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THE WAVELET GALERKIN OPERATOR

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Abstract. We consider the eigenvalue problem

$$R_{m_0,m_0}h = \lambda h, \quad h \in C(\mathbb{T}), \, |\lambda| = 1,$$

where R_{m_0,m_0} is the wavelet Galerkin operator associated to a wavelet filter m_0. The solution involves the construction of representations of the algebra $\mathfrak{A}_N$ — the C^*-algebra generated by two unitaries U, V satisfying $UVU^{-1} = V^N$ introduced in [13].

Keywords: *Wavelets, Hilbert space, unitary operators, operator algebras, transfer operator, Ruelle operator, spectrum.*

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1. INTRODUCTION

The wavelet Galerkin operator appears in several different contexts such as wavelets (see for example [15], [10], [6], [17], [7]), ergodic theory and g-measures ([14]) or quantum statistical mechanics ([19]). For some of the applications of the Ruelle operator we refer the reader to the book by V. Baladi ([1]). It also bears many different names in the literature: the Ruelle operator, the Perron-Frobenius-Ruelle operator, the Ruelle-Araki operator, the Sinai-Bowen-Ruelle operator, the transfer operator and several others. We used the name wavelet Galerkin operator as suggested in [15], because of its close connection to wavelets that we will be using in the sequel. We will also use the name Ruelle operator and transfer operator.

The Ruelle operator considered in this paper is defined by

$$R_{m_0,m_0'}f(z) = \frac{1}{N}\sum_{w^N=z}\overline{m_0(w)}m_0'(w)f(w), \quad z \in \mathbb{T},$$

where $m_0, m_0' \in L^\infty(\mathbb{T})$ are nonsingular (i.e. they do not vanish on a set of positive measure), $\mathbb{T}$ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, $N \geqslant 2$ is an integer. A large amount of information about this operator is contained in [3]. One of the main objectives of this paper is to do a peripheral spectral analysis for the Ruelle operator, that is to solve the equation

$$R_{m_0,m_0} h = \lambda h, \quad |\lambda| = 1,\, h \in C(\mathbb{T}).$$

The restrictions that we will impose on m_0 are:

(1.1) $m_0 \in \mathrm{Lip}_1(\mathbb{T})$, where $\mathrm{Lip}_1(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} : f \text{ is Lipschitz}\}$;

(1.2) m_0 has a finite number of zeros;

(1.3) $R_{m_0,m_0} 1 = 1$;

(1.4) $m_0(1) = \sqrt{N}$.

In ergodic theory the Ruelle operators are used in the derivation of correlation inequalities (see [20] and [11]) and in understanding the Gibbs measures. The role played by the Ruelle operator in wavelet theory is somewhat similar. It can be used to make a direct connection to the cascade approximation and orthogonality relations.

In the applications to wavelets, the function m_0 is a wavelet filter, i.e., its Fourier expansion

$$m_0(z) = \sum_{k \in \mathbb{Z}} a_k z^k \tag{1.5}$$

yield the masking coefficients of the scaling function φ on $\mathbb{R}$, i.e. the function which results from the the fixed-point problem

$$\varphi(x) = \sqrt{N} \sum_{k \in \mathbb{Z}} a_k \varphi(Nx - k). \tag{1.6}$$

Then the solution φ is used in building a multiresolution for the wavelet analysis. If, for example, conditions can be placed on (1.5) which yield $L^2(\mathbb{R})$-solutions to (1.6), then the closed subspace V_0 spanned by the translates $\{\varphi(x-k) : k \in \mathbb{Z}\}$ is invariant under the scaling operator

$$Uf(x) = \frac{1}{\sqrt{N}} f\Big(\frac{x}{N}\Big), \quad x \in \mathbb{R}, \tag{1.7}$$

i.e. $U(V_0) \subset V_0$. Setting $V_j := U^j(V_0)$ for $j \in \mathbb{Z}$ we get the resolution

$$\cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \cdots$$

from which wavelets can be constructed as in [9].

The cascade operator is defined on $L^2(\mathbb{R})$ from the masking coefficients by:

$$M_a \psi = \sqrt{N} \sum_{n \in \mathbb{Z}} a_n \psi(N \cdot - n).$$

The scaling function φ is then a fixed point for the cascade operator, it satisfies the scaling equation $M_a \varphi = \varphi$.

Now set

$$p(\psi_1,\psi_2)(\mathrm{e}^{\mathrm{i}t}) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{int} \int_{\mathbb{R}} \overline{\psi_1}(x)\psi_2(x-n)\,\mathrm{d}x, \quad \psi_1,\psi_2 \in L^2(\mathbb{R}).$$

The relation between the Ruelle operator R_{m_0,m_0} and the cascade operator M_a is

$$R_{m_0,m_0}(p(\psi_1,\psi_2)) = p(M_a\psi_1, M_a\psi_2),$$

and this makes the transfer operator an adequate tool in the analysis of the orthogonality relations.

One of the fundamental problems in wavelet theory is to give necessary and sufficient conditions on m_0 such that the translates of the scaling function $\{\varphi(\,\cdot - n) : n \in \mathbb{Z}\}$ form an orthonormal set. There are two well known results that answer this question: one due to Lawton ([16]), which says that one such condition is that R_{m_0,m_0} as an operator on continuous function has 1 as a simple eigenvalue, the other, due to A. Cohen ([4]), which says that the orthogonality is equivalent to the fact that m_0 has no nontrivial cycles (a cycle is a set $\{z_1,\ldots,z_p\}$ with $z_1^N = z_2,\ldots,z_{p-1}^N = z_p, z_p^N = z^1$ and $|m_0(z_i)| = \sqrt{N}$ for all $i \in \{1,\ldots,p\}$; the trivial cycle is $\{1\}$).

The peripheral spectral analysis in this paper will elucidate, among other things, why these two conditions are equivalent.

The wavelet theory gives a representation of the algebra $\mathfrak{A}_N$ (i.e. the C^*-algebra generated by two unitary operators U and V subject to the relation $UVU^{-1} = V^N$) on $L^2(\mathbb{R})$. U is the scaling operator in (1.7) and V is the translation by 1 $V : \psi \to \psi(\cdot - 1)$. In fact we also have a representation of $L^\infty(\mathbb{T})$ on $L^2(\mathbb{R})$ given by $\pi(f)\psi = \sum_{n\in\mathbb{Z}} c_n\psi(\,\cdot - n)$, for $f = \sum_{n\in\mathbb{Z}} c_n z^n \in L^\infty(\mathbb{T})$.

The scaling equation (1.6) can be rewritten as

$$U\varphi = \pi(m_0)\varphi.$$

This representation of $\mathfrak{A}_N$ together with the scaling function φ is called the *wavelet representation.*

In [13] it is proved that there is a one-to-one correspondence between positive solutions to $R_{m_0,m_0}h = h$ and representations of $\mathfrak{A}_N$. These representations are in fact given by the unitary $U : \mathcal{H} \to \mathcal{H}$, a representation $\pi : L^\infty(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ satisfying

$$U\pi(f) = \pi(f(z^N))U, \quad f \in L^\infty(\mathbb{T})$$

and $\varphi \in \mathcal{H}$ with $U\varphi = \pi(m_0)\varphi$.

We reproduce here the theorem:

THEOREM 1.1. (i) *Let $m_0 \in L^\infty(\mathbb{T})$, and suppose m_0 does not vanish on a subset of $\mathbb{T}$ of positive measure. Let*

$$(Rf)(z) = \frac{1}{N}\sum_{w^N=z} |m_0(w)|^2 f(w), \quad f \in L^1(\mathbb{T}). \tag{1.8}$$

Then there is a one-to-one correspondence between the data (a) *and* (b) *below, where* (b) *is understood as equivalence classes under unitary equivalence:*

(a) $h \in L^1(\mathbb{T})$, $h \geqslant 0$, *and*

$$R(h) = h. \tag{1.9}$$

(b) $\widetilde{\pi} \in \mathrm{Rep}(\mathfrak{A}_N, \mathcal{H})$, $\varphi \in \mathcal{H}$, *and the unitary* U *from* $\widetilde{\pi}$ *satisfying*

$$U\varphi = \pi(m_0)\varphi. \tag{1.10}$$

(ii) *From* (a) $\Rightarrow$ (b), *the correspondence is given by*

$$\langle \varphi : \pi(f)\varphi \rangle_{\mathcal{H}} = \int_{\mathbb{T}} fh \,\mathrm{d}\mu, \tag{1.11}$$

where μ *denotes the normalized Haar measure on* $\mathbb{T}$.

From (b) $\Rightarrow$ (a), *the correspondence is given by*

$$h(z) = h_{\varphi}(z) = \sum_{n\in\mathbb{Z}} z^n \langle \pi(e_n)\varphi : \varphi \rangle_{\mathcal{H}}. \tag{1.12}$$

(iii) *When* (a) *is given to hold for some* h, *and* $\widetilde{\pi} \in \mathrm{Rep}(\mathfrak{A}_N, \mathcal{H})$ *is the corresponding cyclic representation with* $U\varphi = \pi(m_0)\varphi$, *then the representation is unique from* h *and* (1.11) *up to unitary equivalence: that is, if* $\pi' \in \mathrm{Rep}(\mathfrak{A}_N, \mathcal{H}')$, $\varphi' \in \mathcal{H}'$ *also cyclic and satisfying*

$$\langle \varphi' : \pi'(f)\varphi' \rangle = \int_{\mathbb{T}} fh \,\mathrm{d}\mu \quad \text{and} \quad U'\varphi' = \pi'(m_0)\varphi',$$

then there is a unitary isomorphism W *of* $\mathcal{H}$ *onto* $\mathcal{H}'$ *such that* $W\pi(A) = \pi'(A)W$, $A \in \mathfrak{A}_N$, *and* $W\varphi = \varphi'$.

DEFINITION 1.2. Given h as in Theorem 1.1 call $(\pi, \mathcal{H}, \varphi)$ the *cyclic representation of* $\mathfrak{A}_N$ associated to h.

In the case of the orthogonality of the translates of the scaling function φ, the wavelet representation is in fact the cyclic representation corresponding to the unique fixed point of the Ruelle operator R_{m_0,m_0}, which is the constant function 1.

We will also need the results from [12] which show the connection between solutions to $R_{m_0,m_0'}h = h$ and operators that intertwine these representations. Here are those results:

THEOREM 1.3. *Let* $m_0, m_0' \in L^\infty(\mathbb{T})$ *be non-singular and* $h, h' \in L^1(\mathbb{T})$, $h, h' \geqslant 0$, $R_{m_0,m_0}(h) = h$, $R_{m_0',m_0'}(h') = h'$. *Let* $(\pi, \mathcal{H}, \varphi)$, $(\pi', \mathcal{H}', \varphi')$ *be the cyclic representations corresponding to* h *and* h' *respectively.*

If $h_0 \in L^1(\mathbb{T})$, $R_{m_0,m_0'}(h_0) = h_0$ *and* $|h_0|^2 \leqslant chh'$ *for some* $c > 0$ *then there exists a unique operator* $S : \mathcal{H}' \to \mathcal{H}$ *such that*

$$SU' = US, \quad S\pi'(f) = \pi(f)S, \quad \langle \varphi : \pi(f)S\varphi' \rangle = \int_{\mathbb{T}} fh_0 \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T}).$$

Moreover $\|S\| \leqslant \sqrt{c}$.

