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LOCAL SPECTRAL PROPERTIES OF WEIGHTED SHIFTS

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ABSTRACT. For a large class of operators on Banach spaces, a natural growth condition is shown to guarantee Bishop's property (β). For weighted shifts, this result leads to a sufficient condition in terms of the underlying weight sequence. In the opposite direction, it is shown that every unilateral weighted shift with property (β) has fat local spectra and approximate point spectrum a circle, while bilateral weighted shifts with property (β) have either fat local spectra or spectrum a circle. A useful new tool is the inner local spectral radius, a counterpart of the standard (outer) local spectral radius. For weighted shifts with Dunford's property (C), both the inner and outer spectral radii turn out to be constant.

KEYWORDS: *Weighted shifts, decomposable operators, generalized scalar operators, Bishop's property* (β).

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A bounded linear operator $T \in \mathcal{L}(X)$ on a complex Banach space X is decomposable provided that, for any open cover $\{U_1, \dots, U_n\}$ of the complex plane $\mathbb{C}$, there exist closed, T-invariant subspaces X_j, $1 \leqslant j \leqslant n$, such that $X = X_1 + \cdots + X_n$ and that for each j the restriction of T to X_j has spectrum $\sigma(T|X_j) \subseteq U_j$. This class, introduced by Foiaş in the early 1960s, is quite general, containing for example all compact operators, normal Hilbert space operators, Dunford's spectral operators and generalized scalar operators. For an excellent account of the theory of decomposable operators, see the monographs by Colojoară and Foiaş ([5]) and by Vasilescu ([23]). While decomposable operators do not generally possess a functional calculus beyond the analytic Riesz functional calculus, many of the spectral properties of normal operators hold for this more general class. Moreover, several results initially proven for subnormal operators have been extended to the setting of subdecomposable operators ([8], [16], [18], [25]). Albrecht and Eschmeier ([3]) showed that a property first studied by Bishop in the late 1950's gives an intrinsic characterization of restrictions of decomposable operators. They also showed that an operator has this property, Bishop's property (β), if and only if its adjoint has a decomposition property (δ), introduced in [2], where it was shown that an operator T is decomposable if and only if it enjoys both properties (β) and (δ); equivalently, if and only if both T and T^* have Bishop's property (β).

In this note, we obtain a growth condition that suffices for property (β) and apply it to weighted shifts on $\ell^p(\mathbb{N}_0)$ and $\ell^p(\mathbb{Z})$. We also obtain a necessary condition for property (β) in this setting. The question of determining the local spectral properties of weighted shifts is a natural one and has been addressed by several authors over the last quarter century. Shields's survey article may be used to settle the question of which weighted shifts have the single-valued extension property (see Proposition 2.5 below). The bilateral weighted shifts that are spectral were characterized by Sun, who also obtained results on decomposibility ([22]). $\mathcal{E}(\mathbb{T})$-subscalar weighted shifts were recently characterized by Didas in [7]. Weighted shifts provided the first examples of operators with Dunford's property (C), but without Bishop's property (β) ([15] and [12], 1.6.16) and a co-hyponormal bilateral weighted shift served as an important example in the theory of the localized single-valued extension property ([1]). Additional references for local spectral theory and weighted shifts include [9], [10], [11], [14], and [21].

1. INNER AND OUTER LOCAL SPECTRAL RADII

Let T be a bounded linear operator on a non-zero complex Banach space X. We denote the spectrum and approximate point spectrum of T by $\sigma(T)$ and $\sigma_{\text{ap}}(T)$, respectively. The surjectivity spectrum of T is the set $\sigma_{\text{su}}(T)$ of all $\lambda \in \mathbb{C}$ such that $T-\lambda I$ is not surjective. Notice that $\sigma_{\text{su}}(T) = \sigma_{\text{ap}}(T^*)$ and $\sigma_{\text{su}}(T^*) = \sigma_{\text{ap}}(T)$ ([12], 1.3.1). T has spectral radius $r(T) := \lim\limits_{n\to\infty} \|T^n\|^{1/n}$ and lower bound $\kappa(T) := \inf\{\|Tx\| : \|x\| = 1\}$. By 1.6.1, [12], the sequence $(\kappa(T^n)^{1/n})_{n\geqslant 0}$ converges to its supremum, which we denote by $\iota(T)$. Clearly, $\kappa(T) = \|T^{-1}\|^{-1}$ and therefore $\iota(T) = r(T^{-1})^{-1} = \min\{|\lambda| : \lambda \in \sigma(T)\}$ if T is invertible. More generally, as shown in [13], $\iota(T)$ is always the minimum modulus of $\sigma_{\text{ap}}(T)$.

For an open subset U of the complex plane, let $\mathcal{H}(U,X)$ denote the Fréchet space of all X-valued analytic functions on U. For $T \in \mathcal{L}(X)$, define T_U on $\mathcal{H}(U,X)$ by $T_U f(\lambda) := (T-\lambda)f(\lambda)$. The operator T has the single-valued extension property (SVEP) if T_U is injective for every open subset U of $\mathbb{C}$, and T is said to have Bishop's property (β) in the case that T_U is injective and has closed range for every open set $U \subseteq \mathbb{C}$ ([12], 1.2.6). Dunford's property (C) is intermediate to SVEP and (β): $T \in \mathcal{L}(X)$ has property (C) provided that the analytic subspaces $\mathfrak{X}_T(F) := \{x : x \in \operatorname{ran} T_{\mathbb{C}\setminus F}\}$ are closed in X for every closed set $F \subseteq \mathbb{C}$ ([12], 3.3.4). Since $\mathfrak{X}_T(F)$ is the kernel of the quotient map $x \mapsto x + \operatorname{ran} T_{\mathbb{C}\setminus F}$, property (β) clearly implies (C); property (C) in turn implies SVEP by 1.2.19 and 3.3.4, [12].

For $T \in \mathcal{L}(X)$ and $x \in X$, the local spectrum $\sigma_T(x)$ is defined to be the complement in $\mathbb{C}$ of the open set $\rho_T(x)$ of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U of λ such that $x \in \operatorname{ran} T_U$. By 1.2.16, [12], T has SVEP if and only if $\sigma_T(x) \neq \emptyset$ for every non-zero $x \in X$, and, in this case, $\mathfrak{X}_T(F) = \{x : \sigma_T(x) \subseteq F\}$ for every closed subset F of $\mathbb{C}$. We say that T has fat local spectra if $\sigma_T(x) = \sigma(T)$

for every non-zero $x \in X$. Examples of such operators relevant to weighted shifts include certain multiplication operators on spaces of analytic functions ([16] or [12], 1.6.9) and certain non-surjective isometries ([12], 1.6.8). A typical example is the unilateral right shift on $\ell^2(\mathbb{N})$, unitarily equivalent to multiplication by z on the Hardy space H^2. Clearly, operators with fat local spectra have property (C), and Bourhim and Zerouali ([4]) conjectured that every weighted shift with Dunford's property (C) either has fat local spectra or has spectrum equal to a circle. In Theorem 2.7 below, we show this to be true for shifts with Bishop's property (β).

Let $V(a, \delta)$ and $\nabla(a, \delta)$ denote, respectively, the open and closed discs of radius δ centered at $a \in \mathbb{C}$. For arbitrary $T \in \mathcal{L}(X)$, the (outer) local spectral radius of a vector $x \in X$ is defined to be $r_T(x) := \limsup_{n\to\infty} \|T^n x\|^{1/n}$. The formula for the radius of convergence of a power series implies that $x \in \mathfrak{X}_T(\nabla(0,r))$ if and only if $r_T(x) \leqslant r$ ([12], 3.3.13). Thus

$$r_T(x) = \inf\bigl\{r \geqslant 0 : x \in \mathfrak{X}_T\bigl(\nabla(0,r)\bigr)\bigr\}$$

for all $x \in X$. Moreover, by [12], 3.3.14, $r_T(x) = r(T)$ for all x in a set of second category in X.

The following counterpart of the outer local spectral radius will play a decisive role in this paper. For every $x \in X$, we define the inner spectral radius of T at x to be

$$\iota_T(x) := \sup\bigl\{r \geqslant 0 : x \in \mathfrak{X}_T\bigl(\mathbb{C} \setminus V(0,r)\bigr)\bigr\}.$$

So $\iota_T(x) = 0$ means precisely that $0 \in \sigma_T(x)$. Moreover, $\iota_T(x) = \infty$ only when $x = 0$. Proposition 1.2 (e) below shows that, for operators with SVEP, $\iota_T(x)$ is the minimum modulus of $\sigma_T(x)$.

The next lemma provides a characterization of the analytic subspaces $\mathfrak{X}_T\bigl(\mathbb{C} \setminus V(0,r)\bigr)$ and thus a formula for $\iota_T(x)$ that can be applied in many cases.

LEMMA 1.1. *For every $T \in \mathcal{L}(X)$ and $r > 0$, the set $\mathfrak{X}_T\bigl(\mathbb{C} \setminus V(0,r)\bigr)$ consists of all vectors $x \in X$ for which there exists a sequence $(a_n)_{n\geqslant 0}$ in X such that:*

(i) $Ta_0 = x$;

(ii) $a_n = Ta_{n+1}$ *for every* $n \geqslant 0$; *and*

(iii) $\liminf_{n\to\infty} \|a_n\|^{-1/n} \geqslant r$.

