' are sectors in the equivalence class ω with a common base point, and so by Lemma 9.7 in [13], $T = T'$. It follows that $s_{k+l}^x(\omega), s_{k'+l}^y(\omega)$ are both in $S_l^u(\omega) \cap T$, and hence are equal. Thus $m(x, y; \omega) = k' - k$, and $m(x, y; \omega)$ is clearly independent of the choice of $u \in [x, \omega) \cap [y, \omega)$. ∎

1.1.2. LEMMA. *Let x be a vertex of $\mathfrak{X}$, and let C be a chamber containing x. Then for $\omega_0 \in \Omega$, there exists an apartment A which contains C and the sector $S = [x, \omega_0)$.*

Proof. By Lemma 9.4 from [13], given the chamber C and sector $[x, \omega_0)$, there exists an apartment A containing a subsector $S' \subset [x, \omega_0)$ and the chamber C. Note that as $x \in C$, one has $x \in A$.

Choose a sector S'' in A with base vertex x and parallel to S'. Then S'' is equivalent to $[x, \omega_0)$ and so $S'' = [x, \omega_0)$, by uniqueness of the sector with base vertex x representing the boundary point ω_0. $\blacksquare$

The next lemma is a generalisation of Corollary 2.3 from [5].

1.1.3. PROPOSITION. *For $x, y \in \mathfrak{X}^0$, and $\omega \in \Omega$, one has*

$$s_k^x(\omega) \in [x, \omega) \cap [y, \omega)$$

if $k_i \geqslant d(x, y)$ for $1 \leqslant i \leqslant n$.

Proof. Set $r = d(x, y)$, and let $k = (r, r, \ldots, r)$. An easy consequence of Lemma 1.1.1 is that $z \in [x, \omega)$ implies $[z, \omega) \subset [x, \omega)$, and so it is sufficient to show that $s_k^x(\omega) \in [x, \omega) \cap [y, \omega)$.

To proceed inductively, the case $d(x, y) = 1$ is established first. By Lemma 1.1.2, there exists an apartment A containing both y and $S = [x, \omega)$. As S is a sector, there exists a type rotating isomorphism $\varphi : A \to \Sigma$ such that $\varphi(S) = \mathcal{S}_0$ with $\varphi(x) = 0$.

Since x, y are adjacent in A, $\varphi(y) = (y_1, y_2, \ldots, y_n)$, where $y_i \in \{-1, 1, 0\}$, $0 \leqslant i \leqslant n$. Next, define the type rotating isomorphism $\phi : A \to \Sigma$ by

$$\phi(z) = \varphi(z) - \varphi(y).$$

This map takes y to the origin in Σ, and $(\phi)^{-1}(\mathcal{S}_0)$ is a sector. Moreover, for $z \in s_k^x(\omega)$, $\phi(z) = \varphi(z) - \varphi(y) = ((k_1 - y_1), (k_2 - y_2), \ldots, (k_n - y_n))$. Thus $\phi(z) \in \mathcal{S}_0$ if and only if $k_i \geqslant y_i$ for all $1 \leqslant i \leqslant n$.

It follows that $\phi^{-1}(\mathcal{S}_0) = [y, \omega)$. Moreover, as $(1, 1, \ldots, 1) \geqslant (y_1, \ldots, y_n)$, one has that $s_{(1,1,\ldots,1)}^x(\omega) \in [x, \omega) \cap [y, \omega)$. This proves the case for $d(x, y) = 1$.

In general, given $s \in \mathbb{Z}_+, s > 1$, suppose that the statement of the lemma is true for all y' such that $d(x, y') \leqslant s - 1$, and let $y \in S_k(x)$ with $d(x, y) = s$. Without loss of generality, suppose that $k_1 \geqslant 1$ and set $k' = (k_1 - 1, k_2, \ldots, k_n)$. Let z be the unique element in $\text{conv}(x, y) \cap S_{k'}(x)$ and note that $d(z, y) = 1$ and $d(z, x) = d(y, x) - 1 = s - 1$. Hence by the inductive hypothesis

$$a = s_{(s-1,s-1,\ldots,s-1)}^x(\omega) \in [x, \omega) \cap [z, \omega).$$

Then for some $t = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_+^n$, one has $a = s_t^z(\omega)$ and $m_i(x, z; \omega) = (t_i - (s - 1))$. As $t_i \in \mathbb{Z}_+$, it follows that $t_i + 1 \geqslant d(y, z) = 1$. By the inductive hypothesis, this implies that

$$s_{(t_1+1,\ldots,t_n+1)}^z(\omega) \in [z, \omega) \cap [y, \omega).$$

Writing $t' = t - m(x, z; \omega)$,

$$s_{(t_1+1,t_2+1,\ldots,t_n+1)}^z(\omega) = s_{(t'_1+1,t'_2+1,\ldots,t'_n+1)}^x = s_{(s,\ldots,s)}^x(\omega) \in [x, \omega) \cap [y, \omega).$$

The result follows. $\blacksquare$

1.1.4. DEFINITION. Given $y \in \mathfrak{X}^0$, the topology on Ω is given by the basis of open sets $\{\Omega_y^x\}_{x \in \mathfrak{X}^0}$, where

$$\Omega_y^x = \{\omega \in \Omega : x \in [y, \omega]\}.$$

The topology so defined is independent of the choice of y. See below for details. Note that for $y \in \mathfrak{X}^0$, and $k \in \mathbb{Z}_+^n$, the boundary Ω can be expressed as the disjoint union

$$\Omega = \bigcup_{x \in S_k(y)} \Omega_y^x.$$

There is a natural class of Borel measures on Ω. Namely, for a fixed $y \in \mathfrak{X}^0$ and a basic open set Ω_y^x with $x \in S_k(y)$, let

$$\nu_y(\Omega_y^x) = \frac{1}{|S_k(y)|}.$$

$|S_k| = |S_k(y)|$ is independent of y and its actual value was determined in Corollary 2.7 of [4]. Specifically, let q be the order of the $\widetilde{A}_n$ building. Also, for $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n$, index the non-zero entries of k by $\{i : k_i \geqslant 1\} = \{j_1, \ldots, j_t\}$, and set $j_0 = 0$ and $j_{t+1} = n + 1$. Then

$$(1.1) \qquad |S_k| = q^{-\sum_{v=1}^{t} j_v(j_{v+1} - j_v)} \begin{bmatrix} n+1 \\ j_1 - j_0, \ldots, j_{t+1} - j_t \end{bmatrix}_q \cdot q^{\sum_{i=1}^{n} i(n+1-i)k_i},$$

where $[\cdots]_q = [n+1]_q/([j_1 - j_0]_q \cdots [j_{t+1} - j_t]_q)$, and $[k]_q = (q^k - 1) \cdots (q - 1)$.

