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MORE RELATIVE ANGULAR DERIVATIVES

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Abstract. The notion of the angular derivative of a holomorphic self-map b of the unit disk has been generalized to that of an angular derivative of b relative to an inner function u. In this paper, we provide three more conditions equivalent to the existence of an angular derivative of b relative to u, and use these notions to provide a generalization of Julia's lemma.

Keywords: *Angular derivative, Hardy space, Aleksandrov measure.*

MSC (2000): 46E22, 46E30.

1. INTRODUCTION

Relative angular derivatives came into being as a generalization of the notion of the angular derivative of a holomorphic function on the unit disk. In this paper, we will further analyze the notion of an angular derivative of a holomorphic self-map of the unit disk relative to a nonconstant inner function.

Let b be a holomorphic self-map of the unit disk, that is, an analytic function on the unit disk $\mathbb{D}$ of the complex plane with $|b| < 1$ on $\mathbb{D}$. We will take u to be our nonconstant inner function — a holomorphic function on $\mathbb{D}$ with $|u| = 1$ almost everywhere on $\partial\mathbb{D}$. This notation will remain fixed.

The analysis and definition of relative angular derivatives comes primarily from the viewpoint of the Aleksandrov measures μ_λ and ν_λ ($\lambda \in \partial\mathbb{D}$), which we derive from our functions b and u, respectively. These measures are defined and discussed in Section 2. Throughout this paper, we will use m to denote the usual normalized Lebesgue measure on the unit circle. We will also use the notation μ^{ac} and μ^{s} to denote the absolutely continuous and singular parts of the measure μ ($= \mu_1$) with respect to m (similarly for $\mu_\lambda^{\mathrm{ac}}$ and μ_λ^{s}). For any function f on the unit disk $\mathbb{D}$, f_r will denote the function on the boundary $\partial\mathbb{D}$ such that $f_r(\mathrm{e}^{\mathrm{i}\theta}) = f(r\mathrm{e}^{\mathrm{i}\theta})$ for $r < 1$.

The relationship between Aleksandrov measures and angular derivatives has recently been developed by many people. The most direct connection comes essentially from [4], and is developed more in Chapter 7 of [3] and Section VI-7 of [6], THEOREM 1.1. *The function b has angular derivative at a point $z_0 \in \partial\mathbb{D}$ (where $|b(z_0)| = 1$) exactly where its corresponding Aleksandrov measure μ_λ has an atom, and $\mu_\lambda(\{z_0\}) = 1/|b'(z_0)|$.*

Others can be found in [9], in which the author has used Aleksandrov measures to introduce the notion of a relative angular derivative, [2], in which Aleksandrov measures are used to give a very nice way to describe the essential norm of a composition operator, and [8], in which the author has also used Aleksandrov measures to generalize other theorems which give properties of composition operators and composition operator differences in terms of angular derivatives.

In this paper we aim to develop further this type of useful generalization of angular derivatives by studying the more broad category of relative angular derivatives.

Section 2 contains some background material about the Aleksandrov measures. In Section 3, we are reminded how the angular derivative was generalized to become the relative angular derivative. It is in this section that we present the theorem from [9] which lists six conditions, all of which are equivalent, any one of which can be used as a definition of a relative angular derivative.

The first goal of this paper, in Section 4, is to use the properties of the family of Aleksandrov measures, $\{\mu_\lambda\}_{\lambda\in\partial\mathbb{D}}$, to prove that three other conditions are each equivalent to the six which define a relative angular derivative. These new conditions arise by consideration of the behavior of the boundary values of the generalized difference quotient relative to the measures μ_λ for $\lambda \in \partial\mathbb{D}$.

In Section 5, we see how other results, such as Julia's lemma, can be generalized by methods similar to those used to create relative angular derivatives.

2. THE ALEKSANDROV MEASURES

For $\lambda \in \partial\mathbb{D}$, the function $\mathrm{Re}(\frac{\lambda+b}{\lambda-b})$ is positive, and, as the real part of an analytic function, harmonic (on the disk, $\mathbb{D}$). It is thus the Poisson integral of a positive measure on $\partial\mathbb{D}$, which we will call μ_λ. We have, then,

$$\mathrm{Re}\Big(\frac{\lambda+b(z)}{\lambda-b(z)}\Big) = \int\limits_{\partial\mathbb{D}} P(\theta,z)\,\mathrm{d}\mu_\lambda(\mathrm{e}^{\mathrm{i}\theta}) = P\mu_\lambda(z)$$

and the Herglotz integral representation,

$$\frac{\lambda+b(z)}{\lambda-b(z)} = \int\limits_{\partial\mathbb{D}} H(\theta,z)\,\mathrm{d}\mu_\lambda(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{i}\,\mathrm{Im}\frac{\lambda+b(0)}{\lambda-b(0)}.$$

Note that for $z \in \mathbb{D}$, the Poisson kernel, $P(\theta,z) = \frac{1-|z|^2}{|\mathrm{e}^{\mathrm{i}\theta}-z|^2}$, is the real part of the Herglotz kernel, $H(\theta,z) = \frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}$. The measure μ_1 we shall simply call μ. The measure ν is similarly defined to correspond with the inner function u.

