J. OPERATOR THEORY
51(2004), 115–139

SPECTRAL PROBLEMS FOR SOME INDEFINITE CASES OF CANONICAL DIFFERENTIAL EQUATIONS

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Communicated by Florian-Horia Vasilescu

Abstract. The method of operator identities is used to investigate inverse problems of spectral theory for canonical systems of differential equations in some indefinite cases. The theory is extended in a different direction by including both the continuous and discrete cases at the same time. A generalization of the matrix string equation is obtained as an example.

Keywords: *Canonical differential equation, inverse problem, spectral data, operator identity, string equation, Kreĭn space, Pontryagin space.*

MSC (2000): Primary 34A55; Secondary 47E05, 47A48, 47A57, 47B50.

1. INTRODUCTION AND PRELIMINARIES

The study of canonical differential equations [12] has generally proceeded under positivity assumptions, which play an important role in key constructions in the spectral theory of such equations. In [8] the authors considered a special case of difference equations and showed that without positivity, inverse and direct problems can be treated if appropriate modifications are made in the classical theory. We now consider the continuous case and obtain analogous results. We present these results in a form that allows discrete as well as continuous components.

In the classical case, a canonical differential equation is an equation of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \mathrm{i}zJ\mathcal{H}(x)Y, \tag{1.1}$$

with boundary conditions

$$D_2Y_1(0,z) + D_1Y_2(0,z) = 0, \tag{1.2}$$

where $\mathcal{H}(x)$ is nonnegative and locally integrable on an interval $[0, l)$ with $2m \times 2m$ matrix values, z is a complex parameter,

$$J = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \quad Y(x,z) = \begin{bmatrix} Y_1(x,z) \\ Y_2(x,z) \end{bmatrix}, \tag{1.3}$$

$Y_1(x,z), Y_2(x,z)$ have $m \times m$ matrix values, and D_1, D_2 are $m \times m$ matrices such that

$$D_1 D_2^* + D_2 D_1^* = 0 \quad \text{and} \quad D_1 D_1^* + D_2 D_2^* = I_m. \tag{1.4}$$

To add a discrete part, we may also allow the existence of points

$$0 < x_1 < x_2 < \cdots < l$$

having no limit point in $[0, l)$, such that (1.1) holds in the subintervals determined by the points, and $Y(x,z)$ is left-continuous at each x_k and

$$Y(x_k + 0, z) - Y(x_k, z) = \mathrm{i}zJs(x_k)Y(x_k, z) \tag{1.5}$$

for certain nonnegative matrices $s(x_k)$. Particular cases of such equations include many special systems ([12]). The discrete case (1.5) with no continuous part (1.1) is related to Jacobi systems (Chapter 8 in [11]). In [8] the authors generalized the discrete theory to indefinite cases by allowing the matrices $s(x_k)$ to be selfadjoint but not necessarily nonnegative. In this paper, we also admit a nontrivial continuous part (1.1), where $\mathcal{H}(x)$ is selfadjoint but not necessarily nonnegative.

Section 1 concludes with a summary of the integration theory needed in what follows. In Section 2 we introduce a notion of spectral data which is suitable for a large class of indefinite problems.

The main results of the paper are in Section 3 and center around the operator identity

$$AS - SA^* = \mathrm{i}(\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*).$$

In Theorem 3.1 we show how to construct many identities of this type. A continual factorization construction relative to a chain of invariant subspaces of A^* is given in Theorem 3.2. With the aid of this result we describe a solution of the inverse problem in Theorem 3.5, which is the central result of the paper. Theorem 3.5 treats continuous and discrete problems at the same time, a feature that is new even in the definite case. This approach to the inverse problem generalizes results in [12]. Indefinite discrete problems are treated by the authors ([8]).

In Section 4 we discuss concrete examples of the theory which use the function

$$\mathcal{H}(x) = \begin{bmatrix} q(x) \\ p(x) \end{bmatrix} \mathbf{j} \begin{bmatrix} q^*(x) & p^*(x) \end{bmatrix}, \tag{1.6}$$

where $p(x)$ and $q(x)$ are $m \times m$ matrix-valued functions satisfying conditions which we shall describe later and $\mathbf{j}$ is an invertible $m \times m$ matrix such that $\mathbf{j} = \mathbf{j}^* = \mathbf{j}^{-1}$. In this case, (1.1) is equivalent to an indefinite form of the generalized matrix string equation ([12]):

$$\frac{\mathrm{d}}{\mathrm{d}x} \left\{ A(x) \frac{\mathrm{d}}{\mathrm{d}x} \left[p^{*-1}(x) Y \right] \right\} = zp(x)\,\mathbf{j}\,Y.$$

Here $A(x) = A^*(x)$, $p(x)$, and $Y = Y(x,z)$ have $m \times m$ matrix values and z is a complex parameter. Other special cases lead to de Branges' theory of Hilbert spaces of entire functions ([1], [2] in the definite case) and to Pontryagin space generalizations due to Kaltenbäck and Woracek ([5], [6]) and Kaltenbäck ([4]). We hope to discuss these connections in a future work.

PRELIMINARIES ON INTEGRATION THEORY. In what follows we use integrals of the form $\int_a^b f(t)[\,\mathrm{d}M(t)]g(t)$, where $f(t), M(t), g(t)$ are matrix-valued functions on a closed and bounded real interval $[a,b]$, say of orders $p \times m$, $m \times n$, and $n \times q$. The integrals are defined as

$$\int_a^b f(t)[\,\mathrm{d}M(t)]g(t) = \lim \sum f(t_k^*)[M(t_k) - M(t_{k-1})]g(t_k^*),$$

where the t_k are the division points of a finite partition of $[a,b]$, t_k^* is a point in the k-th subinterval, and the limit is taken as the mesh of the partition tends to zero. Thus

$$\int_a^b f(t)[\,\mathrm{d}M(t)]g(t) = \left[\sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} \int_a^b f_{i\alpha}(t) g_{\beta k}(t)\,\mathrm{d}M_{\alpha\beta}(t) \right]_{p\times q},$$

where $f_{i\alpha}(t), g_{\beta k}(t), M_{\alpha\beta}(t)$ are the entries of the matrices $f(t), g(t), M(t)$. The integrals exist, for example, if f and g are continuous and M is of bounded variation (that is, the entries of $M(t)$ are of bounded variation, or, equivalently, there is a constant $C > 0$ such that $\sum \|M(t_k) - M(t_{k-1})\| \leqslant C$ for all finite partitions of $[a,b]$ with division points t_k, where $\|\cdot\|$ is the operator norm). The integrals also exist if f and g are piecewise continuous and bounded, and M is both continuous and of bounded variation.

Integrals are estimated in the operator norm using scalar majorants. For example, let

$$\mu_M(t) = \sup \sum \|M(t_k) - M(t_{k-1})\| + \text{const.},$$

where the supremum is over all finite partitions of $[a,t]$ by points t_k. Then

$$\|M(y) - M(x)\| \leqslant \mu_M(y) - \mu_M(x), \quad a \leqslant x \leqslant y \leqslant b,$$

and

$$\left\| \int_a^b f(t)[\,\mathrm{d}M(t)]g(t) \right\| \leqslant \int_a^b \|f(t)\|\ \|g(t)\|\ \mathrm{d}\mu_M(t)$$

whenever the integrals exist. The integration by parts formula

$$\int_a^b f(t)[\,\mathrm{d}M(t)] = f(b)M(b) - f(a)M(a) - \int_a^b [\,\mathrm{d}f(t)]M(t)$$

holds if both f and M are continuous and of bounded variation on $[a,b]$. Suppose f and g are continuous and M is of bounded variation on $[a,b]$. Put

$$L_f(x) = \int_a^x f(t)[\,\mathrm{d}M(t)] \quad \text{and} \quad R_g(x) = \int_a^x [\,\mathrm{d}M(t)]g(t),$$

$a \leqslant x \leqslant b$. Then

$$\int_a^b f(t)[\,\mathrm{d}M(t)]g(t) = \int_a^b f(t)[\mathrm{d}R_g(t)] = \int_a^b [\mathrm{d}L_f(t)]g(t).$$

Integrals over non-closed intervals are interpreted in the improper sense. The notion of a scalar majorant has an obvious modification for these cases.

We also consider similar integrals where the values of $M(x)$ are operators on a finite-dimensional Hilbert space, and the values of $f(x)$ and $g(x)$ are operators between this space and possibly infinite-dimensional Hilbert spaces. Such integrals are interpreted in the weak sense and thereby reduced to integrals of matrix-valued functions as above.

Let $L^2_m(0,l)$ be the Hilbert space of measurable m-dimensional vector-valued functions on a finite interval $(0,l)$ which are square integrable with respect to Lebesgue measure.

2. SPECTRAL DATA

The usual notion of spectral data ([12]) is too restrictive for the indefinite case, and we must modify this notion for our generalization. The modification follows [8], which adapts an idea from interpolation theory ([9]) and uses the Kreĭn and Langer integral representation of a generalized Nevanlinna function ([7]).

We use an integral form of (1.1). Combined with the boundary conditions (1.4) and discrete part (1.5), this can be written formally as

$$Y(x,z) = Y(0,z) + \mathrm{i}zJ\int_0^x [\mathrm{d}B(t)]Y(t,z) + \mathrm{i}zJ\sum_{x_k<x} s(x_k)Y(x_k,z), \tag{2.1}$$

$$D_2Y_1(0,z) + D_1Y_2(0,z) = 0. \tag{2.2}$$

The associated monodromy matrix is determined by the equation

$$W(x,z) = I_{2m} + \mathrm{i}zJ\int_0^x [\mathrm{d}B(t)]W(t,z) + \mathrm{i}zJ\sum_{x_j<x} s(x_j)W(x_j,z). \tag{2.3}$$

Equations of the type (1.1) are included by choosing $B(x) = \int_0^x \mathcal{H}(t)\,\mathrm{d}t$. More generally, throughout the paper, $B(x)$ denotes a continuous function on a bounded interval $[0,l)$ which has selfadjoint $2m\times 2m$ matrix values, and which is of bounded variation on every compact subinterval. Let x_k be points in $[0,l)$ such that

$$0 < x_1 < x_2 < \cdots < l.$$

If the set of such points is infinite, we assume that $x_k \to l$. For each x_k, let $s(x_k)$ be a $2m \times 2m$ matrix satisfying

$$s(x_k) = s^*(x_k), \quad s(x_k)Js(x_k) = 0. \tag{2.4}$$

We remark that the general form of a matrix satisfying (2.4) is

$$s(x_k) = [\, p_1(k) \quad p_2(k)\,]^{\mathrm{t}} j_k \,[\, p_1^*(k) \quad p_2^*(k)\,]\,,$$

where j_k is an invertible $m \times m$ matrix such that $j_k = j_k^* = j_k^{-1}$ and $p_1(k)$ and $p_2(k)$ are $m \times m$ matrices such that

$$p_1^*(k)p_2(k) + p_2^*(k)p_1(k) = 0.$$

See Proposition 3.1 from [8] for a slightly more precise result. The second relation in (2.4) is used in the proof of the Lagrange relation (2.8).

