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WEAK SEQUENTIAL CONVERGENCE IN THE DUAL OF OPERATOR IDEALS

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Dedicated to the memory of Karim Seddighi

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Abstract. By giving some necessary and sufficient conditions for the dual of operator subspaces to have the Schur property, we improve the results of Brown, Ülger and Saksman-Tylli in the Banach space setting. In particular, under some conditions on Banach spaces X and Y, we show that for a subspace $\mathcal{M}$ of operator ideal $\mathcal{U}(X,Y)$, $\mathcal{M}^*$ has the Schur property iff all point evaluations $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\}$ and $\widetilde{\mathcal{M}}_1(y^*) = \{T^*y^* : T \in \mathcal{M}_1\}$ are relatively norm compact, where $x \in X$, $y^* \in Y^*$ and $\mathcal{M}_1$ is the closed unit ball of $\mathcal{M}$.

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1. INTRODUCTION

A Banach space X has the Schur property if every weakly null sequence in X converges in norm. The simplest Banach space with this property is l_1. Following a work of S.W. Brown ([1]), A. Ülger ([10]) proved that if $\mathcal{M}^*$ the dual of a closed subspace $\mathcal{M}$ of $K(H)$ has the Schur property, then for all $x \in H$, the point evaluations $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\}$ and $\widetilde{\mathcal{M}}_1(x) = \{T^*x : T \in \mathcal{M}_1\}$ are relatively norm compact in H. This result has been generalized for the reflexive Banach spaces by E. Saksman and H.O. Tylli ([9]). Conversely, S.W. Brown ([1]), E. Saksman and H.O. Tylli ([9]), have proved that the relatively compactness of all point evaluations is also sufficient for the Schur property of $\mathcal{M}^*$, where $\mathcal{M}$ is the closed subspace of $K(H)$ or $K(l_p)$ with $1 < p < \infty$. Here we study the Schur property of the dual of closed subspaces of Banach operator ideals between Banach spaces and improve the results of [1], [9] and [10] to larger classes of Banach spaces and operators between them.

The notations and terminologies concerning Banach spaces are standard. Throughout this article H is a Hilbert space and X, Y and Z denote arbitrary Banach spaces. The closed unit ball of a Banach space X is denoted by X_1 and X^* is the dual of X. The duality between X and X^* is denoted by $\langle x, x^* \rangle$ and T^* refers to the adjoint of the operator T. $(\mathcal{U}, A)$ is always a (Banach) operator ideal $\mathcal{U}$ with norm A and its components are denoted by $\mathcal{U}(X, Y)$. For arbitrary Banach spaces X and Y we use $L(X,Y), W(X,Y)$ and $K(X,Y)$ for Banach spaces of all bounded linear, weakly compact and compact linear operators between Banach spaces X and Y respectively, and $K_{w^*}(X^*, Y)$ is the space of all compact weak*-weak continuous operators from X^* to Y. The abbreviation $K(X)$ is used for $K(X,X)$. The projective tensor product of X and Y is denoted by $X \widetilde{\otimes}_\pi Y$. We refer the reader to [3], [4] and [5] for undefined terminologies.

2. Necessary conditions for the Schur property

In this section we prove that for a closed subspace $\mathcal{M} \subseteq \mathcal{U}(X,Y)$, the Schur property of $\mathcal{M}^*$ imply the relatively compactness of all point evaluations $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\}$ and $\widetilde{\mathcal{M}_1}(y^*) = \{T^* y^* : T \in \mathcal{M}_1\}$, provided that one of the following is satisfied:

(1) X^* and Y are weakly sequentially complete (wsc);
(2) X^{**} and Y^* contain no copy of l_1.

In order to prove a key result of this section we give a necessary and sufficient condition for Banach spaces whose duals have the Schur property.