The following are some properties of the Aleksandrov measures defined above:

(i) all positive Borel measures on $\partial\mathbb{D}$ are associated with functions in this way;

(ii) the absolutely continuous part of μ is given by $\frac{1-|b|^2}{|1-b|^2}$ times the normalized Lebesgue measure (on $\partial\mathbb{D}$);

(iii) the measure μ is singular if and only if b is an inner function, i.e., $|b|=1$ almost everywhere on $\partial\mathbb{D}$;

(iv) for μ_λ^{s}-a.e. $\xi\in\partial\mathbb{D}$ we have $P\mu_\lambda(\xi)=\infty$ and thus $b(\xi)=\lambda$;

3. RELATIVE ANGULAR DERIVATIVES

In [9], the relative angular derivatives were defined by extending the notion of the angular derivative of the function b by replacing the identity function z by an arbitrary (nonconstant) inner function u in the denominator of the standard difference quotient, $\frac{b(z)-b(z_0)}{z-z_0}$. The behavior of this generalized difference quotient, $\frac{1-b(z)}{1-u(z)}$, was then studied.

The main theorem from [9] which provided the basis for the definition of the relative angular derivative was:

Theorem 3.1. *The following conditions are equivalent:*

(i) $\nu\ll\mu$ *and* $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\in L^2(\mu)$*;*

(ii) $\frac{1-b}{1-u}k_0^u\in\mathcal{H}(b)$ *(the de Branges-Rovnyak space);*

(iii) $\frac{1-b}{1-u}k_w^u\in\mathcal{H}(b)$ *for all* $w\in\mathbb{D}$*;*

(iv) $\int\limits_{\partial\mathbb{D}}\left|\frac{1-b_r}{1-u_r}\right|\mathrm{d}\nu$ *stays bounded as* $r\nearrow 1$*;*

(v) $\frac{1-b}{1-u}\in H^2$ *and* $\frac{1-b}{1-u}\in H^2(\mu^{\mathrm{ac}})$*;*

(vi) $\frac{1-b}{1-u}\in H^2$ *and* $\frac{1-b}{1-u}\in H^2(\mu)$*.*

If any of the above hold, then we say that b has an angular derivative relative to u.

Our goal now is to continue the study of the boundary values of the generalized difference quotient, $\frac{1-b}{1-u}$, relative to the measures μ_λ, and to show

Theorem 3.2. *The following are equivalent, and each is equivalent to the six conditions in Theorem* 3.1*:*

(i) $\frac{1-b}{1-u}\in H^2(\mu^{\mathrm{ac}})$ *and there is a function* $h\in L^1(m)$ *such that for almost every* $\lambda\in\partial\mathbb{D}$ *we have* $\int\limits_{\partial\mathbb{D}}\left|\frac{1-b}{1-u}\right|^2\mathrm{d}\mu_\lambda<h(\lambda)$*;*

(ii) *there is a function* $h\in L^1(m)$ *such that for almost every* $\lambda\in\partial\mathbb{D}$ *we have* $\int\limits_{\partial\mathbb{D}}\frac{\mathrm{d}\mu_\lambda}{|1-u|^2}<\frac{h(\lambda)}{|1-\lambda|^2}$*;*

(iii) *there is a constant* C *such that for all* $\lambda\in\partial\mathbb{D}\setminus\{1\}$, $\int\limits_{\partial\mathbb{D}}\frac{\mathrm{d}\mu_\lambda}{|1-u|^2}<\frac{C}{|1-\lambda|^2}$*.*

These three new conditions, then, will work as well as the original six, to define the notion of relative angular derivative.

4. THE FAMILY OF ALEKSANDROV MEASURES $\{\mu_\lambda\}_{\lambda\in\partial\mathbb{D}}$

In this section we will explore the relationship between our conditions for b to have an angular derivative relative to u and the family of Aleksandrov measures $\{\mu_\lambda\}_{\lambda\in\partial\mathbb{D}}$. We will make use of a theorem of A.B. Aleksandrov, from [1]:

THEOREM 4.1.

$$\int\limits_{\partial\mathbb{D}} \mu_\lambda \,\mathrm{d}m(\lambda) = m,$$

where by this we mean that any f integrable with respect to Lebesgue measure is defined μ_λ-a.e. and integrable with respect to μ_λ for almost every $\lambda \in \partial\mathbb{D}$, and

$$\int\limits_{\partial\mathbb{D}} f \,\mathrm{d}m = \int\limits_{\partial\mathbb{D}} \Big(\int\limits_{\partial\mathbb{D}} f \,\mathrm{d}\mu_\lambda \Big) \,\mathrm{d}m(\lambda).$$