Let $J, Y(x,z), D_1, D_2$ be as in (1.3) and (1.4). The meaning of (2.3) is that for a fixed complex parameter z, $W(x,z)$ is a $2m \times 2m$ matrix-valued piecewise continuous function of x in $[0,l)$ such that

$$W(x,z) = I_{2m} + \mathrm{i}zJ \int_0^x [\mathrm{d}B(t)]W(t,z), \quad 0 \leqslant x \leqslant x_1, \tag{2.5}$$

and for all $r = 1, 2, \ldots,$

$$W(x_r + 0, z) = [I_{2m} + \mathrm{i}zJs(x_r)]W(x_r, z), \tag{2.6}$$

$$W(x,z) = W(x_r + 0, z) + \mathrm{i}zJ \int_{x_r}^x [\mathrm{d}B(t)]W(t,z), \quad x_r < x \leqslant x_{r+1}. \tag{2.7}$$

A similar meaning is attached to (2.1). Thus if $W(x,z)$ is a solution of (2.3), then $Y(x,z) = W(x,z)[\, D_1^* \quad D_2^* \,]^{\mathrm{t}}$ satisfies (2.1), and this solution satisfies (2.2) by (1.4).

THEOREM 2.1. (i) *The equation* (2.3) *has a unique solution* $W(x,z)$, *and this solution is an entire function of* z *for each* x *in* $[0,l)$.

(ii) *The Lagrange relation*

$$\int_0^\xi W^*(x,u)[\mathrm{d}B(x)]W(x,z) + \sum_{x_j < \xi} W^*(x_j,u)s(x_j)W(x_j,z) = \frac{W^*(\xi,u)JW(\xi,z) - J}{\mathrm{i}(z - \overline{u})} \tag{2.8}$$

holds for all ξ *in* $[0,l)$ *and all complex* z *and* u.

Proof. (i) By the method of successive approximations, (2.5) has a unique solution, and this solution is a continuous function of x in $[0,x_1]$ for each fixed z and an entire function of z for each fixed x. In the same way, we solve (2.7) with $W(x_1 + 0, z)$ replaced by the right side of (2.6), and so on.

(ii) First consider $0 \leqslant \xi \leqslant x_1$. Using (2.5) we may set

$$R(x) = \mathrm{i}z \int_0^x [\mathrm{d}B(t)]W(t,z) = J[W(x,z) - I_{2m}],$$

$$L(x) = -\mathrm{i}\overline{u} \int_0^x W^*(t,u)[\mathrm{d}B(t)] = [W^*(x,u) - I_{2m}]J,$$

and obtain

$$
\begin{aligned}
\mathrm{i}z\int_0^\xi W^*(x,u)[\mathrm{d}B(x)]W(x,z) &= \int_0^\xi W^*(x,u)[\mathrm{d}R(x)] = \int_0^\xi W^*(x,u)[\mathrm{d}JW(x,z)],\\
-\mathrm{i}\overline{u}\int_0^\xi W^*(x,u)[\mathrm{d}B(x)]W(x,z) &= \int_0^\xi [\mathrm{d}L(x)]W(x,z) = \int_0^\xi [\mathrm{d}W^*(x,u)J]W(x,z),
\end{aligned}
$$

and hence (2.8), because

$$
\begin{aligned}
\int_0^\xi W^*(x,u)[\mathrm{d}JW(x,z)] + \int_0^\xi [\mathrm{d}W^*(x,u)]JW(x,z) &= W^*(x,u)JW(x,z)\Big|_{x=0}^{\xi}\\
&= W^*(\xi,u)JW(\xi,z) - J.
\end{aligned}
$$

Suppose (2.8) has been proved for $\xi \leqslant x_r$, and consider $\xi \in (x_r, x_{r+1}]$. The function $W(x,z)$, considered on the interval $[x_r, x_{r+1}]$, has a discontinuity at the left endpoint. The function

$$
\widetilde{W}(x,z) = \begin{cases} W(x_r+0,z) & x = x_r,\\ W(x,z) & x_r < x \leqslant x_{r+1}, \end{cases}
$$

is continuous on $[x_r, x_{r+1}]$. Since by (2.7),

$$
\widetilde{W}(x,z) = \widetilde{W}(x_r,z) + \mathrm{i}zJ\int_{x_r}^{x} [\mathrm{d}B(t)]\widetilde{W}(t,z), \quad x_r \leqslant x \leqslant x_{r+1},
$$

in the same way as above, we can show that

$$
\begin{aligned}
\int_{x_r}^{\xi} \widetilde{W}^*(x,u)[\mathrm{d}B(x)]\widetilde{W}(x,z) &= \frac{1}{\mathrm{i}(z-\overline{u})}\widetilde{W}^*(x,u)J\widetilde{W}(x,z)\Big|_{x=x_r}^{\xi}\\
&= \frac{W^*(\xi,u)JW(\xi,z) - W^*(x_r+0,u)JW(x_r+0,z)}{\mathrm{i}(z-\overline{u})}.
\end{aligned}
$$

Since $B(x)$ is continuous at $x = x_r$,

$$\begin{aligned}
&\int_0^{\xi} W^*(x,u)[\mathrm{d}B(x)]W(x,z) + \sum_{x_k<\xi} W^*(x_k,u)s(x_k)W(x_k,z) \\
&= \Big[\int_0^{x_r} W^*(x,u)[\mathrm{d}B(x)]W(x,z) + \sum_{k=1}^{r-1} W^*(x_k,u)s(x_k)W(x_k,z) \Big] \\
&\quad + \int_{x_r}^{\xi} \widetilde{W}^*(x,u)[\mathrm{d}B(x)]\widetilde{W}(x,z) + W^*(x_r,u)s(x_r)W(x_r,z) \\
&= \frac{W^*(x_r,u)JW(x_r,z) - J}{\mathrm{i}(z-\overline{u})} \\
&\quad + \frac{W^*(\xi,u)JW(\xi,z) - W^*(x_r+0,u)JW(x_r+0,z)}{\mathrm{i}(z-\overline{u})} + W^*(x_r,u)s(x_r)W(x_r,z) \\
&= \frac{W^*(\xi,u)JW(\xi,z) - J}{\mathrm{i}(z-\overline{u})},
\end{aligned}$$

where the last equality is obtained from (2.6) and (2.4). ∎

Given a boundary problem (2.1)–(2.2), define an operator V by $V : f(x) \to F(u)$,

$$\begin{aligned}
F(u) = &\int_0^{l} [\,D_1 \quad D_2\,]\, W^*(x,\overline{u})[\mathrm{d}B(x)]f(x) \\
&+ \sum_{x_k<l} [\,D_1 \quad D_2\,]\, W^*(x_k,\overline{u})s(x_k)f(x_k)
\end{aligned} \tag{2.9}$$

on the set $\operatorname{dom} V$ of piecewise continuous functions $f(x)$ on $[0,l)$ with values in $\mathbb{C}^{2m}$ having compact support and only a finite number of simple discontinuities in $(0,l)$. The transform $F(u)$ of any $f(x)$ in $\operatorname{dom} V$ is a $\mathbb{C}^m$-valued entire function.

We introduce topological notions in $\operatorname{dom} V$ by embedding $\operatorname{dom} V$ in a Hilbert space. To do this, we suppose that $B(x)$ is written in the form

$$B(x) = B_+(x) - B_-(x), \tag{2.10}$$

where $B_\pm(x)$ are nondecreasing continuous functions on $[0,l)$. In a similar way write

$$s(x_k) = s_+(x_k) - s_-(x_k), \tag{2.11}$$

where the matrices $s_\pm(x_k)$ are nonnegative for all k. Then $\operatorname{dom} V$ has a natural embedding in a Hilbert space $L^2_{2m}(B_\pm, s_\pm)$ determined by the inner product

$$\begin{aligned}
\langle f(\cdot), g(\cdot)\rangle = &\int_0^{l} g^*(x)[\mathrm{d}B_+(x)]f(x) + \sum_{x_k<l} g^*(x_k)s_+(x_k)f(x_k) \\
&+ \int_0^{l} g^*(x)\,[\mathrm{d}B_-(x)]f(x) + \sum_{x_k<l} g^*(x_k)s_-(x_k)f(x_k).
\end{aligned}$$

We omit the standard details of the construction of such a Hilbert space. The decompositions (2.10) and (2.11) are not unique, but any choice will suit our purpose. No matter how (2.10) and (2.11) are chosen, a dense set in $\operatorname{dom} V$ consists of functions of the form

$$f(x) = \sum_{j=1}^{p} \chi_{[0,\xi_j)}(x) f_j,$$

where $f_j \in \mathbb{C}^{2m}$ and $\xi_j \in (0,l)$, $j = 1,\dots,p$.