THEOREM 1.4. *Let* $m_0, m_0', h, h', (\pi, \mathcal{H}, \varphi), (\pi', \mathcal{H}', \varphi')$ *be as in Theorem* 1.3. *Suppose* $S : \mathcal{H}' \to \mathcal{H}$ *is a bounded operator that satisfies*

$$SU' = US, \quad S\pi'(f) = \pi(f)S, \quad f \in L^\infty(\mathbb{T}).$$

Then there exists a unique $h_0 \in L^1(\mathbb{T})$ *such that*

$$R_{m_0,m_0'} h_0 = h_0 \quad \text{and} \quad \langle \varphi : S\pi'(f)\varphi' \rangle = \int_{\mathbb{T}} f h_0 \, \mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T}).$$

Moreover, $|h_0|^2 \leqslant \|S\|^2 h h'$ *almost everywhere on* $\mathbb{T}$.

These theorems indicate the correspondence between intertwining operators and the fixed points of the Ruelle operator. This correspondence projects a C^*-algebra structure on the eigenspace corresponding to the eigenvalue 1 (Theorem 2.7, Corollary 2.8), and this algebra is in fact abelian (Theorem 2.3).

To find the solutions $R_{m_0,m_0} h = h$ we construct the representation associated to the function $h_{\max} = 1$. Then, if we can compute the commutant, the solutions will follow from Theorems 1.3 and 1.4.

We will see how each cycle of m_0 gives rise to a representation of $\mathfrak{A}_N$, hence to a positive solution for $R_{m_0,m_0} h = h$ (Proposition 2.13). The representation we are looking for (the one associated to $h_{\max} = 1$) will be a direct sum of the representations constructed for the cycles of m_0 (Theorem 2.16).

The solution of the eigenvalue problem mentioned in the abstract is given in Theorem 2.5 and Corollary 2.18.

2. PERIPHERAL SPECTRAL ANALYSIS

We begin this section by analysing the intertwining operators a little bit further. We will see that the commutator of the cyclic representation associated to a positive h with $R_{m_0,m_0} h = h$ is abelian and we will find the eigenfunction h that corresponds to the composition of two intertwining opertors that correspond to h_1 and h_2 respectively.

In Corrolary 3.9 of [13] it is proved that the cyclic representation $(\mathcal{H}_h, \pi_h, \varphi_h)$ corresponding to some $h \geqslant 0$ with $R_{m_0,m_0} h = h$ is given by:

$$\mathcal{H}_h := \left\{ (\xi_0, \ldots, \xi_n, \ldots) : \sup_n \int_{\mathbb{T}} R^n_{m_0,m_0}(|\xi_n|^2 h) \, \mathrm{d}\mu < \infty, R_{m_0,m_0}(\xi_{n+1} h) = \xi_n h \right\},$$

$$\pi_h(f)(\xi_0, \ldots \xi_n, \ldots) = (f(x)\xi_0, \ldots, f(z^N)\xi_n, \ldots), \quad f \in L^\infty(\mathbb{T}),$$

$$U_h(\xi_0, \ldots, \xi_n, \ldots) = (m_0(z)\xi_1, \ldots, m_0(z^{N^n})\xi_{n+1}, \ldots),$$

$$\langle (\xi_0, \ldots, \xi_n, \ldots) : (\eta_0, \ldots \eta_n, \ldots) \rangle = \lim_{n \to \infty} \int_{\mathbb{T}} R^n_{m_0,m_0}(\overline{\xi_n} \eta_n h) \, \mathrm{d}\mu$$

and

$$\varphi_h = (1, 1, \ldots, 1, \ldots).$$

Also, we have the subspaces $H_0^h \subset H_1^h \subset \cdots \subset H_n^h \subset \cdots \subset \mathcal{H}_h$ whose union is dense in $\mathcal{H}_h$ where $H_n^h := \{(\xi_0, \ldots, \xi_n, \ldots) \in \mathcal{H}_h : \xi_{n+k}(z) = \xi_n(z^{N^k}), \text{ for } k \geqslant 0\}$. The set $\mathcal{V}_n^h := \{U_h^{-n} \pi_h(f) \varphi_h : f \in L^\infty(\mathbb{T})\}$ is dense in H_n^h for all $n \geqslant 0$ and $U_h^n H_n^h = H_0^h$.

SOME NOTATIONS. If m_0 and h are as in Theorem 1.1, then, we denote by $(\mathcal{H}_h, \pi_h, \varphi_h)$ the cyclic representation associated to h.

If m_0, m_0', h, h' and h_0 are as in Theorem 1.3 then denote by S_{h,h',h_0} the intertwining operator from $\mathcal{H}_{h'}$ to $\mathcal{H}_h$ given by the aforementioned theorem.

Sometime we will omit the subscripts.

LEMMA 2.1. *Let $P_{H_0^h}$ be the projection onto the subspace H_0^h. Then $P_{H_0^h} S_{h,h',h_0} P_{H_0^{h'}}$ is multiplication by $\frac{h_0}{h}$ on $H_0^{h'}$ i.e.*

$$P_{H_0^h} S_{h,h',h_0} P_{H_0^{h'}} (\xi(z), \xi(z^N), \ldots, \xi(z^{N^n}), \ldots) = \Big(\xi(z)\frac{h_0(z)}{h(z)}, \xi(z^N)\frac{h_0(z^N)}{h(z^N)}, \ldots, \xi(z^{N^n})\frac{h_0(z^{N^n})}{h(z^{N^n})}, \ldots\Big).$$

Proof. Denote $S\varphi_{h'} = (\varphi_0^S, \ldots, \varphi_n^S, \ldots)$. Then for all $f \in L^\infty(\mathbb{T})$

$$\int_{\mathbb{T}} f h_0 \, \mathrm{d}\mu = \langle (1, 1, \ldots, 1, \ldots) : \pi_h(f)(\varphi_0^S, \ldots, \varphi_n^S, \ldots)\rangle$$

$$= \lim_{n\to\infty} \int_{\mathbb{T}} R^n_{m_0,m_0}(f(z^{N^n})\varphi_n^S h) \, \mathrm{d}\mu = \lim_{n\to\infty} \int_{\mathbb{T}} f(z)\varphi_0^S h \, \mathrm{d}\mu = \int_{\mathbb{T}} f\varphi_0^S h \, \mathrm{d}\mu,$$

thus $\varphi_0^S = \frac{h_0}{h}$. Consider again an $f \in L^\infty(\mathbb{T})$ arbitrary.

$$\begin{aligned} P_{H_0^h} S P_{H_0^{h'}} \pi_{h'}(f)\varphi_{h'} &= P_{H_0^h} S \pi_{h'}(f)\varphi_{h'} = P_{H_0^h} \pi_h(f) S\varphi_{h'} \\ &= P_{H_0^h} (f(z)\varphi_0^S, \ldots, f(z^{N^n})\varphi_n^S, \ldots) \\ &= (f(z)\varphi_0^S, \ldots, f(z^{N^n})\varphi_0^S(z^{N^n}), \ldots). \end{aligned}$$

This calculation shows that $P_{H_0^h} S P_{H_0^{h'}}$ is multiplication by $\frac{h_0}{h}$ on $\mathcal{V}_0^{h'}$, so, by density, on $H_0^{h'}$. ∎

LEMMA 2.2. *$P_{H_n^h} S_{h,h',h_0} P_{H_n^{h'}}$ converges to S_{h,h',h_0} in the strong operator topology.*

Proof. Let $\xi \in \mathcal{H}_{h'}$. Then

$$\begin{aligned} \|P_{H_n^h} S P_{H_n^{h'}} \xi - S\xi\| &\leqslant \|P_{H_n^h} S P_{H_n^{h'}} \xi - P_{H_n^h} S\xi\| + \|P_{H_n^h} S\xi - S\xi\| \\ &\leqslant \|P_{H_n^h}\| \, \|S\| \, \|P_{H_n^{h'}} \xi - \xi\| + \|P_{H_n^h} S\xi - S\xi\| \to 0 \quad \text{as } n \to \infty \end{aligned}$$

because the subspaces H_n^h form an increasing sequence whose union is dense in $\mathcal{H}_h$ (and similarly for $H_n^{h'}$). ∎

THEOREM 2.3. *The commutant $\pi_h(\mathfrak{A}_n)'$ is abelian.*

Proof. Consider $S_1, S_2 \in \pi_h(\mathfrak{A}_n)'$. Then, according to Theorem 1.4, $S_1 = S_{h_1}$, $S_2 = S_{h_2}$, for some h_1, h_2 with $R_{m_0,m_0} h_i = h_i$, $|h_i| \leqslant c_i h$, $i \in \{1, 2\}$. Let $\xi \in \mathcal{H}_h$. It has a decomposition $\xi = \xi_0 + \eta$ with $\xi_0 \in H_0^h$ and $\eta \in {H_0^h}^\perp$. Using Lemma 2.1

$$\begin{aligned} (P_{H_0^h} S_1 P_{H_0^h})(P_{H_0^h} S_2 P_{H_0^h})(\xi) &= P_{H_0^h} S_1 P_{H_0^h} S_2 \xi_0 = P_{H_0^h} S_2 P_{H_0^h} S_1 \xi_0 \\ &= (P_{H_0^h} S_2 P_{H_0^h})(P_{H_0^h} S_1 P_{H_0^h})\xi. \end{aligned}$$

Since $P_{H_n^h} = U^{-n} P_{H_0^h} U^n$ it follows that $P_{H_n^h} S_1 P_{H_n^h}$ and $P_{H_n^h} S_2 P_{H_n^h}$ also commute. Lemma 2.2 can be used to get $S_1 S_2$ as the strong limit of $(P_{H_n^h} S_1 P_{H_n^h})(P_{H_n^h} S_2 P_{H_n^h})$. Similarly for $S_2 S_1$. And as the limit is unique we must have $S_1 S_2 = S_2 S_1$. ∎

Next, suppose we have two intertwining operators $S_1 : \mathcal{H}_h \to \mathcal{H}_{h'}$, $S_2 : \mathcal{H}_{h'} \to \mathcal{H}_{h''}$ which come from h_1 and h_2 respectively. Then S_2S_1 is also an intertwining operator so it must come from some h_3. We want to find the relation between h_1, h_2 and h_3.