We will also need the converse to this theorem,

THEOREM 4.2. *If the nonnegative Borel function f is defined μ_λ-a.e. and integrable with respect to μ_λ for almost every $\lambda \in \partial\mathbb{D}$, and*

$$\int\limits_{\partial\mathbb{D}} \Big(\int\limits_{\partial\mathbb{D}} f \,\mathrm{d}\mu_\lambda \Big) \,\mathrm{d}m(\lambda) < \infty,$$

then f is integrable with respect to Lebesgue measure and

$$\int\limits_{\partial\mathbb{D}} f \,\mathrm{d}m = \int\limits_{\partial\mathbb{D}} \Big(\int\limits_{\partial\mathbb{D}} f \,\mathrm{d}\mu_\lambda \Big) \,\mathrm{d}m(\lambda).$$

Proof. Consider the functions f_M (for any positive real M), defined by

$$f_M(z) = \begin{cases} [c]l f(z), & \text{if } f(z) < M, \\ M, & \text{otherwise.} \end{cases}$$

The function f_M is clearly integrable with respect to Lebesgue measure (on $\partial\mathbb{D}$) since it is bounded, so we have, by the theorem of Aleksandrov,

$$\int\limits_{\partial\mathbb{D}} f_M \,\mathrm{d}m = \int\limits_{\partial\mathbb{D}} \Big(\int\limits_{\partial\mathbb{D}} f_M \,\mathrm{d}\mu_\lambda \Big) \,\mathrm{d}m(\lambda) \leqslant \int\limits_{\partial\mathbb{D}} \Big(\int\limits_{\partial\mathbb{D}} f \,\mathrm{d}\mu_\lambda \Big) \,\mathrm{d}m(\lambda) = K$$

for some positive number K. Hence $\int\limits_{\partial\mathbb{D}} f_M \,\mathrm{d}m \leqslant K$ for all M. This gives us that f is integrable (with integral $\leqslant K$, even), then we can use the theorem of Aleksandrov again to get

$$\int\limits_{\partial\mathbb{D}} f \,\mathrm{d}m = \int\limits_{\partial\mathbb{D}} \Big(\int_{\partial\mathbb{D}} f \,\mathrm{d}\mu_\lambda \Big) \,\mathrm{d}m(\lambda).$$

This completes the proof. ∎

When we apply Aleksandrov's theorem to the boundary function of $\left|\frac{1-b}{1-u}\right|^2$, we get, if this function is integrable,

$$\int_{\partial\mathbb{D}} \left(\int_{\partial\mathbb{D}} \left|\frac{1-b}{1-u}\right|^2 \mathrm{d}\mu_\lambda \right) \mathrm{d}m(\lambda) = \int_{\partial\mathbb{D}} \left|\frac{1-b}{1-u}\right|^2 \mathrm{d}m.$$

I.e., the function h defined for almost every $\lambda \in \partial\mathbb{D}$ by

$$\int_{\partial\mathbb{D}} \left|\frac{1-b}{1-u}\right|^2 \mathrm{d}\mu_\lambda = h(\lambda)$$

is an L^1 function. Similarly, applying the converse of Aleksandrov's theorem, if the function h as defined above is in L^1, then we know that $\left|\frac{1-b}{1-u}\right|^2$ is integrable with respect to Lebesgue measure, so $\frac{1-b}{1-u} \in L^2$, which gives us $\frac{1-b}{1-u} \in H^2$, since $\frac{1-b}{1-u} \in N^+$. This and part (v) of Theorem 3.1 give us

Theorem 4.3. *The function b has an angular derivative relative to u if and only if there is some $h \in L^1(m)$ such that*

$$\int_{\partial\mathbb{D}} \left|\frac{1-b}{1-u}\right|^2 \mathrm{d}\mu_\lambda = h(\lambda) \quad \textit{and} \quad \frac{1-b}{1-u} \in H^2(\mu^{\mathrm{ac}}).$$

This is part (i) of Theorem 3.2. Note: The condition $\frac{1-b}{1-u} \in H^2(\mu^{\mathrm{ac}})$ is equivalent to $\frac{1-b}{1-u} \in L^2(\mu^{\mathrm{ac}})$, which is the same as

$$\int_{\partial\mathbb{D}} \frac{1-|b|^2}{|1-u|^2} \mathrm{d}m < \infty.$$

Next, we need

Theorem 4.4. *Assume that $\int_{\partial\mathbb{D}} \frac{1-|b|^2}{|1-u|^2} \mathrm{d}m < \infty$. Then there are positive constants c_1 and c_2 (independent of $\lambda \in \partial\mathbb{D}\backslash\{1\}$) such that, when the term on the right hand side is finite,*

$$\int_{\partial\mathbb{D}} \frac{|1-\lambda|^2}{|1-u|^2} \mathrm{d}\mu_\lambda < c_1 \int_{\partial\mathbb{D}} \left|\frac{1-b}{1-u}\right|^2 \mathrm{d}\mu_\lambda$$

and

$$\int_{\partial\mathbb{D}} \left|\frac{1-b}{1-u}\right|^2 \mathrm{d}\mu_\lambda < c_2 \int_{\partial\mathbb{D}} \frac{|1-\lambda|^2}{|1-u|^2} \mathrm{d}\mu_\lambda.$$