Moreover, if T is injective or has SVEP, *then, for each $x \in \mathfrak{X}_T\bigl(\mathbb{C} \setminus V(0,r)\bigr)$, there exists a unique sequence $(a_n)_{n\geqslant 0}$ with properties* (i), (ii), *and* (iii), *and this sequence satisfies $\liminf_{n\to\infty} \|a_n\|^{-1/n} = \iota_T(x)$. More generally, if $T \in \mathcal{L}(X)$ is arbitrary, and if $x \in X$ is such that there exists a sequence $(a_n)_{n\geqslant 0}$ satisfying* (i) *and* (ii), *and also, for some constant $c > 0$,*

(iii′) $\|a_n\| \leqslant c \inf\{y : T^{n+1} y = x\}$ *for every* $n \geqslant 0$,

then $\liminf_{n\to\infty} \|a_n\|^{-1/n} = \iota_T(x)$.

Proof. Clearly, $x \in \mathfrak{X}_T\bigl(\mathbb{C} \setminus V(0,r)\bigr)$ if and only if there is a power series $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ such that $(T - \lambda) f(\lambda) = x$ for all $|\lambda| < r$. The series has radius of convergence $\liminf_{n\to\infty} \|a_n\|^{-1/n}$, and, by the uniqueness of power series representations, the equation $(T-\lambda)f(\lambda)=x$ is satisfied on a neighborhood of 0 if and only if (i) and (ii) hold. This establishes the first assertion, and the second claim is a straightforward consequence of this characterization. In fact, the uniqueness assertion for $(a_n)_{n\geqslant 0}$ holds whenever T has SVEP at 0 in the sense of [1], but this level of generality will not be needed here. Finally, if $x\in X$ and $(a_n)_{n\geqslant 0}$ satisfy (i), (ii) and (iii)$'$ for some $c>0$, then, for any sequence $(b_n)_{n\geqslant 0}$ satisfying (i) and (ii), we see that $\|a_n\|\leqslant c\|b_n\|$ for every $n\geqslant 0$. Thus

$$\liminf_{n\to\infty}\|b_n\|^{-1/n}\leqslant \liminf_{n\to\infty} c^{1/n}\|a_n\|^{-1/n}=\liminf_{n\to\infty}\|a_n\|^{-1/n}\leqslant \iota_T(x).$$

It follows that $\liminf\limits_{n\to\infty}\|a_n\|^{-1/n}=\iota_T(x)$. ∎

For convenience, we define the generalized range of $T\in\mathcal{L}(X)$ to be $T^\infty X:=\bigcap\limits_{n\geqslant 0} T^nX$.

Proposition 1.2. *For every $T\in\mathcal{L}(X)$ and $x\in X$, the following assertions hold:*

(i) $\iota_T(x)=0$ *whenever* $x\notin T^\infty X$;

(ii) *if T is injective, then* $\iota_T(x)=\liminf\limits_{n\to\infty}\|T^{-n}x\|^{-1/n}$ *whenever* $x\in T^\infty X$;

(iii) $r_T(Tx)\leqslant r_T(x)$ *and* $\iota_T(Tx)\geqslant\iota_T(x)$;

(iv) *if T is injective, then* $\iota_T(Tx)=\iota_T(x)$;

(v) *if* $\sigma_T(x)\neq\emptyset$, *then* $\iota_T(x)\leqslant\min\{|\lambda|:\lambda\in\sigma_T(x)\}$ *and* $r_T(x)\geqslant\max\{|\lambda|:\lambda\in\sigma_T(x)\}$; *if T has* SVEP, *then equalities obtain for all non-zero* $x\in X$;

(vi) *if T is injective or has* SVEP, *then* $x\in\mathfrak{X}_T\big(\mathbb{C}\setminus V\big(0,\iota_T(x)\big)\big)$;

(vii) *if T is injective or has* SVEP, *then either* $\iota_T(x)=0$ *or* $\iota_T(x)\geqslant\iota(T)$.

Proof. If $\iota_T(x)>0$, then $x\in T^\infty X$ by the preceding lemma, and so (i) holds. (ii) is clear from the last part of Lemma 1.1, while (iii) follows from the fact that, for any closed subset F of the plane, $\mathfrak{X}_T(F)$ is T-invariant. Moreover, for every $r>0$, $T\mathfrak{X}_T\big(\mathbb{C}\setminus V(0,r)\big)=\mathfrak{X}_T\big(\mathbb{C}\setminus V(0,r)\big)$ by 3.3.1, [12]. Thus, if T is injective, then $x\in\mathfrak{X}_T\big(\mathbb{C}\setminus V(0,r)\big)$ if and only if $Tx\in\mathfrak{X}_T\big(\mathbb{C}\setminus V(0,r)\big)$. (iv) is now immediate.

The definition of $\iota_T(x)$ and the fact that $x\in\mathfrak{X}_T\big(\nabla\big(0,r_T(x)\big)\big)$ imply that $\sigma_T(x)\subseteq\{\lambda:\iota_T(x)\leqslant|\lambda|\leqslant r_T(x)\}$ for all $x\in X$. Thus $\iota_T(x)\leqslant\min\{|\lambda|:\lambda\in\sigma_T(x)\}$ and $r_T(x)\geqslant\max\{|\lambda|:\lambda\in\sigma_T(x)\}$ if $\sigma_T(x)\neq\emptyset$. It is well known and easy to see that $r_T(x)=\max\{|\lambda|:\lambda\in\sigma_T(x)\}$ if T has SVEP ([12], 3.3.13). Similarly, if T has SVEP and $\min\{|\lambda|:\lambda\in\sigma_T(x)\}>0$, then, by definition, $\iota_T(x)\geqslant\min\{|\lambda|:\lambda\in\sigma_T(x)\}$. Thus (v) is established. Moreover, if T has SVEP, then $\mathfrak{X}_T\big(\bigcap\limits_k F_k\big)=\bigcap\limits_k\mathfrak{X}_T(F_k)$ for any collection of closed sets $F_k\subseteq\mathbb{C}$ by 3.3.2, [12], and so, for every vector $x\in X$, $x\in\mathfrak{X}_T\big(\mathbb{C}\setminus V\big(0,\iota_T(x)\big)\big)$ in this case. If T is injective, then $x\in\mathfrak{X}_T\big(\mathbb{C}\setminus V\big(0,\iota_T(x)\big)\big)$ by Lemma 1.1 (vi) is proved.

If T is injective and $0<\iota_T(x)<\infty$, then $x\in T^\infty X$ and $\frac{\|x\|}{\|T^{-n}x\|}\geqslant\kappa(T^n)$ for each $n\geqslant 0$. Therefore $\iota_T(x)=\liminf\limits_{n\to\infty}\Big(\frac{\|x\|}{\|T^{-n}x\|}\Big)^{1/n}\geqslant\lim\limits_{n\to\infty}\kappa(T^n)^{1/n}=\iota(T)$, by part (ii) above. If T has SVEP and $0<\iota_T(x)<\infty$, then $\sigma_T(x)\neq\emptyset$ and, by (v), $\iota_T(x)=\min\{|\lambda|:\lambda\in\sigma_T(x)\}=\min\{|\lambda|:\lambda\in\partial\sigma_T(x)\}\geqslant\iota(T)$, since

$\partial\sigma_T(x) \subseteq \sigma_{\mathrm{ap}}(T)$ ([12], 3.1.12) and $|\lambda| \geqslant \iota(T)$ for all $\lambda \in \sigma_{\mathrm{ap}}(T)$ ([12], 1.6.2). This completes the proof of (vii). ∎

For any operator $T \in \mathcal{L}(X)$, the set $\{x : \sigma_T(x) = \sigma_{\mathrm{su}}(T)\}$ is of second category in X by 1.3.2, [12], and since $\partial\sigma(T) \subseteq \sigma_{\mathrm{su}}(T)$, it follows that $r_T(x) = r(T)$ for all x in a set of second category ([12], 3.3.14). One would expect some symmetry between the inner and outer local spectral radii, and the following corollary shows this to be the case up to a point.