Theorem 2.1. *For each Banach space X, the following are equivalent:*
(i) *X^* has the Schur property;*
(ii) *$L(X,Y) = K(X,Y)$, for every* wsc *Banach space Y;*
(iii) *$W(X,Y) = K(X,Y)$, for every Banach space Y.*

Proof. (i) $\Rightarrow$ (ii) Assume that X^* has the Schur property. Then in particular, X contains no copy of l_1. If $T \in L(X,Y)$ and $(x_n) \subseteq X_1$ is an arbitrary sequence in X_1, then by Rosenthal's l_1-theorem ([3]), (x_n) has a weakly Cauchy subsequence (x_{n_k}). Thus (Tx_{n_k}) is weakly Cauchy and so is weakly convergent in Y. This shows that T is weakly compact, therefore T^* is also weakly compact. But by our hypothesis on X^*, T^* and so T is compact. Hence $L(X,Y) = K(X,Y)$.

(ii) $\Rightarrow$ (iii) Since, by Davis-Figiel-Johnson-Pełczynski's theorem ([3]), every weakly compact operator factors through a reflexive (and so wsc) Banach space, this implication is clear.

(iii) $\Rightarrow$ (i) Let (x_n^*) be a weakly null sequence in X^* and define $T : X \to c_0$ by $Tx = (x_n^*(x))_n$. By representation of weakly compact operators into c_0 ([3], p. 114), T is weakly compact and therefore by our hypothesis it is compact. But again by representation of compact operators into c_0, the sequence (x_n^*) converges in norm to zero. ∎

REMARK 2.2. The proof of theorem shows that the conditions of theorem are equivalent to the following condition that: For all reflexive Banach space Y, $L(X,Y) = K(X,Y)$. This is also equivalent to $W(X,c_0) = K(X,c_0)$. Note that the weak sequentially completeness condition in part (ii) of theorem is necessary. In fact if $Y = c_0$, by using the Josefson-Nissenzweig theorem ([3]), for every infinite dimensional Banach space X there exists a non compact bounded linear operator $T : X \to c_0$.

We deduce now an improvement of Theorem 1 of [10] and Theorem 4 of [9] from Theorem 2.1.

THEOREM 2.3. *Suppose that X^* and Y are* wsc *and $\mathcal{M}$ is a closed linear subspace of $\mathcal{U}(X,Y)$. If $\mathcal{M}^*$ has the Schur property, then all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively (norm) compact in Y and X^* respectively, for all $x \in X$ and $y^* \in Y^*$.*

Proof. For each $x \in X$ and $y^* \in Y^*$, consider the point evaluation operators $\varphi_x : \mathcal{M} \to Y$ and $\psi_{y^*} : \mathcal{M} \to X^*$ by $\varphi_x(T) = Tx$ and $\psi_{y^*}(T) = T^*y^*$. It is clear that these operators are bounded and by Theorem 2.1 are compact. So $\mathcal{M}_1(x) = \varphi_x(\mathcal{M}_1) \subseteq Y$ and $\widetilde{\mathcal{M}}_1(y^*) \subseteq X^*$ are relatively compact. ∎

If X and Y are Banach lattices, X contains no complemented copy of l_1 and Y contains no copy of c_0, then X^* and Y are wsc ([7], V.II) and we can apply Theorem 2.3 for closed subspace $\mathcal{M} \subseteq \mathcal{U}(X,Y)$. As another corollary, if instead of X and Y, the closed subspace $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ is a Banach lattice, we have the following corollary:

COROLLARY 2.4. *Suppose that X contains no complemented copy of l_1 and Y contains no copy of c_0. If $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ is a Banach lattice such that $\mathcal{M}^*$ has the Schur property, then all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact.*

Proof. Since X^* and Y contain no copy of c_0, by Theorem I.2 (c) of [6], there are wsc Banach lattices Z and Z', bounded operators $R : \mathcal{M} \to Z$, $S : Z \to Y$, $R' : \mathcal{M} \to Z'$ and $S' : Z' \to X^*$ such that $\varphi_x = SR$ and $\psi_{y^*} = S'R'$. Since Z and Z' are wsc, by Theorem 2.1, the operators R and R' and so φ_x and ψ_{y^*} are compact, for all $x \in X$ and $y^* \in Y^*$. ∎

We recall that an operator is completely continuous if it takes weakly convergent sequences into norm convergent sequences.