Unlike the topology on Ω, the value of the measure ν_y is dependent on the choice of $y \in \mathfrak{X}^0$. However, as shown by the following lemmas, the set of measures $\{\nu_y\}_{y \in \mathfrak{X}^0}$ is absolutely continuous.

The next lemma is generalized from Lemma 2.4 of [5].

1.1.5. LEMMA. *Let* $y \in S_k(x)$. *Suppose that* $z \in S_l(x) \cap S_{l'}(y)$, *where* $l_i \geqslant d(x, y)$ *for all* i. *Then* $\Omega_x^z \subset \Omega_y^z$. *Moreover, if* $m(x, y; \omega) = (m_1, \ldots, m_n)$, *as in Lemma* 1.1.1, *then* $m_i(x, y; \omega) = l'_i - l_i$ *for all* $\omega \in \Omega_x^z$.

Proof. Let $\omega \in \Omega_x^z$. Then $z = s_l^x(\omega)$, and so z is an element of $[x, \omega] \cap [y, \omega]$ by Lemma 1.1.3 and choice of $l \in \mathbb{Z}_+^n$. Thus $\omega \in \Omega_y^z$. Moreover, by the proof of Lemma 1.1.1, $m_i(x, y; \omega) = l'_i - l_i$. ∎

1.1.6. LEMMA. *The topology on* Ω *does not depend on the vertex* $y \in \mathfrak{X}^0$ *chosen in Definition* 1.1.4. *For any* $x, y \in \mathfrak{X}^0$, *the measures* ν_x, ν_y *are mutually absolutely continuous, and the Radon Nikodym derivative of* ν_y *with respect to* ν_x *is given by*

$$(1.2) \qquad \frac{d\nu_y}{d\nu_x}(\omega) = q^{-\sum_{i=1}^{n} i(n+1-i)m_i}$$

for $\omega \in \Omega$, *where* $m_i = m_i(x, y; \omega)$.

Proof. Let $x, y \in \mathfrak{X}^0$. In view of the preceding results, the proof that topology is independent of the base vertex y proceeds exactly as in the case $n = 2$ ([5], Lemma 2.5).

Now fix $\omega \in \Omega$. Choose $k \in \mathbb{Z}_+^n$ such that $k_i \geqslant d(x,y)$ and $k_i + m_i(x,y;\omega) \geqslant d(x,y)$. Set $z = s_k^x(\omega) = s_{k'}^y(\omega)$, where $k' = k + m(x,y;\omega)$. Lemma 1.1.5 implies that $\Omega_x^z = \Omega_y^z$. Moreover it follows from (1.1) that

$$\nu_y(\Omega_x^z) = (|S_{k'}|)^{-1} = \left(q^{\sum_{i=1}^{n} i(n+1-i)m_i} |S_k| \right)^{-1} = q^{-\sum_{i=1}^{n} i(n+1-i)m_i} \nu_x(\Omega_x^z).$$

Since $\{\Omega_x^z : z \in [x,\omega),\, d(x,z) \geqslant d(x,y)\}$ is a basic family of neighbourhoods of ω, the result follows. ∎

1.1.7. REMARK. Equation (1.2) is precisely ([4], Equation (1.6)) and its proof is outlined in Section 4 of [4].

1.2. $\widetilde{A}_n$ GROUPS. Let Π be a finite projective geometry of dimension n and order q. If $n > 2$ then Π is the Desarguesian projective geometry $\Pi(V)$, where V is a vector space of dimension $n+1$ over a finite field of order q. Let $\dim(u)$ denote the dimension of the subspace u of V. In the Desarguesian case the points and lines of Π are the one- and two-dimensional subspaces of V respectively. We shall extend this notation to the non Desarguesian case, so that an element u of a projective plane Π satisfies $\dim u = 1$ if it is a point and $\dim u = 2$ if it is a line. Let λ be an involution of Π such that $\dim(\lambda(u)) = n + 1 - \dim(u) \pmod{n+1}$. An $\widetilde{A}_n$ triangle presentation T compatible with λ is defined as follows. Let T be a set of triples $\{(u,v,w) : u,v,w \in \Pi\}$ which satisfy the following properties:

(1) Given $u, v \in \Pi$, then $(u,v,w) \in T$ for some $w \in \Pi$ if and only if $\lambda(u)$ and v are distinct and incident;

(2) If $(u,v,w) \in T$, then $(v,w,u) \in T$;

(3) If $(u,v,w_1) \in T$ and $(u,v,w_2) \in T$, then $w_1 = w_2$;

(4) If $(u,v,w) \in T$, then $(\lambda(w),\lambda(v),\lambda(u)) \in T$;

(5) If $(u,v,w) \in T$, then $\dim(u) + \dim(v) + \dim(w) \equiv 0 \pmod{n+1}$.

The group associated with this triangle presentation is given by

$$\Gamma_T = \left\langle \{a_v\}_{v \in \Pi(x)} \,\middle|\, \begin{array}{ll} (1)\, a_{\lambda(v)} = a_v^{-1} & \text{for all } v \in \Pi \\ (2)\, a_u a_v a_w = 1 & \text{for all } (u,v,w) \in T \end{array} \right\rangle.$$

The Cayley graph of Γ_T, with respect to the generators $\{a_u\}_{u \in \Pi}$ is the 1-skeleton of an $\widetilde{A}_n$ building $\mathfrak{X}$, and Γ_T acts simply transitivly on the vertices of the building in a type rotating manner. Conversely any group Γ acting on an $\widetilde{A}_n$ building in this way arises as $\Gamma = \Gamma_T$ for some triangle presentation T ([2], pp. 45–46). Unless otherwise specified, a generator a_u of Γ will be identified with the corresponding element $u \in \Pi$.

1.1.8. REMARK. The type rotating hypothesis in the definition of an $\widetilde{A}_n$ group has been removed and the appropriate notion of triangle presentation studied in the Ph.D. Thesis of T. Svenson ([17]), thereby generalising the results of [2].

For the rest of this article, the $\widetilde{A}_n$ group Γ will be assumed to act on $\mathfrak{X}$ by left translation with Γ being identified with the vertex set $\mathfrak{X}^0$. The identity element 1 of Γ is a preferred vertex of $\mathfrak{X}$ of type 0, and we write $S_k = S_k(1)$ for $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n$. The group Γ acts naturally on the boundary Ω.