COROLLARY 1.3. *Let $T \in \mathcal{L}(X)$.*

(i) *if T is surjective, but not injective, then $\iota(T) = 0 < \iota_T(x)$ for every $x \in X$;*

(ii) *if T is bounded below, but not surjective, then $\iota(T) > 0$, but $\{x : \iota_T(x) = 0\}$ is of second category in X;*

(iii) *otherwise, $\{x : \iota_T(x) = \iota(T)\}$ is a set of second category in X.*

In particular, if both T and T^* have SVEP, *then $\{x : \iota_T(x) = \iota(T)\}$ is a set of second category.*

Proof. If T is not injective, then $\iota(T) = 0$, and, since $\sigma_T(x) \subseteq \sigma_{\mathrm{su}}(T)$, it follows that $\iota_T(x) > 0$ for every x whenever T is surjective. If T is bounded below, but not surjective, then $\iota(T) > 0$ by [13], and the generalized range $T^\infty X$ is a proper closed subspace of X and therefore of first category in X. (ii) now follows from Proposition 1.2 (i). If neither (i) nor (ii) holds, then T is invertible or $0 \in \sigma_{\mathrm{ap}}(T) \cap \sigma_{\mathrm{su}}(T)$. In the first case, it follows by Proposition 1.2 (ii) that

$$\iota_T(x) = r_{T^{-1}}(x)^{-1} = \min\{|z| : z \in \sigma(T)\} = \iota(T)$$

for every x such that $\sigma_{T^{-1}}(x) = \sigma_{\mathrm{su}}(T^{-1})$. In the case that $0 \in \sigma_{\mathrm{ap}}(T) \cap \sigma_{\mathrm{su}}(T)$, we have $\iota(T) = 0$ and so again $\iota(T) = \iota_T(x)$ for every x such that $\sigma_T(x) = \sigma_{\mathrm{su}}(T)$. By 1.3.2, [12], this completes the proof of (iii). If T and T^* both have SVEP, then $\sigma(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{su}}(T)$ by 1.3.2, [12], and the last statement follows. ∎

The final conclusion of the preceding corollary applies, in particular, to all decomposable operators, but, in general, the result may fail to hold quite dramatically. For instance, if $T \in \mathcal{L}(X)$ is such that $0 \in \sigma(T) \setminus \sigma_{\mathrm{ap}}(T)$ and T has fat local spectra, then, by Proposition 1.2 (v) and [13], $\iota_T(x) = 0 < \iota(T)$ for every $x \neq 0$. The unweighted unilateral right shift, T, on $\ell^2(\mathbb{N})$, provides a simple example. Since T is an isometry with trivial generalized range, $\iota(T) = 1$, while $\iota_T(x) = 0$ for every non-zero $x \in \ell^2(\mathbb{N})$. See Section 2 below for further details.

The preceding example may also be used to illustrate that, in remarkable contrast to the outer local spectral radius, its inner counterpart need not be invariant under extensions of the given operator to a larger Banach space. Indeed, the unweighted bilateral right shift, S, on $\ell^2(\mathbb{Z})$ may be canonically viewed as an extension of the unilateral right shift, T, on $\ell^2(\mathbb{N})$. The operator S is unitary, hence decomposable, and satisfies $\iota_S(x) = 1$ for all non-zero $x \in \ell^2(\mathbb{Z})$ by Proposition 1.2 (v), while $\iota_T(x) = 0$ for all non-zero $x \in \ell^2(\mathbb{N})$. On the other hand, if the operator $T \in \mathcal{L}(X)$ is injective or has SVEP, and if $x \in X$ satisfies $\iota_T(x) > 0$, then Lemma 1.1 ensures that $\iota_S(x) = \iota_T(x)$ for every bounded linear extension S of T.

Corollary 1.4. *Suppose that $T \in \mathcal{L}(X)$ has Dunford's property* (C) *and that x is a cyclic vector for T. Then $r_T(x) = r(T)$ and either $\iota_T(x) = 0$ or $\iota_T(x) = \iota(T)$. Moreover, $\iota_T(x) > 0$ if and only if T is invertible.*

Proof. If $F = \{\lambda : \iota_T(x) \leqslant |\lambda| \leqslant r_T(x)\}$, then the closed space $\mathfrak{X}_T(F)$ contains $\{T^n x : n \geqslant 0\}$ by Proposition 1.2 (iii) and (v). Since the linear span of the latter set is dense in X, it follows that $\mathfrak{X}_T(F) = X$ and therefore that $\sigma(T) = \sigma_{\mathrm{su}}(T) \subseteq F$ by 1.3.2, [12]. In particular, $r(T) \leqslant r_T(x)$. Since $r_T(u) \leqslant r(T)$ for all $u \in X$, we conclude that $r_T(x) = r(T)$. Moreover, if $\iota_T(x) > 0$, then T is invertible, $\iota(T) = \min\{|\lambda| : \lambda \in \sigma(T)\} \geqslant \iota_T(x)$, and so $\iota_T(x) = \iota(T)$ by Proposition 1.2 (vii). Conversely, if T is invertible, then clearly $0 \in \rho_T(x)$ and therefore $\iota_T(x) > 0$. ∎

Unlike $(\|T^n\|^{1/n})_{n\geqslant 1}$ and $(\kappa(T^n)^{1/n})_{n\geqslant 1}$, the sequences $(\|T^n x\|^{1/n})_{n\geqslant 1}$ and $(\|T^{-n}x\|^{-1/n})_{n\geqslant 1}$ are generally not convergent. We provide an elementary example in the next section. On the other hand, the following result establishes convergence for a large class of operators.

Proposition 1.5. *If $T \in \mathcal{L}(X)$ has property Bishop's (β), then, for every $x \in X$, $r_T(x) = \lim_{n\to\infty} \|T^n x\|^{1/n}$, and either $\iota_T(x) = 0$ or there is a unique sequence $(a_n)_{n\geqslant 0}$ in X satisfying*

(i) $Ta_0 = x$;

(ii) $a_n = Ta_{n+1}$ *for every* $n \geqslant 0$; *and*

(iii) $\liminf_{n\to\infty} \|a_n\|^{-1/n} > 0$.

In the latter case, the sequence $(\|a_n\|^{-1/n})_{n\geqslant 1}$ is convergent with limit $\iota_T(x)$. Moreover, if T is injective, then $\iota_T(x) = \lim_{n\to\infty} \|T^{-n}x\|^{-1/n}$ for every $x \in T^\infty X$.

Proof. The first statement is due to Atzmon (see [12], 3.3.17). If $\iota_T(x) > 0$, then, since T has SVEP, Lemma 1.1 implies that there is a unique sequence $(a_n)_{n\geqslant 0}$ in X satisfying (i), (ii) and (iii) above, and $\liminf_{n\to\infty} \|a_n\|^{-1/n} = \iota_T(x)$. It remains to show that $\limsup_{n\to\infty} \|a_n\|^{-1/n} \leqslant \iota_T(x)$. Since conditions (i), (ii) and (iii) are preserved under similarity and hold for any decomposable extension of T, we may, without loss of generality, assume that T is decomposable by 2.4.4, [12]. Let $s > \iota_T(x)$, define Q to be the quotient mapping, $Q : X \to X/\mathfrak{X}_T(\mathbb{C} \setminus V(0,s))$, and let S be the continuous linear map on $X/\mathfrak{X}_T(\mathbb{C} \setminus V(0,s))$ defined by $SQ = QT$. Then $Qx \neq 0$ and, by the decomposability of T and 1.2.22, [12], $\sigma(S) \subseteq \nabla(0,s)$. Since

$$\|Qx\| = \|QT^{n+1}a_n\| = \|S^{n+1}Qa_n\| \leqslant \|S^{n+1}\|\,\|a_n\|$$

for every n, we have $\|a_n\|^{-1/n} \leqslant \|S^{n+1}\|^{1/n}\,\|Qx\|^{-1/n}$, and therefore

$$\limsup_{n\to\infty} \|a_n\|^{-1/n} \leqslant s$$

by the spectral radius formula for S. Since $s > \iota_T(x)$ is arbitrary, the main result follows. This, along with Proposition 1.2 (ii), implies the last assertion of the proposition if $\iota_T(x) > 0$.

If T has property (β) and is also injective, then T is similar to the restriction of an injective decomposable operator, $R \in \mathcal{L}(Y)$, on some Banach space Y ([12], 2.4.3 and 2.4.4). Suppose that $J \in \mathcal{L}(X,Y)$ is bounded below and satisfies $JT = RJ$. If $x \in T^{\infty}X$ has inner local spectral radius $\iota_T(x) = 0$, then $\iota_R(Jx) = 0$ as well. Indeed, $\{y \in Y : R^{n+1}y = Jx\} = \{JT^{-(n+1)}x\}$ for every $n \geqslant 0$, and therefore, by Lemma 1.1,

$$\iota_R(Jx) = \liminf_{n\to\infty} \|JT^{-(n+1)}x\|^{-1/n} = \liminf_{n\to\infty} \|T^{-n}x\|^{-1/n} = \iota_T(x) = 0.$$

This shows that, also in the case where $\iota_T(x) = 0$, one may assume that T is decomposable. A repetition of the argument in the preceding paragraph now leads to $\limsup\limits_{n\to\infty} \|T^{-n}x\|^{-1/n} \leqslant s$ for every $s > 0$, and the last assertion is established. ∎

If T is injective, then $\iota_T(T^m x) = \iota_T(x)$ for every $x \in X$ and $m \geqslant 0$ by Proposition 1.2 (iv). Under the assumption of (β), we obtain the analogous result for outer spectral radii.