As in [8], we use certain forms which depend on complex numbers λ_k and associated $m \times m$ matrix-valued polynomials $R_k(\lambda)$ such that $R_k(0) = 0$, $k = 1,\dots,\nu$. If $F(z)$ and $G(z)$ are entire functions whose values are $m \times m$ matrices, we set

$$\begin{aligned} \mathfrak{F}_k(F(z),G(z)) &= \operatorname*{Res}_{\lambda=\lambda_k} \Big[G^*(\overline{\lambda}) R_k\Big(\frac{1}{\lambda-\lambda_k}\Big) F(\lambda)\Big], \\ \widehat{\mathfrak{F}}_k(F(z),G(z)) &= \operatorname*{Res}_{\lambda=\overline{\lambda}_k} \Big[G^*(\overline{\lambda}) R_k^*\Big(\frac{1}{\overline{\lambda}-\lambda_k}\Big) F(\lambda)\Big]. \end{aligned} \tag{2.12}$$

Explicitly, if $R_k(\lambda) = \tau_{k1}\lambda + \tau_{k2}\lambda^2 + \cdots + \tau_{k\rho_k}\lambda^{\rho_k}$ these expressions are given by

$$\begin{aligned} \mathfrak{F}_k(F(z),G(z)) = \Big\{ & G^*(\overline{\lambda})\tau_{k1}F(\lambda) + [G^*(\overline{\lambda})\tau_{k2}F(\lambda)]' + \frac{1}{2!}[G^*(\overline{\lambda})\tau_{k3}F(\lambda)]'' \\ & + \cdots + \frac{1}{(\rho_k-1)!}[G^*(\overline{\lambda})\tau_{k\rho_k}F(\lambda)]^{(\rho_k-1)} \Big\}\Big|_{\lambda=\lambda_k}, \\ \widehat{\mathfrak{F}}_k(F(z),G(z)) = \Big\{ & G^*(\overline{\lambda})\tau_{k1}^*F(\lambda) + [G^*(\overline{\lambda})\tau_{k2}^*F(\lambda)]' + \frac{1}{2!}[G^*(\overline{\lambda})\tau_{k3}^*F(\lambda)]'' \\ & + \cdots + \frac{1}{(\rho_k-1)!}[G^*(\overline{\lambda})\tau_{k\rho_k}^*F(\lambda)]^{(\rho_k-1)} \Big\}\Big|_{\lambda=\overline{\lambda}_k}. \end{aligned}$$

Definition 2.2. (i) We define *spectral data* for the boundary problem (2.1)–(2.2) as a tuple

$$\boldsymbol{\tau} = \{\tau(u); \mathfrak{F}_1, \dots, \mathfrak{F}_\nu\}, \tag{2.13}$$

consisting of an $m \times m$ selfadjoint matrix-valued function $\tau(u)$ on $(-\infty,\infty)$ which is of bounded variation on all compact subintervals, together with forms $\mathfrak{F}_1,\dots,\mathfrak{F}_\nu$ of the type (2.12), such that

$$\begin{aligned} & \int_0^l g^*(x)\,[\mathrm{d}B(x)] f(x) + \sum_{x_k<l} g^*(x_k)s(x_k)f(x_k) \\ & \qquad = \int_{-\infty}^{\infty} G^*(u)[\mathrm{d}\tau(u)]F(u) + \sum_{k=1}^{\nu}[\mathfrak{F}_k(F(z),G(z)) + \widehat{\mathfrak{F}}_k(F(z),G(z))] \end{aligned} \tag{2.14}$$

for all $f(x)$ and $g(x)$ in $\operatorname{dom} V$ and their transforms $F(u)$ and $G(u)$. For emphasis, we say that the boundary problem (2.1)–(2.2) has $\boldsymbol{\tau}$ as spectral data in the *strong sense* when this condition is satisfied.

(ii) If (2.14) holds for all $f(x)$ and $g(x)$ in a dense subset of $\operatorname{dom} V$, we say that the boundary problem (2.1)–(2.2) has spectral data $\boldsymbol{\tau}$ in the *weak sense.*

The integral on the right side of (2.14) is interpreted as

$$\int_{-\infty}^{\infty} = \lim_{\substack{c_1 \to -\infty \\ c_2 \to +\infty}} \int_{c_1}^{c_2},$$

and its existence is part of the condition in Definition 2.2.

Proposition 2.3. *If the boundary problem* (2.1)–(2.2) *has spectral data* (2.13) *in the weak sense and $B(x)$ and $\tau(x)$ are nondecreasing, then the problem has spectral data* (2.13) *in the strong sense as well.*

Proof. Suppose that (2.14) holds for all $f(x)$ in the dense set $\mathfrak{D}$ in $\operatorname{dom} V$. Then given any $f(x)$ in $\operatorname{dom} V$ we may choose an approximating sequence $f_1(x)$, $f_2(x), \ldots$ in $\mathfrak{D}$. If $V : f(x) \to F(u)$ and $V : f_n(x) \to F_n(u)$ for all n, then

$$\lim_{n\to\infty} F_n(u) = F(u)$$

uniformly on compact sets. By assumption (2.14) holds for functions in $\mathfrak{D}$, and hence for any $m, n = 1, 2, \ldots,$

$$\begin{aligned}
&\int_{-\infty}^{\infty} [F_m^*(u) - F_n^*(u)][\mathrm{d}\tau(u)][F_m(u) - F_n(u)] \\
&= \int_0^l [f_m^*(x) - f_n^*(x)]\,[\mathrm{d}B(x)]\,[f_m(x) - f_n(x)] \\
&+ \sum_{x_k < l} [f_m^*(x_k) - f_n^*(x_k)] s(x_k) [f_m(x_k) - f_n(x_k)] \\
&- \sum_{k=1}^{\nu} [\mathfrak{F}_k(F_m(z) - F_n(z), F_m(z) - F_n(z)) + \widehat{\mathfrak{F}}_k(F_m(z) - F_n(z), F_m(z) - F_n(z))].
\end{aligned}$$

Letting $m \to \infty$, we obtain an identity which implies that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} [F^*(u) - F_n^*(u)]\,[d\tau(u)]\,[F(u) - F_n(u)] = 0.$$

It is now straightforward to verify that (2.14) holds for all $f(x)$ and $g(x)$ in the domain of V. ∎

3. OPERATOR IDENTITIES AND THE INVERSE PROBLEM

The inverse problem is to find a boundary problem (2.1)–(2.2) which has a given tuple $\boldsymbol{\tau} = \{\tau; \mathfrak{F}_1, \dots, \mathfrak{F}_\nu\}$ as spectral data. Solutions are not unique, and, following [12], to find them we impose extra conditions that restrict the search. These conditions are formulated in terms of an operator identity

$$AS - SA^* = \mathrm{i}(\Phi_1\Phi_2^* + \Phi_2\Phi_1^*), \quad A, S \in \mathfrak{L}(\mathfrak{H}),\ \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H}),\ S = S^*, \tag{3.1}$$

where $\mathfrak{H}$ is a Hilbert space and $\mathfrak{G} = \mathbb{C}^m$. We assume that A is a Volterra operator, that is, A is compact and the origin is the only point in its spectrum. In outline, the strategy is first to choose operators A and Φ_2. Then using $\boldsymbol{\tau}$ we construct $S = S^*$ and Φ_1 so that the operator identity (3.1) is satisfied (see Theorem 3.1). As in the definite case by choosing A and Φ_2 we define the class of systems in which we search for a solution of the inverse problem. The main results of this section, Theorems 3.2 and 3.5, exploit the relationship between invariant subspaces and factorization to obtain a solution of the inverse problem within the chosen class of systems.

Theorem 3.1. *Let $A \in \mathfrak{L}(\mathfrak{H})$ be a Volterra operator on a Hilbert space $\mathfrak{H}$. Let $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$, where $\mathfrak{G} = \mathbb{C}^m$. Suppose that $\boldsymbol{\tau} = \{\tau; \mathfrak{F}_1, \dots, \mathfrak{F}_\nu\}$ is given as in Definition 2.2, and assume that $\tau(u)$ has a scalar majorant $\mu_\tau(u)$ satisfying*

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu_\tau(u)}{1+u^2} < \infty \tag{3.2}$$

and

$$\int_{-\infty}^{\infty} \left\| \Phi_2^*(I - uA^*)^{-1}h \right\|^2 \mathrm{d}\mu_\tau(u) < \infty, \quad h \in \mathfrak{H}. \tag{3.3}$$

Define $S \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_1 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ by

$$\begin{aligned} S = &\int_{-\infty}^{\infty} (I - Au)^{-1}\Phi_2[\mathrm{d}\tau(u)]\Phi_2^*(I - A^*u)^{-1} \\ &+ \sum_{k=1}^{\nu} [\mathfrak{F}_k(\Phi_2^*(I - A^*z)^{-1}, \Phi_2^*(I - A^*z)^{-1}) \\ &+ \widehat{\mathfrak{F}}_k(\Phi_2^*(I - A^*z)^{-1}, \Phi_2^*(I - A^*z)^{-1})] \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \Phi_1 = &-\mathrm{i} \int_{-\infty}^{\infty} \Big[A(I - Au)^{-1} + \frac{u}{u^2+1}I\Big]\Phi_2[\mathrm{d}\tau(u)] \\ &- \mathrm{i}\sum_{k=1}^{\nu} [\mathfrak{F}_k(I, \Phi_2^*A^*(I - zA^*)^{-1}) + \widehat{\mathfrak{F}}_k(I, \Phi_2^*A^*(I - zA^*)^{-1})]. \end{aligned} \tag{3.5}$$

Then the integrals in (3.4) *and* (3.5) *exist in the weak sense, and the operators* A, S, Φ_1, Φ_2 *satisfy* (3.1).

In the simplest case where $R_k(z) = \tau_k z$ for some operators τ_k on $\mathfrak{G}$, $k = 1, \dots, \nu$, the formulas (3.4) and (3.5) take the form

$$S = \int_{-\infty}^{\infty} (I - Au)^{-1} \Phi_2 [\mathrm{d}\tau(u)] \Phi_2^* (I - A^* u)^{-1} + \sum_{k=1}^{\nu} [(I - \lambda_k A)^{-1} \Phi_2 \tau_k \Phi_2^* (I - \lambda_k A^*)^{-1} + (I - \overline{\lambda}_k A)^{-1} \Phi_2 \tau_k^* \Phi_2^* (I - \overline{\lambda}_k A^*)^{-1}]$$

and

$$\Phi_1 = -\mathrm{i} \int_{-\infty}^{\infty} \Big[A(I - Au)^{-1} + \frac{u}{u^2+1} I \Big] \Phi_2 [\mathrm{d}\tau(u)] - \mathrm{i} \sum_{k=1}^{\nu} [A(I - \lambda_k A)^{-1} \Phi_2 \tau_k + A(I - \overline{\lambda}_k A)^{-1} \Phi_2 \tau_k^*].$$

Proof of Theorem 3.1. The integral in (3.4) converges by (3.3). The proof of convergence of the integral in (3.5) also uses (3.2) and is similar to an argument in p. 2 of [11].