If u_1, u_2 are elements of Π we denote by $u_1 \vee u_2$ their *join*; that is their least upper bound in the lattice of subspaces of Π. If $\Pi = \Pi(V)$ is Desarguesian then $u_1 \vee u_2 = \Pi$ means simply that $u_1 + u_2 = V$. On the other hand, if Π is a non Desarguesian plane and u_1 is a point and u_2 is a line of Π, then $u_1 \vee u_2 = \Pi$ means that u_1 and u_2 are not incident.

By Lemma 2.2 of [2], every word in Γ can be expressed uniquely in *normal form*

$$x = u_1 u_2 \cdots u_l,$$

where $\dim(u_i) \leqslant \dim(u_{i+1})$ and $u_i^{-1} \vee u_{i+1} = \Pi$. Moreover, $x \in S_k$, where $k_j = |\{u_i : \dim(u_i) = j\}|$.

Recall from the proof of Theorem 2.5 of [2] that if $x \in \mathfrak{X}^0$ then the projective geometry of neighbours of x is $\{xu : u \in \Pi\}$ and $\tau(xu) = \tau(x) + \dim u \pmod{n+1}$. Moreover, xu and xu' are adjacent vertices if and only if u and u' are incident in Π (that is, $u \subset u'$ or $u' \subset u$). In particular, a chamber of $\mathfrak{X}$ containing the vertex x has the form

$$\{x, xu_1, xu_2, \ldots, xu_n\}$$

where $\dim u_i = i$ and $u_1 \subset u_2 \subset \cdots \subset u_n$ is a complete flag in Π.

For more information on $\widetilde{A}_n$ groups, the reader is referred to [2].

2. AN ERGODIC MEASURE PRESERVING SUBGROUP OF THE FULL GROUP

The action of an $\widetilde{A}_n$ group Γ on the boundary Ω of the corresponding $\widetilde{A}_n$ building is measure-theoretically ergodic with respect to each of the measures ν_y, $y \in \mathfrak{X}$. For the classification of the action it will be necessary to show that the full group $[\Gamma]$ (defined below) contains a countable measure preserving subgroup $K_0 \subset [\Gamma]$ which acts ergodically on Ω.

The following two lemmas are straightforward generalisations of Lemma 4.6 from [12] and Lemma 4.7 from [12], respectively.

2.1. LEMMA. *Let K be a group which acts on Ω. If K acts transitively on the collection of sets $\{\Omega_1^x : x \in S_k\}$ for every $k \in \mathbb{Z}_+^n$, then K acts ergodically on Ω.*

Proof. Observe first that K preserves ν_1 since $\nu_1(\Omega_1^x)$ is independent of x. Suppose that $X_0 \subseteq \Omega$ is a Borel set which is invariant under K and such that $\nu_1(X_0) > 0$. It will be shown that $\nu_1(\Omega \setminus X_0) = 0$, thus establishing the ergodicity of the action.

Define a new measure μ by $\mu(X) = \nu_1(X \cap X_0)$ for each Borel set $X \subseteq \Omega$. Now, for each $g \in K$,

$$\mu(gX) = \nu_1(gX \cap X_0) = \nu_1(X \cap g^{-1}X_0) \leqslant \nu_1(X \cap X_0) + \nu_1(X \cap (g^{-1}X_0 \setminus X_0))$$

$$= \nu_1(X \cap X_0) = \mu(X).$$

Similarly, $\mu(gX) \leqslant \mu(g^{-1}gX) = \mu(X)$. Therefore μ is K-invariant.

For each $x, y \in S_k$ there exists a $g \in K$ such that $g\Omega_1^x = \Omega_1^y$ by transitivity. Thus $\mu(\Omega_1^x) = \mu(\Omega_1^y)$. Since Ω is the union of $|S_k|$ disjoint sets Ω_1^x, $y \in S_k$, each of which has equal measure, one has that

$$\mu(\Omega_1^x) = \frac{c}{|S_k|} \quad \text{for each } x \in S_k,$$

where $c = \mu(X_0) = \nu_1(X_0) > 0$. Thus $\mu(\Omega_1^x) = c\nu_1(\Omega_1^x)$ for every vertex $x \in \mathfrak{X}$.

Since the sets Ω_1^x generate the Borel σ-algebra, it follows that $\mu(X) = c\nu_1(X)$ for each Borel set X. Therefore

$$\nu_1(\Omega \setminus X_0) = c^{-1}\mu(\Omega \setminus X_0) = c^{-1}\nu_1((\Omega \setminus X_0) \cap X_0) = 0,$$

thus proving ergodicity. ∎

2.2. LEMMA. *Assume that $K \leqslant \operatorname{Aut}(\Omega)$ acts transitively on the collection of sets $\{\Omega_1^x : x \in S_m\}$ for every $m \in \mathbb{Z}_+^n$. Then there is a countable subgroup K_0 of K which also acts transitively on the collection of sets $\{\Omega_1^x : x \in S_m\}$ for every S_m, $m \in \mathbb{Z}_+^n$.*

Proof. For each pair $x, y \in S_m$, there exists an element $k \in K$ such that $k\Omega_1^y = \Omega_1^x$. Choose one such element $k \in K$ and label it $k_{x,y}$. Since S_m is finite, there are a finite number of elements $k_{x,y} \in K$ for each S_m. There are countably many sets S_m, so the set $\{k_{x,y} : x, y \in S_m, m \in \mathbb{Z}_+^n\}$ is countable. Hence the group

$$K_0 = \langle k_{x,y} \mid x, y \in S_m, m \in \mathbb{Z}_+^n \rangle \leqslant K$$

is countable and satisfies the required condition. ∎

2.3. DEFINITION. Given a group Γ acting on a measure space Ω, define the *full group*, $[\Gamma]$, of Γ by

$$[\Gamma] = \{T \in \operatorname{Aut}(\Omega) : T\omega \in \Gamma\omega \text{ for almost every } \omega \in \Omega\}.$$

The set $[\Gamma]_0$ of measure preserving maps in $[\Gamma]$ is then given by

$$[\Gamma]_0 = \{T \in [\Gamma] : \nu_y \circ T = \nu_y, \, y \in \mathfrak{X}^0\}.$$

It will be shown that there is a countable group K_0 of *measure-preserving* automorphisms of Ω such that:

(1) K_0 acts ergodically on Ω;
(2) $K_0 \leqslant [\Gamma]$.

By the lemmas above and the definition of $[\Gamma]$, it is enough to find for each $k \in \mathbb{Z}_+^n$ and $x, y \in S_k$, an automorphism $g \in \operatorname{Aut}(\Omega)$ such that $g(\Omega_1^x) = \Omega_1^y$ and $g\omega \in \Gamma\omega$ for almost all $\omega \in \Omega$.