COROLLARY 1.6. *If $T \in \mathcal{L}(X)$ has Bishop's property (β), then $r_T(T^m x) = r_T(x)$ for every $x \in X$ and $m \geqslant 0$.*

Proof. For all $x \in X$ and for all natural numbers m and n, we have that

$$\|T^n(T^m x)\|^{1/n} = \big(\|T^{n+m}x\|^{1/(n+m)}\big)^{(n+m)/n}.$$

By Proposition 1.5, $\|T^{n+m}x\|^{1/(n+m)} \to r_T(x)$ as $n \to \infty$, and therefore $r_T(T^m x) = r_T(x)$. ∎

2. LOCAL SPECTRA OF WEIGHTED SHIFTS

We refer the reader to Shields's article ([20]) for a survey of the theory of weighted shifts on Hilbert spaces; for unilateral weighted shifts on $\ell^p(\mathbb{N})$, $p \neq 2$, see [12], 1.6 as well. Below, we collect some of the basic facts essentially from [20]. Let $\mathbb{K}$ stand for either the set of integers $\mathbb{Z}$ or the set $\mathbb{N}_0 = \{0,1,2,\ldots\}$. Given a bounded sequence $(\omega_n)_{n\in\mathbb{K}}$ of strictly positive weights, we define the corresponding weighted right shift on $\ell^p(\mathbb{K})$, $1 \leqslant p \leqslant \infty$, by $T(x_n)_{n\in\mathbb{K}} = (\omega_{n-1}\, x_{n-1})_{n\in\mathbb{K}}$, where $x_{-1} = \omega_{-1} = 0$ if $\mathbb{K} = \mathbb{N}_0$. T is injective since $\omega_n > 0$ for all $n \in \mathbb{K}$, and $\|T\| = \sup\limits_{n\in\mathbb{K}} \omega_n < \infty$ since $(\omega_n)_{n\in\mathbb{K}}$ is bounded. Weighted shifts acting on $\ell^p(\mathbb{N}_0)$ and $\ell^p(\mathbb{Z})$ are referred to as unilateral and bilateral, respectively. If $p < \infty$, then the weighted shift T is characterized by its action on the canonical basis for $\ell^p(\mathbb{K})$: $Te_n = \omega_n e_{n+1}$ for all $n \in \mathbb{K}$. For every $n \in \mathbb{K}$, define

$$\alpha_n = \begin{cases} \omega_0 \cdots \omega_{n-1} & \text{if } n > 0; \\ 1 & \text{if } n = 0; \\ (\omega_n \cdots \omega_{-1})^{-1} & \text{if } n < 0 \text{ and } \mathbb{K} = \mathbb{Z}. \end{cases}$$

So $T^n e_k = \alpha_{n+k}/\alpha_k\, e_{n+k}$ for all $n \geqslant 0$ and $k \in \mathbb{K}$. Notice that, for every $n \geqslant 0$,

$$\|T^n\| = \sup_{k\in\mathbb{K}} \frac{\alpha_{n+k}}{\alpha_k} \quad \text{and} \quad \kappa(T^n) = \inf_{k\in\mathbb{K}} \frac{\alpha_{n+k}}{\alpha_k}.$$

If T is unilateral, then $\sigma_{\mathrm{ap}}(T) = \{\lambda : \iota(T) \leqslant |\lambda| \leqslant r(T)\}$ and $\sigma(T) = \{\lambda : |\lambda| \leqslant r(T)\}$. To describe the spectrum in the bilateral case, we first observe that the adjoint of a bilateral right shift may be viewed as a bilateral left shift and hence is always injective. Consequently, every bilateral shift T with $\iota(T) > 0$ is invertible, since this condition ensures that T is bounded below and hence that T^* is surjective. As in Theorem 5 of [20] for the case $p = 2$, it then follows that the spectrum of an arbitrary bilateral shift T on $l^p(\mathbb{K})$ is given by $\sigma(T) = \{\lambda : \iota(T) \leqslant |\lambda| \leqslant r(T)\}$. The approximate point spectrum is described in terms of the quantities

$$\iota^-(T) = \lim_{n\to\infty} \left(\inf_{k<0} \frac{\alpha_k}{\alpha_{k-n}} \right)^{1/n}, \quad \iota^+(T) = \lim_{n\to\infty} \left(\inf_{k\geqslant 0} \frac{\alpha_{n+k}}{\alpha_k} \right)^{1/n},$$

$$r^-(T) = \lim_{n\to\infty} \left(\sup_{k<0} \frac{\alpha_k}{\alpha_{k-n}} \right)^{1/n}, \quad r^+(T) = \lim_{n\to\infty} \left(\sup_{k\geqslant 0} \frac{\alpha_{n+k}}{\alpha_k} \right)^{1/n}.$$

If $r^-(T) < \iota^+(T)$, then $\sigma_{\mathrm{ap}}(T) = \{\lambda : \iota^-(T) \leqslant |\lambda| \leqslant r^-(T)\} \cup \{\lambda : \iota^+(T) \leqslant |\lambda| \leqslant r^+(T)\}$; otherwise, $\sigma_{\mathrm{ap}}(T) = \sigma(T)$. Moreover, T has spectral radius $r(T) = \max\{r^-(T), r^+(T)\}$, and $\iota(T) = \min\{\iota^-(T), \iota^+(T)\}$. The proofs of these facts for shifts on $\ell^p(\mathbb{K})$, $1 \leqslant p < \infty$, follow just as in the case $p = 2$; see [19] and 1.6.15, [12].

We now identify the inner and outer local spectral radii of weighted shifts in terms of the underlying weight sequence. In the unilateral case, $T^\infty \ell^p(\mathbb{N}_0) = \{0\}$, and so $\iota_T(x) = 0$ for every $x \neq 0$ in $\ell^p(\mathbb{N}_0)$ by Proposition 1.2 (i). Since $T^{*(n+1)} e_n = 0$, $r_{T^*}(e_n) = 0$ for all $n \geqslant 0$, and $\alpha_{n+1}^{-1} = \min\{\|x\| : e_0 = T^{*(n+1)} x\}$. Thus, by Lemma 1.1,

$$\iota_{T^*}(e_0) = \liminf_{n\to\infty} \alpha_n^{1/n} \tag{2.1}$$

and

$$r_T(e_0) = \limsup_{n\to\infty} \alpha_n^{1/n}. \tag{2.2}$$

If T is a bilateral shift on $\ell^p(\mathbb{Z})$, $1 \leqslant p < \infty$, then, by the canonical identification of right and left bilateral shifts, T^* is a bilateral shift on $\ell^q(\mathbb{Z})$, where $q = \frac{p}{p-1}$. Indeed, if $Te_n = \omega_n e_{n+1}$ for all $n \in \mathbb{Z}$, then $T^* f_n = \omega_n^* f_{n+1}$, where $f_n = e_{-n}$ and $\omega_n^* = \omega_{-n-1}$. Also, $T^{*n} f_k = \frac{\alpha_{n+k}^*}{\alpha_k^*} f_{n+k}$ where $\alpha_n^* = \alpha_{-n}^{-1}$ for all n. Therefore, by Proposition 1.2 (ii),

$$\iota_{T^*}(e_0) = \liminf_{n\to\infty} \alpha_n^{1/n} \tag{2.3}$$

and

$$r_T(e_0) = \limsup_{n\to\infty} \alpha_n^{1/n}; \tag{2.4}$$

$$\iota_T(e_0) = \liminf_{n\to\infty} \alpha_{-n}^{-1/n} \tag{2.5}$$

and

$$r_{T^*}(e_0) = \limsup_{n\to\infty} \alpha_{-n}^{-1/n}. \tag{2.6}$$

Notice that the inferior and superior limits that occur in (2.1)–(2.6) are precisely the parameters used by Shields in his analysis of the point spectra of a weighted shift and its adjoint ([20]).

Now it is a simple matter to construct an example of an injective operator $T \in \mathcal{L}(X)$ and a vector $x \in T^\infty X$ for which $(\|T^n x\|^{1/n})_{n\geqslant 1}$ and $(\|T^{-n}x\|^{-1/n})_{n\geqslant 1}$ are both divergent.

EXAMPLE 2.1. Define a sequence $(\alpha_k)_{k\in\mathbb{Z}}$ by

$$\alpha_k = \begin{cases} \exp(-2^n) & \text{if } 2^n \leqslant k < 2^{n+1}; \\ 1 & \text{if } k = 0; \\ \exp(2^n) & \text{if } -2^{n+1} < k \leqslant -2^n. \end{cases}$$

The corresponding bilateral shift T on $\ell^2(\mathbb{Z})$ is bounded, and, by (2.3)–(2.6), it satisfies

$$\iota_{T^*}(e_0) = \liminf_{n\to\infty} \alpha_n^{1/n} = \mathrm{e}^{-1} < \mathrm{e}^{-1/2} = \limsup_{n\to\infty} \alpha_n^{1/n} = r_T(e_0),$$

and

$$\iota_T(e_0) = \liminf_{n\to\infty} \alpha_{-n}^{-1/n} = \mathrm{e}^{-1} < \mathrm{e}^{-1/2} = \limsup_{n\to\infty} \alpha_{-n}^{-1/n} = r_{T^*}(e_0).$$

It follows from Proposition 1.5 that T does not have property (β). Evidently, the same phenomenon occurs for every decreasing null sequence $(\alpha_k)_{k\in\mathbb{N}}$ for which the sequence $(\alpha_k^{1/k})_{k\in\mathbb{N}}$ fails to be convergent.

We refer to a sequence $x = (x_n)_{n\in\mathbb{K}}$ as finitely supported provided that $x_n = 0$ for all but finitely many $n \in \mathbb{K}$.

