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ONE PARAMETER SEMIGROUPS OF ENDOMORPHISMS OF FACTORS OF TYPE II_1

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Abstract. We study invariants for continuous semigroups of $*$-endomorphisms of type II_1-factors. An index is defined, based on R. Powers's notion of the boundary representation, and computed for all known examples on the hyperfinite II_1-factor $\mathcal{R}$, as well as for examples on $L(F_\infty)$. We also introduce the analogue of W. Arveson's continuous tensor product system associated with an E_0-semigroup, and show that it is a complete invariant under cocycle conjugacy.

Keywords: *$*$-endomorphisms, E_0-semigroups, Hilbert modules, Product systems, II_1-factors.*

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1. INTRODUCTION

An E_0-semigroup of a von Neumann algebra $\mathcal{A}$ is a w*-continuous one parameter semigroup $\alpha = \{\alpha_t : t \geqslant 0\}$ of unit preserving $*$-endomorphisms of $\mathcal{A}$. Their study was initiated by R. Powers in [20] as a first step towards the development of an index theory for unbounded derivations. There is by now a highly developed theory for E_0-semigroups of factors of type I_∞, though the situation is quite complicated and many questions remain unanswered (see [3]–[6], [21], [26]–[27], [2] and references therein). The study of E_0-semigroups of factors of type II_1 is still at a nascent state; the same is true, we should remark, of the theory for one-parameter groups of automorphisms (of the hyperfinite II_1-factor), despite significant progress due to the work of Y. Kawahigashi ([17], [18]).

Examples of E_0-semigroups on the hyperfinite II_1 factor $\mathcal{R}$ may be given using second quantization and the trace representation of the CAR algebra, and on $L(F_\infty)$ using Voiculescu's free analogue of the Gaussian functor. Using free and tensor product constructions one may obtain more examples on other factors. We are interested in the classification (up to cocycle conjugacy) of E_0-semigroups of II_1-factors, especially of $\mathcal{R}$. In this paper we introduce an index for an E_0-semigroup α of a II_1-factor $\mathcal{M}$ based on R. Powers's notion of the boundary representation, and another invariant, the product system of Hilbert modules associated with α, which is the analogue of W. Arveson's product system associated with an E_0-semigroup of $\mathcal{B}(\mathcal{H})$.

Given an E_0-semigroup α on a II_1-factor $\mathcal{M}$ acting standardly on $L^2(\mathcal{M})$, its left boundary representation is a $*$-representation of the domain $\mathcal{D}(\delta_\alpha)$ of the generator of α on the deficiency space M of the intertwining semigroup of isometries defined by the trace vector. We use this to define the index of α as the coupling constant $\dim_{\mathcal{M}}(M_{\mathrm{normal}})$ where M_{normal} is the submodule of M corresponding to the w*-continuous part of the boundary representation. This index is invariant under bounded perturbations of the generator and subadditive under tensoring (additive when the boundary representation is normal). It is apriori real valued, though in all examples it is an integer. For the flow on $\mathcal{R}$ arising from the second quantization of the translation semigroup of multiplicity n on the CAR algebra via the trace representation, the index is $2n$. All integer values may be obtained using the Clifford algebra rather than the CAR algebra. We believe that the index is invariant under cocycle conjugacy and that therefore CAR/Clifford flows of different indices are not cocycle conjugate. This would be in contrast to the case of reversible flows arising from second quantization of bilateral shifts and which are all cocycle conjugate by a result of Kawahigashi ([17]). One can show however, using R. Powers's idea of the relative commutant index, that the tensor powers of a CAR flow are pairwise non cocycle conjugate (again in contrast to the corresponding case of automorphism groups).