THEOREM 2.4. *If $S_{h_1} : \mathcal{H}_h \to \mathcal{H}_{h'}$ and $S_{h_2} : \mathcal{H}_{h'} \to \mathcal{H}_{h''}$ are intertwining operators then, if $S_{h_3} = S_{h_2}S_{h_1}$. We have for all $f \in L^\infty(\mathbb{T})$:*

$$\int_{\mathbb{T}} |f(z)|^2 R^n_{m_0,m_0}\Big(\Big|\frac{h_1}{h'}\frac{h_2}{h''} - \frac{h_3}{h''}\Big|^2 h''\Big)\,\mathrm{d}\mu \to 0.$$

Proof. We begin with a calculation. For $f \in L^\infty(\mathbb{T})$:

$$\begin{aligned} P_{H_n^{h'}} S_1 P_{H_n^h}(U_h^{-n}\pi_h(f)\varphi_h) &= U_{h'}^{-n} P_{H_0^{h'}} U_{h'}^n S_1 U_h^{-n} P_{H_0^h} U_h^n U_h^{-n}\pi_h(f)\varphi_h \\ &= U_{h'}^{-n} P_{H_0^{h'}} S_1 P_{H_0^h}\pi_h(f)\varphi_h = U_{h'}^{-n}\pi_{h'}\Big(f\frac{h_1}{h'}\Big)\varphi_{h'}. \end{aligned}$$

For the second equality we used the fact that S_1 is intertwining and for the last one, Lemma 2.1.

$$\begin{aligned} &(P_{H_n^{h''}} S_2 P_{H_n^{h'}})(P_{H_n^{h'}} S_1 P_{H_h^n})(U_h^{-n}\pi_h(f)\varphi_h) \\ &\quad = (P_{H_n^{h''}} S_2 P_{H_n^{h'}})U_{h'}^{-n}\pi_{h'}\Big(f\frac{h_1}{h'}\Big)\varphi_{h'} = U_{h''}^{-n}\pi_{h''}\Big(f\frac{h_1}{h'}\frac{h_2}{h''}\Big)\varphi_{h''}. \end{aligned} \tag{2.1}$$

Similarly

$$(P_{H_n^{h''}} S_2 S_1 P_{H_n^h})(U_h^{-n}\pi_h(f)\varphi_h) = U_{h''}^{-n}\pi_h{}''\Big(f\frac{h_3}{h''}\Big)\varphi_h{}''. \tag{2.2}$$

Using (2.1), (2.2) and the notation $m_0^{(n)}(z) := m_0(z)m_0(z^N)\cdots m_0(z^{N^{n-1}})$, we have

$$\begin{aligned} &\|(P_{H_n^{h''}} S_2 P_{H_n^{h'}})(P_{H_n^{h'}} S_1 P_{H_n^h})(\pi_h(f)\varphi_h) - (P_{H_n^{h''}} S_2 S_1 P_{H_n^h})(\pi_h(f)\varphi_h)\|_{\mathcal{H}_{h''}} \\ &\quad = \|(P_{H_n^{h''}} S_2 P_{H_n^{h'}})(P_{H_n^{h'}} S_1 P_{H_n^h})U_h^{-n}\pi_h(f(z^{N^n})m_0^{(n)})\varphi_h \\ &\qquad\quad - (P_{H_n^{h''}} S_2 S_1 P_{H_n^h})U_h^{-n}\pi_h(f(z^{N^n})m_0^{(n)})\varphi_h\|_{\mathcal{H}_{h''}} \\ &\quad = \Big\|U_{h''}^{-n}\Big(\pi_{h''}\Big(f(z^{N^n})m_0^{(n)}(z)\frac{h_1}{h'}\frac{h_2}{h''}\Big)\Big)\varphi_{h''} \\ &\qquad\quad - U_{h''}^{-n}\pi_{h''}\Big(f(z^{N^n})m_0^{(n)}(z)\frac{h_3}{h''}\Big)\varphi_{h''}\Big\|_{\mathcal{H}_{h''}} \\ &\quad = \int_{\mathbb{T}} |f(z^{N^n})|^2 |m_0^{(n)}(z)|^2\Big|\frac{h_1}{h'}\frac{h_2}{h''} - \frac{h_3}{h''}\Big|^2 h''\,\mathrm{d}\mu \\ &\quad = \int_{\mathbb{T}} |f(z)|^2 R^n_{m_0,m_0}\Big(\Big|\frac{h_1}{h'}\frac{h_2}{h''} - \frac{h_3}{h''}\Big|^2 h''\Big)\,\mathrm{d}\mu. \end{aligned}$$

But, by Lemma 2.2, the first term in this chain of equalities converges to 0 for all $f \in L^\infty(\mathbb{T})$ so we obtain the desired conclusion. ∎

COROLLARY 2.5. *If $S_{h_1}, S_{h_2} \in \pi_h(\mathfrak{A}_N)'$, $S_{h_3} = S_{h_1}S_{h_2}$ and $h \in L^\infty(\mathbb{T})$ then*

$$\int_{\mathbb{T}} |g| \left| R^n_{m_0,m_0}\Big(\frac{h_1h_2}{h}\Big) - h_3 \right| \mathrm{d}\mu \to 0, \quad g \in L^\infty(\mathbb{T}).$$

Proof. We will need the following inequality

$$|R^n_{m_0,m_0}(\xi h)|^2 \leqslant R^n_{m_0,m_0}(|\xi|^2 h)h. \tag{2.3}$$

This can be proved using Schwartz's inequality:

$$\begin{aligned}|R^n_{m_0,m_0}(\xi h)|^2 &= \Big|\frac{1}{N^n}\sum_{w^{N^n}=z}|m_0^{(n)}(w)|^2\xi(w)h(w)\Big|^2\\ &\leqslant \Big(\frac{1}{N^n}\sum_{w^{N^n}=z}|m_0^{(n)}(w)|^2|\xi(w)|^2h(w)\Big)\Big(\frac{1}{N^n}\sum_{w^{N^n}=z}|m_0^{(n)}(w)|^2h(w)\Big)\\ &= R^n_{m_0,m_0}(|\xi|^2h)R^n_{m_0,m_0}h = R^n_{m_0,m_0}(|\xi|^2h)h.\end{aligned}$$

Now, take $g \in L^\infty(\mathbb{T})$ and $f = gh^{1/2}$ in Theorem 2.4 ($h = h' = h''$). We have:

$$\begin{aligned}&\Big(\int_{\mathbb{T}} |g| \left|R^n_{m_0,m_0}\Big(\frac{h_1h_2}{h}\Big) - h_3\right| \mathrm{d}\mu\Big)^2 \leqslant \int_{\mathbb{T}} |g|^2 \left|R^n_{m_0,m_0}\Big(\frac{h_1h_2}{h}\Big) - h_3\right|^2 \mathrm{d}\mu\\ &= \int_{\mathbb{T}} |g|^2 \left|R^n_{m_0,m_0}\Big(\Big(\frac{h_1}{h}\frac{h_2}{h} - \frac{h_3}{h}\Big)h\Big)\right|^2 \mathrm{d}\mu \leqslant \int_{\mathbb{T}} |g|^2 h R^n_{m_0,m_0}\Big(\left|\frac{h_1}{h}\frac{h_2}{h} - \frac{h_3}{h}\right|^2 h\Big)\,\mathrm{d}\mu\\ &= \int_{\mathbb{T}} |f|^2 R^n_{m_0,m_0}\Big(\left|\frac{h_1}{h}\frac{h_2}{h} - \frac{h_3}{h}\right|^2 h\Big)\,\mathrm{d}\mu \to 0. \quad \blacksquare\end{aligned}$$

In the sequel, we consider intertwining operators that correspond to continuous eigenfunctions h. We will prove that if h_1 and h_2 are continuous and $S_{h_3} = S_{h_1}S_{h_2}$ then h_3 must be also continuous. The fundamental result needed here is from [3]:

THEOREM 2.6. *Let m_0 be a function on $\mathbb{T}$ satisfying $m_0 \in \mathrm{Lip}_1(\mathbb{T})$, $R_{m_0,m_0}1 \leqslant 1$ and consider the restriction of R_{m_0,m_0} to $\mathrm{Lip}_1(\mathbb{T})$ going into $\mathrm{Lip}_1(\mathbb{T})$. It follows that R_{m_0,m_0} has at most a finite number $\lambda_1, \dots, \lambda_p$ of eigenvalues of modulus 1, $|\lambda_i| = 1$, and R has a decomposition*

$$R_{m_0,m_0} = \sum_{i=1}^{p} \lambda_i T_{\lambda_i} + S, \tag{2.4}$$

where T_{λ_i} and S are bounded operators from $\mathrm{Lip}_1(\mathbb{T})$ to $\mathrm{Lip}_1(\mathbb{T})$, T_{λ_i} have finite-dimensional range, and

$$T^2_{\lambda_i} = T_{\lambda_i}, \quad T_{\lambda_i}T_{\lambda_j} = 0 \text{ for } i \neq j, \quad T_{\lambda_i}S = ST_{\lambda_i} = 0, \tag{2.5}$$

and there exist positive constants M, h such that

$$\|S^n\|_{\mathrm{Lip}_1(\mathbb{T})\to\mathrm{Lip}_1(\mathbb{T})} \leqslant \frac{M}{(1+h)^n} \tag{2.6}$$

for $n = 1, 2, \ldots$. *Furthermore* $\|R_{m_0,m_0}\|_{\infty\to\infty} \leqslant 1$, *and there is a constant* M *such that*

$$\|S^n\|_{\infty\to\infty} \leqslant M \tag{2.7}$$

for $n = 1, 2, \ldots$.

Finally, the operators T_{λ_i} *and* S *extend to bounded operators* $C(\mathbb{T}) \to C(\mathbb{T})$, *and the properties* (2.4) *and* (2.5) *still hold for this extension. Moreover*

$$\lim_{n\to\infty} S^n f = 0, \quad T_{\lambda_i}(f) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_i^{-k} R^k_{m_0,m_0}(f), \quad f \in C(\mathbb{T}).$$

Proof. Everything is contained in [3], Theorem 3.4.4, Proposition 4.4.4 and its proof. ∎

THEOREM 2.7. *Assume* m_0 *is Lipschitz,* $R_{m_0,m_0}1 \leqslant 1$, $h \geqslant 0$ *is continuous,* $R_{m_0,m_0}h = h$. *If* $S_{h_1}, S_{h_2} \in \pi_h(\mathfrak{A}_N)'$, *with* h_1, h_2 *continuous and* $S_{h_3} = S_{h_1}S_{h_2}$ *then* h_3 *is also continuous and* $h_3 = T_1\left(\frac{h_1h_2}{h}\right) = \lim_{n\to\infty} R^n_{m_0,m_0}\left(\frac{h_1h_2}{h}\right)$, *uniformly.*

Proof. By Corollary 2.5 we have:

$$\int_{\mathbb{T}} gR^n\Big(\frac{h_1h_2}{h}\Big)\,\mathrm{d}\mu \to \int_{\mathbb{T}} gh_3\,\mathrm{d}\mu, \quad g \in L^\infty(\mathbb{T}). \tag{2.8}$$

Also, observe that $\frac{h_1h_2}{h}$ is continuous because $|h_1| \leqslant c_1h$, $|h_2| \leqslant c_2h$ for some positive constants c_1, c_2, and if $x_0 \in \mathbb{T}$ with $h(x_0) = 0$ then $h_1(x_0) = 0$, $h_2(x_0) = 0$ and $|\frac{h_1h_2}{h}| \leqslant c_2h_1$ Relation (2.8) implies that for all $g \in L^\infty(\mathbb{T})$

$$\int_{\mathbb{T}} \frac{1}{m}\sum_{n=0}^{m-1} R^n\Big(\frac{h_1h_2}{h}\Big)\,\mathrm{d}\mu \to \int_{\mathbb{T}} gh_3\,\mathrm{d}\mu$$

However, by Theorem 2.6, we have

$$\frac{1}{m}\sum_{n=0}^{m-1} R^n\Big(\frac{h_1h_2}{h}\Big) \to T_1\Big(\frac{h_1h_2}{h}\Big), \text{ uniformly.}$$

Therefore $h_3 = T_1\left(\frac{h_1h_2}{h}\right)$.