THEOREM 2.5. *Suppose that X^{**} and Y^* contain no copy of l_1 and $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ is a closed subspace. Then for the following assertions:*

(i) *$\mathcal{M}^*$ has the Schur property;*

(ii) *for every Banach space Z, every bounded linear operator $T : Z \to \mathcal{M}^*$ is completely continuous;*

(iii) *the natural restriction operator $R : \mathcal{U}(X,Y)^* \to \mathcal{M}^*$ is completely continuous;*

(iv) *all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact;*

the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) *are valid.*

Proof. Since $\mathcal{M}^*$ has the Schur property iff the identity operator on $\mathcal{M}^*$ is completely continuous, the implication (i) $\Leftrightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii) is obvious. For the proof of (iii) $\Rightarrow$ (iv), note that $\|T\| \leqslant A(T)$ for all $T \in \mathcal{U}(X,Y)$ and so the operator $\psi : X^{**}\widetilde{\otimes}_\pi Y^* \to \mathcal{U}(X,Y)^*$ defined by

$$v \mapsto \Big(T \mapsto \operatorname{tr}(T^{**}v) := \sum_{n=1}^{\infty} \langle T^{**}x_n^{**}, y_n^*\rangle\Big)$$

is linear and continuous, where $T \in \mathcal{U}(X,Y)$ and $v \in X^{**}\widetilde{\otimes}_\pi Y^*$ has a representation $v = \sum\limits_{n=1}^{\infty} x_n^{**} \otimes y_n^*$. So the operator $\varphi = R \circ \psi$ defined by $\langle \varphi(v), T\rangle = \operatorname{tr}(T^{**}v)$ is completely continuous.

Fix now an arbitrary element $x \in X$ and define the operator $U_x : Y^* \to X^{**}\widetilde{\otimes}_\pi Y^*$ by $U_x(y^*) = x \otimes y^*$. Since $\varphi \circ U_x : Y^* \to \mathcal{M}^*$ is completely continuous and Y^* contains no copy of l_1, by Rosenthal's l_1- theorem, $\varphi \circ U_x$ is compact. But $\varphi_x^* = \varphi \circ U_x$. So φ_x is compact and $\mathcal{M}_1(x)$ is relatively compact in Y.

Similarly, for $y^* \in Y^*$ the operator $(\psi_{y^*})^* = \varphi \circ V_{y^*} : X^{**} \to \mathcal{M}^*$ is completely continuous and so is compact, where $V_{y^*} : X^{**} \to X^{**}\widetilde{\otimes}_\pi Y^*$ via $V_{y^*}(x^{**}) = x^{**} \otimes y^*$. This shows that $\widetilde{\mathcal{M}}_1(y^*)$ is also relatively compact in X^*.

3. SUFFICIENT CONDITIONS FOR THE SCHUR PROPERTY

In this section by the technique given in [1], we improve the results of [1] and [9]. In particular we show that an analogous result of Corollary 4 of [9] is valid for the l_p-direct sum and c_0-direct sum of finite dimensional Banach spaces.

If V is a complemented subspace of a Banach space X, the projection of X onto V is denoted by P_V and $P_{V'} = I - P_V$ is the projection onto complementary subspace V' of V. If $\sum\limits_{n=1}^{\infty} \oplus X_n$ and $\sum\limits_{n=1}^{\infty} \oplus Y_n$ are Schauder decompositions of X and Y respectively, and $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ is a closed subspace, we say that $\mathcal{M}$ has the $\mathcal{P}$-property if for all integers m_0 and n_0 and every operators $T, S \in \mathcal{M}$,

$$\|P_W T P_V + P_{W'} S P_{V'}\| \leqslant \max\{\|P_W T P_V\|, \|P_{W'} S P_{V'}\|\},$$

where $V = X_1 \oplus \cdots \oplus X_{m_0}$ and $W = Y_1 \oplus \cdots \oplus Y_{n_0}$. Finally, if $\sum\limits_{n=1}^{\infty} \oplus X_n$ is a shrinking Schauder decomposition for X ([7]), we denote the corresponding Schauder decomposition of X^* by $\sum\limits_{n=1}^{\infty} \oplus X'_n$.