In Section 3 we consider continuous tensor product systems of Hilbert modules associated with E_0-semigroups of II_1-factors as the main invariant. Given a factor $\mathcal{M}$ of type II_1 acting in the standard way on the Hilbert space $L^2(\mathcal{M})$ and an E_0-semigroup $\alpha = \{\alpha_t : t \geqslant 0\}$ of $\mathcal{M}$, we consider, for $t > 0$, the intertwining space E_t for α_t in the standard representation of $\mathcal{M}$. E_t is a bimodule over the commutant $\mathcal{M}'$ of $\mathcal{M}$, with the $\mathcal{M}'$ actions given by left and right multiplication. Moreover the formula $\langle S, T\rangle = S^*T$ defines an $\mathcal{M}'$-valued inner product on E_t making it into a right Hilbert $\mathcal{M}'$-module. We describe the structure of the family $\{E_t : t > 0\}$ as follows:

(i) E_t is a full, self-dual right Hilbert module over the II_1 factor $\mathcal{M}'$. The w*-algebra $\mathcal{A}(E_t)$ of all bounded module maps (with adjoint) from E_t into itself is naturally isomorphic to $\alpha_t(\mathcal{M})'$. (In particular we get a $*$-isomorphism φ_t of $\mathcal{M}'$ into $\mathcal{A}(E_t)$).

(ii) The map $E_s \times E_t \ni (S, T) \to ST \in E_{s+t}$ induces an isometric isomorphism of $E_s \otimes_{\varphi_t} E_t$ onto a w*-dense Hilbert submodule of E_{s+t}.

Given an E_0-semigroup of a II_1 factor $\mathcal{M}$, we associate with it the set $\mathcal{E}_\alpha = \{(T, t) : T \in E_t,\ t > 0\}$. The previous remarks imply that $\mathcal{E}_\alpha$ has the structure of what we call a continuous tensor product system of Hilbert modules over the type II_1 factor $\mathcal{N} = \mathcal{M}'$. We obtain the desirable stability property: if α and β are two E_0-semigroups of the II_1 factor $\mathcal{M}$, then they are cocycle conjugate if and only if their product systems $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are isomorphic.

The objective is to look for isomorphism invariants for product systems of Hilbert modules. For example one can consider the set of all semigroups of isometries in $\mathcal{E}_\alpha$ that have scalar inner product with the canonical semigroup defined by the trace vector. Those generate a product system of Hilbert spaces whose Arveson index is computable and, we believe, an isomorphism invariant. The automorphism group of the product system is an isomorphism invariant and we have evidence that its computation in the case of the CAR/Clifford flows will show that flows with different indices are not cocycle conjugate. We mention that there is also a notion of a covariant representation of a product system of Hilbert modules over a II_1-factor, and one may carry out, to some extent, a program similar to Arveson's representation theory of product systems of Hilbert spaces. In particular, given a product system $\mathcal{E}$ one may define a spectral C^*-algebra $\mathcal{A}(\mathcal{E})$ as, roughly speaking, the C^*-algebra generated by the left regular representation of $\mathcal{E}$. We will take up these issues elsewhere.

We finally mention a few problems concerning dilations and extensions of E_0-semigroups. It is true that all known examples of E_0-semigroups on a II_1-factor $\mathcal{M}$ have extensions to E_0-semigroups of $\mathcal{B}(L^2(\mathcal{M}))$ which are, in fact, in standard form ([2]). It would be very interesting to know whether this is generally the case and to what extend such extensions are cocycle conjugacy invariant. Dilations to automorphism groups of II_1-factors also exist for all examples; using the result in [7] one can show that such dilations generally exist, but on type II_∞ factors. Finally, in relation to R. Bhat's theorem ([9]), we note that there exist semigroups of completely positive maps on II_1 factors that have no dilations to E_0-semigroups of II_1-factors. For example, the maps $\varphi_t(\lambda(w)) = \mathrm{e}^{-t|w|}\lambda(w)$ where $w \in F_n$ is a word in the free group of n generators $a_1, \dots, a_n$, $|\cdot|$ is the length function, and λ is the left regular representation, extend to completely positive maps on $L(F_n)$, and $\{\varphi_t : t \geqslant 0\}$ is a CP semigroup; (the crucial fact here, proven by U. Haagerup in [15], is that the length function is conditionally negative definite). But a dilation of $\{\varphi_t : t \geqslant 0\}$ to an E_0-semigroup of a II_1-factor would have no normal invariant states, whence such a dilation does not exist.