To complete the proof, by linearity it is sufficient to assume that S is one of the terms of (3.4) and Φ_1 is the corresponding term of (3.5). For the integral terms, (3.1) follows as in the definite case.

Suppose that S and Φ_1 are given by

$$\begin{aligned} S &= \mathfrak{F}_k(\Phi_2^*(I - A^* z)^{-1}, \Phi_2^*(I - A^* z)^{-1}) \\ &\quad + \widehat{\mathfrak{F}}_k(\Phi_2^*(I - A^* z)^{-1}, \Phi_2^*(I - A^* z)^{-1}) \\ &= \operatorname*{Res}_{\lambda=\lambda_k} \Big[(I - A\lambda)^{-1} \Phi_2 R_k\Big(\frac{1}{\lambda - \lambda_k}\Big) \Phi_2^* (I - A^*\lambda)^{-1} \Big] \\ &\quad + \operatorname*{Res}_{\lambda=\overline{\lambda}_k} \Big[(I - A\lambda)^{-1} \Phi_2 R_k^*\Big(\frac{1}{\overline{\lambda} - \lambda_k}\Big) \Phi_2^* (I - A^*\lambda)^{-1} \Big], \\ \Phi_1 &= -\mathrm{i}[\mathfrak{F}_k(I, \Phi_2^* A^*(I - zA^*)^{-1}) + \widehat{\mathfrak{F}}_k(I, \Phi_2^* A^*(I - zA^*)^{-1})] \\ &= -\mathrm{i} \operatorname*{Res}_{\lambda=\lambda_k} \Big[A(I - A\lambda)^{-1} \Phi_2 R_k\Big(\frac{1}{\lambda - \lambda_k}\Big) \Big] \\ &\quad - \mathrm{i} \operatorname*{Res}_{\lambda=\overline{\lambda}_k} \Big[A(I - A\lambda)^{-1} \Phi_2 R_k^*\Big(\frac{1}{\overline{\lambda} - \lambda_k}\Big) \Big]. \end{aligned}$$

For any operator-valued function $F(\lambda)$ which is holomorphic in a deleted neighborhood of λ_1,

$$\Big[\operatorname*{Res}_{\lambda=\lambda_1} F(\lambda) \Big]^* = \operatorname*{Res}_{\lambda=\overline{\lambda}_1} F^*(\overline{\lambda}).$$

Using this formula we verify (3.1) by straightforward calculations. ∎

Operator identities are useful in factorization problems. As a preliminary, we recall the most basic result in a form which is convenient for our present purpose.

Theorem 3.2. (Main Factorization Theorem, [12], p. 21.) *Let $\widetilde{A}, \widetilde{B}, \widetilde{S} \in \mathfrak{L}(\widetilde{\mathfrak{H}})$ and $\Pi_1, \Pi_2, \Gamma_1, \Gamma_2 \in \mathfrak{L}(\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{H}})$, where $\widetilde{\mathfrak{H}}$ and $\widetilde{\mathfrak{G}}$ are Hilbert spaces. Assume that $\widetilde{S}$ is invertible, and*

$$\widetilde{A}\widetilde{S} - \widetilde{S}\widetilde{B} = \Pi_1\Pi_2^*, \quad \widetilde{S}\Gamma_1 = \Pi_1, \quad \Gamma_2^*\widetilde{S} = \Pi_2^*.$$

Let $\widetilde{W}(z)$ be a holomorphic function such that, in a neighborhood of infinity,

$$\widetilde{W}(z) = I - \Gamma_2^*(\widetilde{A} - zI)^{-1}\Pi_1, \quad \widetilde{W}^{-1}(z) = I + \Pi_2^*(\widetilde{B} - zI)^{-1}\Gamma_1.$$

Suppose that $P_2\widetilde{A}P_2 = \widetilde{A}P_2$ and $P_1\widetilde{B}P_1 = \widetilde{B}P_1$, where P_1 and P_2 are the orthogonal projections onto the components $\widetilde{\mathfrak{H}}_1$ and $\widetilde{\mathfrak{H}}_2$ of a decomposition $\widetilde{\mathfrak{H}} = \widetilde{\mathfrak{H}}_1 \oplus \widetilde{\mathfrak{H}}_2$. Write

$$\widetilde{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

and

$$\widetilde{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad \widetilde{T} = \widetilde{S}^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

relative to this decomposition. Then if S_{11} is invertible, so is T_{22}, and

$$S_{11}^{-1} = T_{11} - T_{12}T_{22}^{-1}T_{21}, \quad T_{22}^{-1} = S_{22} - S_{21}S_{11}^{-1}S_{12}.$$

Moreover $\widetilde{W}(z) = \widetilde{W}_2(z)\widetilde{W}_1(z)$, where

$$\widetilde{W}_1(z) = I - \Pi_2^*P_1S_{11}^{-1}(A_{11} - zI)^{-1}P_1\Pi_1, \quad \widetilde{W}_2(z) = I - \Gamma_2^*P_2(A_{22} - zI)^{-1}T_{22}^{-1}P_2\Gamma_1.$$

A family of projections on a Hilbert space $\mathfrak{H}$ is called a *chain* if it is totally ordered with respect to inclusion of ranges. It is called an *eigenchain* for an operator $A \in \mathfrak{L}(\mathfrak{H})$ if the range of each projection in the chain is invariant under A.

We use the Main Factorization Theorem in a continual form. Two versions of this result are given in Theorems 3.2 and Theorem 3.4. These results generalize constructions in pp. 40–41 of [12].

Theorem 3.3. *Let A, S, Φ_1, Φ_2 satisfy* (3.1) *where $\mathfrak{H}$ is a Hilbert space, $\mathfrak{G} = \mathbb{C}^m$, and A is a Volterra operator. Let $\{P_x\}_{0 \leqslant x \leqslant l}$, $P_0 = 0$, $P_l = I$, be an eigenchain for A^*. We assume that the chain is strongly continuous except at points $0 < x_1 < x_2 < \cdots < l$ which have no limit point in $[0, l)$, and at such points the chain is strongly continuous from the left. Set*

$$S_x = P_xSP_x|\mathfrak{H}_x, \quad \mathfrak{H}_x = P_x\mathfrak{H}, \quad 0 \leqslant x \leqslant l.$$

Assume:

(i) *The operators S_x are invertible, and there is a constant C such that $\|S_x^{-1}\| \leqslant C$, $0 \leqslant x \leqslant l$. Moreover $S_x^{-1}P_x$ is strongly continuous on $[0, l)$ except perhaps at the points x_k, where it is strongly left continuous and a strong limit from the right exists.*

(ii) *Define a function $B_1(x)$ on $[0, l)$ whose values are selfadjoint $\mathbb{C}^{2m} \times \mathbb{C}^{2m}$ matrices by*

$$B_1(x) = \Pi^*P_xS_x^{-1}P_x\Pi, \quad \Pi = [\,\Phi_1 \quad \Phi_2\,].$$

Then $B_1(x)$ *is of bounded variation on all compact subintervals of* $[0,l)$.

(iii) *There is a constant* $M > 0$ *such that*

$$\|(P_{x+\Delta x} - P_x)A(P_{x+\Delta x} - P_x)\| \leqslant M\,\Delta x$$

whenever $0 \leqslant x < x + \Delta x < l$.
Then $B_1(x) = B(x)+\Sigma(x)$, *where* $B(x)$ *is continuous on* $[0,l)$ *and* $\Sigma(x)$ *is constant on the intervals determined by the points* x_k *and left continuous at every such point. The jump* $s(x_k) = B_1(x_k+0)-B_1(x_k)$ *at* x_k *satisfies* (2.4) *with* J *defined by* (1.3). *The formula*

$$W(x,z) = I_{2m} + \mathrm{i}zJ\Pi^* P_x S_x^{-1} P_x (I - zA)^{-1}\Pi, \tag{3.6}$$

defines the unique solution of (2.3) *with this choice of* $B(x)$ *and* $s(x_k)$.

See Section 4 for a concrete case in which the conditions of Theorem 3.2 are met.

Proof. By (i), the function $B_1(x)$ is continuous on $[0,l)$ except for possible simple discontinuities at the points x_k, and it is left continuous at these points. Thus $B_1(x) = B(x) + \Sigma(x)$ as in the theorem. For fixed x, $W(x,z)$ is an entire function of z because A is a Volterra operator. For fixed z, $W(x,z)$ is continuous on $[0,l)$ except for possible simple discontinuities at the points x_k, and it is left continuous at these points by (i). For any x in $[0,l)$, set $A_x = P_x A P_x|\mathfrak{H}_x$. Since $P_x A P_x = P_x A$, we have $(I - zA_x)^{-1}P_x = P_x(I - zA)^{-1}$ and hence

$$W(x,z) = I_{2m} + \mathrm{i}zJ\Pi^* P_x S_x^{-1}(I - zA_x)^{-1}P_x\Pi. \tag{3.7}$$

Consider any points $0 \leqslant x < x + \Delta x = y < l$. We shall apply the Main Factorization Theorem with $\widetilde{\mathfrak{H}} = \mathfrak{H}_y$, $\widetilde{\mathfrak{G}} = \mathfrak{G} \oplus \mathfrak{G}$, $\mathfrak{G} = \mathbb{C}^m$, $\widetilde{A} = A_y$, $\widetilde{B} = A_y^*$, $\widetilde{S} = S_y^*$, and $\widetilde{W}(z) = W(y,1/z) = I_{2m} - \mathrm{i}J\Pi^* P_y S_y^{-1}(A_y - zI)^{-1}P_y\Pi$. Choose $\Pi_1 = P_y\Pi$, $\Gamma_1 = S_y^{-1}P_y\Pi$, $\Pi_2^* = \mathrm{i}J\Pi^* P_y|\mathfrak{H}_y$, $\Gamma_2^* = \mathrm{i}J\Pi^* P_y S_y^{-1}$. The conditions of the Main Factorization Theorem are met relative to the decomposition $\widetilde{\mathfrak{H}} = \widetilde{\mathfrak{H}}_1 \oplus \widetilde{\mathfrak{H}}_2$, where $\widetilde{\mathfrak{H}}_1 = \mathfrak{H}_x$ and $\widetilde{\mathfrak{H}}_2 = \mathfrak{H}_y \ominus \mathfrak{H}_x$. In the notation of that result, $A_{11} = A_x$, $S_{11} = S_x$, and

$$S_x^{-1} = T_{11} - T_{12}T_{22}^{-1}T_{21}, \quad T_{22}^{-1} = S_{22} - S_{21}S_x^{-1}S_{12}. \tag{3.8}$$