Proof. We will prove this theorem in two stages. First, consider the statement of the theorem above for integrals with respect to the singular part of μ_λ, or μ_λ^{s}. Since $b(\xi) = \lambda$ for μ_λ^{s}-a.e. $\xi \in \partial\mathbb{D}$, the two integrals, $\int_{\partial\mathbb{D}} \left|\frac{1-b}{1-u}\right|^2 \mathrm{d}\mu_\lambda^{\mathrm{s}}$ and $\int_{\partial\mathbb{D}} \frac{|1-\lambda|^2}{|1-u|^2} \mathrm{d}\mu_\lambda^{\mathrm{s}}$ are equal (whenever either is $< \infty$).

In order to prove the theorem for the same integrals with respect to the absolutely continuous part of μ_λ, or $\mu_\lambda^{\mathrm{ac}}$, we will introduce the functions $f = \frac{1-|b|^2}{|1-u|^2}$ and $g = \frac{1-\lambda}{\lambda - b}$. We now have

$$\int_{\partial\mathbb{D}} \frac{|1-\lambda|^2}{|1-u|^2}\,\mathrm{d}\mu_\lambda^{\mathrm{ac}} = \int_{\partial\mathbb{D}} \frac{|1-\lambda|^2}{|1-u|^2}\frac{1-|b|^2}{|\lambda-b|^2}\,\mathrm{d}m = \int_{\partial\mathbb{D}} f|g|^2\,\mathrm{d}m$$

and

$$\int_{\partial\mathbb{D}} \left|\frac{1-b}{1-u}\right|^2 \mathrm{d}\mu_\lambda^{\mathrm{ac}} = \int_{\partial\mathbb{D}} \left|\frac{1-b}{1-u}\right|^2 \frac{1-|b|^2}{|\lambda-b|^2}\,\mathrm{d}m = \int_{\partial\mathbb{D}} f\left|\frac{1-b}{\lambda-b}\right|^2 \mathrm{d}m = \int_{\partial\mathbb{D}} f|1+g|^2\,\mathrm{d}m.$$

For the theorem, we are assuming $\int_{\partial\mathbb{D}} \frac{1-|b|^2}{|1-u|^2}\,\mathrm{d}m < \infty$, or $\int_{\partial\mathbb{D}} f\,\mathrm{d}m < \infty$. Theorem 4.4 then follows from the use of the triangle inequality in $L^2(f\,\mathrm{d}m)$. ∎

Finally, we can use this theorem to prove

THEOREM 4.5. *The function b has an angular derivative relative to u if and only if there is some function $h \in L^1(m)$ such that for almost every $\lambda \in \partial\mathbb{D}$ we have*

$$\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \frac{h(\lambda)}{|1-\lambda|^2}.$$

Proof. The proof is straightforward now. If b has an angular derivative relative to u, then, by Theorems 4.3 and 4.4, we have some function $h \in L^1(m)$ such that $\int_{\partial\mathbb{D}} \frac{|1-\lambda|^2}{|1-u|^2}\,\mathrm{d}\mu_\lambda < h(\lambda)$, which proves one direction of the theorem. To prove the other direction, assume there is a function $h \in L^1(m)$ with $\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \frac{h(\lambda)}{|1-\lambda|^2}$ for almost all $\lambda \in \partial\mathbb{D}$. Then pick any λ for which $h(\lambda)$ is defined and the above holds, and we then have, since $|\lambda - b|^{-1} \geqslant \frac{1}{2}$,

$$\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \frac{h(\lambda)}{|1-\lambda|^2} \to \int_{\partial\mathbb{D}} \frac{1-|b|^2}{|1-u|^2|\lambda-b|^2}\,\mathrm{d}m < \frac{h(\lambda)}{|1-\lambda|^2}$$
$$\to \int_{\partial\mathbb{D}} \frac{1-|b|^2}{|1-u|^2}\,\mathrm{d}m < \text{ some constant.}$$

This, plus Theorems 4.3 and 4.4 give us what we want, and our theorem is proved. This is part (ii) of the Theorem 3.2. ∎

Sarason, in Section IV-17 of [6], proves an even stronger version of part of this last theorem. He proves (the equivalent of):

Theorem 4.6. *If b is a nonextreme point of the unit ball of H^∞, and b has an angular derivative relative to u, then there is a positive constant C such that for all $\lambda \in \partial\mathbb{D} \setminus \{1\}$,*

$$\int\limits_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \frac{C}{|1-\lambda|^2}.$$

He shows, in fact, that we may take C to be $2\left(\frac{1+|u(0)|}{1-|u(0)|}\right)\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)}$.