Next we want to prove that $R^n\left(\frac{h_1h_2}{h}\right) \to h_3$ uniformly. By Proposition 4.4.4, [3], this is equivalent to $T_{\lambda_i}\left(\frac{h_1h_2}{h}\right) = 0$ for $\lambda_i \neq 1$.

From (2.8) it follows, using Theorem 2.6, that

$$\sum_{\lambda_1\neq 1} \lambda_i^n \int_{\mathbb{T}} gT_{\lambda_i}\Big(\frac{h_1h_2}{h}\Big)\,\mathrm{d}\mu \to 0 \tag{2.9}$$

for all $g \in L^\infty(\mathbb{T})$.

But $T_{\lambda_i}\left(\frac{h_1h_2}{h}\right)$ are eigenvectors corresponding to different eigenvalues so, some are 0 and the rest are linearly independent. For all i with $T_{\lambda_i}\left(\frac{h_1h_2}{h}\right) \neq 0$ we can find a $g_i \in L^\infty(\mathbb{T})$ such that $\int\limits_{\mathbb{T}} g_i T_{\lambda_i}\left(\frac{h_1h_2}{h}\right) \mathrm{d}\mu = 1$ and $\int\limits_{\mathbb{T}} g_i T_{\lambda_j}\left(\frac{h_1h_2}{h}\right) \mathrm{d}\mu = 0$ for $\lambda_j \neq \lambda_i$ (this can be obtain from the fact that $L^\infty(\mathbb{T})$ is the dual of $L^1(\mathbb{T})$ which contains the vectors $T_{\lambda_i}\left(\frac{h_1h_2}{h}\right)$). Then, if we use (2.9) for g_i, we get that $\lambda_i^n \to 0$ whenever $T_{\lambda_i}\left(\frac{h_1h_2}{h}\right) \neq 0$, $\lambda_i \neq 1$, which is clearly absurd unless all $T_{\lambda_i}\left(\frac{h_1h_2}{h}\right)$ are 0, for $\lambda_i \neq 1$. Thus, as we have mentioned before, this implies that $R^n\left(\frac{h_1h_2}{h}\right) \to h_3$. ∎

COROLLARY 2.8. *If $h \in C(\mathbb{T})$, $h \geqslant 0$, $R_{m_0,m_0}h = h$ then the space*

$$\{h_0 \in C(\mathbb{T}) : R_{m_0,m_0}h_0 = h_0, |h_0| \leqslant ch\}$$

is a finite dimensional abelian C^*-algebra under the pointwise addition and multiplication by scalars, complex conjugation and the product given by $h_1 * h_2$ defined by $S_{h_1 * h_2} = S_{h_1}S_{h_2}$.

Proof. Everything follows from Theorem 2.7 and Theorem 2.3. For the finite dimensionality see [3] or [7]. ∎

REMARK 2.9. When $h = 1$ the C^*-algebra structure given in Corollary 2.8 is the same as the one introduced in Theorem 5.5.1, [3].

Now we will show how each m_0-cycle (see Definition 2.10 below) gives rise to a continuous solution $h \geqslant 0$, $R_{m_0,m_0}h = h$. In the end we will see that any eigenfunction $R_{m_0,m_0}h = h$ is a linear combination of eigenfunctions coming from such cycles.

DEFINITION 2.10. Let $m_0 \in C(\mathbb{T})$. An *m_0-cycle* is a set $\{z_1, \ldots, z_p\}$ contained in $\mathbb{T}$ such that $z_i^N = z_{i+1}$ for $i \in \{1, \ldots, p-1\}$, $z_p^N = z_1$ and $|m_0(z_i)| = \sqrt{N}$ for $i \in \{1, \ldots, p\}$.

First, we consider the eigenfunction that corresponds to the cycle $\{1\}$. This appears in many instances and it is the one that defines the scaling function in the theory of multiresolution approximations (see [9], [3]).

PROPOSITION 2.11. *Let $m_0 \in \mathrm{Lip}_1(\mathbb{T})$ with $m_0(1) = \sqrt{N}$, $R_{m_0,m_0}1 = 1$. Define*

$$\varphi_{m_0,1}(x) = \prod_{k=1}^{\infty} \frac{m_0\left(\frac{x}{N^k}\right)}{\sqrt{N}}, \quad x \in \mathbb{R}.$$

(i) *$\varphi_{m_0,1}$ is a well defined, continuous function and it belongs to $L^2(\mathbb{R})$.*

(ii) *If $h_{m_0,1} = \mathrm{Per}\,|\varphi_{m_0,1}|^2$ is Lipschitz (trigonometric polynomial if m_0 is one), where*

$$\mathrm{Per}(f)(x) := \sum_{k \in \mathbb{Z}} f(x + 2k\pi), \quad x \in [0, 2\pi],\ f : \mathbb{R} \to \mathbb{C}.$$

Also $R_{m_0,m_0}h_{m_0,1} = h_{m_0,1}$, $h_{m_0,1}(1) = 1$, $h_{m_0,1}$ is 0 on every m_0-cycle disjoint of $\{1\}$.

(iii) *If* $U_1 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(U_1\xi)(x) = \sqrt{N}\xi(Nx)$ *and* $\pi_1(f) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $\pi_1(f)(\xi) = f\xi$ *for all* $f \in L^\infty(\mathbb{T})$, *then* $(U_1, \pi_1, \varphi_{m_0,1})$ *define the cyclic representation corresponding to* $h_{m_0,1}$.

(iv) *The commutant of the representation from* (iii) *is* $\{M_f : f \in L^\infty(\mathbb{R}),\ f(Nx) = f(x) \text{ a.e.}\}$, *where* M_f *is the operator of multiplication by* f.

(v) $h_{m_0,1}$ *is minimal, in the sense that if* $0 \leqslant h' \leqslant ch_{m_0,1}$, $c > 0$, h' *continuous and* $R_{m_0,m_0}h' = h'$ *then there exists a* $\lambda \geqslant 0$ *such that* $h' = \lambda h_{m_0,1}$.

(vi) *If* $h \geqslant 0$ *is continuous,* $R_{m_0,m_0}h = h$ *and* $h(1) = 1$ *then* $h \geqslant h_{m_0,1}$.

Proof. (i) See [9] or [3].

(ii) See Theorem 5.1.1 and Lemma 5.5.6 in [3].

For (iv) see [12]. Also, in [12] it is proved that we are dealing with a representation of $\mathfrak{A}_N$ (it is the Fourier transform of the wavelet representation mentioned in the introduction). We only need to check that $\varphi_{m_0,1}$ is cyclic for this representation.

Consider P the projection onto the subspace generated by $\pi_1(\mathfrak{A}_N)\varphi_{m_0,1}$. We prove first that P commutes with the representation. Take $A \in \pi_1(\mathfrak{A}_N)$, A selfadjoint. If $B \in \pi_1(\mathfrak{A}_N)$ then $A(B\varphi_{m_0,1}) \in \pi_1(\mathfrak{A}_N)\varphi_{m_0,1}$ so $PA(B\varphi_{m_0,1}) = A(B\varphi_{m_0,1})$. So $PAP = AP$. Then

$$AP = PAP = (PAP)^* = (AP)^* = PA,$$

so P commutes with A, and since any member of $\pi_1(\mathfrak{A}_N)$ is a linear combination of selfadjoint operators from this set, it follows that P lies in the commutant of the representation. Then, by (iv), $P = M_f$ for some $f \in L^\infty(\mathbb{R})$ with $f(Nx) = f(x)$ a.e. As P is a projection $f^2 = f = \overline{f}$ so $f = \chi_A$ for some subset A of the real line. But $P\varphi_{m_0,1} = \varphi_{m_0,1}$ so $\varphi_{m_0,1}\chi_A = \varphi_{m_0,1}$ a.e. Since $\varphi_{m_0,1}(0) = 1$ and $\varphi_{m_0,1}$ is continuous, it follows that A contains a neighbourhood of 0. This, coupled with the fact that $\chi_A(Nx) = \chi_A(x)$ a.e., imply that $\chi_A = 1$ a.e. so P is the identity and thus $\pi_1(\mathfrak{A}_N)\varphi_{m_0,1}$ is dense, which means exactly that $\varphi_{m_0,1}$ cyclic.

(v) Consider h' as mentioned in the hypothesis. Then h' induces a member of the commutant $S_{h'}$. By (iv), $S_{h'} = M_{f_{h'}}$ for some $f_{h'} \in L^\infty(\mathbb{R})$ with $f_{h'}(Nx) = f_{h'}(x)$ a.e. We have

$$\langle \varphi_{m_0,1} : S_{h'}\pi_1(f)\varphi_{m_0,1}\rangle = \int_{\mathbb{T}} f h' \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T}),$$

which implies that

$$h' = \mathrm{Per}(\overline{\varphi_{m_0,1}} S_{h'}\varphi_{m_0,1}) = \mathrm{Per}(f_{h'}|\varphi_{m_0,1}|^2).$$

We prove that $f_{h'}$ is continuous at 0.

$$h'(x) = f_{h'}(x)|\varphi_{m_0,1}|^2(x) + \sum_{k\neq 0} f_{h'}(x + 2k\pi)|\varphi_{m_0,1}|^2(x + 2k\pi). \tag{2.10}$$

As

$$h_{m_0,1}(x) = |\varphi_{m_0,1}|^2(x) + \sum_{k\neq 0} |\varphi_{m_0,1}|^2(x + 2k\pi)$$

and $h_{m_0,1}(0) = |\varphi_{m_0,1}|^2(0) = 1$ and $h_{m_0,1}$, $\varphi_{m_0,1}$ are continuous, it follows that

$$\sum_{k\neq 0} |\varphi_{m_0,1}|^2(x+2k\pi) \to 0 \quad \text{as } x \to 0.$$

Then, as $x \to 0$,

$$\Big\| \sum_{k\neq 0} f_{h'}(x+2k\pi)|\varphi_{m_0,1}|^2(x+2k\pi)\Big\| \leqslant \|f_{h'}\|_\infty \sum_{k\neq 0} |\varphi_{m_0,1}|^2(x+2k\pi) \to 0.$$

Using this in (2.10) we obtain that $\lim_{x\to 0} f_{h'}(x) = h'(0)$. But $f_{h'}(Nx) = f_{h'}(x)$ a.e. so $f_{h'} = h'(0)$ a.e. which implies that $h' = h'(0)h_{m_0,1}$.