By (3.7), the identity $\widetilde{W}(1/z) = \widetilde{W}_2(1/z)\widetilde{W}_1(1/z)$ can be written in the form

$$W(y,z) = V(x,y,z)W(x,z), \tag{3.9}$$

where

$$V(x,y,z) = I_{2m} + z\Gamma_2^* P_2(I - zA_{22})^{-1}T_{22}^{-1}P_2\Gamma_1 \tag{3.10}$$

for every complex z. By the first relation in (3.8),

$$
\begin{aligned}
B_1(y) - B_1(x) &= \Pi^* P_y S_y^{-1} P_y \Pi - \Pi^* P_x S_x^{-1} P_x \Pi \\
&= \Pi^* P_y \left[S_y^{-1} - P_1 S_x^{-1} P_1 \right] P_y \Pi \\
&= \Pi^* P_y \left\{ \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} - \begin{bmatrix} S_x^{-1} & 0 \\ 0 & 0 \end{bmatrix} \right\} P_y \Pi \\
&= \Pi^* P_y \begin{bmatrix} T_{12} T_{22}^{-1} T_{21} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} P_y \Pi \\
&= \Pi^* P_y \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & T_{22}^{-1} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} P_y \Pi \\
&= \Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \Pi.
\end{aligned} \tag{3.11}
$$

Thus

$$
\begin{aligned}
&[V(x,y,z) - I_{2m}] - \mathrm{i}zJ[B_1(y) - B_1(x)] \\
&\quad = z\Gamma_2^* P_2 (I - zA_{22})^{-1} T_{22}^{-1} P_2 \Gamma_1 - \mathrm{i}zJ\Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \Pi \\
&\quad = z\Gamma_2^* P_2 \big[(I - zA_{22})^{-1} - I \big] T_{22}^{-1} P_2 \Gamma_1 \\
&\quad = z^2 \Gamma_2^* P_2 A_{22} (I - zA_{22})^{-1} T_{22}^{-1} P_2 \Gamma_1 \\
&\quad \overset{\text{def}}{=} R(x, \Delta x).
\end{aligned} \tag{3.12}
$$

We have shown that

$$
\begin{aligned}
&W(x + \Delta x, z) - W(x, z) \\
&\qquad = \{ \mathrm{i}zJ[B_1(x + \Delta x) - B_1(x)] + R(x, \Delta x) \} W(x, z).
\end{aligned} \tag{3.13}
$$

We now verify (2.3), that is, the identities (2.5)–(2.7) hold for every fixed complex number z. Clearly $W(0,z) = I_{2m}$. On $[0, x_1]$, $B_1(x) = B(x)$ is continuous and of bounded variation by (i) and (ii). Suppose $0 \leqslant x < x + \Delta x = y \leqslant x_1$. We show that in (3.13),

$$
\|R(x, \Delta x)\| = \mathrm{o}(\Delta x), \quad \Delta x \to 0, \tag{3.14}
$$

uniformly in x, that is, for every number $\varepsilon > 0$ there is a $\delta > 0$ such that $\|R(x, \Delta x)\| < \varepsilon \, \Delta x$ whenever $0 \leqslant x < x + \Delta x = y \leqslant x_1$ and $0 < \Delta x < \delta$. By (i), $\|z^2 \Gamma_2^* P_2\| = \|z^2 \mathrm{i} J \Pi^* P_y S_y^{-1} P_2\| \leqslant K_1$ for some constant $K_1 = K_1(z)$. By (iii), $\|A_{22}\| = \|(P_{x+\Delta x} - P_x) A (P_{x+\Delta x} - P_x)\| \leqslant M \Delta x$. From $(I - zA_{22})^{-1}(P_y - P_x) = (P_y - P_x)(I - zA)^{-1}(P_y - P_x)$ we get $\|(I - zA_{22})^{-1}\| \leqslant K_2$ for some $K_2 = K_2(z)$. By the second relation in (3.8) and (i), $\|T_{22}^{-1}\| \leqslant K_3$ for a constant K_3. It remains to estimate $\|P_2 \Gamma_1\|$. Since $P_2 \Gamma_1 = (P_y - P_x) P_y S_y^{-1} P_y \Pi = (P_y - P_x)[P_y S_y^{-1} P_y \Pi - P_x S_x^{-1} P_x \Pi]$, we have

$$
\|P_2 \Gamma_1\| \leqslant \|P_y S_y^{-1} P_y \Pi - P_x S_x^{-1} P_x \Pi\|.
$$

By the second part of (i) and the finite dimensionality of $\widetilde{\mathfrak{G}} = \mathfrak{G} \oplus \mathfrak{G}$, $P_x S_x^{-1} P_x \Pi$ is norm-continuous on $[0, x_1]$. Hence $\|P_2 \Gamma_1\|$ can be made as small as we wish by choosing $\Delta x = y - x$ sufficiently small. This yields (3.14). Then using (3.13) and (3.14) we verify (2.5) by examining the partial sums of the integral.

We obtain (2.6) with $r = 1$ and $s(x_1) = B_1(x_1 + 0) - B_1(x_1)$ by choosing $x = x_1$ in (3.13) and estimating $R(x, \Delta x)$ in a similar way; the only change is to use the simpler estimate $\|P_2\Gamma_1\| \leqslant K_4$ for the last part. It is clear that $s^*(x_1) = s(x_1)$. We show that $s(x_1)Js(x_1) = 0$. Using (3.11) with $x = x_1$ and $y = x_1 + \Delta x$, we set

$$s_{\Delta x}(x_1) \stackrel{\text{def}}{=} B_1(x_1 + \Delta x) - B_1(x_1) = \Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \Pi.$$

Then by (3.1),

$$\begin{aligned} s_{\Delta x}(x_1)&Js_{\Delta x}(x_1) \\ &= \Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \Pi J \Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \Pi \\ &= \Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \cdot \frac{AS - SA^*}{\mathrm{i}} \cdot P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \Pi \\ &= -\mathrm{i}\Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \cdot AS \cdot P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \Pi \\ &\qquad + \mathrm{i}\Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \cdot SA^* \cdot P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \Pi. \end{aligned}$$

In the central parts of the last two terms, replace P_yA and A^*P_y by P_yAP_y and $P_yA^*P_y$. This allows a cancellation of terms S_y and S_y^{-1}, yielding

$$\begin{aligned} s_{\Delta x}(x_1)Js_{\Delta x}(x_1) = -\mathrm{i}\Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} [P_y A P_2] T_{22}^{-1} P_2 S_y^{-1} P_y \Pi \\ + \mathrm{i}\Pi^* P_y S_y^{-1} P_2 T_{22}^{-1} [P_2 A^* P_y] S_y^{-1} P_2 T_{22}^{-1} P_2 S_y^{-1} P_y \Pi. \end{aligned}$$

Since $P_yAP_2 = P_{x_1+\Delta x}A(P_{x_1+\Delta x} - P_{x_1}) = (P_{x_1+\Delta x} - P_{x_1})A(P_{x_1+\Delta x} - P_{x_1})$, we have $\|P_yAP_2\| \to 0$ and $\|P_2A^*P_y\| \to 0$ as $\Delta x \to 0$. Hence $s(x_1)Js(x_1) = \lim_{\Delta x\to 0} s_{\Delta x}(x_1)Js_{\Delta x}(x_1) = 0$, and so $s(x_1)$ satisfies (2.4) with $k = 1$.

In the same way as above, we show that for any small positive number ε,

$$W(x,z) = W(x_1 + \varepsilon, z) + \mathrm{i}zJ \int_{x_1+\varepsilon}^{x} [\mathrm{d}B(t)]W(t,z)$$

for $x_1 + \varepsilon < x \leqslant x_2$. Since $B(t)$ is continuous and $W(x,z)$ has at most a simple discontinuity at x_1, this yields (2.7) for $r = 1$. We continue in the same manner for the points $x_2, x_3, \ldots$. ∎

We show that the conditions of Theorem 3.2 are satisfied in a special case, namely, whenever S is factorable ([3], [10]).

DEFINITION 3.4. Let $\mathfrak{H}$ be a Hilbert space. An operator $S \in \mathfrak{L}(\mathfrak{H})$ is called *left factorable* with respect to a chain $\mathfrak{P}$ of projections on $\mathfrak{H}$ if $S = LU$, where $L, U \in \mathfrak{L}(\mathfrak{H})$ are invertible operators such that for every $P \in \mathfrak{P}$,

$$PL^{\pm 1}P = PL^{\pm 1} \quad \text{and} \quad PU^{\pm 1}P = U^{\pm 1}P.$$

Suppose that $S \in \mathfrak{L}(\mathfrak{H})$ is left factorable with respect to a chain $\mathfrak{P}$, and let $S = LU$ as above. Given $P \in \mathfrak{P}$, let S_P, L_P, U_P be the compressions of S, L, U to $P\mathfrak{H}$, that is, $S_P = PSP|P\mathfrak{H}$, $L_P = PLP|P\mathfrak{H}$, $U_P = PUP|P\mathfrak{H}$. These operators are invertible, and their inverses are given by $L_P^{-1} = PL^{-1}P|P\mathfrak{H}$, $U_P^{-1} = PU^{-1}P|P\mathfrak{H}$, $S_P^{-1} = U_P^{-1}L_P^{-1} = PU^{-1}PL^{-1}P|P\mathfrak{H}$. We omit the elementary proofs.