2. THE BOUNDARY REPRESENTATION AND THE INDEX

DEFINITION 2.1. An E_0-semigroup of a von Neumann algebra $\mathcal{M}$ is a one parameter family $\alpha = \{\alpha_t : t \geqslant 0\}$, such that:

(i) α_t is a unit preserving, normal $*$-endomorphism of $\mathcal{M}$ $\forall t \geqslant 0$;
(ii) $\alpha_0 = \mathrm{id}_{\mathcal{M}}$ and $\alpha_{t+s} = \alpha_t \circ \alpha_s$ $\forall t, s \geqslant 0$;
(iii) the function $t \to \rho(\alpha_t(x))$ is continuous $\forall x \in M$, $\forall \rho \in \mathcal{M}_*$;
(iv) $\alpha_t(\mathcal{M}) \subsetneq \mathcal{M}$ for some (hence, for all) $t > 0$.

The generator δ, defined by $\delta(x) = \mathrm{w}^*\text{-}\lim\limits_{t\to 0+} \frac{\alpha_t(x)-x}{t}$, is a w*-closed, w*-densely defined $*$-derivation, whose domain (a weakly dense $*$-subalgebra of $\mathcal{M}$) we denote by $\mathcal{D}(\delta)$. Bounded perturbations of δ give rise to E_0-semigroups on $\mathcal{M}$ which are cocycle perturbations of α in the following sense: if $\delta' = \delta + \delta_0$, where δ_0 is a bounded derivation on $\mathcal{M}$, then δ' is the generator of an E_0-semigroup $\beta = \{\beta_t : t \geqslant 0\}$ and there is a *norm-continuous* unitary α-cocycle $\{U_t : t \geqslant 0\}$ in $\mathcal{M}$, such that $\beta_t = \mathrm{Ad}\, U_t \circ \alpha_t$. The cocycle $\{U_t : t \geqslant 0\}$ is obtained (using, for example, Dyson's expansion theorem) as a solution to the equation $\frac{\mathrm{d}U_t}{\mathrm{d}t} = \imath U_t \alpha_t(h)$, where h is a self-adjoint element in $\mathcal{M}$ such that $\delta_0(\cdot) = \imath[h, \cdot]$ (see [10], Proposition 3.1.6, Corollary 3.2.47, Proposition 5.4.1.)

In keeping with A. Connes' definition for automorphism groups (and with the theory of E_0-semigroups of $B(\mathcal{H})$), we consider more general perturbations by relaxing the continuity condition on the cocycle.

DEFINITION 2.2. (i) Two E_0-semigroups α and β acting on the von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ respectively, are conjugate, if there is a $*$-isomorphism $\theta : \mathcal{M} \to \mathcal{N}$ such that $\theta \circ \alpha_t = \beta_t \circ \theta$, $\forall t \geqslant 0$.

(ii) If α is an E_0-semigroup of $\mathcal{M}$, then an E_0-semigroup β of $\mathcal{M}$ is a cocycle perturbation of α, if there is a strongly continuous family of unitaries $\{U_t : t \geqslant 0\} \subset \mathcal{M}$, such that $U_{t+s} = U_t \alpha_t(U_s)$ $\forall t, s \geqslant 0$ (cocycle condition), and $\beta_t = \operatorname{Ad} U_t \circ \alpha_t$.

(iii) Two E_0-semigroups are cocycle conjugate if one is conjugate to a cocycle perturbation of the other.

There is also a weaker equivalence relation, that of stable conjugacy: two E_0-semigroups α and β are stably conjugate, if $\alpha \otimes \mathrm{id}_\infty$ and $\beta \otimes \mathrm{id}_\infty$ are cocycle conjugate, where id_∞ denotes the trivial (semi)group on the type I_∞ factor. It is not hard to see that cocycle conjugacy and stable conjugacy coincide in the case where $\mathcal{M} = B(\mathcal{H})$ but this is most likely not the case when $\mathcal{M}$ is a II_1-factor (cf. [18]).

In [20] R. Powers defined a certain birepresentation associated with an E_0-semigroup α of a II_1-factor $\mathcal{M}$. We briefly recall the definition below. Since the trace is α_t-invariant, the formula

$$S_t x\xi_0 = \alpha_t x \xi_0, \quad x \in \mathcal{M}$$

defines a strongly continuous semigroup of isometries of $\mathcal{B}(L^2(\mathcal{M}))$ that intertwines α in the sense that $\alpha_t(x)S_t = S_t x$, $\forall x \in \mathcal{M}$, $\forall t \geqslant 0$. The generator $-d$ of $\{S_t : t \geqslant 0\}$ is a maximal skew-symmetric operator whose deficiency space can be identified with the Hilbert space $(\mathcal{D}(d^*)/\mathcal{D}(d), \langle \cdot\,, \cdot \rangle_*)$ with the inner product given by

$$\langle [\xi], [\eta] \rangle_* = \frac{1}{2}\langle d^*\xi, \eta\rangle + \frac{1}{2}\langle \xi, d^*\eta\rangle, \quad \xi, \eta \in \mathcal{D}(d^*),$$