D. Suarez [11] has recently shown that Theorem 4.6 above is true in the case where b is extreme (for a slightly larger constant C), if we consider only the absolutely continuous part of μ_λ. This is similar to what Sarason did in proving the theorem, except that in the case where b is nonextreme, μ_λ is absolutely continuous for almost all λ, so the proof in the absolutely continuous case is sufficient to give us the result for all μ_λ. When b is extreme, μ_λ does not necessarily have to be absolutely continuous for any λ, which can happen when, for example, b is inner.

Sarason [7] then showed that this stronger version is true for arbitrary holomorphic self-maps of the disk, by proving:

Theorem 4.7. *If b has an angular derivative relative to u, then there is a positive constant C such that for all $\lambda \in \partial\mathbb{D} \setminus \{1\}$,*

$$\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \frac{C}{|1-\lambda|^2}.$$

The following is Sarason's proof, which he has allowed to be presented here: First, we use a lemma by Poltoratskii in [5].

Lemma 4.8. *For each $\lambda \in \partial\mathbb{D}$, $\mathcal{H}(b) \subset L^2(\mu_\lambda)$, and the inclusion operator has norm at most 2.*

Next we have the following theorem:

Theorem 4.9. *If b has an angular derivative relative to u, then for $\lambda \in \partial\mathbb{D} \setminus \{1\}$, the function $\frac{1}{1-u}$ is well defined almost everywhere with respect to μ_λ (by nontangential limits), and it belongs to $L^2(\mu_\lambda)$, with*

$$\left\|\frac{1}{1-u}\right\|^2_{L^2(\mu_\lambda)} \leqslant \frac{20}{|1-\lambda|^2}\left(\frac{1+|u(0)|}{1-|u(0)|}\right)^2\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu_\lambda)}.$$

Proof. Fix λ in $\partial\mathbb{D} \setminus \{1\}$. We have

$$\begin{aligned} V_b\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) &= (1-b)K\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mu\right) = (1-b)K(\nu) \\ &= \frac{1-b}{1-u}V_u(1) = \frac{1-\overline{u(0)}u}{1-\overline{u(0)}}\left(\frac{1-b}{1-u}\right). \end{aligned} \tag{4.1}$$

Since $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$, we have $V_b\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \in \mathcal{H}(b)$, so by the lemma, $\frac{1-\overline{u(0)}u}{1-\overline{u(0)}}\left(\frac{1-b}{1-u}\right)$ is in $L^2(\mu_\lambda)$ and thus well defined almost everywhere with respect to μ_λ. Hence so is the function $\frac{1}{1-u}$, because b is almost never 1 with respect to μ_λ.

The lemma also tells us that $\frac{1-\overline{u(0)}u}{1-\overline{u(0)}}\left(\frac{1-b}{1-u}\right)$ has norm in $L^2(\mu_\lambda)$ of at most $2\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\|_{L^2(\mu)}$. Hence the function $\frac{1-b}{1-u}$ is in $L^2(\mu_\lambda)$, with norm not greater than $2\left(\frac{1+|u(0)|}{1-|u(0)|}\right)\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|_{L^2(\mu_\lambda)}$. Since $b=\lambda$ almost everywhere with respect to μ_λ^{s}, we can conclude that

$$\left\|\frac{1}{1-u}\right\|^2_{L^2(\mu_\lambda^{\mathrm{s}})} \leqslant \frac{4}{|1-\lambda|^2}\left(\frac{1+|u(0)|}{1-|u(0)|}\right)^2\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu_\lambda)}.$$

By equation (4.1) and the lemma,

$$\begin{aligned}4\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu_\lambda)} &\geqslant \left(\frac{1-|u(0)|}{1+|u(0)|}\right)^2\int_{\partial\mathbb{D}}\left|\frac{1-b}{1-u}\right|^2\mathrm{d}\mu \geqslant \left(\frac{1-|u(0)|}{1+|u(0)|}\right)^2\int_{\partial\mathbb{D}}\left|\frac{1-b}{1-u}\right|^2\mathrm{d}\mu^{\mathrm{ac}}\\ &= \left(\frac{1-|u(0)|}{1+|u(0)|}\right)^2\int_{\partial\mathbb{D}}\left|\frac{1-b}{1-u}\right|^2\frac{1-|b|^2}{|1-b|^2}\,\mathrm{d}m = \left(\frac{1-|u(0)|}{1+|u(0)|}\right)^2\int_{\partial\mathbb{D}}\frac{1}{|1-u|^2}(1-|b|^2)\,\mathrm{d}m.\end{aligned}$$