Theorem 3.5. *The conclusions of Theorem* 3.2 *hold if the hypotheses* (i) *and* (ii) *are replaced by the condition that S is left factorable with respect to the chain* $\{P_x\}_{0\leqslant x\leqslant l}$.

Proof. In the notation of Theorem 3.2, let $S = LU$ where L and U are invertible and $P_x L^{\pm 1} P_x = P_x L^{\pm 1}$ and $P_x U^{\pm 1} P_x = U^{\pm 1} P_x$ for $0 \leqslant x \leqslant l$. Condition (i) is easily checked using the representation $S_x^{-1} = P_x U^{-1} P_x L^{-1} P_x | \mathfrak{H}_x$. To prove (ii), note that $B_1(x) = \Pi^* P_x U^{-1} P_x L^{-1} P_x \Pi$, where the domain of Π is finite-dimensional. Hence it is sufficient to show that for any $h \in \mathfrak{H}$, the vector-valued functions $f(x) = P_x L^{-1} P_x h$ and $g(x) = P_x U^{*-1} P_x h$ are of bounded variation on $[0, l]$ relative to the norm of $\mathfrak{H}$. This is evident from the identities

$$P_y L^{-1} P_y h - P_x L^{-1} P_x h = (P_y - P_x) L^{-1} h,$$
$$P_y U^{*-1} P_y h - P_x U^{*-1} P_x h = (P_y - P_x) U^{*-1} h,$$

which hold whenever $0 \leqslant x < y \leqslant l$. ∎

The next result provides a solution of the inverse problem: given $\boldsymbol{\tau}$ and operators A and Φ_2, we construct S and Φ_1 by Theorem 3.1, and then Theorem 3.2 yields a system having $\boldsymbol{\tau}$ as spectral data.

Theorem 3.6. *Let* $\boldsymbol{\tau} = \{\tau(t); \mathfrak{F}_1, \ldots, \mathfrak{F}_\nu\}$ *be given as in Definition* 2.2. *Let* A, S, Φ_1, Φ_2 *satisfy* (3.1) *where $\mathfrak{H}$ is a Hilbert space,* $\mathfrak{G} = \mathbb{C}^m$*, and A is a Volterra operator. Suppose moreover that S is given by* (3.4)*, that is, it is constructed from $\boldsymbol{\tau}$ as in Theorem* 3.1. *If* $\{P_x\}_{0\leqslant x\leqslant l}$ *is an eigenchain for A^* satisfying the conditions of Theorem* 3.2, *then the boundary problem* (2.1)–(2.2) *determined by Theorem* 3.2 *with* $[\,D_1 \quad D_2\,] = [\,0 \quad I_m\,]$ *has spectral data $\boldsymbol{\tau}$ in the weak sense.*

By Proposition 2.3, if $B(x)$ and $\tau(x)$ are nondecreasing, the boundary problem has the spectral data $\boldsymbol{\tau}$ in the strong sense as well. We hope to discuss this case in a future work.

Proof. By linearity it is enough to verify (2.14) when $f(x) = \chi_{[0,\xi)}(x) f$ and $g(x) = \chi_{[0,\eta)}(x) g$, where $\xi, \eta \in (0, l)$ and $f, g \in \mathbb{C}^{2m}$. Write $F_\xi(z)$ and $G_\eta(z)$ for the corresponding entire functions defined by (2.9) with $[\,D_1 \quad D_2\,] = [\,0 \quad I_m\,]$. Then by (2.8) and (3.6),

$$\begin{aligned} F_\xi(z) &= [\,0 \quad I_m\,] \left\{ \int_0^\xi W^*(x, \overline{z})[\mathrm{d}B(x)] + \sum_{x_k < \xi} W^*(x_k, \overline{z}) s(x_k) \right\} f \\ &= [\,0 \quad I_m\,] \frac{W^*(\xi, \overline{z}) J - J}{-\mathrm{i}z} f = \Phi_2^* P_\xi (I - zA^*)^{-1} P_\xi S_\xi^{-1} P_\xi \Pi f \\ &= \Phi_2^* (I - zA^*)^{-1} P_\xi S_\xi^{-1} P_\xi \Pi f. \end{aligned}$$

The last equality holds by the assumption that $\{P_x\}_{0\leqslant x\leqslant l}$ is an eigenchain for A^*. Similarly, $G_\eta(z) = \Phi_2^*(I - zA^*)^{-1} P_\eta S_\eta^{-1} P_\eta \Pi g$. Thus the right side of (2.14) is

$$\int_{-\infty}^{\infty} G_\eta^*(u)[\mathrm{d}\tau(u)]F_\xi(u) + \sum_{k=1}^{\nu}[\mathfrak{F}_k(F_\xi(z), G_\eta(z)) + \widehat{\mathfrak{F}}_k(F_\xi(z), G_\eta(z))]$$

$$\begin{aligned}
&= g^*\Pi^* P_\eta S_\eta^{-1} P_\eta \Big\{ \int_{-\infty}^{\infty} (I - Au)^{-1}\Phi_2[\mathrm{d}\tau(u)]\Phi_2^*(I - A^*u)^{-1} \\
&\qquad + \sum_{k=1}^{\nu} \mathfrak{F}_k(\Phi_2^*(I - A^*z)^{-1}, \Phi_2^*(I - A^*z)^{-1}) \\
&\qquad + \widehat{\mathfrak{F}}_k(\Phi_2^*(I - A^*z)^{-1}, \Phi_2^*(I - A^*z)^{-1}) \Big\} P_\xi S_\xi^{-1} P_\xi \Pi f \\
&= g^*\Pi^* P_\eta S_\eta^{-1} P_\eta \, S \, P_\xi S_\xi^{-1} P_\xi \Pi f \\
&= g^*\Pi^* P_\zeta S_\zeta^{-1} P_\zeta \Pi f,
\end{aligned}$$

where $\zeta = \min(\xi, \eta)$. The left side of (2.14) is

$$\begin{aligned}
\int_0^l g^*\chi_{[0,\eta)}(x)[\mathrm{d}B(x)]\chi_{[0,\xi)}(x) f + \sum_{x_k<\zeta} g^* s(x_k) f &= g^*\Big\{ \int_0^\zeta [\mathrm{d}B(x)] + \sum_{x_k<\zeta} s(x_k) \Big\} f \\
&= g^*[B(\zeta) + \Sigma(\zeta)] f = g^* B_1(\zeta) f = g^*\Pi^* P_\zeta S_\zeta^{-1} P_\zeta \Pi f.
\end{aligned}$$

Thus (2.14) is satisfied, and the result follows. ∎

Example 3.7. To illustrate Theorem 3.5 with a concrete example, choose the data $\boldsymbol{\tau} = \{\tau(u)\}$,

$$\tau(u) = \frac{1}{2\pi}\,\mathbf{j}\,u + \sigma(u), \quad -\infty < u < \infty,$$

where $\mathbf{j} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and

$$\sigma(u) = \begin{cases} 0 & u \leqslant \lambda_0, \\ \tau_0 & u > \lambda_0; \end{cases} \qquad \tau_0 = \begin{bmatrix} 0 & a \\ \bar{a} & 0 \end{bmatrix},$$

for some real number λ_0 and complex number a. In Theorem 3.1 choose $\mathfrak{H} = L_2^2(0, l)$ for any $l > 0$, $\mathfrak{G} = \mathbb{C}^2$, and

$$A : f(x) \to \mathrm{i} \int_0^x f(t)\,\mathrm{d}t, \tag{3.15}$$

$$\Phi_2 : g \to g, \tag{3.16}$$

for all $f(x)$ in $\mathfrak{H}$ and g in $\mathfrak{G}$. The operators S and Φ_1 defined by (3.4) and (3.5) are given by

$$S : f(x) \to \mathbf{j}\, f(x) + \int_0^l \mathrm{e}^{\mathrm{i}\lambda_0(x-t)}\tau_0 f(t)\,\mathrm{d}t,$$

$$\Phi_1 : g \to q(x)g, \quad q(x) = \frac{1}{2}\mathbf{j} - \mathrm{i}\Big(\frac{\mathrm{e}^{\mathrm{i}\lambda_0 x}-1}{\lambda_0} + \frac{\lambda_0}{\lambda_0^2+1}\Big)\tau_0,$$

for all $f(x)$ in $\mathfrak{H}$ and g in $\mathfrak{G}$. In Theorem 3.2, let P_ξ be the projection onto the subspace $\mathfrak{H}_\xi$ of functions supported on $(0,\xi)$. Then $S_\xi : f(x) \to \mathbf{j}\, f(x) + \int_0^\xi \mathrm{e}^{\mathrm{i}\lambda_0(x-t)}\tau_0 f(t)\,\mathrm{d}t$ for all $f(x)$ in $\mathfrak{H}_\xi$. This operator is invertible with inverse

$$S_\xi^{-1} : f(x) \to \mathbf{j}\, f(x) + \int_0^\xi \mathrm{e}^{\mathrm{i}\lambda_0(x-t)}\rho_0(\xi) f(t)\,\mathrm{d}t, \quad \rho_0(\xi) = \frac{1}{1+\xi^2|a|^2}\begin{bmatrix} -\xi|a|^2 & a \\ \overline{a} & \xi|a|^2 \end{bmatrix}$$

for all $f(x)$ in $\mathfrak{H}_\xi$. All conditions of Theorem 3.2 are met. Thus according to Theorem 3.5, the boundary problem (2.1)–(2.2) with $B(x)$ constructed as in Theorem 3.2 and $[\,D_1 \quad D_2\,] = [\,0 \quad I_m\,]$ has spectral data $\boldsymbol{\tau}$ in the weak sense. Explicitly,

$$B(x) = \int_0^x \begin{bmatrix} q^*(t) \\ I_2 \end{bmatrix} \mathbf{j}\,[\,q(t) \quad I_2\,]\,\mathrm{d}t + Q^*(x)\rho_0(x)Q(x)$$

where $Q(x) = \int_0^x [\,q(t) \quad I_2\,]\,\mathrm{e}^{-\mathrm{i}\lambda_0 t}\,\mathrm{d}t$. Alternatively, the boundary problem (1.1)–(1.2) with $\mathcal{H}(x) = [\,q^*(x) \quad I_2\,]^{\mathrm{t}}\,\mathbf{j}\,[\,q(x) \quad I_2\,] + \frac{\mathrm{d}}{\mathrm{d}x}[Q^*(x)\rho_0(x)Q(x)]$ has spectral data $\boldsymbol{\tau}$ in the weak sense.