Theorem 3.1. *Let X and Y have monotone shrinking finite dimensional Schauder decompositions (*FDD*) $\sum\limits_{n=1}^{\infty} \oplus X_n$ and $\sum\limits_{n=1}^{\infty} \oplus Y_n$ respectively, and $\mathcal{M}$ be a closed subspace of $K_{w^*}(X^*, Y)$ which has the $\mathcal{P}$-property. If all point evaluations $\mathcal{M}_1(x^*)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact in Y and X respectively, then $\mathcal{M}^*$ has the Schur property.*

For the proof of this theorem we need two lemmas.

Lemma 3.2. *Let X and Y have Schauder decompositions $\sum_{n=1}^{\infty} \oplus X_n$ and $\sum_{n=1}^{\infty} \oplus Y_n$ respectively, such that the decomposition of X is shrinking. If $K_1, \dots, K_n \in K_{w^*}(X^*, Y)$ and $\varepsilon > 0$, then there are integers m_0 and n_0 such that*

$$\|P_{W'} K_i\| \leqslant \varepsilon \quad \textit{and} \quad \|K_i P_{V'}\| \leqslant \varepsilon, \quad i = 1, 2, \dots, n,$$

where $V = X_1' \oplus \dots \oplus X_{m_0}'$ and $W = Y_1 \oplus \dots \oplus Y_{n_0}$, V' and W' are complementary subspaces of V and W in X^* and Y respectively.

Proof. Without loss of generality, we may assume that $n = 1$ and $K = K_1 \in K_{w^*}(X^*, Y)$. Set $C = \sup \|P_{W'}\|$, where the supremum is taken over all $W' = \sum_{n>N} \oplus Y_n$, for $N \geqslant 1$.

If $\{z_1, \dots, z_l\}$ is an $\varepsilon/2C$-covering of $K(X_1^*)$ in Y and each z_i has a representation $z_i = \sum_{k=1}^{\infty} y_{i_k}$, choose an integer $n_0 > 0$ such that $\Big\| \sum_{k>n_0} y_{i_k} \Big\| < \varepsilon/2$, for all $1 \leqslant i \leqslant l$.

Now for each $x^* \in X_1^*$ and suitable $1 \leqslant i \leqslant l$,

$$\|P_{W'} K x^*\| \leqslant \|P_{W'}(Kx^* - z_i)\| + \|P_{W'} z_i\| \leqslant \|P_{W'}\| \, \|Kx^* - z_i\| + \Big\| \sum_{k>n_0} y_{i_k} \Big\| < \varepsilon.$$

This shows that $\|P_{W'} K\| < \varepsilon$ where $W = Y_1 \oplus \dots \oplus Y_{n_0}$.

Since $K^* : Y^* \to X$ is compact and $\sum_{n=1}^{\infty} \oplus X_n$ is a decomposition of X, we can deduce that there exists an integer m_0 such that $\|PK^*\| < \varepsilon$ where P is the canonical projection of X onto $\sum_{k>m_0} \oplus X_k$. Set $V = X_1' \oplus \dots \oplus X_{m_0}'$. Since $P_{V'} = P^*$ we have