(vi) This is contained also in [3] but here is a different proof. Consider $f \in L^\infty(\mathbb{T})$, arbitrary. Define

$$\varphi_n(x) = f(x)\chi_{[-N^n\pi, N^n\pi]} h^{1/2}\Big(\frac{x}{N^n}\Big) \prod_{k=1}^{n} \frac{m_0\big(\frac{x}{N^k}\big)}{\sqrt{N}}.$$

Clearly $\varphi_n(x) \to f(x)\varphi_{m_0,1}$, $x \in \mathbb{R}$ and

$$\begin{aligned}\int_{\mathbb{R}} |\varphi_n(x)|^2\,\mathrm{d}x &= \int_{-N^n\pi}^{N^n\pi} |f|^2(x) h\Big(\frac{x}{N^n}\Big)\prod_{k=1}^{n}\frac{|m_0|^2\big(\frac{x}{N^k}\big)}{N}\,\mathrm{d}x\\ &= \int_{-\pi}^{\pi} |f|^2(N^n y)h(y)\prod_{k=0}^{n-1}|m_0|^2(N^k x)\,\mathrm{d}y\\ &= \int_{-\pi}^{\pi} R^n_{m_0,m_0}(h(y)|f|^2(N^n y))\,\mathrm{d}y = \int_{-\pi}^{\pi}|f|^2(y)h(y)\,\mathrm{d}y.\end{aligned}$$

Using Fatou's lemma we obtain:

$$\int_{\mathbb{R}} |f(x)|^2|\varphi_{m_0,1}|^2\,\mathrm{d}y = \int_{\mathbb{R}} \liminf_n |\varphi_n|^2\,\mathrm{d}x \leqslant \liminf_n \int_{\mathbb{R}} |\varphi_n(x)|^2\,\mathrm{d}x = \int_{\mathbb{T}} |f|^2 h\,\mathrm{d}\mu$$

and after periodization

$$\int_{\mathbb{T}} |f|^2 h_{m_0,1}\,\mathrm{d}\mu \leqslant \int_{\mathbb{T}} |f|^2 h\,\mathrm{d}\mu.$$

As f was arbitrary this shows that $h_{m_0,1} \leqslant h$. ∎

Now we generalize a little bit, by considering a cycle $\{z_0\}$ where $z_0^N = z_0$.

PROPOSITION 2.12. *Let $m_0 \in \mathrm{Lip}_1(\mathbb{T})$, $z_0 \in \mathbb{T}$ with $z_0^N = z_0$, $m_0(z_0) = \sqrt{N}\mathrm{e}^{\mathrm{i}\theta_0}$, $R_{m_0,m_0}1 = 1$. Define*

$$\varphi_{m_0,z_0}(x) = \prod_{k=1}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\theta_0}\alpha_{z_0}(m_0)\big(\frac{x}{N^k}\big)}{\sqrt{N}}, \quad x \in \mathbb{R},$$

where $\alpha_\rho(f)(z) = f(\rho z)$ for $z, \rho \in \mathbb{T}$ and $f \in L^\infty(\mathbb{T})$.

(i) φ_{m_0,z_0} *is a well defined continuous function that belongs to* $L^2(\mathbb{R})$.

(ii) $h_{m_0,z_0} := \alpha_{z_0^{-1}}(\mathrm{Per}\,|\varphi_{m_0,z_0}|^2)$ *is Lipschitz (trigonometric polynomial if* m_0 *is one),* $R_{m_0,m_0}h_{m_0,z_0} = h_{m_0,z_0}$, $h_{m_0,z_0}(z_0) = 1$, h_{m_0,z_0} *is* 0 *on every* m_0*-cycle disjoint of* $\{z_0\}$.

(iii) *If* $U_{z_0} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $U_{z_0}\xi = \mathrm{e}^{\mathrm{i}\theta_0}U_1\xi$ *and* $\pi_{z_0}(f)(\xi) = \pi_1(\alpha_{z_0}(f))(\xi)$ *for* $f \in L^\infty(\mathbb{T})$, *then* $(U_{z_0}, \pi_{z_0}, \varphi_{m_0,z_0})$ *define the cyclic representation corresponding to* h_{m_0,z_0}.

(iv) *The commutant of this representation is* $\{M_f : f \in L^\infty(\mathbb{R}),\ f(Nx) = f(x)$ *a.e.*$\}$.

(v) h_{m_0,z_0} *is minimal (see Proposition* 2.11 (v)*).*

(vi) *If* $h \geqslant 0$ *is continuous,* $R_{m_0,m_0}h = h$ *and* $h(z_0) = 1$ *then* $h \geqslant h_{m_0,z_0}$.

Proof. Consider $m_0' := \mathrm{e}^{-\mathrm{i}\theta_0}\alpha_{z_0}(m_0)$. We check that m_0' satisfies the hypotheses of Proposition 2.11. Clearly m_0' is Lipschitz, $m_0'(1) = \sqrt{N}$,

$$R_{m_0',m_0'}1(z) = \frac{1}{N}\sum_{w^N=z}|\alpha_{z_0}(m_0)|^2(w) = \frac{1}{N}\sum_{w^N=z}|m_0|^2(z_0w)$$
$$= \frac{1}{N}\sum_{y^N=z_0z}|m_0|^2(y) = R_{m_0,m_0}(z_0z) = 1.$$

Thus we can apply Proposition 2.11 to m_0'.

(i) $\varphi_{m_0,z_0} = \varphi_{m_0',1}$ and everything follows.

(ii) $h_{m_0,z_0} = \alpha_{z_0^{-1}}(h_{m_0',1})$

$$\begin{aligned}R_{m_0,m_0}h_{m_0,z_0}(z) &= \frac{1}{N}\sum_{w^N=z}|m_0|^2(w)\alpha_{z_0^{-1}}(h_{m_0',1})(w)\\ &= \frac{1}{N}\sum_{w^N=z}|m_0|^2(w)h_{m_0',1}(wz_0^{-1})\\ &= \frac{1}{N}\sum_{y^N=zz_0^{-1}}|m_0|^2(yz_0)h_{m_0',1}(y)\\ &= R_{m_0',m_0'}h_{m_0',1}(zz_0^{-1}) = h_{m_0,z_0}(z).\end{aligned}$$

Also $h_{m_0,z_0}(z_0) = h_{m_0',1}(z_0^{-1}z_0) = 1$ and, if C is an m_0-cycle disjoint of $\{z_0\}$ then $z_0^{-1}C$ is an m_0'-cycle disjoint of $\{1\}$ and again Proposition 2.11 applies.

(iii) and (iv) can also be deduced from Proposition 2.11. The relation

$$U_{z_0}\pi_{z_0}(f) = \pi_{z_0}(f(z^N))U_{z_0}$$

follows from the identity $\alpha_{z_0}(f(z^N)) = \alpha_{z_0}(f)(z^N)$.

(v) If h' is as given, then $\alpha_{z_0}(h')$ satisfies: $0 \leqslant \alpha_{z_0}(h') \leqslant c\alpha_{z_0}(h_{m_0,z_0}) = ch_{m_0',1}$ and $R_{m_0',m_0'}\alpha_{z_0}(h') = \alpha_{z_0}(R_{m_0,m_0}h') = \alpha_{z_0}(h')$. Then, by Proposition 2.11, $\alpha_{z_0}(h') = \lambda h_{m_0',1}$ for some $\lambda \geqslant 0$ so $h' = \lambda h_{m_0,z_0}$.

(vi) The argument is similar to the one used in (v). ∎

Using Proposition 2.12 we are now able to prove that each m_0-cycle gives rise to a continuous solution for $R_{m_0,m_0}h = h$.

PROPOSITION 2.13. *Let $m_0 \in \mathrm{Lip}_1(\mathbb{T})$, $R_{m_0,m_0}1 = 1$ and let $C = \{z_1, z_2 = z_1^N, \dots, z_p = z_{p-1}^N\}$, $z_p^N = z_1$, be an m_0-cycle, $m_0(z_k) = \sqrt{N}\mathrm{e}^{\mathrm{i}\theta_k}$ for $k \in \{1, \dots, p\}$. Denote by $\theta_C = \theta_1 + \dots + \theta_p$. Define*

$$\varphi_{k,m_0,C}(x) = \prod_{k=1}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\theta_C}\alpha_{z_k}(m_0^{(p)})\left(\frac{x}{N^{kp}}\right)}{\sqrt{N^p}}, \quad k \in \{1, \dots, p\}.$$

(i) *$\varphi_{k,m_0,C}$ is a well defined continuous function that belongs to $L^2(\mathbb{R})$.*

(ii) *Define $g_{k,m_0,C} = \alpha_{z_k^{-1}}(\mathrm{Per}\,|\varphi_{k,m_0,C}|^2)$ for all $k \in \{1, \dots, p\}$. Then $g_{k,m_0,C}$ is Lipschitz (trigonometric polynomial if m_0 is one). Also*

$$R_{m_0,m_0}^p g_{k,m_0,C} = g_{k,m_0,C} \quad \textit{and} \quad R_{m_0,m_0} g_{k,m_0,C} = g_{k+1,m_0,C}$$

(we will use the notation $\mathrm{mod}\,p$ that is $z_{p+1} = z_1$, $g_{p+2,m_0,C} = g_{2,m_0,C}$ etc.),
$g_{k,m_0,C}(z_j) = \delta_{kj}$, $g_{k,m_0,C}$ is 0 on every m_0-cycle disjoint of C.

(iii) *Define $h_{m_0,C} = \sum_{k=1}^{p} g_{k,m_0,C}$. Then $h_{m_0,C}$ is Lipschitz (trigonometric polynomial if m_0 is one). Also $R_{m_0,m_0}h_{m_0,C} = h_{m_0,C}$, $h_{m_0,C}(z_k) = 1$ for all $k \in \{1, \dots, p\}$ and $h_{m_0,C}$ is 0 on every m_0-cycle disjoint of C.*

(iv) *$h_{m_0,C}$ is minimal.*

(v) *If $h \geqslant 0$ is continuous, $R_{m_0,m_0}h = h$ and h is 1 on C then $h \geqslant h_{m_0,C}$.*

(vi) *If $U_C : L^2(\mathbb{R})^p \to L^2(\mathbb{R})^p$,*

$$U_C(\xi_1, \dots, \xi_p) = (\mathrm{e}^{\mathrm{i}\theta_1}U_1\xi_2, \dots, \mathrm{e}^{\mathrm{i}\theta_{p-1}}U_1\xi_p, \mathrm{e}^{\mathrm{i}\theta_p}U_1\xi_1)$$

and for $f \in L^\infty(\mathbb{T})$, $\pi_C(f) : L^2(\mathbb{R})^p \to L^2(\mathbb{R})^p$,

$$\pi_C(f)(\xi_1, \dots, \xi_p) = (\pi_1(\alpha_{z_1}(f))(\xi_1), \dots, \pi_1(\alpha_{z_p}(f))(\xi_p));$$

then $(U_C, \pi_C, (\varphi_{1,m_0,C}, \dots, \varphi_{p,m_0,C}))$ is the cyclic representation corresponding to $h_{m_0,C}$.