Hence,

$$\int_{\partial\mathbb{D}}\frac{1}{|1-u|^2}(1-|b|^2)\,\mathrm{d}m \leqslant 4\left(\frac{1+|u(0)|}{1-|u(0)|}\right)^2\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)}. \tag{4.2}$$

Again, by the lemma,

$$\left\|\frac{1-b}{1-u}\right\|_{L^2(\mu_\lambda^{\mathrm{ac}})} \leqslant \left\|\frac{1-b}{1-u}\right\|_{L^2(\mu_\lambda)} \leqslant 2\left(\frac{1+|u(0)|}{1-|u(0)|}\right)\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|_{L^2(\mu)},$$

which can be written

$$\int_{\partial\mathbb{D}}\left|\frac{1-b}{1-u}\right|^2\frac{1-|b|^2}{|\lambda-b|^2}\,\mathrm{d}m \leqslant 4\left(\frac{1+|u(0)|}{1-|u(0)|}\right)^2\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)}. \tag{4.3}$$

Since $\frac{1-b}{\lambda-b}=1+\frac{1-\lambda}{\lambda-b}$, we have

$$\frac{1}{|\lambda-b|^2} \leqslant \frac{2}{|1-\lambda|^2}\left(1+\left|\frac{1-b}{\lambda-b}\right|^2\right).$$

This, in conjunction with equations (4.2) and (4.3) gives

$$\begin{aligned}\left\|\frac{1}{1-u}\right\|^2_{L^2(\mu_\lambda^{\mathrm{ac}})} &= \int_{\partial\mathbb{D}}\frac{1}{|1-u|^2}\frac{1-|b|^2}{|\lambda-b|^2}\,\mathrm{d}m\\ &\leqslant \frac{2}{|1-\lambda|^2}\left[\int_{\partial\mathbb{D}}\frac{1}{|1-u|^2}(1-|b|^2)\,\mathrm{d}m + \int_{\partial\mathbb{D}}\left|\frac{1-b}{1-u}\right|^2\frac{1-|b|^2}{|\lambda-b|^2}\,\mathrm{d}m\right]\\ &\leqslant \frac{16}{|1-\lambda|^2}\left(\frac{1+|u(0)|}{1-|u(0)|}\right)^2\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)}.\end{aligned}$$

Combining the estimates for the squares of the norm of $\frac{1}{1-u}$ in $L^2(\mu_\lambda^{\mathrm{s}})$ and $L^2(\mu_\lambda^{\mathrm{ac}})$, we obtain the desired inequality in Theorem 4.9. ∎

This is then sufficient to prove Theorem 4.7, which we can also state as:

THEOREM 4.10. *If b has an angular derivative relative to u, then the measure $\frac{1}{|1-u|^2}\mu_\lambda$ is finite (for $\lambda \in \partial\mathbb{D} \setminus \{1\}$), with the square of its norm no greater than $\frac{20}{|1-\lambda|^2}\left(\frac{1+|u(0)|}{1-|u(0)|}\right)^2 \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)}$.*

When we put Theorem 4.7 together with Theorem 4.5 we get the following somewhat unexpected theorem:

THEOREM 4.11. *There is a function h in $L^1(m)$ with*

$$\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \frac{h(\lambda)}{|1-\lambda|^2}$$

for almost every $\lambda \in \partial\mathbb{D}$ if and only if there is a constant C such that for all $\lambda \in \partial\mathbb{D} \setminus \{1\}$,

$$\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \frac{C}{|1-\lambda|^2}.$$

We might want an even stronger result than the one above. In the case of ordinary angular derivatives, we have the theorem in Section VI-10 of [6]:

THEOREM 4.12. *If, for the point $z_0 \in \partial\mathbb{D}$, there is some $\lambda \in \partial\mathbb{D}$ such that $\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-\overline{z_0}z|^2} < \infty$, then b has an angular derivative at the point z_0.*

The generalization of this theorem would be the following:

CONJECTURE 4.13. *If, for some inner function u, there is some $\lambda \in \partial\mathbb{D}$ such that $\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \infty$, then b has an angular derivative relative to u.*

This conjecture, together with Theorem 4.7, would give us the following somewhat surprising result:

CONJECTURE 4.14. *If, for some inner function u, there is some $\lambda \in \partial\mathbb{D}$ such that $\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \infty$, then there is a constant C such that for all $\lambda \in \partial\mathbb{D} \setminus \{1\}$,*

$$\int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu_\lambda}{|1-u|^2} < \frac{C}{|1-\lambda|^2}.$$

5. GENERALIZATIONS OF JULIA'S LEMMA

For a holomorphic self-map of the disk b with an angular derivative at the point z_0, we have (see [6], VI-6) the theorem known as Julia's Lemma, which says that for all $z \in \mathbb{D}$

$$\frac{|b(z)-b(z_0)|^2}{1-|b(z)|^2} \leqslant c\frac{|z-z_0|^2}{1-|z|^2}$$

for the constant $c = |b'(z_0)|$.