(and where, of course, $[\xi]$ denotes the class of $\xi \in \mathcal{D}(d^*)$ in the quotient). It can be shown that the operators $x \in \mathcal{D}(\delta_\alpha)$ (acting on the left) leave both $\mathcal{D}(d)$ and $\mathcal{D}(d^*)$ invariant and that the map

$$\pi_\alpha^{\mathrm{l}} : \mathcal{D}(\delta_\alpha) \to \mathcal{B}(\mathcal{D}(d^*)/\mathcal{D}(d)), \quad \pi_\alpha^{\mathrm{l}}(x)([\xi]) = [x\xi]$$

is a norm continuous $*$-representation of $\mathcal{D}(\delta_\alpha)$. We call this the left boundary representation of α. Considering the right action of $\mathcal{M}$, one obtains a right boundary $*$-antirepresentation

$$\pi_\alpha^{\mathrm{r}} : \mathcal{D}(\delta_\alpha) \to \mathcal{B}(\mathcal{D}(d^*)/\mathcal{D}(d)), \quad \pi_\alpha^{\mathrm{r}}(x)([\xi]) = [\xi x].$$

The pair $(\pi_\alpha^{\mathrm{l}}, \pi_\alpha^{\mathrm{r}})$ is the boundary birepresentation of α.

We use the left boundary representation to tentatively define an index for E_0-semigroups of II_1-factors as follows: letting p be the largest projection in $\pi_\alpha^{\mathrm{l}}(\mathcal{D}(\delta_\alpha))'$, such that the subrepresentation of π_α^{l} corresponding to p is normal, we obtain (by extension), a normal representation of $\mathcal{M}$ on $p\mathcal{D}(d^*)/\mathcal{D}(d)$. We define the index as the coupling constant

$$\operatorname{ind}(\alpha) = \dim_{\mathcal{M}}(p\mathcal{D}(d^*)/\mathcal{D}(d)).$$

We record the main properties of this index in the following proposition.

PROPOSITION 2.3. (i) *The index is invariant under conjugacy and under bounded perturbations of the generator.*

(ii) $\mathrm{ind}(\alpha\otimes\beta) \leqslant \mathrm{ind}\ (\alpha)+\mathrm{ind}(\beta)$ *with equality when* π^l_α *and* π^l_β *are normal.*

Proof. (i) is straightforward while (ii) follows from the proof of Lemma 4.4 in [20]. ∎

We next turn to examples.

EXAMPLE 2.4. Let $\mathcal{K}$ be a real Hilbert space and consider the Clifford algebra $\mathcal{U}_\mathcal{K}$ over $\mathcal{K}$; this is the C^*-algebra generated by $\mathbb{1}$ and self-adjoint operators $u(f)$, $f\in\mathcal{K}$, satisfying the relations:

$$\begin{array}{ll} u(\lambda f+g)=\lambda u(f)+u(g), & \forall\lambda\in\mathbb{R};\\ u(f)u(g)+u(g)u(f)=\langle f,g\rangle\mathbb{1}, & \forall f,g\in\mathcal{K}. \end{array}$$

Then $\mathcal{U}_\mathcal{K}$ is isomorphic to the UHF-algebra of type 2^∞. The GNS construction with respect to its normalized trace τ gives rise to the hyperfinite II_1 factor: $\pi_\tau(\mathcal{U}_\mathcal{K})''=\mathcal{R}$. Every strongly continuous semigroup of isometries $\{S_t : t\geqslant 0\}$ of $\mathcal{K}$ induces a semigroup of endomorphisms $\alpha=\{\alpha_t : t\geqslant 0\}$ of $\mathcal{U}_\mathcal{K}$ defined on monomials by

$$\alpha_t(u(f_1)u(f_2)\cdots u(f_k))=u(S_tf_1)u(S_tf_2)\cdots u(S_tf_k)$$

and extended linearly and continuously. As each α_t leaves the trace invariant, it is an easy matter to show that it is extended to an endomorphism, also denoted by α_t, of $\mathcal{R}$ and that $\alpha=\{\alpha_t : t\geqslant 0\}$ is an E_0-semigroup of $\mathcal{R}$.