Identify a simplex in $\mathfrak{X}$ with its vertex set, and recall from Section 1.2 that a chamber of $\mathfrak{X}$ containing the vertex x is of the form

$$\{x, xu_1, xu_2, \ldots, xu_n\}$$

where $\dim u_i = i$ and $u_1 \subset u_2 \subset \cdots \subset u_n$ is a complete flag in Π.

2.4. LEMMA. *Let $C = \{1, p_1, p_2, \ldots, p_n\}$ be a chamber in $\mathfrak{X}$ with base vertex the identity element 1 of Γ, where p_i are generators of Γ and $\dim p_i = i$. There are q chambers $C' = \{x, p_1 \ldots, p_n\}$ in $\mathfrak{X}$ meeting C in the face $\{p_1, \ldots, p_n\}$. The vertex x opposite 1 in $C \cup C'$ has the normal form $x = p_1 u_n$, where $\dim u_n = n$ and $p_1^{-1} \vee u_n = \Pi$. Thus $x \in S_{(1,0,\ldots,0,1)}$.*

Equivalently, $x = p_n u_1'$, where $\dim u_1' = 1$ and $p_n \vee (u_1')^{-1} = \Pi$.
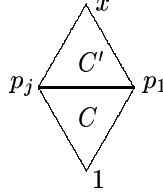
Figure 1

Proof. Consider the projective geometry of the neighbours of p_1. For $2 \leqslant i \leqslant n$ there exists $u_{i-1} \in \Pi_{i-1}$ such that

$$p_i = p_1 u_{i-1} \quad \text{and} \quad u_{i-1} \subset u_{j-1} \quad \text{for } i \leqslant j.$$

Now choose $u_n \in \Pi_n$ such that $u_{n-1} \subset u_n$ and $u_n \neq p_1^{-1}$. There exist q such choices for u_n. One then has that for all $2 \leqslant i \leqslant n$, $u_{i-1} \subset u_n$, and hence $p_i = p_1 u_{i-1}$ is adjacent to $p_1 u_n$. Thus $C' = \{p_1, p_2, \ldots, p_n, p_1 u_n\}$ is a chamber of $\mathfrak{X}$ and $p_1 u_n$ is the vertex x opposite 1 in $C \cup C'$. Clearly $p_1^{-1} \vee u_n = \Pi$, so $x = p_1 u_n$ is the normal form expressing x as a word of minimal length.

It now follows that $x \in S_{(1,0,\ldots,0,1)}$, and a similar argument proves the final statement. ∎

2.5. LEMMA. *Let $x \in S_k$ and $y \in S_{k'}$, $k, k' \in \mathbb{Z}_+^n$, where $k = (k_1, \ldots, k_n)$ and $k' = (k_1', \ldots, k_n')$. Then there exists an automorphism φ of Ω such that:*
   (i) *$\varphi \in [\Gamma]$, the full group of Γ;*
   (ii) *φ is almost everywhere a bijection from Ω_1^x onto Ω_1^y;*
   (iii) *φ is the identity on $\Omega \setminus (\Omega_1^x \cup \Omega_1^y)$.*
   Moreover, if $k = k'$ then φ is measure preserving.

Proof. Let $\delta = e_1 + e_n = (1, 0, \ldots, 0, 1)$ and consider the set of all vertices $x_1 \in S_{k+\delta}$ such that $x \in \mathrm{conv}\{1, x_1\}$. For such a vertex x_1, one has that $x_1 \in S_\delta(x)$ and $\Omega_1^{x_1} = \Omega_x^{x_1}$. Thus Ω_1^x is a (disjoint) union of sets of the form $\Omega_1^{x_1} = \Omega_x^{x_1}$, where $x_1 \in S_\delta(x) \cap S_{k+\delta}$ is constructed using the procedure of Lemma 2.4.

Similarly, Ω_1^y is a disjoint union of sets of the form $\Omega_1^{y_1} = \Omega_y^{y_1}$, where $y_1 \in S_\delta(y) \cap S_{k'+\delta}$. Refer to Figure 2 below.

It is therefore enough to show that for every such x_1, y_1, there is a bijection $\varphi : \Omega_x^{x_1} \to \Omega_y^{y_1}$ which coincides pointwise with the action of Γ almost everywhere on $\Omega_x^{x_1}$. That is, for almost each $\omega \in \Omega_x^{x_1}$, there exists $g \in \Gamma$ such that $\varphi\omega = g\omega$.

Fix such x_1, y_1. Choose $x_2 \in S_{e_n}(x_1) \cap S_{k+\delta+e_n}$. Also choose $v_n \in S_{e_n}$ such that
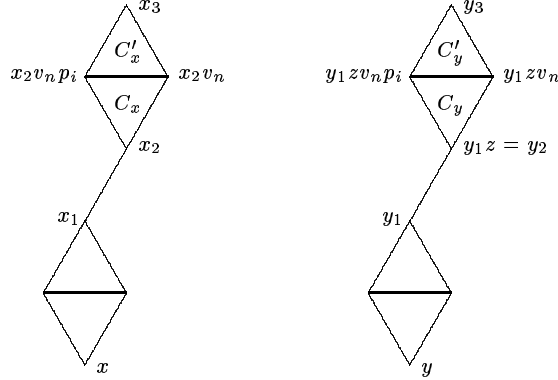
$$x_2 v_n \in S_{2e_n}(x_1) \cap S_{k+\delta+2e_n}.$$

Figure 2

Since $y_1 \in S_{k'+\delta}$, it has normal form

$$y_1 = u_1 \cdots u_l, \quad \text{where } u_l \in S_{e_n}.$$

We now show that there exists $z \in S_{e_n}$ such that

(2.1) $$u_l^{-1} \vee z = \Pi \quad \text{and} \quad z^{-1} \vee v_n = \Pi.$$

To prove the claim, it is necessary to make use of the identification of the generators of Γ with elements of the finite projective space Π. Set $\Pi_r = \{x \in \Pi : \dim x = r\} = S_{e_r}$.

Now, $u_l^{-1} \in \Pi_1$, and $v_n \in \Pi_n$. Therefore

(a) $|\{z \in \Pi_n : u_l^{-1} \vee z \neq \Pi\}| = |\{z \in \Pi_n : u_l^{-1} \subset z\}| = 1+q+q^2+\cdots+q^{n-1}$.
(b) $|\{z \in \Pi_n : z^{-1} \vee v_n \neq \Pi\}| = |\{z \in \Pi_n : z^{-1} \subset v_n\}| = 1+q+q^2+\cdots+q^{n-1}$.