We wish to generalize this lemma, just as we did the standard difference quotient, by replacing the identity function by an arbitrary inner function. Assuming as we did earlier that $b(z_0) = 1$, we replace z by $u(z)$ (where we here require $u(z_0) = 1$, too), and write

$$\frac{|1-b(z)|^2}{1-|b(z)|^2} \leqslant c\frac{|1-u(z)|^2}{1-|u(z)|^2}.$$

We ask if the above is true (for all $z \in \mathbb{D}$) for a suitable constant c, where here we will now assume that b has an angular derivative relative to u, that is, that $\nu \ll \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$.

To answer this, we note that

$$P\mu(z) = \int\limits_{\partial\mathbb{D}} P(\theta, z)\,\mathrm{d}\mu = \mathrm{Re}\left(\frac{1+b(z)}{1-b(z)}\right) = \frac{1-|b(z)|^2}{|1-b(z)|^2},$$

and similarly for ν and u, so that we can rewrite the previous inequality as

$$\frac{1}{P\mu(z)} \leqslant c\frac{1}{P\nu(z)} \quad \text{or} \quad \frac{P\nu(z)}{P\mu(z)} \leqslant c.$$

This leads us to

THEOREM 5.1. *For b a holomorphic self-map of the disk, and u an inner function, if b has an angular derivative relative to u, then there is a constant c such that*

$$\frac{|1-b(z)|^2}{1-|b(z)|^2} \leqslant c\frac{|1-u(z)|^2}{1-|u(z)|^2} \tag{5.1}$$

for all $z \in \mathbb{D}$ if and only if $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^\infty(\mu)$, in which case we may take c to be $\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|_\infty$.

Proof. The proof of this theorem is based on writing the inequality as $\frac{P\nu(z)}{P\mu(z)} \leqslant c$, where $\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mu$, so we can write the inequality now as

$$\frac{P\nu(z)}{P\mu(z)} = \frac{P\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\nu(z)}{P\mu(z)} = \frac{\int\limits_{\partial\mathbb{D}} P(\theta,z)\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu}{\int_{\partial\mathbb{D}} P(\theta,z)\,\mathrm{d}\mu} \leqslant c.$$

Since $P(\theta, z)$ is real and positive, we know that if $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^\infty(\mu)$, then

$$\int\limits_{\partial\mathbb{D}} P(\theta,z)\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu \leqslant \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|_\infty \int\limits_{\partial\mathbb{D}} P(\theta,z)\,\mathrm{d}\mu,$$

so for the choice of $c = \| \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \|_\infty$, the inequality (5.1) holds.

To prove the opposite direction, we assume that $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \notin L^\infty(\mu)$, i.e., that for any constant c, there is a set of positive μ^{s}-measure on which $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} > c$. We will then have, since the Poisson kernel is an approximate identity, for μ^{s}-a.e. ξ in such a set,

$$\frac{P\nu(z)}{P\mu(z)} = \frac{P\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mu(z)}{P\mu(z)} \to \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\xi) > c$$

as $z \to \xi$ nontangentially. Hence no constant c can make the desired inequality true for all $z \in \mathbb{D}$. This completes the proof of the theorem. ∎

We can look at the restriction of the above theorem in the same special case we were studying earlier, i.e., where $u(z) = \overline{z_0}z$ for some $z_0 \in \partial\mathbb{D}$. Then we have

Theorem 5.2. (Julia's Lemma) *For a holomorphic self-map of the disk b with an angular derivative at the point z_0,*

$$\frac{|b(z) - b(z_0)|^2}{1 - |b(z)|^2} \leqslant c\frac{|z - z_0|^2}{1 - |z|^2}$$

for all $z \in \mathbb{D}$ where we can take $c = |b'(z_0)|$.

Proof. This is an application of Theorem 5.1 with the function $\overline{b(z_0)}b$ and $u(z) = \overline{z_0}z$. Again, we get $\nu = \delta_{z_0}$, and since b has an angular derivative at z_0, μ has an atom at z_0, and we have $\nu \ll \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$. In fact, by a result in the proof of Theorem 15 in [9], we have $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(z_0) = |b'(z_0)|$, and is zero at all other points, so $\left\| \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right\|_\infty = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(z_0) = |b'(z_0)|$, and the theorem applies. The theorem gives us directly that for all $z \in \mathbb{D}$

$$\frac{|1 - \overline{b(z_0)}b(z)|^2}{1 - |\overline{b(z_0)}b(z)|^2} \leqslant \left\| \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right\|_\infty \left(\frac{|1 - \overline{z_0}z|^2}{1 - |\overline{z_0}z|^2} \right)$$

and this, given that $|b(z_0)| = 1$ (since b has an angular derivative at z_0) is easily seen to be equivalent to

$$\frac{|b(z) - b(z_0)|^2}{1 - |b(z)|^2} \leqslant |b'(z_0)|\frac{|z - z_0|^2}{1 - |z|^2}.$$