If $\mathcal{K}=\mathcal{L}^2(0,\infty)\otimes\mathfrak{M}$, where $\mathfrak{M}$ is an n-dimensional real Hilbert space, and $\{S_t : t\geqslant 0\}$ is the unilateral shift on $\mathcal{K}$, the resulting E_0-semigroup on $\mathcal{R}$ is called the Clifford flow of rank n ([20]).

REMARK 2.5. One may use the algebra of the Canonical Anticommutation Relations over a complex Hilbert space in the above construction. More specifically, let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{A}_\mathcal{H}$ the CAR algebra over $\mathcal{H}$, i.e. the C^*-algebra generated by $\mathbb{1}$ and operators $a(f)$ satisfying the relations

$$\begin{array}{ll} a(\lambda f+g)=\lambda a(f)+a(g), & \forall\lambda\in\mathbb{C},\\ a(f)a(g)+a(g)a(f)=0, & \forall f,g\in\mathcal{H},\\ a(f)a(g)^*+a(g)^*a(f)=\langle f,g\rangle\mathbb{1}. & \end{array}$$

The GNS representation of $\mathcal{A}_\mathcal{H}$ with respect to its trace gives again rise to the hyperfinite II_1 factor: $\mathcal{R}=\pi_\tau(\mathcal{A}_\mathcal{H})''$. As before, strongly continuous semigroups $\{S_t : t\geqslant 0\}$ of isometries of $\mathcal{H}$ give rise to E_0-semigroups of $\mathcal{R}$, defined by $\beta_t(a(f))=a(S_tf)$. These are however conjugate to Clifford flows. More specifically we have the following:

Proposition 2.6. *Let $\mathcal{H} = \mathcal{L}^2(0,\infty) \otimes \mathfrak{N}$ where $\mathfrak{N}$ is an n-dimensional complex Hilbert space and let $\{S_t : t \geqslant 0\}$ be the unilateral shift (of multiplicity n) on $\mathcal{H}$. If β is the E_0-semigroup of $\mathcal{R}$ constructed from $\{S_t : t \geqslant 0\}$ using the CAR algebra $\mathcal{A}_{\mathcal{H}}$ as in the previous paragraph, then β is conjugate to the Clifford flow of rank $2n$.*

Proof. Let $\mathfrak{M}$ be a $2n$-dimensional real Hilbert space and J a linear operator on $\mathfrak{M}$ such that $\langle J\xi, \eta\rangle = -\langle \xi, J\eta\rangle$ and $J^2 = -1$. $\mathfrak{M}$, with the obvious complex structure and inner product defined by $\langle \xi, \eta\rangle_{\mathbb{C}} = \langle \xi, \eta\rangle + \imath\langle \xi, J\eta\rangle$ is an n-dimensional complex Hilbert space $\mathfrak{N}$. J gives rise to an operator on the real Hilbert space $\mathcal{K} = \mathcal{L}^2(0,\infty) \otimes \mathfrak{M}$, denoted again by J, with the properties: $\langle Jf, g\rangle = -\langle f, Jg\rangle$, $J^2 = -1$ which induces on $\mathcal{K}$ a complex structure with which it becomes identical to $\mathcal{H} = \mathcal{L}^2(0,\infty) \otimes \mathfrak{N}$. The unilateral shift (of multiplicity $2n$) on $\mathcal{K}$ commutes with J, so it is identified with the shift $\{S_t : t \geqslant 0\}$ on $\mathcal{H}$.

Now, the Clifford algebra $\mathcal{U}_{\mathcal{K}}$ is identical to the CAR algebra $\mathcal{A}_{\mathcal{H}}$ since the operators

$$a(f) = \frac{u(f) - \imath u(Jf)}{\sqrt{2}}, \quad a(f)^* = \frac{u(f) + \imath u(Jf)}{\sqrt{2}}, \quad f \in \mathcal{H},$$

satisfy the canonical anticommutation relations and the operators $u(f)$ may be recovered from the $a(f)$'s via $u(f) = \frac{a(f)+a(f)^*}{\sqrt{2}}$. Finally, since S_t commutes with J, the Clifford flow $\alpha = \{\alpha_t : t \geqslant 0\}$ of rank $2n$, acts on $\mathcal{A}_{\mathcal{H}}$ in the expected way: $\alpha_t(a(f)) = a(S_t f)$. ∎

Proposition 2.7. *If α is the Clifford flow of rank n on $\mathcal{R}$, then* $\operatorname{ind}(\alpha) = n$.