Also,

$$|\Pi_n| = 1 + q + q^2 + \cdots + q^n > 2(1 + q + \cdots + q^{n-1}).$$

Hence there exists $z \in \Pi_n$ such that (2.1) holds.

It follows that the word $y_1 z v_n = u_1 \cdots u_l z v_n$ is in normal form and hence that

$$y_1 z v_n \in S_{k'+\delta+2e_n}.$$

Moreover, $y_2 = y_1 z \in S_{k'+\delta+e_n}$.

It will now be shown that the chambers C_x, C_y can be constructed which lie as indicated in (the two dimensional) Figure 3. By this it is meant, for example, that if $\omega \in \Omega$ and $C_x \subset S_{x_2}(\omega)$ then $S_{x_2}(\omega) \subset S_{x_1}(\omega) \subset S_x(\omega)$. In fact, $C_x = x_2 C$ and $C_y = y_1 z C$, where C is the chamber based at 1, as illustrated in Figure 3 (in two dimensions).
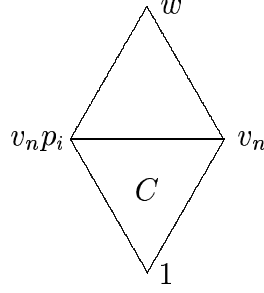
Figure 3

The vertices $x_2 v_n p_i$, $2 \leqslant i \leqslant n$, will be constructed from a flag $v_n^{-1} \subset p_2 \subset p_3 \subset \cdots \subset p_n$, where $p_i \in \Pi_i$. For the following argument, note that if $b \in \Pi_{r-1}$, where $r \geqslant 2$, then

$$|\{a \in \Pi_r : a \supset b\}| = 1 + q + \cdots + q^{n-r+1} \geqslant 1 + q.$$

There are at least $1+q$ elements $p_2 \in \Pi_2$ such that $p_2 \supset v_n^{-1}$. By reference to both Lemma 2.4 and the proof of Proposition 2.7 in [4], there is precisely one such p_2 such that $|x_2 v_n p_2| < |x_2 v_n|$. (In fact, in that case $x_2 v_n p_2 \in S_{k+\delta+2e_n-e_{n-1}}$.)

Similarly, there is precisely one $p_2 \in \Pi_2$, $p_2 \supset v_n^{-1}$ such that $|y_1 z v_n p_2| < |y_1 z v_n|$.

One can therefore choose $p_2 \in \Pi_2$ with $p_2 \supset v_n^{-1}$ such that $|x_2 v_n p_2| \geqslant |x_2 v_n|$ and $|y_1 z v_n p_2| \geqslant |y_1 z v_n|$. Moreover, since $v_n p_2$ is then adjacent to 1, these inequalities are in fact equalities.

This process is now continued. There are at least $1 + q$ elements $p_3 \in \Pi_3$ such that $p_3 \supset p_2$ and at most two of them satisfy either $|x_2 v_n p_3| < |x_2 v_n|$ or $|y_1 z v_n p_2| < |y_1 z v_n|$. Thus we may choose $p_3 \supset p_2$ such that $|x_2 v_n p_3| = |x_2 v_n|$ and $|y_1 z v_n p_3| = |y_1 z v_n|$.

Continue in this way to obtain a flag

$$v_n^{-1} \subset p_2 \subset p_3 \subset \cdots \subset p_n$$

such that the vertex set of the chamber C is

$$\{1, v_n, v_n p_2, v_n p_3, \ldots, v_n p_n\}.$$

Then $C_x = x_2 C$ and $C_y = y_2 C$.

Now choose, by Lemma 2.4, a vertex w (one of q possible) of a chamber C'_x which meets C_x in the face $C_x \setminus \{1\}$. Thus $w \in S_\delta$, and $x_3 = x_2 w \in S_{k+2\delta+e_n}$. Also $y_3 = y_2 w \in S_{k'+2\delta+e_n}$. (Recall that, by definition, $y_2 = y_1 z$.) Moreover, $y_2 x_2^{-1} C_x = C_y$.

It has now been shown that

$$\Omega_{x_2}^{x_3} \subset \Omega_x^{x_1}, \quad \Omega_{y_2}^{y_3} \subset \Omega_y^{y_1} \quad \text{and} \quad y_2 x_2^{-1} \Omega_{x_2}^{x_3} = \Omega_{y_2}^{y_3}.$$

Therefore one can define the map φ on $\Omega_{x_2}^{x_3}$ by

$$\varphi \omega = y_2 x_2^{-1} \omega.$$

Now recall that $x \in S_k$, $y \in S_{k'}$ and $x_1 \in S_{k+\delta}$, $y_1 \in S_{k'+\delta}$ were fixed, and that $x_2 \in S_{e_n}(x_2) \cap S_{k+\delta+e_n}$ was chosen. The set $\Omega_x^{x_1}$ is a disjoint union of sets

of the form $\Omega_{x_2}^{x_3}$ where $x_3 \in S_\delta(x_2)$. Let K denote the number of such sets. This number is independent of the choice of x, x_1 and k by Lemma 2.4 of [4], (or by the fact that Γ acts simply transitively on $\mathfrak{X}^0$).

The definition $\varphi\omega = y_2 x_2^{-1}\omega$ in the above choice of $\Omega_{x_2}^{x_3}$ therefore leaves φ undefined on a proportion $1 - (1/K))$ of $\Omega_x^{x_1}$. However, where φ is defined it coincides with the action of an element of Γ, namely $y_2 x_2^{-1}$.

Now repeat the process on each of the $K-1$ sets of the form $\Omega_{x_2}^{x_3}$ where φ has not been defined. As before, φ can be defined except on a proportion $(1 - (1/K))$ of each such set, and φ can therefore be defined everywhere except on a proportion $(1 - (1/K))^2$ of the original set $\Omega_x^{x_1}$.

Continuing in this manner, one sees that at the nth step, φ has been defined everywhere except on a proportion $(1 - (1/K))^n$ of $\Omega_x^{x_1}$.

Since $(1 - (1/K))^n \to 0$ as $n \to \infty$, φ is defined almost everywhere on $\Omega_x^{x_1}$ and satisfies the required properties. If $k = k'$ then it is clear from the construction that φ is measure preserving. ∎

2.6. REMARK. This result extends the Proposition 4.9 of [12]. Moreover, for $n = 2$ this proof deals with the case $q = 2$, which was left open in [12]. Thus the hypothesis $q \geqslant 3$ in the main result Theorem 4.10 of [12] is not in fact necessary.