PROPOSITION 2.2. *For every weighted shift T on $\ell^p(\mathbb{K})$, $1 \leqslant p < \infty$, the following assertions hold for all non-zero $x \in \ell^p(\mathbb{K})$:*

(i) *$r_T(e_0) \leqslant r_T(x) \leqslant r(T)$ and $\iota_T(x) \leqslant \iota_T(e_0)$; moreover, if $\mathbb{K} = \mathbb{Z}$, then $\iota(T) \leqslant \iota_T(x)$;*

(ii) *$r_T(x) = r_T(e_0)$, and $\iota_T(x) = \iota_T(e_0)$ whenever x is finitely supported.*

If T is bilateral, then the following also obtain:

(iii) *$\iota_T(x) \leqslant r_{T^*}(e_0)$ and $\iota_{T^*}(e_0) \leqslant r_T(x)$, and*

(iv) *$\iota_{T^*}(x^*) \leqslant r_T(e_0)$ and $\iota_T(e_0) \leqslant r_{T^*}(x^*)$ for all non-zero $x^* \in \ell^q(\mathbb{K})$, where $q = p/(p-1)$.*

Proof. First we show that $r_T(e_n) = r_T(e_0)$ and $\iota_T(e_n) = \iota_T(e_0)$ for every $n \in \mathbb{K}$, and therefore, if $X = \ell^p(\mathbb{K})$, then $\operatorname{span}\{e_n\}_{n\in\mathbb{K}} \subseteq \mathfrak{X}_T\big(\mathbb{C} \setminus V(0, \iota_T(e_0))\big) \cap \mathfrak{X}_T\big(\nabla(0, r_T(e_0))\big)$ by Proposition 1.2 (vi). This, together with (i), will prove (ii). The sequence $(\iota_T(e_n))_{n\in\mathbb{K}}$ is constant by Proposition 1.2 (i) and (iv), and $(r_T(e_n))_{n\in\mathbb{K}}$ is decreasing by Proposition 1.2 (iii). To see that the latter sequence is actually constant, fix $m \in \mathbb{K}$, let $\varepsilon > 0$, and choose N so that $\|T^n e_{m+1}\|^{1/n} \leqslant r_T(e_{m+1}) + \varepsilon$ for all $n \geqslant N$. For $n \geqslant N$, it follows that

$$\begin{aligned} \|T^{n+1} e_m\|^{1/(n+1)} &= \Big(\frac{\alpha_{m+1}}{\alpha_m}\Big)^{1/(n+1)} (\|T^n e_{m+1}\|^{1/n})^{n/(n+1)} \\ &\leqslant \Big(\frac{\alpha_{m+1}}{\alpha_m}\Big)^{1/(n+1)} (r_T(e_{m+1}) + \varepsilon)^{n/(n+1)}, \end{aligned}$$

and so $r_T(e_m) = \limsup\limits_{n\to\infty} \|T^{n+1} e_m\|^{1/(n+1)} \leqslant r_T(e_{m+1}) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that $r_T(e_m) \leqslant r_T(e_{m+1})$, and so the sequence $(r_T(e_m))_{m\in\mathbb{K}}$ is constant.

Now, let P_m denote the canonical projection onto $\operatorname{span}(e_m)$ for each $m \in \mathbb{K}$. Then, for arbitrary $x \in \ell^p(\mathbb{K})$,

$$\|T^n x\|^{1/n} = \Big(\sum_{k\in\mathbb{K}} \|T^n P_k x\|^p\Big)^{1/(np)} \geqslant \|T^n P_m x\|^{1/n}$$

for all $m, n \in \mathbb{N}$. Thus, for any m such that $P_m x \neq 0$, it follows that $r_T(x) \geqslant r_T(e_m) = r_T(e_0)$. By Proposition 1.2 (ii), the estimate $\iota_T(x) \leqslant \iota_T(e_0)$ follows similarly if T is bilateral and $x \in T^\infty X$; otherwise, $\iota_T(x) = 0$. For T bilateral, $\sigma(T) = \{\lambda : \iota(T) \leqslant |\lambda| \leqslant r(T)\}$. Hence, if $\iota(T) > 0$, then T is invertible, and therefore, by Proposition 1.2 (ii), $\iota_T(x) = r_{T^{-1}}(x)^{-1} \geqslant r(T^{-1})^{-1} = \iota(T)$ for all non-zero $x \in l^p(\mathbb{Z})$. (i) is established.

If T is bilateral, then T^* is also a bilateral shift, and so the argument above gives $r_{T^*}(e_n) = r_{T^*}(e_0)$ and $\iota_{T^*}(e_n) = \iota_{T^*}(e_0)$ for every $n \in \mathbb{Z}$. If $r_{T^*}(e_0) < \iota_T(x)$ or if $r_T(x) < \iota_{T^*}(e_0)$, then $\langle x, e_n\rangle = 0$ for all $n \in \mathbb{K}$ by 2.5.1, [12], thus $x = 0$, the desired contradiction. This proves (iii), and the last statement follows similarly. ∎

For weighted shifts with property (C), we obtain the following improvement of Corollary 1.4.

Corollary 2.3. *Let T be a weighted shift on $\ell^p(\mathbb{K})$, $1 \leqslant p < \infty$. If T has property* (C)*, then $r_T(x) = r(T)$ for all $x \neq 0$, and either $\iota_T(x) = 0$ for every $x \neq 0$ or $\iota_T(x) = \iota(T)$ for every $x \neq 0$.*

Proof. If $F = \{\lambda : \iota_T(e_0) \leqslant |\lambda| \leqslant r_T(e_0)\}$ and $X = \ell^p(\mathbb{K})$, then the dense set $\operatorname{span}\{e_n\}_{n\in\mathbb{K}}$ is contained in $\mathfrak{X}_T(F)$ by Propositions 1.2 (v) and 2.2 (ii). Therefore, if T has property (C), then $X = \mathfrak{X}_T(F)$. Thus, by Proposition 2.2 (i), $r_T(x) = r(T)$ and $\iota_T(x) = \iota_T(e_0)$ for all $x \neq 0$. If $\iota_T(e_0) > 0$, then it follows from $\sigma(T) \subseteq F$ that T is invertible with $\iota(T) = \min\{|\lambda| : \lambda \in \sigma(T)\} \geqslant \iota_T(e_0) > 0$. Thus $\iota_T(e_0) = \iota(T)$ by Propositions 1.2 (vii) or 2.2 (i). ∎

As a consequence of Corollary 2.3, we obtain an elementary proof of a result due to Sun ([22]).

Corollary 2.4. *Let T be a weighted shift on $\ell^p(\mathbb{K})$, $1 \leqslant p < \infty$. If T is decomposable, then $\sigma(T) = \{\lambda : |\lambda| = r(T)\}$.*

Proof. Without loss of generality, we may assume that $r(T) > 0$. For any r_1 and r_2 such that $0 < r_1 < r_2 < r(T)$, we have $\mathbb{C} = V(0, r_2) \cup (\mathbb{C} \setminus \nabla(0, r_1))$, and so, letting $X = \ell^p(\mathbb{K})$, we have

$$X = \mathfrak{X}_T(\nabla(0, r_2)) + \mathfrak{X}_T(\mathbb{C} \setminus V(0, r_1)) = \mathfrak{X}_T(\mathbb{C} \setminus V(0, r_1))$$

by the previous corollary. It follows that $\sigma(T) \subseteq \mathbb{C} \setminus V(0, r_1)$, and, since r_1 was arbitrary, $0 < r_1 < r(T)$, we must have $\sigma(T) \subseteq \{\lambda : |\lambda| = r(T)\}$. Because the spectrum of T is circularly symmetric, the claim is established. ∎

Since the spectrum of a unilateral shift is a disc centered at the origin, the preceding corollary implies that a unilateral shift is decomposable if and only if it is quasinilpotent, a fact already observed in 1.6.14, [12].

If T is a weighted shift on $\ell^p(\mathbb{K})$ for $1 \leqslant p < \infty$, then, as in the Hilbert space case in Theorems 8 and 9, [20], a simple computation shows that every eigenvector for an eigenvalue λ of the adjoint T^* is a multiple of the sequence $k_\lambda := (\lambda^n/\alpha_n)_{n\in\mathbb{K}}$. In particular, it follows that $k_\lambda \in \ell^q(\mathbb{K})$ where $q = p/(p-1)$, and thus, by the root test and formulas (2.1), (2.3), and (2.6), $r_{T^*}(e_0) \leqslant |\lambda| \leqslant \iota_{T^*}(e_0)$. Conversely, if $r_{T^*}(e_0) < \iota_{T^*}(e_0)$, then $k_\lambda \in \ell^q(\mathbb{K})$ and $(T^* - \lambda)k_\lambda = 0$ for all $\lambda \in \mathbb{C}$ for which $r_{T^*}(e_0) < |\lambda| < \iota_{T^*}(e_0)$. Moreover, the function $\lambda \mapsto k_\lambda$ is analytic on the open annulus $\{\lambda : r_{T^*}(e_0) < |\lambda| < \iota_{T^*}(e_0)\}$. These observations are central to the following results.