Remark 3.8. In Example 3.7, we can replace $q(x)$ by $q(x)+\mathrm{i}C$ for any 2×2 matrix C such that $C^* = C$, and the same conclusions hold. For then A, S, Φ_1, Φ_2 satisfy (3.1) and the hypotheses of Theorem 3.2 are met.

Example 3.9. We modify Example 3.7 by splitting the discontinuity of $\tau(u)$ at the point λ_0 into a part in the upper half-plane and a part in the lower half-plane. Thus we now construct a solution of the inverse problem for the data

$$\boldsymbol{\tau} = \{\tau(u); \mathfrak{F}_1\},$$

where $\tau(u) = \frac{1}{2\pi}\mathbf{j}\,u$, $-\infty < u < \infty$,

$$\mathfrak{F}_1(F(z),G(z)) = G^*(\overline{\lambda}_1)\tau_1 F(\lambda_1), \quad \widehat{\mathfrak{F}}_1(F(z),G(z)) = G^*(\lambda_1)\tau_1^* F(\overline{\lambda}_1),$$

and

$$\mathbf{j} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tau_1 = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.$$

Here we assume that λ_1 and a are fixed complex numbers such that $\lambda_1 - \overline{\lambda}_1 = \mathrm{i}\eta_1 \neq 0$. As in Example 3.7 take $\mathfrak{H} = L_2^2(0,l)$ for any $l > 0$ and $\mathfrak{G} = \mathbb{C}^2$, and define

A and Φ_2 by (3.15) and (3.16). The operators S and Φ_1 defined by (3.4) and (3.5) are given by

$$S : f(x) \to \mathbf{j}\, f(x) + \int_0^l [e^{i\lambda_1(x-t)}\tau_1 + e^{i\overline{\lambda}_1(x-t)}\tau_1^*] f(t)\, dt,$$

$$\Phi_1 : g \to q_1(x)g, \quad q_1(x) = \frac{1}{2}\mathbf{j} - i\Big(\frac{e^{i\lambda_1 x} - 1}{\lambda_1}\tau_1 + \frac{e^{i\overline{\lambda}_1 x} - 1}{\overline{\lambda}_1}\tau_1^*\Big),$$

for all $f(x)$ in $\mathfrak{H}$ and g in $\mathfrak{G}$. Calculations as in Example 3.7 produce weak-sense solutions of the inverse problem with

$$B(x) = \int_0^x \begin{bmatrix} q_1^*(t) \\ I_2 \end{bmatrix} \mathbf{j}\, [\, q_1(t) \quad I_2\,]\, dt + Q_1^*(x)\rho_1(x)Q_1(x),$$

or

$$\mathcal{H}(x) = \begin{bmatrix} q_1^*(x) \\ I_2 \end{bmatrix} \mathbf{j}\, [\, q_1(x) \quad I_2\,] + \frac{d}{dx}\Big[Q_1^*(x)\rho_1(x)Q_1(x)\Big],$$

with $q_1(x)$ as above and

$$Q_1(x) = \int_0^x \begin{bmatrix} e^{-\overline{\lambda}_1 t} & 0 \\ 0 & e^{-\lambda_1 t} \end{bmatrix} [\, q_1(t) \quad I_2\,]\, dt,$$

$$\rho_1(x) = \frac{1}{1 + \gamma_1(x)\gamma_2(x)|a|^2} \begin{bmatrix} -\gamma_1(x)|a|^2 & a \\ \overline{a} & \gamma_2(x)|a|^2 \end{bmatrix},$$

where $\gamma_1(x) = \frac{e^{\eta_1 x} - 1}{\eta_1}$, and $\gamma_2(x) = \frac{1 - e^{-\eta_1 x}}{\eta_1}$. Thus for $\lambda_1 \to \lambda_0$,

$$q_1(x) \to q(x) + iC \quad \text{and} \quad Q_1(x) \to Q(x),$$

where $C^* = C$. That is, the solution in Example 3.9 passes into the form obtained in Example 3.7 in the limit as $\lambda_1 \to \lambda_0$ (see Remark 3.8).

4. GENERALIZED MATRIX STRING EQUATION

Throughout this section, $p(x)$ and $q(x)$ denote continuously differentiable $m \times m$ matrix-valued functions on $[0, l]$ such that $p(x)$ has invertible values. Let $\mathbf{j}$ be an invertible $m \times m$ matrix such that $\mathbf{j} = \mathbf{j}^* = \mathbf{j}^{-1}$.

THEOREM 4.1. *Let $\mathfrak{H} = L_m^2(0, l)$ and $\mathfrak{G} = \mathbb{C}^m$. Then the operators $A, S \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ defined by*

$$S : f(x) \to \mathbf{j}\, f(x),$$

$$A : f(x) \to i \int_0^x [p^*(x)q(t) + q^*(x)p(t)]\, \mathbf{j}\, f(t)\, dt,$$

$$\Phi_1 : g \to q^*(x)g, \quad \Phi_2 : g \to p^*(x)g,$$

satisfy (3.1), *and A is Volterra. The hypotheses of Theorems* 3.2 *and* 3.4 *are met if P_x is the projection onto the subspace of functions supported on* $(0,x)$ *for each x (the set of points x_k in Theorem* 3.2 *is empty). The function $B_1(x) = B(x)$ in Theorem* 3.2 *is given by*

$$B(x) = \int_0^x \mathcal{H}(x)\,\mathrm{d}x, \quad \mathcal{H}(x) = \begin{bmatrix} q(x) \\ p(x) \end{bmatrix} \mathbf{j}\,[\,q^*(x) \quad p^*(x)\,],$$

$0 \leqslant x < l$. *The solution of*

$$\frac{\mathrm{d}}{\mathrm{d}x} W(x,z) = \mathrm{i}zJ\mathcal{H}(x)W(x,z), \quad W(0,z) = I_{2m}, \tag{4.1}$$

provided by formula (3.6) *in Theorem* 3.2 *is given by*

$$\begin{aligned} W(x,z) = I_{2m} &+ \mathrm{i}z \int_0^x J\mathcal{H}(t)\,\mathrm{d}t + (\mathrm{i}z)^2 \int_0^x\int_0^t J\mathcal{H}(t)J\mathcal{H}(u)\,\mathrm{d}u\,\mathrm{d}t \\ &+ (\mathrm{i}z)^3 \int_0^x\int_0^t\int_0^u J\mathcal{H}(t)J\mathcal{H}(u)J\mathcal{H}(s)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}t + \cdots \end{aligned} \tag{4.2}$$

for all x in $[0,l)$ *and all complex z.*

We omit the straightforward proof and only remark that (4.2) coincides with the solution of (4.1) obtained by successive approximations.

The assumption that $p(x)$ has invertible values is not always needed, but it is convenient in some places. In particular, it guarantees that the function $\mathcal{H}(x)$ constructed in Theorem 4.1 satisfies

$$\operatorname{rank} \mathcal{H}(x) = m$$

for all x.

The next result generalizes Theorem 1.2, p. 156, from [12].

THEOREM 4.2. (i) *In the situation of Theorem* 4.1, *for any matrices D_1 and D_2 satisfying* (1.4) *the function*

$$Y(x,z) = [\,q^*(x) \quad p^*(x)\,]\,W(x,z) \begin{bmatrix} D_1^* \\ D_2^* \end{bmatrix} \tag{4.3}$$

satisfies

$$Y(x,z) = \mathrm{i}z \int_0^x [q^*(x)p(t) + p^*(x)q(t)]\,\mathbf{j}\,Y(t,z)\,\mathrm{d}t + q^*(x)D_1^* + p^*(x)D_2^* \tag{4.4}$$

and is the unique solution of this equation.

(ii) *Assume in addition that*

$$q^*(x)p(x) + p^*(x)q(x) = 0 \tag{4.5}$$

for all x, and that the values of

$$r(x) = \frac{\mathrm{d}}{\mathrm{d}x}[p^{*-1}(x)q^*(x)]$$

are invertible. Set $A(x) = -\mathrm{i}r^{-1}(x)$*. Then* (4.4) *is equivalent to the system*

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big\{A(x)\frac{\mathrm{d}}{\mathrm{d}x}[p^{*-1}(x)Y(x,z)]\Big\} = zp(x)\,\mathbf{j}\,Y(x,z) \tag{4.6}$$

with the boundary conditions

$$Y(0,z) = q^*(0)D_1^* + p^*(0)D_2^*, \quad \frac{\mathrm{d}}{\mathrm{d}x}[p^{*-1}(x)Y(x,z)]\Big|_{x=0} = r(0)D_1^*. \tag{4.7}$$

Equation (4.4) is the *generalized string equation.*

Proof. (i) Write $W(x,z) = [W_{ik}(x,z)]_{i,k=1}^2$, where the values of $W_{ik}(x,z)$ are $m\times m$ matrices. Then setting

$$\varphi_1(x,z) = p^*(x)W_{21}(x,z) + q^*(x)W_{11}(x,z),$$
$$\varphi_2(x,z) = p^*(x)W_{22}(x,z) + q^*(x)W_{12}(x,z),$$

we can write (4.3) in the form

$$Y(x,z) = \varphi_1(x,z)D_1^* + \varphi_2(x,z)D_2^*. \tag{4.8}$$

By (4.1), $\frac{\mathrm{d}W_{1k}(x,z)}{\mathrm{d}x} = \mathrm{i}zp(x)\,\mathbf{j}\,\varphi_k(x,z)$, $\frac{\mathrm{d}W_{2k}(x,z)}{\mathrm{d}x} = \mathrm{i}zq(x)\,\mathbf{j}\,\varphi_k(x,z)$, and hence

$$W_{1k}(x,z) = \mathrm{i}z\int_0^x p(t)\,\mathbf{j}\,\varphi_k(t,z)\,\mathrm{d}t + W_{1k}(0,z),$$
$$W_{2k}(x,z) = \mathrm{i}z\int_0^x q(t)\,\mathbf{j}\,\varphi_k(t,z)\,\mathrm{d}t + W_{2k}(0,z).$$

We deduce from the definitions of $\varphi_1(x,z)$ and $\varphi_2(x,z)$ that

$$\varphi_1(x,z) = \mathrm{i}z\int_0^x [q^*(x)p(t) + p^*(x)q(t)]\,\mathbf{j}\,\varphi_1(t,z)\,\mathrm{d}t + q^*(x),$$
$$\varphi_2(x,z) = \mathrm{i}z\int_0^x [q^*(x)p(t) + p^*(x)q(t)]\,\mathbf{j}\,\varphi_2(t,z)\,\mathrm{d}t + p^*(x).$$

Hence by (4.8), (4.3) satisfies (4.4). The solution of (4.4) is unique by standard estimates.