We can now prove the following.

2.7. PROPOSITION. *There exists a countable subgroup K_0 of $[\Gamma]$ such that:*
 (i) *K_0 is measure preserving;*
 (ii) *K_0 acts ergodically on Ω.*

Proof. It suffices to take the group generated by the automorphisms of the form φ defined in Lemma 2.5, with $k = k'$ then use Lemma 2.2 to extract a countable subgroup K_0. Finally, Lemma 2.1 shows that the action of K_0 is ergodic. ∎

2.8. LEMMA. *Let Γ be a countable group acting on a measure space (Ω, μ). Suppose that the action of the full group $[\Gamma]$ is ergodic. Then so is the action of Γ.*

Proof. Let S be a measurable subset of Ω such that $\mu(gS \setminus S) = 0$ for all $g \in \Gamma$. Let $k \in [\Gamma]$. It will be shown that $\mu(kS \setminus S) = 0$. For each $g \in \Gamma$, let

$$S_g = \{\omega \in S : k\omega = g\omega\},$$

which is a measurable subset of S. Since $k \in [\Gamma]$ it follows that

$$S = S_0 \cup \bigcup_{g \in \Gamma} S_g,$$

where S_0 has measure zero.

Then

$$\mu(kS \setminus S) \leqslant \sum_{g \in \Gamma} \mu(kS_g \setminus S) = \sum_{g \in \Gamma} \mu(gS_g \setminus S) \leqslant \sum_{g \in \Gamma} \mu(gS \setminus S) = 0.$$

Thus $\mu(kS \setminus S) = 0$ for all $k \in [\Gamma]$. Since the action of $[\Gamma]$ is ergodic it follows that S is either null or conull with respect to the measure μ. Therefore the action of Γ is ergodic. ∎

2.9. COROLLARY. *The action of Γ on Ω is ergodic.*

Proof. This follows from Proposition 2.7 and Lemma 2.8. ∎

3. CLASSIFICATION OF THE ACTION OF Γ ON Ω

3.1. Having shown that the action of an $\widetilde{A}_n$ group on its boundary Ω is ergodic, we now show that it is type III$_\lambda$, where $0 \leqslant \lambda \leqslant 1$, and the value of λ depends on the *ratio set*.

3.1.1. DEFINITION. Let Γ be a countable group of automorphisms of the measure space (Ω, ν). Following Krieger, define the *ratio set* $r(\Gamma)$ to be the set of $\lambda \in [0, \infty)$ such that for every $\varepsilon > 0$ and Borel set $\mathcal{E}$ with $\nu(\mathcal{E}) > 0$, there exists a $g \in \Gamma$ and a Borel set $\mathcal{F}$ such that $\nu(\mathcal{F}) > 0$, $\mathcal{F} \cup g\mathcal{F} \subset \mathcal{E}$ and

$$\left| \frac{\mathrm{d}\nu \circ g}{\mathrm{d}\nu}(\omega) - \lambda \right| < \varepsilon$$

for all $\omega \in \mathcal{F}$.

3.1.2. REMARK. The ratio set $r(\Gamma)$ depends only on the quasi-equivalence class of the measure ν ([9], Section I-3, Lemma 14). It also depends only on the full group in the sense that

$$[H] = [G] \Rightarrow r(H) = r(G).$$

3.1.3. PROPOSITION. *Let $\mathfrak{X}$ be a locally finite, thick $\widetilde{A}_n$ building of order q, and let Γ be a countable group of type rotating automorphisms of $\mathfrak{X}$. Fix a vertex $O \in \mathfrak{X}^0$ of type 0, and suppose that for each $0 \leqslant i \leqslant n$ there exists an element $g_i \in \Gamma$ such that $d(g_i O, O) = 1$ and $g_i O$ is a vertex of type i. Also, suppose that there exists a countable subgroup K of $[\Gamma]_0$ whose action on Ω is ergodic. Then*

$$r(\Gamma) = \begin{cases} \{q^n : n \in \mathbb{Z}\} \cup \{0\} & \text{for } n \text{ odd,} \\ \{q^{2n} : n \in \mathbb{Z}\} \cup \{0\} & \text{for } n \text{ even.} \end{cases}$$

Proof. By Remark 3.1.2, it is sufficient to prove the statement for some group H such that $[H] = [\Gamma]$. In particular, since $[\Gamma] = [\langle \Gamma, K \rangle]$ for any subgroup K of $[\Gamma]_0$, we may assume without loss of generality that $K \leqslant \Gamma$.

Let $\nu = \nu_O$. For $g_i \in \Gamma$ as in the statement of the lemma, let $x = g_i O$, and note that $\nu_x = \nu \circ g_i^{-1}$. If $m(O, x; \omega) = (m_1, m_2, \ldots, m_n)$ then by Lemma 1.1.6,

$$\frac{\mathrm{d}\nu \circ g_i^{-1}}{\mathrm{d}\nu}(\omega) = \frac{\mathrm{d}\nu_x}{\mathrm{d}\nu}(\omega) = q^{-\sum_{i=1}^{n} i(n+1-i)m_i}.$$

Then for $\omega \in \Omega_O^x$, one has that $m(O, x; \omega) = (0, \ldots, 0, -1, 0, \ldots, 0)$, where the -1 is in the ith place. Thus

$$(3.1) \qquad \frac{\mathrm{d}\nu_x}{\mathrm{d}\nu}(\omega) = q^{i(n+1-i)} \quad \text{for } \omega \in \Omega_O^x.$$

Let $\mathcal{E} \subset \Omega$ be a Borel set with $\nu(\mathcal{E}) > 0$. Then by the ergodicity of K, there exist $k_1, k_2 \in K$ such that the set

$$\mathcal{F} = \{\omega \in \mathcal{E} : k_1\omega \in \Omega_O^x \text{ and } k_2 g_i^{-1} k_1 \omega \in \mathcal{E}\}$$

has positive measure.

Next, let $t = k_2 g_i^{-1} k_1 \in \Gamma$. By construction, $\mathcal{F} \cup t\mathcal{F} \subset \mathcal{E}$. Moreover, since K is measure preserving,

$$\frac{\mathrm{d}\nu \circ t}{\mathrm{d}\nu}(\omega) = \frac{\mathrm{d}\nu \circ g_i^{-1}}{\mathrm{d}\nu}(k_1\omega) = \frac{\mathrm{d}\nu_{x_i}}{\mathrm{d}\nu}(k_1\omega) = q^{i(n+1-i)} \quad \text{for all } \omega \in \mathcal{F}$$

by (3.1), and since $k_i\omega \in \Omega_O^x$. Hence $q^{i(n+1-i)} \in r(\Gamma)$ for $1 \leqslant i \leqslant n$.