(vii) *The commutant of this representation is*

$$\{M_{f_1} \oplus \dots \oplus M_{f_p} : f_k \in L^\infty(\mathbb{R}), f_{k+1}(Nx) = f_k(x) \text{ a.e., for } k \in \{1, \dots, p\}\}.$$

Proof. Let $m_0' := m_0^{(p)}$. Observe that

$$\begin{aligned} m_0'(z_i) = m_0^{(p)}(z_i) &= m_0(z_i)m_0(z_i^N)\cdots m_0(z_i^{N^{p-1}}) \\ &= m_0(z_1)m_0(z_2)\cdots m_0(z_p) = \sqrt{N^p}\mathrm{e}^{\mathrm{i}\theta_C}. \end{aligned}$$

(i) Note that $R_{m_0^{(p)},m_0^{(p)}} = R_{m_0,m_0}^p$ so $R_{m_0^{(p)},m_0^{(p)}}1 = 1$. Thus (i) follows from Proposition 2.12 (i) (replace N by N^p when working with $m_0^{(p)}$).

(ii) If $y_1, y_2 = y_1^N, \dots, y_q = y_{q-1}^N, y_1 = y_q^N$ is an m_0-cycle, then $\{y_i\}$ is an $m_0^{(p)}$-cycle. Therefore, all assertions in (ii), except the one that relates $g_{k,m_0,C}$ and $g_{k+1,m_0,C}$, follow from Proposition 2.12 (ii).

We check now (vi). U_C is unitary as a composition of unitary operators. For $f \in L^\infty(\mathbb{T})$ we have:

$$\begin{aligned} U_C\pi_C(f)(\xi_1,\dots,\xi_p) &= (\mathrm{e}^{\mathrm{i}\theta_1}\pi_1(\alpha_{z_2}(f)(z^N))U_1\xi_2,\dots,\\ &\qquad \mathrm{e}^{\mathrm{i}\theta_{p-1}}\pi_1(\alpha_{z_p}(f)(z^N))U_1\xi_p, \mathrm{e}^{\mathrm{i}\theta_p}\pi_1(\alpha_{z_1}(f)(z^N))U_1\xi_1)\\ &= (\mathrm{e}^{\mathrm{i}\theta_1}\pi_1(\alpha_{z_1}(f(z^N)))U_1\xi_2,\dots,\\ &\qquad \mathrm{e}^{\mathrm{i}\theta_{p-1}}\pi_1(\alpha_{z_{p-1}}(f(z^N)))U_1\xi_p, \mathrm{e}^{\mathrm{i}\theta_p}\pi_1(\alpha_{z_1}(f(z^N)))U_1\xi_1)\\ &= \pi_C(f(z^N))U_C(\xi_1,\dots,\xi_p). \end{aligned}$$

Here we used that $\alpha_{z_{i+1}}(f)(z^N) = \alpha_{z_i}(f(z^N))$.

We must check also that

$$U_C(\varphi_{1,m_0,C},\dots,\varphi_{p,m_0,C}) = \pi_C(m_0)(\varphi_{1,m_0,C},\dots,\varphi_{p,m_0,C}).$$

To do this observe that

$$\begin{aligned} \alpha_{z_1}(m_0^{(p)})(z) &= \alpha_{z_1}(m_0(z))\alpha_{z_1}(m_0(z^N))\cdots\alpha_{z_1}(m_0(z^{N^{p-1}}))\\ &= \alpha_{z_1}(m_0)(z)\alpha_{z_2}(m_0)(z^N)\cdots\alpha_{z_p}(m_0)(z^{N^{p-1}}). \end{aligned}$$

Thus

$$\begin{aligned} \varphi_{1,m_0,C}(x) = &\frac{\mathrm{e}^{-\mathrm{i}\theta_p}\alpha_{z_p}(m_0)\big(\frac{x}{N}\big)}{\sqrt{N}}\frac{\mathrm{e}^{-\mathrm{i}\theta_{p-1}}\alpha_{z_{p-1}}(m_0)\big(\frac{x}{N^2}\big)}{\sqrt{N}}\cdots\frac{\mathrm{e}^{-\mathrm{i}\theta_1}\alpha_{z_1}(m_0)\big(\frac{x}{N^p}\big)}{\sqrt{N}}\cdots\\ &\frac{\mathrm{e}^{-\mathrm{i}\theta_p}\alpha_{z_p}(m_0)\big(\frac{x}{N^{p+1}}\big)}{\sqrt{N}}\frac{\mathrm{e}^{-\mathrm{i}\theta_{p-1}}\alpha_{z_{p-1}}(m_0)\big(\frac{x}{N^{p+2}}\big)}{\sqrt{N}}\cdots\frac{\mathrm{e}^{-\mathrm{i}\theta_1}\alpha_{z_1}(m_0)\big(\frac{x}{N^{2p}}\big)}{\sqrt{N}}\cdots \end{aligned}$$

so

$$\varphi_{1,m_0,C}(x) = \prod_{k=1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\theta_{1-k}}\alpha_{z_{1-k}}(m_0)\big(\frac{x}{N^k}\big)}{\sqrt{N}}.$$

Similarly

$$\varphi_{i,m_0,C}(x) = \prod_{k=1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\theta_{i-k}}\alpha_{z_{i-k}}(m_0)\big(\frac{x}{N^k}\big)}{\sqrt{N}} \qquad \text{for } i \in \{1,\dots,p\}.$$

Using this formula we obtain:

$$\begin{aligned} U_1\varphi_{i+1,m_0,C} = \sqrt{N}\varphi_{i+1,m_0,C}(Nx) &= \mathrm{e}^{-\mathrm{i}\theta_i}\alpha_{z_i}(m_0)\prod_{k=2}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\theta_{i+1-k}}\alpha_{z_{i+1-k}}(m_0)\big(\frac{x}{N^{k-1}}\big)}{\sqrt{N}}\\ &= \mathrm{e}^{-\mathrm{i}\theta_i}\alpha_{z_i}(m_0)\varphi_{i,m_0,C} \end{aligned}$$

which shows that $U_C(\varphi_{1,m_0,C},\dots,\varphi_{p,m_0,C}) = \pi_C(m_0)(\varphi_{1,m_0,C},\dots,\varphi_{p,m_0,C})$. Next we compute the commutant. Consider $A : L^2(\mathbb{R})^p \to L^2(\mathbb{R})^p$ commuting with the representation. Let P_i be the projection onto the i-th component, and let $A_{ij} = P_iAP_j$. Note that $U_C^p(\xi_1,\dots,\xi_p) = (\mathrm{e}^{-\mathrm{i}\theta_C}U_1^p\xi_1,\dots,\mathrm{e}^{-\mathrm{i}\theta_C}U_1^p\xi_p)$.

Also, since $z_i^{N^p} = z_i$, $z_i = \frac{2\pi k_i}{N^p-1}$ for some integer k_i. Take any $\frac{2\pi}{N^p-1}$-periodic essentially bounded function, g. Then $\alpha_{z_i}(g) = g$ so $\pi_C(g)(\xi_1,\dots,\xi_p) = (\pi_1(g)\xi_1,\dots,\pi_p(g)\xi_p)$. Then P_i commute with U_C^p and $\pi_C(g)$ so A_{ij} commute with

U_1^p and $\pi_1(g)$ and, using the argument in [12] (proof of Theorem 4.1), (see also the proof of Lemma 2.14 below), it follows that $A_{ij} = M_{f_{ij}}$ for some $f_{ij} \in L^\infty(\mathbb{R})$.

Since A and $\pi_C(f)$ commute for all $f \in L^\infty(\mathbb{T})$, we have for $i \in \{1, \dots, p\}$

$$\sum_{j=1}^{p} f_{ij}\pi_1(\alpha_{z_j}(f))\xi_j = \pi_1(\alpha_{z_i}(f)) \sum_{j=1}^{p} f_{ij}\xi_j.$$

Fix k and take $\xi_j = 0$ for all $j \neq k$, then $f_{ik}\pi_1(\alpha_{z_k}(f))\xi_k = \pi_1(\alpha_{z_i}(f))f_{ik}\xi_k$ so $f_{ik} = 0$ for $i \neq k$. Then, since A commutes with U we have

$$\begin{aligned}(&\mathrm{e}^{\mathrm{i}\theta_1}\sqrt{N} f_{22}(Nx)\xi_2(Nx), \dots, \mathrm{e}^{\mathrm{i}\theta_{p-1}}\sqrt{N} f_{pp}(Nx)\xi_p(Nx), \mathrm{e}^{\mathrm{i}\theta_p}\sqrt{N} f_{11}(Nx))\xi_{11} \\ &= (\mathrm{e}^{\mathrm{i}\theta_1} f_{11}(x)\sqrt{N}\xi_2(Nx), \dots, \mathrm{e}^{\mathrm{i}\theta_{p-1}}\sqrt{N} f_{p-1p-1}(x)\xi_p(Nx), \mathrm{e}^{\mathrm{i}\theta_p}\sqrt{N} f_{pp}(Nx)).\end{aligned}$$

Therefore,

$$f_{22}(Nx) = f_{11}(x) \text{ a.e.,} \quad f_{33}(Nx) = f_{22}(x) \text{ a.e.,} \dots, f_{11}(Nx) = f_{pp}(x) \text{ a.e.}$$

and (vii) follows.

The cyclicity of $(\varphi_{1,m_0,C}, \dots, \varphi_{p,m_0,C})$ follows as in the proof of Proposition 2.11 (iii).

We check that $R_{m_0,m_0} g_{i,m_0,C} = g_{i+1,m_0,C}$. Take $f \in L^\infty(\mathbb{T})$. We have:

$$\begin{aligned}\int_{\mathbb{T}} f g_{i+1,m_0,C}\,\mathrm{d}\mu &= \langle \varphi_{i+1,m_0,C} : \pi_1(\alpha_{z_{i+1}}(f))\varphi_{i+1,m_0,C}\rangle \\ &= \langle U_1\varphi_{i+1,m_0,C} : U_1\pi_1(\alpha_{z_{i+1}}(f))\varphi_{i+1,m_0,C}\rangle \\ &= \langle \mathrm{e}^{-\mathrm{i}\theta_i}\pi_1(\alpha_{z_i}(m_0))\varphi_{i,m_0,C} : \mathrm{e}^{-\mathrm{i}\theta_i}\pi_1(\alpha_{z_{i+1}}(f)(z^N))\pi_1(\alpha_{z_i}(m_0))\varphi_{i,m_0,C}\rangle \\ &= \langle \varphi_{i,m_0,C} : \pi_1(\alpha_{z_i}(f(z^N))\alpha_{z_i}(|m_0|^2))\varphi_{i,m_0,C}\rangle \\ &= \int_{\mathbb{T}} f(z^N)|m_0|^2 g_{i,m_0,C}\,\mathrm{d}\mu = \int_{\mathbb{T}} f(z) R_{m_0,m_0} g_{i,m_0,C}\,\mathrm{d}\mu.\end{aligned}$$

Hence $R_{m_0,m_0} g_{i,m_0,C} = g_{i+1,m_0,C}$.

(iii) follows from (ii).