$$\|K P_{V'}\| = \|K^{**} P^*\| = \|PK^*\| < \varepsilon. \quad \blacksquare$$

Lemma 3.3. *Let X and Y have shrinking* FDD*s $\sum_{n=1}^{\infty} \oplus X_n$ and $\sum_{n=1}^{\infty} \oplus Y_n$ respectively. Let m_0 and n_0 be arbitrary integers, $V = X_1' \oplus \dots \oplus X_{m_0}'$ and $W = Y_1 \oplus \dots \oplus Y_{n_0}$ and $\varepsilon > 0$ be given. If $\mathcal{M} \subseteq K_{w^*}(X^*, Y)$ is a closed subspace such that all point evaluations $\mathcal{M}_1(x^*)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are relatively compact, then there exists a norm closed subspace $\mathcal{G}$ of $\mathcal{M}$ of finite codimension such that $\|GP_V\| \leqslant \varepsilon$ and $\|P_W G\| \leqslant \varepsilon$ for all $G \in \mathcal{G}_1$.*

Proof. We first construct a norm closed subspace $\mathcal{E}$ of $\mathcal{M}$ of finite codimension such that $\|GP_V\| \leqslant \varepsilon$, for all $G \in \mathcal{E}_1$. For each $1 \leqslant i \leqslant m_0$, let $\{x_{ij}^* : 1 \leqslant j \leqslant n_i\}$ be a normalized basis of X_i' and choose a constant $C > 0$ such that for all $x^* = \sum_{i=1}^{\infty} x_i^* \in X_1^*$ with $x_i^* = \sum_{j=1}^{n_i} c_{ij} x_{ij}^*$; $\sum_{j=1}^{n_i} |c_{ij}| \leqslant C$, for all $1 \leqslant i \leqslant m_0$. Fix $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_i$. By assumption the point evaluation operator $\varphi_{ij} : \mathcal{M} \to Y$ defined by $\varphi_{ij}(T) = Tx_{ij}^*$ is compact. Set $\eta = \varepsilon/(m_0 C(K_Y + 2))$, where K_Y denotes the decomposition constant of $\sum_{n=1}^{\infty} \oplus Y_n$

and choose an η-covering $\{z_1,\dots,z_l\}$ of $\varphi_{ij}(\mathcal{M}_1)$. If $z_k=\sum\limits_{n=1}^{\infty} y_{k_n}$ we can choose an integer p such that $\Big\|\sum\limits_{n>p} y_{k_n}\Big\|<\eta$, for all $1\leqslant k\leqslant l$. Let $Z_{ij}:=\sum\limits_{n>p}\oplus Y_n$, then $\sup\{\|y\|: y\in Z_{ij}\cap\varphi_{ij}(\mathcal{M}_1)\}\leqslant\varepsilon/m_0C$.

It is easy to check that

$$\mathcal{E}:=\bigcap_{i=1}^{m_0}\bigcap_{j=1}^{n_i}\varphi_{ij}^{-1}(Z_{ij})$$

is norm closed and of finite codimension in $\mathcal{M}$. Let now $G\in\mathcal{E}_1$. Then $\|Gx^*_{ij}\|\leqslant\varepsilon/m_0C$, for all $1\leqslant j\leqslant n_i$ and $1\leqslant i\leqslant m_0$. If $x^*=\sum\limits_{i=1}^{\infty}x^*_i\in X^*_1$ then

$$\|GP_Vx^*\|=\Big\|G\sum_{i=1}^{m_0}x^*_i\Big\|\leqslant\sum_{i=1}^{m_0}\Big\|G\Big(\sum_{j=1}^{n_i}c_{ij}x^*_{ij}\Big)\Big\|\leqslant\sum_{i=1}^{m_0}\sum_{j=1}^{n_i}|c_{ij}|\,\|Gx^*_{ij}\|\leqslant\varepsilon.$$

Thus $\|GP_V\|\leqslant\varepsilon$.