Since the action of Γ on Ω is ergodic, $r(\Gamma) - \{0\}$ forms a group. It is now possible to determine the generator of $r(\Gamma) - \{0\}$.

Suppose that n is odd. Then for $i \in \{1, 2\}$, one has that $q^n, q^{2(n-1)} \in r(\Gamma)$. As $n, 2(n-1)$ are coprime for n odd, and as $r(\Gamma) - \{0\}$ forms a group, it follows that $q \in r(\Gamma)$, and hence

$$r(\Gamma) = \{q^n : n \in \mathbb{Z}\} \quad \text{for } n \text{ odd}.$$

Suppose that n is even. As before, $q^n, q^{2(n-1)} \in r(\Gamma)$. Moreover, as highest common factor of $n, 2(n-1)$ is 2 for n even, and as $r(\Gamma)$ forms a group, it follows that $q^2 \in r(\Gamma)$. Finally, as $i(n+1-i)$ is even for all i if n is even, it follows that for $g \in \Gamma$, and $x = g^{-1}O$,

$$\frac{\mathrm{d}\nu \circ g}{\mathrm{d}\nu} = \frac{\mathrm{d}\nu_x}{\mathrm{d}\nu} = q^{-\sum i(n+1-i)m_i} \in \{q^{2n} : n \in \mathbb{Z}\}.$$

Thus

$$r(\Gamma) = \{q^{2n} : n \in \mathbb{Z}\} \quad \text{for } n \text{ even}. \quad \blacksquare$$

3.1.4. Proposition. *Let Γ be an $\widetilde{A}_n$ group. Then the action of Γ on Ω is amenable.*

Proof. This is a straightforward generalization of the case $n = 2$, proved in Proposition 3.14 from [12]. $\blacksquare$

3.2. Proof of Theorem 0.1. This follows from Proposition 2.7, Corollary 2.9, Proposition 3.1.3 and Theorem 3.1.4. $\blacksquare$

3.3. Proof of Theorem 0.3. The proof of Theorem 0.3 is now easy. Let $\mathfrak{X}$ be the affine building of $G = \mathrm{PGL}(n+1, \mathbb{Q}_p)$. By Proposition VI.9F of [1], the boundary Ω of $\mathfrak{X}$ is isomorphic to $\mathrm{PGL}(n+1, \mathbb{Q}_p)/B$ as a topological G-space. The measure μ on G/B is (up to equivalence) the natural quasi-invariant Borel measure on G/B.

The vertex set $\mathfrak{X}^0$ of $\mathfrak{X}$ is identified with $\mathrm{PGL}(n+1, \mathbb{Q}_p)/\mathrm{PGL}(n+1, \mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Let $O = \mathrm{PGL}(n+1, \mathbb{Z}_p) \in \mathfrak{X}^0$. It follows from Proposition 3.1 of [15] that $\mathrm{PGL}(n+1, \mathbb{Z}_p)$ acts transitively on each set $S_k(O)$. Since the vertex set $\mathfrak{X}^0 = \mathrm{PGL}(n+1, \mathbb{Q}_p)/\mathrm{PGL}(n+1, \mathbb{Z}_p)$ is a discrete space and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ it follows that $\mathrm{PGL}(n+1, \mathbb{Z})$ also acts transitively on $S_k(O)$ for each $k \in \mathbb{Z}_+^n$. Moreover $\mathrm{PGL}(n+1, \mathbb{Z})$ stabilizes O. Therefore $\mathrm{PGL}(n+1, \mathbb{Z})$ acts ergodically on Ω by Lemma 2.1. The group $\mathrm{PGL}(n+1, \mathbb{Q})$ also acts transitively on $\mathfrak{X}^0$. By our previous computation of Radon-Nikodym derivatives and the argument of Proposition 4.4 from [12] the type of the action is as stated.

Note that in this argument there is no need to consider the full group, since $\mathrm{PGL}(n+1, \mathbb{Z})$ is already a measure preserving ergodic subgroup of $\mathrm{PGL}(n+1, \mathbb{Q})$. Thus the proof is considerably simpler than the proof of Theorem 0.1. $\blacksquare$

4. FREENESS OF THE ACTION ON THE BOUNDARY

A simple modification of Proposition 4.12 from [12] shows that if $\mathbb{F}$ is a (possibly non commutative) local field then the action of $\mathrm{PGL}(n+1,\mathbb{F})$ on its Furstenberg boundary Ω is measure-theoretically free. If $\mathfrak{X}$ is a thick, locally finite affine building of type $\widetilde{A}_n$, where $n \geqslant 3$, then $\mathfrak{X}$ is the building of $\mathrm{PGL}(n+1,\mathbb{F})$ for some such local field ([13], p. 137). All known type-rotating $\widetilde{A}_n$ groups, with $n \geqslant 3$, embed in $\mathrm{PGL}(n+1,\mathbb{F})$ and act upon the building of $\mathrm{PGL}(n+1,\mathbb{F})$ in the canonical way. For such groups Γ, the action on Ω is therefore measure theoretically free.

The case of $\widetilde{A}_2$ groups is more interesting because the associated $\widetilde{A}_2$ building may not be the affine building of a linear group. In fact this is the case for the buildings of many of the groups constructed in [6]. The boundary action of Γ is nevertheless free.

4.1. PROPOSITION. *Let $\mathfrak{X}$ be an $\widetilde{A}_2$ building, and let Γ be an $\widetilde{A}_2$ group. Then the action of Γ on Ω is measure-theoretically free* (cf. [12], Proposition 3.10).

This result was stated in Proposition 3.10 of [12]. The proof given there contains a gap, because the proof of Case 2 of Lemma 3.7 from [12] is not complete. Our purpose is to fill this gap (Lemma 4.3 below). A similar argument applies to the $\widetilde{A}_n$ case, where $n \geqslant 3$. In view of the comments at the beginning of this section, we have confined ourselves to the $\widetilde{A}_2$ case.