PROPOSITION 2.5. *Let T be a weighted shift on $\ell^p(\mathbb{K})$, $1 \leqslant p < \infty$.*

(i) *If T is a unilateral shift, then T has* SVEP *and $\sigma_T(x) = \sigma_T(e_0) = \{\lambda : |\lambda| \leqslant r_T(e_0)\}$ for all finitely supported $x \neq 0$. T^* has* SVEP *if and only if $\iota_{T^*}(e_0) = 0$.*

(ii) *If T is a bilateral shift, then T has* SVEP *if and only if $\iota_T(e_0) \leqslant r_T(e_0)$. Equivalently, T fails to have* SVEP *if and only if $\sigma_T(x) = \emptyset$ for every finitely supported x. If T has* SVEP*, then $\sigma_T(x) = \sigma_T(e_0) = \{\lambda : \iota_T(e_0) \leqslant |\lambda| \leqslant r_T(e_0)\}$ for all finitely supported $x \neq 0$.*

(iii) *If T is a bilateral shift, then T^* has* SVEP *if and only if $\iota_{T^*}(e_0) \leqslant r_{T^*}(e_0)$. Finally, either T or T^* has* SVEP.

Proof. Clearly, every unilateral weighted shift has empty point spectrum and therefore SVEP. The stipulated characterizations of SVEP for T^* in both the unilateral and bilateral case are immediate from the discussion preceding this proposition, and a similar argument shows that a bilateral shift T fails to have SVEP if and only if $\iota_T(e_0) > r_T(e_0)$. By Propositions 1.2 (v) and 2.2 (ii), this happens precisely when $\sigma_T(x) = \emptyset$ for all finitely supported $x \in \ell^p(\mathbb{Z})$. The fact that at least one of the shifts T and T^* has SVEP is now clear from the formulas (2.3)–(2.6).

We establish the formula for the local spectrum only in the bilateral case, since the proof in the unilateral case is similar and, in fact, easier. So suppose that T is a bilateral shift for which $\iota_T(e_0) \leqslant r_T(e_0)$, and let $x \in \ell^p(\mathbb{Z})$ be non-zero and finitely supported. Since T has SVEP, Propositions 1.2 (v) and 2.2 (ii) ensure that $\sigma_T(x) \subseteq \{\lambda : \iota_T(e_0) \leqslant |\lambda| \leqslant r_T(e_0)\}$. Evidently, equality holds when $\iota_T(e_0) = r_T(e_0) = 0$. Hence it remains to be seen that the two sets $\rho_T(x)$ and $\{\lambda : \iota_T(e_0) \leqslant |\lambda| \leqslant r_T(e_0)\}$ are disjoint provided that either $\iota_T(e_0) < r_T(e_0)$ or $\iota_T(e_0) = r_T(e_0) > 0$. Our proof of this fact is inspired by an argument given in [4]. Let $x = \sum\limits_{n=N}^{M} x_n e_n$ where $x_N, x_M \neq 0$, and let f be the unique analytic function on $\rho_T(x)$ such that $(T-\lambda)f(\lambda) = x$ for every $\lambda \in \rho_T(x)$. If $F_n(\lambda) = \langle f(\lambda), e_n\rangle$, then each F_n is analytic on $\rho_T(x)$, $f(\lambda) = \sum\limits_{n=-\infty}^{\infty} F_n(\lambda)e_n$ and

$$\lambda F_n(\lambda) + x_n = \omega_{n-1}F_{n-1}(\lambda)$$

for all integers n. Since $\omega_n\alpha_n = \alpha_{n+1}$ for every $n \in \mathbb{Z}$, it follows by induction that, for every $k \geqslant 1$,

$$\begin{aligned} F_{N-k}(\lambda) &= \Big(F_N(\lambda) + \frac{x_N}{\lambda}\Big)\frac{\alpha_{N-k}}{\alpha_N}\,\lambda^k; \\ F_{N+k}(\lambda) &= -\alpha_{N+k}\lambda^{-k}\Big(\sum_{\nu=1}^{k}\frac{x_{N+\nu}}{\alpha_{N+\nu}}\,\lambda^{\nu-1} - \frac{1}{\alpha_N}\,F_N(\lambda)\Big); \\ F_{M+k}(\lambda) &= F_M(\lambda)\,\frac{\alpha_{M+k}}{\alpha_M}\,\lambda^{-k}. \end{aligned}$$

Define $f_-(\lambda) = \sum_{k=1}^{\infty}\frac{\alpha_{N-k}}{\alpha_N}\,\lambda^k\,e_{N-k}$ and $f_+(\lambda) = \sum_{k=0}^{\infty}\frac{\alpha_{M+k}}{\alpha_M}\,\lambda^{-k}\,e_{M+k}$. Then, for every $\lambda \in \rho_T(x)$, we obtain

$$\begin{aligned} \|f(\lambda)\|^p &\geqslant \sum_{k=1}^{\infty}|F_{N-k}(\lambda)|^p + \sum_{k=0}^{\infty}|F_{M+k}(\lambda)|^p \\ &= \Big|F_N(\lambda) + \frac{x_N}{\lambda}\Big|^p\sum_{k=1}^{\infty}\Big|\frac{\alpha_{N-k}}{\alpha_N}\,\lambda^k\Big|^p + |F_M(\lambda)|^p\sum_{k=0}^{\infty}\Big|\frac{\alpha_{M+k}}{\alpha_M}\,\lambda^{-k}\Big|^p \\ &= \Big|F_N(\lambda) + \frac{x_N}{\lambda}\Big|^p\|f_-(\lambda)\|^p + |F_M(\lambda)|^p\|f_+(\lambda)\|^p. \end{aligned}$$

But $\|f_-(\lambda)\|^p < \infty$ only if $|\lambda| \leqslant \Big(\limsup_{k\to\infty}\Big(\frac{\alpha_{N-k}}{\alpha_N}\Big)^{1/k}\Big)^{-1} = \iota_T(e_N) = \iota_T(e_0)$, and $\|f_+(\lambda)\|^p < \infty$ only if $|\lambda| \geqslant \limsup_{k\to\infty}\Big(\frac{\alpha_{M+k}}{\alpha_M}\Big)^{1/k} = r_T(e_M) = r_T(e_0)$ by Proposition 2.2 (ii). Thus $F_N(\lambda) + \frac{x_N}{\lambda} = 0$ for every $\lambda \in \rho_T(x)$ with $|\lambda| > \iota_T(e_0)$, and $F_M(\lambda) = 0$ for every $\lambda \in \rho_T(x)$ such that $|\lambda| < r_T(e_0)$. But $F_M(\lambda) = 0$ implies that either $M = N$ or $F_N(\lambda) = \alpha_N\sum_{\nu=1}^{M-N}\frac{x_{N+\nu}}{\alpha_{N+\nu}}\,\lambda^{\nu-1}$, and so we must have $\rho_T(x) \cap \{\lambda : \iota_T(e_0) \leqslant |\lambda| \leqslant r_T(e_0)\} = \emptyset$ in the case that $\iota_T(e_0) < r_T(e_0)$. In the remaining case that $\iota_T(e_0) = r_T(e_0) > 0$ we obtain, by continuity, that $F_N(\lambda) + \frac{x_N}{\lambda} = 0$ and $F_M(\lambda) = 0$ for all $\lambda \in \rho_T(x)$ for which $|\lambda| = \iota_T(e_0) = r_T(e_0)$. As before, this implies that the sets $\rho_T(x)$ and $\{\lambda : \iota_T(e_0) = |\lambda| = r_T(e_0)\}$ are disjoint. ∎

The following elementary observation was made in [4]. For completeness, we provide a proof.

Proposition 2.6. *Let T be a weighted shift on $\ell^p(\mathbb{K})$, $1 \leqslant p < \infty$. If T^* fails to have* SVEP, *then the annulus $\mathcal{A} = \{\lambda : r_{T^*}(e_0) < |\lambda| < \iota_{T^*}(e_0)\}$ is contained in $\sigma_T(x)$ for all non-zero vectors $x \in \ell^p(\mathbb{K})$.*

Proof. Suppose that $x \in \ell^p(\mathbb{K})$ satisfies $\mathcal{A} \cap \rho_T(x) \neq \emptyset$. Since, by Proposition 2.5, T has SVEP, there exists an analytic function $f : \rho_T(x) \to \ell^p(\mathbb{K})$ such that $(T - \lambda)f(\lambda) = x$ for all $\lambda \in \rho_T(x)$. For each $\lambda \in \mathcal{A} \cap \rho_T(x)$, it follows from the observation preceding Proposition 2.5 that

$$\langle k_\lambda, x\rangle = \langle k_\lambda, (T-\lambda)f(\lambda)\rangle = \langle (T^* - \lambda)k_\lambda, f(\lambda)\rangle = \langle 0, f(\lambda)\rangle = 0.$$

Thus, by analyticity, $\langle k_\lambda, x\rangle = 0$ for all $\lambda \in \mathcal{A}$, and therefore $x = 0$, as desired. ∎

The preceding results together with Proposition 1.5 lead to a condition necessary for a weighted shift to have Bishop's property (β). Notice that, by Corollary 2.4, non-quasinilpotent, decomposable bilateral shifts have spectrum a circle, but of course do not have fat local spectra. Thus both cases in the second assertion below obtain.