By a similar method to the previous case, using the finite dimensional decomposition $\sum\limits_{n=1}^{\infty}\oplus Y'_n$ of Y^* and relative compactness of all $\widetilde{\mathcal{M}}_1(y^*)$ in X, we construct a norm closed subspace $\mathcal{F}$ of $\mathcal{M}$ of finite codimension such that $\|G^*P\|\leqslant\varepsilon$, for all $G\in\mathcal{F}_1$; where $P:Y^*\to Y'_1\oplus\cdots\oplus Y'_{n_0}$ is the canonical projection. If $P_W:Y\to Y_1\oplus\cdots\oplus Y_{n_0}$ is the canonical projection, it is straightforward to check that $P=P^*_W$. So $\|P_WG\|=\|G^*P^*_W\|=\|G^*P\|\leqslant\varepsilon$. Now set $\mathcal{G}=\mathcal{E}\cap\mathcal{F}$. ∎

Proof of Theorem 3.1. Since the decompositions of X^* and Y are monotone, by notation of Lemma 3.3, $\|P_V\|=\|P_W\|=1$, $\|P_{V'}\|\leqslant 2$ and $\|P_{W'}\|\leqslant 2$.

Let $(\Gamma_i)\subseteq\mathcal{M}^*$ be a normalized weakly null sequence in $\mathcal{M}^*$. Let (ε_i) be a sequence of positive numbers such that $\sum i\varepsilon_i<\infty$. Suppose that $\Lambda_1=\Gamma_1$, and choose $K_1\in\mathcal{M}_1$ such that $\langle K_1,\Lambda_1\rangle>1/3$. Inductively, assume that $\Lambda_1,\dots,\Lambda_n\in(\Gamma_i)$ and $K_1,\dots,K_n\in\mathcal{M}_1$ have been chosen. To obtain Λ_{n+1} and K_{n+1}, by Lemmas 3.2 and 3.3 we find finite dimensional subspaces V and W of X^* and Y respectively, and norm closed subspace $\mathcal{G}$ of finite codimension in $\mathcal{M}$ such that

$$\begin{array}{lll}\|K_iP_{V'}\|\leqslant\varepsilon_{n+1} \quad\text{and}\quad \|P_{W'}K_i\|\leqslant\varepsilon_{n+1} & \text{for all } i=1,2,\dots,n,\\ \|GP_V\|\leqslant\varepsilon_{n+1} \quad\text{and}\quad \|P_WG\|\leqslant\varepsilon_{n+1} & \text{for all } G\in\mathcal{G}_1.\end{array}$$

By the technique given in the proof of Theorem 1.1 of [1], we can choose $\Lambda_{n+1}\in(\Gamma_i)$ and $K_{n+1}\in\mathcal{M}_1$ such that

$$\begin{array}{lll}|\langle K_i,\Lambda_{n+1}\rangle|<2^{-n-1} & & \text{for } i=1,2,\dots,n,\\ \langle K_{n+1},\Lambda_{n+1}\rangle>\frac{1}{3} & \text{and}\quad \langle K_{n+1},\Lambda_j\rangle=0 & \text{for } j=1,2,\dots,n.\end{array}$$

Also $\|K_{n+1}P_V\|<\varepsilon_{n+1}$ and $\|P_WK_{n+1}\|<\varepsilon_{n+1}$. These properties yield that

$$\Big\|P_W\sum_{i=1}^{n}K_iP_V-\sum_{i=1}^{n}K_i\Big\|\leqslant 4n\varepsilon_{n+1}\quad\text{and}\quad\|P_{W'}K_{n+1}P_{V'}-K_{n+1}\|\leqslant 5\varepsilon_{n+1}.$$