(ii) Write (4.4) in the form

$$p^{*-1}(x)Y(x,z) = \mathrm{i}zp^{*-1}(x)q^*(x)\int_0^x p(t)\,\mathbf{j}\,Y(t,z)\,\mathrm{d}t$$
$$+\,\mathrm{i}z\int_0^x q(t)\,\mathbf{j}\,Y(t,z)\,\mathrm{d}t + p^{*-1}(x)q^*(x)D_1^* + D_2^*.$$

Then differentiate and use (4.5) to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[p^{*-1}(x)Y(x,z)] &= \mathrm{i}zr(x)\int_0^x p(t)\,\mathbf{j}\,Y(t,z)\,\mathrm{d}t + \mathrm{i}zp^{*-1}(x)q^*(x)p(x)\,\mathbf{j}\,Y(x,z) \\ &\quad + \mathrm{i}zq(x)\,\mathbf{j}\,Y(x,z) + r(x)D_1^* \\ &= \mathrm{i}zr(x)\int_0^x p(t)\,\mathbf{j}\,Y(t,z)\,\mathrm{d}t + r(x)D_1^*.\end{aligned}$$

Multiply both sides by $A(x) = -\mathrm{i}r^{-1}(x)$ and differentiate again to obtain (4.6). The boundary conditions (4.7) are immediate from the preceding formulas. ∎

There is a natural notion of spectral data for the generalized string equation (4.4) (or (4.6)–(4.7)). Given an equation (4.4), define an operator $\widetilde{V}$ by

$$\widetilde{V} : \widetilde{f}(x) \to \widetilde{F}(u), \quad \widetilde{F}(u) = \int_0^l Y^*(x,\overline{u})\,\mathbf{j}\,\widetilde{f}(x)\,\mathrm{d}x, \tag{4.9}$$

on the set $\operatorname{dom}\widetilde{V}$ of piecewise continuous functions $\tilde{f}(x)$ on $[0,l)$ with values in $\mathbb{C}^m$ having compact support and only a finite number of simple discontinuities in $(0,l)$. For each $\widetilde{f}(x)$ in $\operatorname{dom}\widetilde{V}$, $\widetilde{F}(u)$ is an entire function of u with values in $\mathbb{C}^m$. We give $\operatorname{dom}\widetilde{V}$ the topology of $L^2_m(0,l)$.

DEFINITION 4.3. (i) By *spectral data* for a generalized string equation (4.4) we mean a tuple $\boldsymbol{\tau} = \{\tau(u); \mathfrak{F}_1, \ldots, \mathfrak{F}_\nu\}$ as in Definition 2.2 such that

$$\begin{aligned}&\int_0^l \widetilde{g}^*(x)\,\mathbf{j}\,\widetilde{f}(x)\,\mathrm{d}x \\ &\quad = \int_{-\infty}^{\infty} \widetilde{G}^*(u)[\mathrm{d}\tau(u)]\widetilde{F}(u) + \sum_{k=1}^{\nu}[\mathfrak{F}_k(\widetilde{F}(z),\widetilde{G}(z)) + \widehat{\mathfrak{F}}_k(\widetilde{F}(z),\widetilde{G}(z))]\end{aligned} \tag{4.10}$$

for all $\widetilde{f}(x)$ and $\widetilde{g}(x)$ in $\operatorname{dom}\widetilde{V}$ and their transforms $\widetilde{F}(u)$ and $\widetilde{G}(u)$ defined by (4.9). In this case, we also say that (4.4) has $\boldsymbol{\tau}$ as spectral data in the *strong sense*.

(ii) If (4.10) holds for all $\widetilde{f}(x)$ and $\widetilde{g}(x)$ in a dense subset of $\operatorname{dom}\widetilde{V}$, we say that (4.4) has $\boldsymbol{\tau}$ as spectral data in the *weak sense*.

Thus when a generalized string equation (4.4) and boundary problem (2.1)–(2.2) are related as in Theorems 4.1 and 4.2, we have two notions of spectral data given by Definitions 2.2 and 4.3. To relate them, notice that there is a natural correspondence between the domains of the operators V and $\widetilde{V}$. Namely, if $f(x)$ is in $\operatorname{dom} V$, then

$$\tilde{f}(x) = [\,q^*(x) \quad p^*(x)\,]\,f(x) \tag{4.11}$$

defines a function in $\operatorname{dom}\widetilde{V}$. Every function in $\operatorname{dom}\widetilde{V}$ occurs in this way. In fact, if $\widetilde{V}f(x)$ is in $\operatorname{dom}\widetilde{V}$ and $s(x) = q^*(x)q(x) + p^*(x)p(x)$, then $s(x)$ has invertible values and $f(x) = [\, q(x) \quad p(x)\,]^{\mathrm{t}}\, s^{-1}(x)\widetilde{f}(x)$ defines an element of $\operatorname{dom} V$ satisfying (4.11). If $f(x)$ and $\tilde{f}(x)$ are related by (4.11), then

$$\int_0^l \tilde{f}^*(x)\tilde{f}(x)\,\mathrm{d}x = \int_0^l f^*(x)\begin{bmatrix} q(x) \\ p(x)\end{bmatrix}[\, q^*(x) \quad p^*(x)\,]\, f(x)\,\mathrm{d}x.$$

Hence we may choose the topology on $\operatorname{dom} V$ so that under the correspondence (4.11), dense sets in $\operatorname{dom} V$ correspond to dense sets in $\operatorname{dom} \widetilde{V}$.

Theorem 4.4. *Let a generalized string equation* (4.4) *and boundary problem* (2.1)–(2.2) *be related as in Theorems* 4.1 *and* 4.2. *Then* (4.4) *has spectral data* $\boldsymbol{\tau}$ *in the strong* (*respectively weak*) *sense as defined in Definition* 4.3 *if and only if* (2.1)–(2.2) *has spectral data* $\boldsymbol{\tau}$ *in the strong (respectively weak) sense as defined in Definition* 2.2.

Proof. Assume that (4.4) has spectral data $\boldsymbol{\tau}$ in the strong sense defined in Definition 4.3. Let $f(x)$ and $g(x)$ be arbitrary elements of $\operatorname{dom} V$, and let

$$\widetilde{f}(x) = [\, q^*(x) \quad p^*(x)\,]\, f(x), \quad \widetilde{g}(x) = [\, q^*(x) \quad p^*(x)\,]\, g(x), \tag{4.12}$$

be the elements of $\operatorname{dom} \widetilde{V}$ obtained from the correspondence (4.11). Let $F(u), G(u)$ and $\widetilde{F}(u), \widetilde{G}(u)$ be the transforms of $f(x), g(x)$ and $\tilde{f}(x), \tilde{g}(x)$ defined by (2.9) and (4.9). By (4.3),

$$\begin{aligned} F(u) &= \int_0^l [\, D_1 \quad D_2\,]\, W^*(x,\overline{u})\mathcal{H}(x) f(x)\,\mathrm{d}x \\ &= \int_0^l [\, D_1 \quad D_2\,]\, W^*(x,\overline{u})\begin{bmatrix} q(x) \\ p(x)\end{bmatrix} \mathbf{j}\, [\, q^*(x) \quad p^*(x)\,]\, f(x)\,\mathrm{d}x \\ &= \int_0^l Y^*(x,\overline{u})\,\mathbf{j}\, \widetilde{f}(x)\,\mathrm{d}x = \widetilde{F}(u), \end{aligned}$$

and similarly, $G(u) = \widetilde{G}(u)$. Since (4.4) has spectral data $\boldsymbol{\tau}$,

$$\begin{aligned} \int_0^l g^*(x)\mathcal{H}(x) f(x)\,\mathrm{d}x &= \int_0^l g^*(x)\begin{bmatrix} q(x) \\ p(x)\end{bmatrix} \mathbf{j}\, [\, q^*(x) \quad p^*(x)\,]\, f(x)\,\mathrm{d}x \\ &= \int_0^l \widetilde{g}^*(x)\,\mathbf{j}\,\widetilde{f}(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \widetilde{G}^*(u)[\mathrm{d}\tau(u)]\widetilde{F}(u) + \sum_{k=1}^{\nu}[\mathfrak{F}_k(\widetilde{F}(z),\widetilde{G}(z)) + \widehat{\mathfrak{F}}_k(\widetilde{F}(z),\widetilde{G}(z))] \\ &= \int_{-\infty}^{\infty} G^*(u)[\mathrm{d}\tau(u)]F(u) + \sum_{k=1}^{\nu}[\mathfrak{F}_k(F(z),G(z)) + \widehat{\mathfrak{F}}_k(F(z),G(z))]. \end{aligned}$$

Thus the boundary problem (2.1)–(2.2) has spectral data $\boldsymbol{\tau}$ in the strong sense as defined in Definition 2.2. Since arbitrary elements $\tilde{f}(x)$ and $\tilde{g}(x)$ of $\operatorname{dom}\widetilde{V}$ have the form (4.12), these steps are reversible, and so the result follows for the strong case.

In the weak case, we repeat the preceding arguments but applied to appropriate dense subsets of $\operatorname{dom} V$ and $\operatorname{dom}\widetilde{V}$ which correspond by means of the relation (4.11). ∎

Acknowledgements. J. Rovnyak was supported by the National Science Foundation Grant DMS-0100437.

L.A. Sakhnovich thanks the University of Virginia for hospitality during visits when this work was carried out.

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Received December 3, 2001.