THEOREM 2.7. *Suppose that T is a weighted shift with Bishop's property (β). Then,*

(i) *if T is unilateral, then $\iota(T) = r(T)$ and T has fat local spectra;*

(ii) *if T is bilateral, then either $\iota(T) = r(T)$ or T has fat local spectra.*

Proof. In the unilateral case, $0 = r_{T^*}(e_0)$ and $\iota_{T^*}(e_0) = \lim\limits_{n\to\infty} \alpha_n^{1/n} = r_T(e_0) = r(T)$ by Proposition 1.5, formulas (2.1) and (2.2), and Corollary 2.3. Thus T has fat local spectra by the last proposition, and $\iota(T) = r(T)$ by 1.6.16, [12], or [15]. If T is bilateral with $\iota(T) < r(T)$, then again $\iota_{T^*}(e_0) = \lim\limits_{n\to\infty} \alpha_n^{1/n} = r_T(e_0) = r(T)$. By Propositions 1.2 (v), 1.5, formulas (2.5) and (2.6), and Corollary 2.3, $\iota(T) = \iota_T(e_0) = \lim\limits_{n\to\infty} \alpha_{-n}^{-1/n} = r_{T^*}(e_0)$. Thus, by Proposition 2.6, the annulus $\mathcal{A} = \{\lambda : \iota(T) < |\lambda| < r(T)\}$ is contained in $\sigma_T(x)$ for every non-zero $x \in \ell^p(\mathbb{Z})$. Since $\mathcal{A}$ is dense in $\sigma(T)$, T has fat local spectra. ∎

As an application of this last theorem, we obtain a result of Williams ([24]).

COROLLARY 2.8. *Every non-normal, hyponormal weighted shift on $\ell^2(\mathbb{K})$ has fat local spectra.*

Proof. If T is hyponormal, then T is the restriction of a decomposable operator ([17]) and consequently has property (β). Because T is non-normal, a well known theorem of Putnam ([6], IV.3.2) gives us that $\sigma(T)$ has positive area; in particular, $\sigma(T)$ does not lie in a circle. Theorem 2.7 therefore implies that T has fat local spectra. ∎

3. GROWTH CONDITIONS

We denote by $\mathcal{E}(\mathbb{C})$ the usual Fréchet algebra of all C^∞-functions on $\mathbb{C}$ with the topology of uniform convergence of all orders of derivatives on the compact subsets of $\mathbb{C}$. An operator $T \in \mathcal{L}(X)$ is called *generalized scalar* if there is a continuous $\mathcal{E}(\mathbb{C})$-functional calculus for T ([12], 1.5). If an operator $T \in \mathcal{L}(X)$ has spectrum contained in the unit circle, $\mathbb{T}$, and if there exist constants $K, s > 0$ so that $\|T^n\| \leqslant K\,|n|^s$ for every non-zero integer n, then Colojoară and Foiaş show that T admits a continuous $\mathcal{E}(\mathbb{T})$-functional calculus; in particular, the operator T is generalized scalar. Moreover, a generalized scalar operator has spectrum $\sigma(T) \subseteq \mathbb{T}$ only if T satisfies the growth condition above ([5], 5.3.4 or [12], 1.5.12). Thus $T \in \mathcal{L}(X)$ is $\mathcal{E}(\mathbb{T})$-scalar if and only if T is invertible and there exist constants K and s such that, for every natural number n,

$$\frac{1}{K\,n^s} \leqslant \kappa(T^n) \leqslant \|T^n\| \leqslant K\,n^s. \tag{3.1}$$

A quasinilpotent generalized scalar operator is necessarily nilpotent ([5], 3.5, and [12], 1.5.10). Thus, by Corollary 2.4, the only possible generalized scalar weighted shifts must be bilateral with positive spectral radius. The growth condition of Colojoară and Foiaş may therefore be used to characterize all generalized scalar weighted shifts. Obviously, we may normalize and restrict our attention to the case $r(T) = 1$.

Proposition 3.1. *Let T be a weighted shift on $\ell^p(\mathbb{Z})$, $1 \leqslant p < \infty$, with spectral radius* 1. *Then T is generalized scalar if and only if T is invertible and there exist positive constants K and s such that, for every natural number n,*

$$\frac{1}{K\, n^s} \leqslant \inf_{k\in\mathbb{Z}} \frac{\alpha_{n+k}}{\alpha_k} \leqslant \sup_{k\in\mathbb{Z}} \frac{\alpha_{n+k}}{\alpha_k} \leqslant K\, n^s.$$

Colojoară and Foiaş ([5], 5.3.2) show further that an invertible operator $T \in \mathcal{L}(X)$ is (strongly) decomposable provided that

$$\sum_{n=-\infty}^{\infty} \frac{\log \|T^n\|}{1+n^2} < \infty.$$

Thus we easily obtain weighted shifts that are decomposable, but not generalized scalar. For example, consider the weighted shift corresponding to $\alpha_n = \mathrm{e}^{\sqrt{|n|}}$ for all $n \in \mathbb{Z}$.

An operator is said to be $\mathcal{E}(\mathbb{T})$-*subscalar* operator if it is the restriction of an $\mathcal{E}(\mathbb{T})$-scalar operator to a closed invariant subspace. Recently, Didas ([7]) has obtained characterizations of these operators. Specifically, he shows that, for a certain class of not necessarily invertible Hilbert space operators, the growth condition (3.1) of Colojoară and Foiaş gives a complete description of $\mathcal{E}(\mathbb{T})$-subscalar operators that, in particular, applies to weighted shifts on $\ell^2(\mathbb{K})$ ([7], 2.2.11 and 4.1.3). While stated only for unilateral shifts, Didas's proof works in the bilateral case as well.

Notice that the growth condition (3.1) applies to the Bergman shift, $Te_n := \sqrt{\frac{n+1}{n+2}}\, e_{n+1}$, on $\ell^2(\mathbb{N}_0)$. T is subnormal with minimal normal extension $M_z f(z) := z\, f(z)$ on $L^2(\mathbb{D}, \pi^{-1}\mathrm{d}A)$. Thus, while the spectrum of the minimal normal extension of T is the closed unit disc $\overline{\mathbb{D}}$, Didas's theorem shows that T has a generalized scalar extension with spectrum in the unit circle. It would be interesting to know if there is a natural $\mathcal{E}(\mathbb{T})$-scalar extension.

By Proposition 3.1, every generalized scalar weighted shift is $\mathcal{E}(r\,\mathbb{T})$-scalar for some $r > 0$, but, in contrast, not all subscalar weighted shifts must be $\mathcal{E}(r\,\mathbb{T})$-subscalar. Indeed, if T is a weighted shift on $\ell^2(\mathbb{K})$, then T is hyponormal if and only if its weight sequence $(\omega_n)_{n\in\mathbb{K}}$ is nondecreasing ([20]). Since hyponormal operators are subscalar ([17]) it is therefore impossible to give a growth condition necessary for a shift to be subscalar or subdecomposable.

Analogous to Didas's theorem and the sufficient condition for decomposability of Colojoară and Foiaş, we obtain a sufficient condition for property (β) for arbitrary Banach space operators. Its proof is an application of Levinson's log-log integrability criterion through Theorems 1.7.1 and 1.7.4 of [12].

THEOREM 3.2. *Let X be a complex Banach space, and suppose that $T \in \mathcal{L}(X)$ satisfies:*

(a) *There exist constants $K > 0$ and $0 < s < 1$ such that*

$$\frac{1}{K\,\mathrm{e}^{n^s}} \leqslant \kappa(T^n) \leqslant \|T^n\| \leqslant K\,\mathrm{e}^{n^s}$$

for all $n \in \mathbb{N}$.

(b) *There is a sequence of closed subspaces Y_n of X such that*

(i) *$X = T^n X \oplus Y_n$ for every $n \geqslant 0$,*

(ii) *the projections with range $T^n X$ and kernel Y_n are uniformly bounded, and*

(iii) *$TY_n \subseteq Y_{n+1}$ for every $n \geqslant 0$.*

Then T has property (β). If, in addition, T is invertible, then T is decomposable.

REMARK 3.3. Clearly, condition (a) ensures that T^n has closed range for every $n \in \mathbb{N}$. On the other hand, if X is a Hilbert space, then condition (b) above is satisfied provided that all powers of T have closed range and the condition considered by Didas ([7], 2.2.11) holds:

(b′) $T^*T^{n+1}X \subseteq T^n X$ for all $n \geqslant 0$.

Indeed, for $Y_n = \ker(T^{*n})$, conditions (b) (i) and (ii) clearly hold. If $x^* \in \ker(T^{*n})$ and $x \in X$ is arbitrary, then

$$\langle x, T^{*(n+1)}Tx^*\rangle = \langle T^*T^{(n+1)}x, x^*\rangle = \langle T^n x', x^*\rangle$$

for some $x' \in X$, and therefore

$$\langle x, T^{*(n+1)}Tx^*\rangle = \langle x', T^{*n}x^*\rangle = 0.$$

Condition (b) (iii) follows.