Hence

$$\Big\|\sum_{i=1}^{n+1} K_i\Big\| \leqslant \Big\|\sum_{i=1}^{n} K_i - P_W \sum_{i=1}^{n} K_i P_V\Big\| + \|K_{n+1} - P_{W'} K_{n+1} P_{V'}\|$$
$$+ \Big\|P_W \sum_{i=1}^{n} K_i P_V + P_{W'} K_{n+1} P_{V'}\Big\|$$
$$\leqslant (4n+5)\varepsilon_{n+1} + \max\Big\{\Big\|\sum_{i=1}^{n} K_i\Big\|, 4\Big\},$$

where the last inequality holds by $\mathcal{P}$-property of $\mathcal{M}$. This shows that the sequence $T_n = \sum_{i=1}^{n} K_i$ is bounded and so has a weak* limit point $T \in \mathcal{M}^{**}$. For each j, choose an integer $n > j$ such that $|\langle T - T_n, \Lambda_j\rangle| < 1/2^j$. Therefore

$$|\langle T, \Lambda_j\rangle| \geqslant \Big|\sum_{i=1}^{j}\langle K_i, \Lambda_j\rangle| - \frac{1}{2^j} \geqslant |\langle K_j, \Lambda_j\rangle\Big| - \sum_{i=1}^{j-1}|\langle K_i, \Lambda_j\rangle| - \frac{1}{2^j} \geqslant \frac{1}{3} - \frac{j}{2^j} > \frac{1}{4},$$

for sufficiently large j. Hence $\langle T, \Lambda_j\rangle$ and so $\langle T, \Gamma_j\rangle$ does not tend to zero. Thus the sequence (Γ_j) does not converge weakly to zero and the proof is completed. ∎

As a corollary, since $K(X,Y)$ isometrically isomorphic to $K_{w^*}(X^{**},Y)$, via $T \mapsto T^{**}$ ([8]), if we replace the role of X in the above theorem by X^*, we obtain an analogous result for closed subspaces of $K(X,Y)$. By a proof similar to that of the Theorem 3.1, one can deduce the following theorem for $K(X,Y)$:

Theorem 3.4. *Let X and Y have monotone shrinking* FDD*s, $\mathcal{M}$ be a closed subspace of $K(X,Y)$ which has the $\mathcal{P}$-property. If all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact in Y and X^* respectively, then $\mathcal{M}^*$ has the Schur property.*

There are several Banach spaces that are (isometrically) isomorphic or (isometrically) embed into $K_{w^*}(X^*,Y)$ ([8]), and one can obtain similar results as Theorem 3.1 for these spaces.

In the following two corollaries we give a large class of Banach spaces such that the space of compact operators between them possesses the $\mathcal{P}$-property.

Corollary 3.5. *Let X be an l_p-direct sum and Y be an l_q-direct sum of finite dimensional Banach spaces and $1 < p \leqslant q < \infty$. If $\mathcal{M}$ is a closed subspace of $K(X,Y)$ such that all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact in Y and X^* respectively, then $\mathcal{M}^*$ has the Schur property.*

Proof. By Theorem 3.4, it is enough to prove that $K(X,Y)$ has the $\mathcal{P}$-property. It is clear that for arbitrary bounded operators $U_1 : X_1 \to Y_1$ and $U_2 : X_2 \to Y_2$, the direct sum operator $U_1 \oplus U_2 : X_1 \oplus_p X_2 \longrightarrow Y_1 \oplus_q Y_2$ has norm equal to $\max\{\|U_1\|, \|U_2\|\}$, where $X_1 \oplus_p X_2$ is the l_p-direct sum of X_1 and X_2.

Now for the bounded linear operators $P_W T P_V|V : V \to W$ (restriction of $P_W T P_V$ to V) and $P_{W'} S P_{V'}|V' : V' \to W'$ we have

$$\|P_W T P_V|V \oplus P_{W'} S P_{V'}|V'\| = \max\{\|P_W T P_V|V\|, \|P_{W'} S P_{V'}|V'\|\}$$
$$\leqslant \max\{\|P_W T P_V\|, \|P_{W'} S P_{V'}\|\}.$$

Since $V \oplus_p V'$ and $W \oplus_q W'$ are isometrically isomorphic to X and Y respectively, and the operator $P_W T P_V | V \oplus P_{W'} S P_{V'} | V'$ as an operator from X to Y is equal to $P_W T P_V + P_{W'} S P_{V'}$, the proof is completed. ∎

By Theorem 2.5 and Corollary 3.5, we can conclude that for a closed subspace $\mathcal{M}$ of $K(X, Y)$ the four assertions of Theorem 2.5 are equivalent, where X and Y are l_p and l_q-direct sum of finite dimensional Banach spaces respectively.