Next we prove that $h_{m_0,C}$ is minimal. Take a continuous h' with $0 \leqslant h' \leqslant h_{m_0,C}$, $R_{m_0,m_0}h' = h'$. Then $R_{m_0^{(p)},m_0^{(p)}}h' = R^p_{m_0,m_0}h' = h'$ and $0 \leqslant h' \leqslant c(g_{1,m_0,C} + \dots + g_{p,m_0,C})$. Now we use the fact that the space $\{g \in C(\mathbb{T}) : R_{m_0^{(p)},m_0^{(p)}} g = g\}$ is a C^*-algebra isomorphic to $C(\{1, \dots, d\})$ for some d (see Corollary 2.8), and by Proposition 2.12 (iv), $g_{i,m_0,C}$ are minimal. It follows that h' can be written uniquely as $h' = \alpha_1 g_{1,m_0,C} + \dots + \alpha_p g_{p,m_0,C}$ with $\alpha_1, \dots, \alpha_p \in \mathbb{C}$ (the uniqueness comes from the fact that $g_{i,m_0,C}$ are linearly independent, which, in turn, is implied by (ii)). Then $R_{m_0,m_0}h' = \alpha_1 g_{2,m_0,c} + \dots + \alpha_{p-1} g_{p,m_0,C} + \alpha_p g_{1,m_0,C}$ so, by uniqueness $\alpha_1 = \alpha_2 = \dots = \alpha_p = \alpha_1$ and $h' = \alpha_1(g_{1,m_0,C} + \dots + g_{p,m_0,C}) = \alpha_1 h_{m_0,C}$.

For (v) we use a similar argument: take h' as given in the hypothesis. Then $R_{m_0^{(p)},m_0^{(p)}}h' = R^p_{m_0,m_0}h' = h'$, $h'(z_i) = 1$ for all i. Using Proposition 2.12 (v), we get $h' \geqslant g_{i,m_0,C}$ for all i.

Now we use again the fact that $\{g \in C(\mathbb{T}) : R_{m_0^{(p)},m_0^{(p)}} g = g\}$ is a C^*-algebra isomorphic to $C(\{1, \dots, d\})$ and $g_{i,m_0,C}$ are minimal, so $h' \geqslant (g_{1,m_0,C} + \dots + g_{p,m_0,C}) = h_{m_0,C}$. ∎

Lemma 2.14. *Consider m_0, m_0' satisfying* (1.1)–(1.4)*y. Let $C : z_1^N = z_2, \ldots, z_p^N = z_1$ be an m_0-cycle and $C' : z_1'^N = z_2', \ldots, z_{p'}'^N = z_1'$ be an m_0'-cycle, $m_0(z_k) = \sqrt{N}\mathrm{e}^{\mathrm{i}\theta_k}$, $m_0'(z_k') = \sqrt{N}\mathrm{e}^{\mathrm{i}\theta_k'}$ for all k. Consider the cyclic representations associated to this cycles as in Proposition* 2.13, *(U_C, π_C, φ_C), $(U_{C'}, \pi_{C'}, \varphi_{C'})$ and let $S : L^2(\mathbb{R})^{p'} \to L^2(\mathbb{R})^p$ be an intertwining operator. Then $S = 0$ if $C \neq C'$. If $C = C'$ and, after relabeling, $z_k = z_k'$ for all k, $p = p'$ then, there exist $f_1, \ldots, f_p \in L^\infty(\mathbb{R})$ such that*

$$S(\xi_1, \ldots, \xi_p) = (f_1\xi_1, \ldots, f_p\xi_p)$$

with

$$\begin{aligned} f_1(x) &= \mathrm{e}^{\mathrm{i}(\theta_1 - \theta_1')} f_2(Nx), \text{ a.e.},\\ &\ldots,\\ f_{p-1}(x) &= \mathrm{e}^{\mathrm{i}(\theta_{p-1} - \theta_{p-1}')} f_p(Nx), \text{ a.e.},\\ f_p(x) &= \mathrm{e}^{\mathrm{i}(\theta_p - \theta_p')} f_1(Nx), \text{ a.e.} \end{aligned}$$

Proof. Note that

$$U_C^p = \mathrm{e}^{\mathrm{i}\theta_C} U_1^p \oplus \cdots \oplus \mathrm{e}^{\mathrm{i}\theta_C} U_1^p,$$

where $\theta_C = \theta_1 + \cdots + \theta_p$. Similarly for $U_{C'}^{p'}$. This shows that U_C^p commutes with the projections P_i onto the i-th component.

We have $SU_{C'}^{pp'} = U_C^{pp'} S$ so $(P_i S P_j) U_{C'}^{pp'} = U_C^{pp'} (P_i S P_j)$, therefore $S_{ij} \mathrm{e}^{\mathrm{i}p\theta'_C} U_1^{pp'} = \mathrm{e}^{\mathrm{i}p'\theta_C} U_1^{pp'} S_{ij}$, where $S_{ij} = P_i S P_j$.

Also, since $z_k^{N^p} = z_k$, z_k has the form $\mathrm{e}^{\mathrm{i}\frac{2l\pi}{m}}$ for all k and similarly for z_k'. If we take $f \in L^\infty(\mathbb{T})$ to be $\frac{2\pi}{mm'}$-periodic, then $\alpha_{z_k}(f) = f$, $\alpha_{z_k'}(f) = f$ for all k, so $\pi_C(\xi_1, \ldots, \xi_p) = (\pi_1(f)\xi_1, \ldots, \pi_1(f)\xi_p)$ and $\pi_{C'}(\xi_1, \ldots, \xi_{p'}) = (\pi_1(f)\xi_1, \ldots, \pi_1(f)\xi_{p'})$, and again

$$S_{ij}\pi_1(f) = \pi_1(f) S_{ij}.$$

Hence S_{ij} commutes with $\pi_1(f) = M_f$ whenever $f \in L^\infty(\mathbb{R})$ is $\frac{2\pi}{mm'}$-periodic.

But then also

$$(U_1^{-pp'} \pi_1(f) U_1^{pp'}) S_{ij} = S_{ij} (U_1^{-pp'} \pi_1(f) U_1^{pp'})$$

and $U_1^{-pp'} \pi_1(f) U_1^{pp'} = M_g$ where $g(N^{pp'}x) = f(x)$ for $x \in \mathbb{R}$ and g is $\frac{2\pi}{mm'} N^{pp'}$-periodic. By induction, it follows that S_{ij} commutes with M_f whenever $f \in L^\infty(\mathbb{R})$ is $\frac{2\pi}{mm'} N^{lpp'}$-periodic, $l \in \mathbb{N}$.

Now take $f \in L^\infty(\mathbb{R})$. Define $f_l(x) = f(x)$ on $[-\frac{\pi}{mm'} N^{lpp'}, \frac{\pi}{mm'} N^{lpp'}]$ and extend it to $\mathbb{R}$ such that f_l is $\frac{2\pi}{mm'} N^{lpp'}$-periodic.

We prove that M_{f_l} converges to M_f in the strong operator topology. Take $\psi \in L^2(\mathbb{R})$.

$$\begin{aligned} \|M_{f_l}\psi - M_f\psi\|_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |f_l - f|^2 |\psi|^2 \,\mathrm{d}x = \int_{|x| \geqslant \frac{\pi}{mm'} N^{lpp'}} |f_l - f|^2 |\psi|^2 \,\mathrm{d}x \\ &\leqslant (2\|f\|_\infty^2) \int_{\mathbb{R}} \chi_{\{|x| \geqslant \frac{\pi}{mm'} N^{lpp'}\}} |\psi|^2 \,\mathrm{d}x \to 0 \quad \text{as } l \to \infty. \end{aligned}$$

Consequently, the limit holds and M_f will commute also with S_{ij}. As f was arbitrary in $L^\infty(\mathbb{R})$, using Theorem IX.6.6 in [8], we obtain that $S_{ij} = M_{f_{ij}}$ for some $f_{ij} \in L^\infty(\mathbb{R})$.

Having this, we rewrite the intertwining properties. First, we have for all $f \in L^\infty(\mathbb{T})$:

$$\sum_{j=1}^{p'} f_{ij}\alpha_{z'_j}(f)\xi_j = \alpha_{z_i}(f)\sum_{j=1}^{p'} f_{ij}\xi_j, \quad i \in \{1, \dots, p\}. \tag{2.11}$$

Fix $k \in \{1, \dots, p'\}$ and take $\xi_j = 0$ for $j \neq k$. Then

$$f_{ik}\alpha_{z'_k}(f)\xi_k = \alpha_{z_i}(f) f_{ik}\xi_k. \tag{2.12}$$

Since $f \in L^\infty(\mathbb{T})$ is arbitrary, it follows that $f_{ik} = 0$ unless $z'_k = z_i$.

If $z'_k = z_i$ then we get $C = C'$. If $C \neq C'$ then $C \cap C' = \emptyset$ so $f_{ij} = 0$ for all i, j and $S = 0$.

It remains to consider the case $C = C'$ and, relabeling $z_k = z'_k$ for all k, $p = p'$. Equation (2.12) implies that $f_{ij} = 0$ for $i \neq j$ so $S(\xi_1, \dots, \xi_p) = (f_1\xi_1, \dots, f_p\xi_p)$ (we used the notation $f_i = f_{ii}$).

The fact that $SU_{C'} = U_C S$ can be rewritten:

$$\begin{aligned} f_1(x)e^{i\theta'_1}\sqrt{N}\xi_2(Nx) &= e^{i\theta_1}\sqrt{N} f_2(Nx)\xi_2(Nx) \\ &\vdots \\ f_{p-1}(x)e^{i\theta'_{p-1}}\sqrt{N}\xi_p(Nx) &= e^{i\theta_{p-1}}\sqrt{N} f_p(Nx)\xi_p(Nx) \\ f_p(x)e^{i\theta'_p}\sqrt{N}\xi_1(Nx) &= e^{i\theta_p}\sqrt{N} f_1(Nx)\xi_1(Nx), \end{aligned}$$

so

$$\begin{aligned} f_1(x) &= e^{i(\theta_1-\theta'_1)} f_2(Nx), \quad \text{a.e.}, \\ &\vdots \\ f_{p-1}(x) &= e^{i(\theta_{p-1}-\theta'_{p-1})} f_p(Nx), \quad \text{a.e.}, \\ f_p(x) &= e^{i(\theta_p-\theta'_p)} f_1(Nx), \quad \text{a.e.} \end{aligned}$$

∎

THEOREM 2.15. *Let m_0 satisfy (1.1)–(1.4). Let $C_1, \dots, C_n$ be the m_0-cycles. Then, each $h \in C(\mathbb{T})$ with $R_{m_0,m_0}h = h$ can be written uniquely as*

$$h = \sum_{i=1}^{n} \alpha_i h_{m_0,C_i}$$

with $\alpha_i \in \mathbb{C}$. Moreover $\alpha_i = h|C_i$. In particular, $1 = \sum_{i=1}^{n} h_{m_0,C_i}$.