Let O be a fixed vertex in $\mathfrak{X}$ and let $n \in \mathbb{Z}_+$. Any sector S based at O has at its base a triangle T_n with apex O consisting of all vertices v in S such that $d(v,O) \leqslant n$. (See Figure 4 below.)
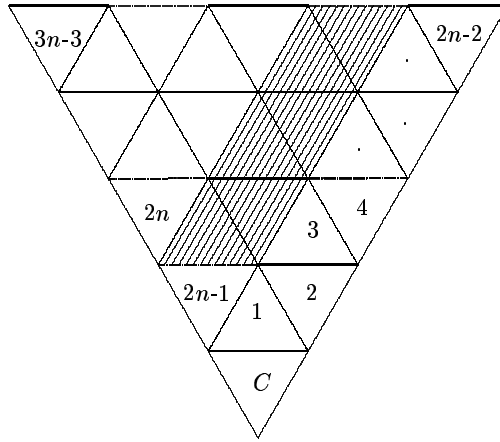

Figure 4

Consider the set $\mathcal{S}_n$ of all such triangles T_n in $\mathfrak{X}$.

4.2. LEMMA. *Let $\mathcal{S}_n$ be as above. Then*

$$|\mathcal{S}_n| = (q^2 + q + 1)(q+1)q^{3n-3}.$$

Proof. First note that there are $(q^2 + q + 1)(q + 1)$ choices for the base chamber C.

The reason for this is that one edge of C containing O is determined by the number of points P in a finite projective plane Π of order q. There are $q^2 + q + 1$ such points P. See Figure 5 below.

Having chosen the point P, there are precisely $q + 1$ possible lines L in Π which are incident with P. There are therefore $(q^2 + q + 1)(q + 1)$ choices of C.

Having chosen C, there are q choices for each of the chambers labeled $1, 2, \ldots,$ $(2n - 2)$ in the figure. Then choose the chamber labeled $(2n - 1)$ (q choices) in Figure 4. This choice then determines the whole shaded region in the figure (which is contained in the convex hull of the chambers already chosen, and hence is uniquely determined). Now choose the chamber labeled $2n$ (q choices) and continue the process until finally chamber $3n - 3$ is chosen. This determines the triangle completely and there are $(q^2 + q + 1)(q+1)q^{3n-3}$ possibilities altogether. ∎
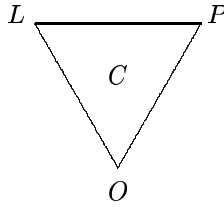

Figure 5

This demonstrates that for each positive integer n, the boundary Ω of $\mathfrak{X}$ is partitioned into $(q^2 + q + 1)(q + 1)q^{3n-3}$ sets $\{\Omega_T : T \in \mathcal{S}_n\}$, where

$$\Omega_T = \{\omega \in \Omega : T \subset [O, \omega)\}.$$

Moreover each of these sets has the same measure ([5]).

The proof of case 2 of Lemma 3.7 from [12] can now be completed.

4.3. LEMMA. *Let W be a wall of $\mathfrak{X}$ and let Σ denote the set of boundary points $\omega \in \Omega$ such that for some vertex v, the sector $[v, \omega)$ lies in an apartment containing W. Then*

$$\nu_O(\Sigma) = 0.$$

Proof. By translating to a parallel sector, one can assume that $v = O$. Also, W is the union of two sector panels, which will be denoted by $[O, W^+), [O, W^-)$.

Given $n \in \mathbb{Z}_+$, let $\mathcal{S}_n^+, \mathcal{S}_n^-, \mathcal{S}_n^\perp$ denote the subsets of $\mathcal{S}_n$ consisting of triangles $T_n^+, T_n^-, T_n^\perp$ respectively, lying in some apartment containing W as illustrated below in Figure 6.
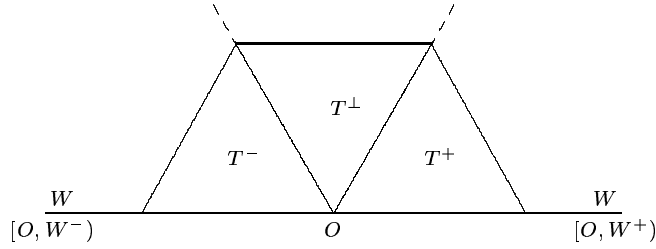
Figure 6

Let $\mathcal{S}_n^W = \mathcal{S}_n^+ \cup \mathcal{S}_n^- \cup \mathcal{S}_n^\perp$. Then

(4.1) $$\Sigma \subset \bigcup_{T \in \mathcal{S}_n^W} \Omega_T.$$

The first step is to calculate the number of triangles in $\mathcal{S}_n^W$.

To do this, the number of possible choices for $T^+ \in \mathcal{S}_n^+$ must be determined. Refer to Figure 7 below.

There are $(q+1)$ possible choices for the chamber c_1. This choice then determines all the other chambers labeled 1 which lie in the convex hull of c_1 and $[0, W^+)$. There are then q choices for the chamber c_2. This choice now determines all chambers labeled 2 which are in the convex hull of c_2 and all the other chambers previously determined.

Continue in this way until the chamber c_n is reached, thereby determining the whole triangle $T^+ \in \mathcal{S}_n^+$. There are thus $(q+1)q^{n-1}$ choices for T^+.

Now each triangle $T^+ \in \mathcal{S}_n^+$ determines a unique pair of triangles $T^- \in \mathcal{S}_n^-$, $T^\perp \in \mathcal{S}_n^\perp$ subject to the condition that $T^-, T^\perp$ lie in the convex hull of $\mathcal{S}_n^+ \cup W$. Conversely such T^- or $T^\perp$ determine T^+ uniquely. Hence the sets $\mathcal{S}_n^+, \mathcal{S}_n^-, \mathcal{S}_n^\perp$ have the same number of elements. It follows that
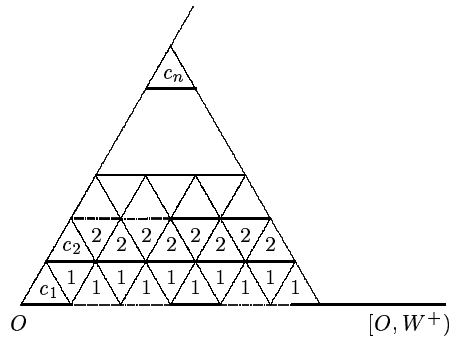
$$|\mathcal{S}_n^W| = 3(q+1)q^{n-1}.$$

Figure 7

Since the sets Ω_T, $T \in \mathcal{S}_n$ have equal measure and partition Ω, it follows from Lemma 4.2 and equation (4.1) that

$$\nu_O(\Sigma) \leqslant \nu_O\left(\bigcup_{T \in \mathcal{S}_n^W} \Omega_T\right) = \frac{3(q+1)q^{n-1}}{(q^2+q+1)(q+1)q^{3n-3}} \to 0 \quad \text{as } n \to \infty.$$

Thus $\nu_O(\Sigma) = 0$.   ∎

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