Proof. By hypothesis (a) and 1.6.2, [12], T satisfies $\sigma_{\mathrm{ap}}(T) \subseteq \partial\mathbb{D}$. Thus, if T is invertible, then $\sigma(T) \subseteq \partial\mathbb{D}$, and the result follows from 5.3.2, [5]. We therefore assume that $0 \in \sigma(T)$, equivalently, that $\sigma(T) = \overline{\mathbb{D}}$.

Define

$$Y = \Big\{(x_n^*)_{n\geqslant 0} \in \bigoplus_{n\geqslant 0} Y_n^{\perp} : T^*x_{n+1}^* = x_n^* \text{ and } \sup_{n\geqslant 0} \mathrm{e}^{-n^s}\|x_n^*\| < \infty\Big\}$$

with norm $\|(x_n^*)_{n\geqslant 0}\|_Y := \sup\limits_{n\geqslant 0} \mathrm{e}^{-n^s}\|x_n^*\|$. By hypotheses (b) (i) and (ii), $X^* = \ker(T^{*n}) \oplus Y_n^{\perp}$, and there is a constant C, independent of n, so that

$$\|x^*\| \leqslant C \sup_{x\neq 0} \frac{|\langle T^n x, x^*\rangle|}{\|T^n x\|} \tag{3.2}$$

for all $n \in \mathbb{N}$ and $x^* \in Y_n^{\perp}$.

We claim that the linear mapping given by $P(x_n^*)_{n\geqslant 0} := x_0^*$ is invertible from Y onto X^*. Clearly, P is continuous. If $(0, x_1^*, x_2^*, \ldots) \in Y$, then $x_n^* \in \ker(T^{*n}) \cap Y_n^{\perp}$ for all $n \geqslant 1$, and it follows that P is injective.

Next, choose $x^* \in X^*$. Since T^* is surjective, there exists a sequence $(x_n^*)_{n\geqslant 1} \in \bigoplus_{n\geqslant 1} Y_n^\perp$ so that $T^{*n}x_n^* = x^*$ for each n. If $x \in X$, write $x = T^n x_n' + y_n$ for some $x_n' \in X$ and $y_n \in Y_n$. Then

$$\langle T^n x_n', T^* x_{n+1}^* \rangle = \langle x_n', T^{*(n+1)} x_{n+1}^* \rangle = \langle x_n', x^* \rangle = \langle T^n x_n', x_n^* \rangle,$$

and, by hypothesis (b) (iii),

$$\langle y_n, T^* x_{n+1}^* \rangle = \langle T y_n, x_{n+1}^* \rangle = 0 = \langle y_n, x_n^* \rangle.$$

Thus $\langle x, T^* x_{n+1}^* \rangle = \langle x, x_n^* \rangle$. Since x is arbitrary in X, it follows that $T^* x_{n+1}^* = x_n^*$ for all $n \geqslant 1$. Finally, if $x \in X$, then

$$|\langle T^n x, x_n^* \rangle| = |\langle x, x^* \rangle| \leqslant \|x\|\,\|x^*\| \leqslant K\,\mathrm{e}^{n^s}\|T^n x\|\,\|x^*\|$$

by (a). Thus, by (3.2), $\|x_n^*\| \leqslant CK\mathrm{e}^{n^s}\|x^*\|$ for all $n \geqslant 1$, and so $(x^*, x_1^*, x_2^*, \ldots) \in Y$; P is surjective.

Writing $P^{-1}(x^*) = (q_n(x^*))_{n\geqslant 0}$, we have that

$$\limsup_{n\to\infty} \|q_n(x^*)\|^{1/n} \leqslant \limsup_{n\to\infty} (CK\mathrm{e}^{n^s}\|x^*\|)^{1/n} \leqslant 1,$$

and so $R_\lambda(x^*) := \sum_{n=1}^{\infty} q_n(x^*)\lambda^{n-1}$ is absolutely summable for all $|\lambda| < 1$. Since P^{-1} is linear and continuous, R_λ is linear and continuous on X^*. A calculation shows that R_λ is a right inverse of $T^* - \lambda$ for $|\lambda| < 1$: if $x^* \in X^*$, then

$$\begin{aligned}(T^* - \lambda)R_\lambda(x^*) &= (T^* - \lambda)\sum_{n=1}^{\infty} q_n(x^*)\lambda^{n-1} = \sum_{n=1}^{\infty} T^* q_n(x^*)\lambda^{n-1} - \sum_{n=1}^{\infty} q_n(x^*)\lambda^n \\ &= x^* + \sum_{n=2}^{\infty} q_{n-1}(x^*)\lambda^{n-1} - \sum_{n=1}^{\infty} q_n(x^*)\lambda^n = x^*.\end{aligned}$$

Clearly, the norms $\|R_\lambda\|$ are uniformly bounded on each compact subset of $\mathbb{D}$. To estimate the growth of $\|R_\lambda\|$ as $|\lambda| \to 1^-$, we introduce, for arbitrary non-zero $\lambda \in \mathbb{D}$, the quantity $m(\lambda) := \Big(\log \frac{1+|\lambda|}{2|\lambda|}\Big)^{1/(s-1)}$, and observe that

$$\begin{aligned}\|R_\lambda\| &\leqslant C\,K\Big(\sum_{n\leqslant m(\lambda)} \mathrm{e}^{n^s}|\lambda|^{n-1} + \sum_{n>m(\lambda)} \mathrm{e}^{n^s}|\lambda|^{n-1})\Big) \\ &\leqslant C\,K\Big(\sum_{n\leqslant m(\lambda)} \mathrm{e}^{n} + \frac{1}{|\lambda|}\sum_{n>m(\lambda)} \Big(\frac{1+|\lambda|}{2}\Big)^n\Big) \\ &\leqslant C\,K\Big(3\mathrm{e}^{m(\lambda)} + \frac{2}{|\lambda|(1-|\lambda|)}\Big),\end{aligned}$$

by the standard formula for the geometric series. Since, by l'Hôpitals rule, $\frac{m(\lambda)^{s-1}}{1-|\lambda|} \to \frac{1}{2}$ and $\frac{m(\lambda)^{1-s}}{\mathrm{e}^{m(\lambda)}} \to 0$ as $|\lambda| \to 1^-$, there exist constants $C_1, \delta > 0$ such that $\|R_\lambda\| \leqslant C_1\mathrm{e}^{m(\lambda)}$ for all $\lambda \in \mathbb{C}$ for which $1 - \delta < |\lambda| < 1$. Noting that $\|R_\lambda\|$ is uniformly bounded for $|\lambda| \leqslant 1 - \delta$, we obtain with suitable constants $C_2 > 0$ and $C_3 > 1$ the estimate

$$\|R_\lambda\| \leqslant C_2 \exp\big(C_3\big|1 - |\lambda|\big|^{1/(s-1)}\big) \tag{3.3}$$

for arbitrary $\lambda \in \mathbb{D}$. For $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, we define $R_\lambda := (T^* - \lambda)^{-1}$ and note that

$$\|R_\lambda\| \leqslant \sum_{n=0}^{\infty} \|T^n\| \, |\lambda|^{-n-1} \leqslant K \sum_{n=0}^{\infty} \mathrm{e}^{n^s} |\lambda|^{-n-1}.$$

Hence, after increasing the constants C_2 and C_3 if necessary, we obtain the estimate (3.3) for all λ in some open neighborhood V of the closed unit disc.

Now, let $\gamma(t) := \exp(-C_3\, t^{1/(s-1)})$ for all $t > 0$. Clearly, γ is increasing on $(0, \infty)$ and satisfies

$$\int_0^1 \log\bigl|\log \gamma(t)\bigr| \, \mathrm{d}t < \infty.$$

Moreover, for all $\lambda \in V$ and $x \in X$, we have $R_\lambda^*(T - \lambda)x = x$ and therefore

$$\gamma\bigl(\bigl|1 - |\lambda|\bigr|\bigr)\|x\| \leqslant \gamma\bigl(\bigl|1 - |\lambda|\bigr|\bigr)\|R_\lambda^*\| \, \|(T - \lambda)x\| \leqslant C_2\|(T - \lambda)x\|$$

by (3.3). Consequently, Theorems 1.7.1 and 1.7.4 of [12] ensure that T has property (β). ∎

Notice that a weighted shift on $\ell^p(\mathbb{K})$ with closed range satisfies condition (b) in the theorem above. Furthermore, every bilateral weighted shift with closed range is invertible. Thus we obtain the following corollary.

COROLLARY 3.4. *Suppose that T is a weighted shift on $\ell^p(\mathbb{K})$, $1 \leqslant p < \infty$. If there exist constants $K > 0$ and $0 < s < 1$ such that*

$$\frac{1}{K\,\mathrm{e}^{n^s}} \leqslant \inf_{k \in \mathbb{K}} \frac{\alpha_{n+k}}{\alpha_k} \leqslant \sup_{k \in \mathbb{K}} \frac{\alpha_{n+k}}{\alpha_k} \leqslant K\,\mathrm{e}^{n^s}$$

for every $n \in \mathbb{N}$, then T has Bishop's property (β). Moreover, in the bilateral case, T is decomposable.

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