Proof. Proposition 2.13 (iii) shows that h_{m_0,C_i} are linearly independent. Since the dimension of $\{h \in C(\mathbb{T} : R_{m_0,m_0}h = h\}$ is n (see [3]), it follows that h_{m_0,C_i} form a basis for this space. So

$$h = \sum_{i=1}^{n} \alpha_i h_{m_0,C_i}$$

for some $\alpha_i \in \mathbb{C}$. An application of Proposition 2.13 (iii) shows that $\alpha_i = h|C_i$. ∎

Theorem 2.16. *Suppose m_0 satisfies the conditions* (1.1)–(1.4). *Let $C_1, \dots, C_n$ be the m_0-cycles. For each i consider $(U_{C_i}, \pi_{C_i}, \varphi_{C_i})$ which give the cyclic representation corresponding to h_{m_0,C_i} (see Proposition* 2.13). *Define*

$$U = U_{C_1} \oplus \cdots \oplus U_{C_n}, \quad \pi = \pi_{C_1} \oplus \cdots \oplus \pi_{C_n}, \quad \varphi = \varphi_{C_1} \oplus \cdots \oplus \varphi_{C_n}.$$

Then (U, π, φ) give the cyclic representation corresponding to the constant function 1. *Each element S in the commutant of this representation has the form $S = S_{C_1} \oplus \cdots \oplus S_{C_n}$, where S_{C_i} is in the commutant of $(U_{C_i}, \pi_{C_i}, \varphi_{C_i})$.*

Proof. Since

$$1 = \sum_{i=1}^{n} h_{m_0,C_i}$$

for the first statement, it is enough to check that φ is cyclic. For this we will need the commutant and then the reasoning is the same as the one in the proofs of Proposition 2.11 (iii) or Proposition 2.13 (vi). But Lemma 2.14 makes it clear that the elements of the commutant have the form mentioned in the hypotesis (see also the proof of Theorem 2.17). We also need to prove that if S is in the commutant, $S = S^2 = S^*$ and $S\varphi = \varphi$ then S is the identity. But,

$$S = S_{C_1} \oplus \cdots \oplus S_{C_n},$$

so $S_{C_i} = S_{C_i}^2 = S_{C_i}^*$ and $S_{C_i}\varphi_{C_i} = \varphi_{C_i}$, and, as φ_{C_i} is cyclic in the corresponding representation, it follows that S_{C_i} is the identity so $S = I$. ∎

Theorem 2.17. *Suppose m_0 satisfies* (1.1)–(1.4). *Let $C_1, \dots, C_n$ be the m_0-cycles, $C_i : z_{1i}, z_{2i} = z_{1i}^N, \dots, z_{p_i\, i} = z_{p_i-1\, i}^N, z_{1i} = z_{p_i\, i}^N$, for $i \in \{1, \dots, n\}$. Let g_{k,m_0,C_i} be as in Proposition* 2.13, *$k \in \{1, \dots, p_i\}$, $i \in \{1, \dots, n\}$.*

If $h \in C(\mathbb{T})$, $h \neq 0$ and $R_{m_0,m_0}h = \lambda h$ for some $\lambda \in \mathbb{T}$, then there exists an $i \in \{1, \dots, n\}$ such that $\lambda^{p_i} = 1$, and there exist $\alpha_i \in \mathbb{C}$, $i \in \{1, \dots, n\}$ such that

$$h = \sum_{i=1}^{n} \alpha_i \Big(\sum_{k=1}^{p_i} \lambda^{-k+1} g_{k,m_0,C_i} \Big)$$

and $\alpha_i = 0$ if $\lambda^{p_i} \neq 1$.

Proof. First note that instead of m_0 we can take $|m_0|$ and the problem remains the same. We have

$$\frac{1}{N} \sum_{w^N = z} \overline{\lambda m_0(w)} m_0(w) h(w) = h(z), \quad z \in \mathbb{T},$$

so $R_{\lambda m_0,m_0}h = h$. Using Theorem 1.3, it follows that h induces an intertwining operator $S : \mathcal{H}_{m_0} \to \mathcal{H}_{\lambda m_0}$, where $(\mathcal{H}_{m_0}, \pi_{m_0}, \varphi_{m_0})$ is the cyclic representation corresponding to the constant function 1 and m_0, and $(\mathcal{H}_{\lambda m_0}, \pi_{\lambda m_0}, \varphi_{\lambda m_0})$ is the cyclic representation corresponding to 1 and λm_0.

Using Theorem 2.16 and proposition 2.13, we see that $\mathcal{H}_{m_0} = \mathcal{H}_{\lambda m_0}$, $\pi_{m_0}(f) = \pi_{\lambda m_0}(f)$, for $f \in L^\infty(\mathbb{T})$, $\varphi_{m_0} = \varphi_{\lambda m_0}$ and $U_{\lambda m_0} = \lambda U_{m_0}$.

The intertwining property of S implies that

$$SU_{m_0} = \lambda U_{m_0} S \quad \text{and} \quad S\pi_{m_0}(f) = \pi_{m_0}(f)S, \quad f \in L^\infty(\mathbb{T}).$$

If P_{C_i} is the projection onto the components corresponding to the cycle C_i then we see that P_{C_i} commutes with both U_{m_0} and $\pi_{m_0}(f)$ for $f \in L^\infty(\mathbb{T})$. Therefore

$$(P_{C_i}SP_{C_j})U_{C_j} = \lambda U_{C_i}(P_{C_i}SP_{C_j}),$$
$$(P_{C_i}SP_{C_j})\pi_{C_j}(f) = \pi_{C_i}(f)(P_{C_i}SP_{C_j}), \quad f \in L^\infty(\mathbb{T}).$$

Using Lemma 2.14, we obtain, $(P_{C_i}SP_{C_j}) = 0$ if $i \neq j$ and for each $i \in \{1,\dots,n\}$ there exist $f_{1i},\dots,f_{p_i\,i} \in L^\infty(\mathbb{R})$ such that

$$(P_{C_i}SP_{C_j})(\xi_1,\dots,\xi_{p_i}) = (f_{1i}\xi_1,\dots,f_{p_i\,i}\xi_{p_i}),$$
$$f_{1i}(x) = \lambda f_{2i}(Nx) \text{ a.e.},$$
$$\dots,$$
$$f_{p_i-1\,i}(x) = \lambda f_{p_i\,i}(Nx) \text{ a.e.},$$
$$f_{p_i\,i}(x) = \lambda f_{1i}(Nx) \text{ a.e.}$$

Also, as $\int_{\mathbb{T}} fh\,\mathrm{d}\mu = \langle \varphi_{m_0} : \pi_{m_0}(f)S\varphi_{m_0}\rangle$, $f \in L^\infty(\mathbb{T})$, after periodization we get

$$h = \sum_{i=1}^{n}\sum_{k=1}^{p_i} \alpha_{z_{ki}^{-1}}(\mathrm{Per}(f_{ki}|\varphi_{k,m_0,C_i}|^2)).$$

We want to prove that each f_{ki} is continuous at 0. Take $i \in \{1,\dots,n\}$, $k \in \{1,\dots,p_i\}$. We know from Proposition 2.13 that g_{k,m_0,C_i} is 1 at z_{ki} and 0 at every other z_{lj}. Then

$$|\alpha_{z_{lj}^{-1}}(\mathrm{Per}(f_{lj}|\varphi_{l,m_0,C_j}|^2))| \leqslant \|f_{lj}\|_\infty g_{l,m_0,C_j},$$

so, this function has limit 0 at z_{ki} for $(l,j) \neq (i,k)$. The argument used in the proof of Proposition 2.11 (v) can be repeated here to obtain that $\lim_{x\to 0} f_{ki}(x) = h(z_{ki})$.

On the other hand we have

$$f_{ki}(N^{p_i}x) = \lambda^{-p_i} f_{ki}(x) \tag{2.13}$$

so if we let $x \to 0$, we obtain $h(z_{ki}) = \lambda^{-p_i}h(z_{ki})$. Consequently, $h(z_{ki}) = f_{ki} = 0$ or $\lambda^{p_i} = 1$. Since $h \neq 0$, there exists an $i \in \{1,\dots,n\}$ with $\lambda^{p_i} = 1$.

For an i with $\lambda^{p_i} \neq 1$ we have $f_{ki} = 0$ for all $k \in \{1,\dots,p_i\}$. Now take an i with $\lambda^{p_i} = 1$. From (2.13) and the fact that f_{ki} is continuous at 0, it follows that f_{ki} is constant. Let $\alpha_i = f_{1i}$. Then $f_{2i} = \lambda^{-1}\alpha_i,\dots,f_{p_i\,i} = \lambda^{-p_i+1}\alpha_i$ and the last assertion of the theorem is proved. ∎

COROLLARY 2.18. *Let m_0 as in Theorem 2.17. For an eigenvalue $\lambda \in \mathbb{T}$ and i with $\lambda^{p_i} = 1$, define $h^\lambda_{m_0,C_i} = \sum_{k=1}^{p_i} \lambda^{-k+1} g_{k,m_0,C_i}$. Then for each eigenvalue $\lambda \in \mathbb{T}$, the eigenfunctions $h^\lambda_{m_0,C_i}$ with $\lambda^{p_i} = 1$ are linearly independent. Moreover, if we define the measures*

$$\nu_i^\lambda = \frac{1}{p_i}\sum_{k=1}^{p_i} \lambda^{k-1}\delta_{z_{ki}}, \quad i \in \{1,\dots,n\}, \lambda \in \mathbb{T}, \lambda^{p_i} = 1,$$

where δ_z is the Dirac measure at z, then

$$T_\lambda(f) = \sum_{\substack{i=1 \\ \lambda^{p_i}=1}}^{n} \nu_i^\lambda(f) h_{m_0,C_i}^\lambda.$$

Proof. First, we see that Theorem 2.17 implies that h_{m_0,C_i}^λ with $\lambda^{p_i} = 1$ span the eigenspace corresponding to the eigenvalue λ. Then we also note that, using Proposition 2.13 (ii) we have:

$$\nu_i^\lambda(h_{m_0,C_j}^\lambda) = \delta_{ij}. \tag{2.14}$$

This shows that h_{m_0,C_i}^λ are linearly independent.

On the other hand we have for all $f \in C(\mathbb{T})$, using the fact that C_i is an m_0-cycle:

$$\begin{aligned}
\nu_i^\lambda(R_{m_0,m_0}(f)) &= \frac{1}{p_i}\sum_{k=1}^{p_i} \lambda^{k-1}\delta_{z_{ki}}(R_{m_0,m_0}(f)) \\
&= \frac{1}{p_i}\sum_{k=1}^{p_i} \lambda^{k-1}\frac{1}{N}\sum_{w^N=z_{ki}} |m_0(w)|^2 f(w) \\
&= \frac{1}{p_i}\sum_{k=1}^{p_i} \lambda^{k-1}\frac{1}{N}\Big(|m_0(z_{k-1\,i})|^2 f(z_{k-1\,i}) + \sum_{\substack{w^N=z_{ki} \\ w\neq z_{k-1\,i}}} |m_0(w)|^2 f(w)\Big) \\
&= \frac{1}{p_i}\sum_{k=1}^{p_i} \lambda^{k-1} f(z_{k-1\,i}) = \lambda\nu_i^\lambda(f).
\end{aligned}$$

Then, according to Theorem 2.6,

$$\nu_i^\lambda(T_\lambda(f)) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\lambda^{-k}\nu_i^\lambda(R_{m_0,m_0}^k(f)) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\lambda^{-k}\lambda^k\nu_i^\lambda(f) = \nu_i^\lambda(f).$$

This, together with (2.14) and the fact that h_{m_0,C_i}^λ form a basis for the eigenspace, imply the last equality of the corollary. ∎

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