This completes the proof of Julia's lemma as a special case of the Theorem 5.1. ∎

Another way to look at our generalization of Julia's lemma is to see that we showed, for appropriate b and u,

$$\frac{\frac{|1-b(z)|^2}{1-|b(z)|^2}}{\frac{|1-u(z)|^2}{1-|u(z)|^2}} = \frac{P\nu(z)}{P\mu(z)}$$

or

$$\frac{\frac{|1-b(z)|}{1-|b(z)|}}{\frac{|1-u(z)|}{1-|u(z)|}} = \frac{\left(\frac{1+|b(z)|}{1+|u(z)|}\right)\left(\frac{P\nu(z)}{P\mu(z)}\right)}{\left|\frac{1-b(z)}{1-u(z)}\right|}. \tag{5.2}$$

The left side of the equality above is the ratio of two terms. The top, $\frac{|1-b(z)|}{1-|b(z)|}$ is, as $z \to \xi$ for some $\xi \in \partial\mathbb{D}$, in some sense, the angle at which $b(z)$ approaches $b(\xi)$. (Note that $b(\xi) = 1$ for ν-a.e. ξ). Similarly, the lower term is the same for u. The right side of the equality approaches 1 as $z \to \xi$ nontangentially for ν-a.e. $\xi \in \partial\mathbb{D}$. This is true because as $z \to \xi$ nontangentially for ν-a.e. $\xi \in \partial\mathbb{D}$, both $b(z)$ and $u(z) \to 1$, so $\frac{1+|b(z)|}{1+|u(z)|} \to 1$. Also, we know $\frac{1-b(z)}{1-u(z)} \to \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\xi)$, which is nonzero for ν-a.e. ξ, and, as we have seen before, $\frac{P\nu(z)}{P\mu(z)} \to \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\xi)$. Our conclusion is

THEOREM 5.3. *If b has an angular derivative relative to u, then for ν-a.e. $\xi \in \partial\mathbb{D}$, as $z \to \xi$ nontangentially, the angle at which $b(z)$ approaches 1 is the same (in the limit) as the angle at which $u(z)$ approaches 1.*

A different way to look at equation (5.2) is to write it as

$$\frac{1-|b(z)|}{1-|u(z)|} = \frac{|\frac{1-b(z)}{1-u(z)}|^2}{\left(\frac{1+|b(z)|}{1+|u(z)|}\right)\left(\frac{P\nu(z)}{P\mu(z)}\right)} \to \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\xi)$$

as $z \to \xi$ nontangentially, for ν-a.e. ξ. This is true since as $z \to \xi$ nontangentially, for ν-a.e. ξ, $\left|\frac{1-b(z)}{1-u(z)}\right| \to \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\xi)$, $\frac{1+|b(z)|}{1+|u(z)|} \to 1$, and $\frac{P\nu(z)}{P\mu(z)} \to \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\xi)$. Since $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^1(\nu)$ this gives us

THEOREM 5.4. *If b has an angular derivative relative to u, then the function on $\partial\mathbb{D}$ whose values are the nontangential limits of $\frac{1-|b|}{1-|u|}$ is in $L^1(\nu)$ and its norm is equal to $\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\|_{L^1(\nu)}$, or $\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\|^2_{L^2(\mu)}$.*

REMARK 5.5. Because of the above theorem, we might want to add the condition

$$\text{the boundary function of } \quad \frac{1-|b|}{1-|u|} \quad \text{is in } L^1(\nu)$$

to the list of conditions equivalent to the assertion that b has an angular derivative relative to u. We cannot do this, however, because the converse of the above theorem does not hold. Pick, for example, $b(z) = -z$ and $u(z) = z$. $\frac{1-|b|}{1-|u|} = 1 \in L^1(\nu)$, (note that again we define the boundary values of $\frac{1-|b|}{1-|u|}$ as the nontangential limits of values in the disk) but because $\nu = \delta_1$ and $\mu = \delta_{-1}$ we do not have $\nu \ll \mu$. In this case, however, $-b$ has an angular derivative relative to u. We can pick b and u, however, so that $\bar{\xi}b$ will not have an angular derivative relative to u for any $\xi \in \partial\mathbb{D}$. Such an example can be found by taking $b(z) = z^3$ and $u(z) = z^2$. Then the boundary function of $\frac{1-|b|}{1-|u|} = \frac{1-|z^3|}{1-|z^2|} = \frac{(1-|z|)(1+|z|+|z|^2)}{(1-|z|)(1+|z|)} = \frac{3}{2}$ (everywhere on $\partial\mathbb{D}$), but μ has atoms at the cube roots of unity, whereas ν has atoms at the square roots of unity, so $\nu \not\ll \mu$, therefore b does not have an angular derivative relative to u, nor, it is clear, would $\bar{\xi}b$ have an angular derivative relative to u for any $\xi \in \partial\mathbb{D}$.

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