Corollary 3.6. *Let X have a monotone shrinking* FDD *and Y be a c_0-direct sum of finite dimensional Banach spaces. If $\mathcal{M}$ is either a closed subspace of $K(X, Y)$ or $K_{w^*}(X^*, Y)$ such that all of the corresponding point evaluations are relatively compact, then $\mathcal{M}^*$ has the Schur property.*

Proof. We will show that $\mathcal{M}$ again has the $\mathcal{P}$-property. For each $x \in X_1$ (respectively $x^* \in X_1^*$) since $P_W T P_V x$ and $P_{W'} S P_{V'} x$ belong to W and W' respectively, according to the definition of c_0-norm in Y,

$$\begin{aligned} \|P_W T P_V x + P_{W'} S P_{V'} x\| &= \max\{\|P_W T P_V x\|, \|P_{W'} S P_{V'} x\|\} \\ &\leqslant \max\{\|P_W T P_V\|, \|P_{W'} S P_{V'}\|\}. \end{aligned}$$

So for each V and W described in the definition of $\mathcal{P}$-property,

$$\|P_W T P_V + P_{W'} S P_{V'}\| \leqslant \max\{\|P_W T P_V\|, \|P_{W'} S P_{V'}\|\}. \quad ∎$$

Remarks 3.7. (i) By Pitt's theorem, $K(l_p, l_q)$ for $p > q > 1$ is reflexive and so has no infinite dimensional closed subspace with (dual) Schur property; but if we apply Corollary 3.5 to $K(l_p, l_q)$ for $1 < p \leqslant q < \infty$, we see that this corollary extends Corollary 4 of [9]. For the non-reflexive case, by Corollary 3.6 one can obtain another improvement of Corollary 4 of [9] to the space $K(l_p, c_0)$ where $1 < p < \infty$.

(ii) Saksman and Tylli ([9]), have constructed a closed subspace $\Delta \subseteq K(l_p \oplus l_q)$, $1 < p < q < \infty$, such that all point evaluations are relatively compact but the dual of Δ does not have the Schur property. This shows that the condition of finite dimension in the above theorems is necessary and secondly, $l_p \oplus l_q$ does not have any representation of the form l_r-direct sum of finite dimensional Banach spaces where $1 < r < \infty$ is arbitrary.

(iii) By Theorem 7 of [10] if H is a Hilbert space and A is a closed subalgebra of $K(H)$ such that all left and right multiplication operators on A to be compact, then A^* has the Schur property. Since its proof is based on the fact that relatively compactness of all point evaluations imply the Schur property of dual subspace, by Corollaries 3.5 and 3.6, we have an analogous result of Theorem 7 of [10] for closed subalgebra A of $K(X)$ where X is either an l_p or c_0-direct sum of finite dimensional Banach spaces with $1 < p < \infty$.

(iv) If H_1 and H_2 are two Hilbert spaces and $\mathcal{M}$ is a closed subspace of $K(H_1, H_2)$, then $\mathcal{M}^*$ has the Schur property iff all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y)$ are relatively compact, for all $x \in H_1$ and $y \in H_2$. In fact, its necessary proof is an application of Theorem 2.3 or Theorem 2.5 and the sufficient proof is completely similar to [1].

(v) The proof of Corollaries 3.5 and 3.6 also shows that $L(X, Y)$ and all its closed subspaces have the $\mathcal{P}$-property, where either X and Y are l_p and l_q-direct sum of Banach spaces respectively, with $1 < p \leqslant q < \infty$ or X has a Schauder decomposition and Y is a c_0-direct sum of Banach spaces. However we do not know what another (closed subspaces of) operator ideals between Banach spaces have the $\mathcal{P}$-property.

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