Proof. We show that the Clifford flow of rank n on $\mathcal{R}$ admits an extension to a completely spatial E_0-semigroup of $\mathcal{B}(L^2(\mathcal{R}))$ of index n. Indeed, let $\mathfrak{M}$ be an n-dimensional real Hilbert space, $\mathfrak{M}_{\mathbb{C}}$ its complexification, $\mathcal{H} = \mathcal{L}^2(0,\infty) \otimes \mathfrak{M}_{\mathbb{C}}$ and $\{S_t : t \geqslant 0\}$ the translation semigroup on $\mathcal{H}$. Let $\mathcal{A}_{\mathcal{H}}$ be the CAR algebra over $\mathcal{H}$ in its Fock representation on $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$, with Fock state $\omega(\,\cdot\,) = \langle \cdot\, \xi_0, \xi_0\rangle$. The map $a(f) \to a(S_t f)$ extends to an E_0-semigroup $\widetilde{\alpha}$ of $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$, the CAR-flow of rank n. Letting $u(f) = \frac{a(f)+a(f)^*}{2}$ for $f \in \mathcal{K} = \mathcal{L}^2(0,\infty) \otimes \mathfrak{M}$, it is straightforward to show that the operators $u(f)$ generate the Clifford algebra over $\mathcal{K}$ in its trace representation, with cyclic trace vector ξ_0. So the von Neumann algebra $\{u(f) : f \in \mathcal{K}\}''$ is equal to $\mathcal{R}$ acting standardly, and the restriction of $\widetilde{\alpha}$ on $\mathcal{R}$ is the Clifford flow of rank n. Since the boundary representation of the CAR flow of rank n is equivalent to a direct sum of n copies of the identity representation, the same is true of π^1_α, its restriction to the domain $\mathcal{D}(\delta_\alpha)$ of the generator of the Clifford flow. ∎

We believe that both the left boundary representation and the index are cocycle conjugacy invariants but we do not have a proof of that at the moment. For the case of the Clifford flows, it would be enough to show that the boundary representation does not depend on the choice of intertwining semigroup of isometries i.e. the analogue of the result in [1]. The difficulty here is that one does not know how two intertwining semigroups of isometries of an E_0-semigroup of a II_1 factor are related to each other.

We next exhibit, using R.T. Powers's idea of the "relative commutant index", a countably infinite family of cocycle conjugacy classes of E_0-semigroups of $\mathcal{R}$. If

α is an E_0-semigroup of a type II_1 factor $\mathcal{M}$, we consider for every $t > 0$ the von Neumann algebra $(\alpha_t(\mathcal{M})' \cap \mathcal{M}) \vee \alpha_t(\mathcal{M})$ generated by $\alpha_t(\mathcal{M})$ and its relative commutant in $\mathcal{M}$. If this is a subfactor of $\mathcal{M}$ we let

$$c_\alpha(t) = [\mathcal{M} : (\alpha_t(\mathcal{M})' \cap \mathcal{M}) \vee \alpha_t(\mathcal{M})].$$

We denote by $\mathcal{I}_\alpha$ the set of all $t > 0$ for which $(\alpha_t(\mathcal{M})' \cap \mathcal{M}) \vee \alpha_t(\mathcal{M})$ is a subfactor.

PROPOSITION 2.8. (i) *The family $(c_\alpha(t))_{t\in\mathcal{I}_\alpha}$ is a stable conjugacy invariant.*

(ii) *If α (respectively β) is an E_0-semigroup of the II_1 factor $\mathcal{M}$ (respectively $\mathcal{N}$) then, for $t \in \mathcal{I}_\alpha \cap \mathcal{I}_\beta$,*

$$c_{\alpha\otimes\beta}(t) = c_\alpha(t) \cdot c_\beta(t).$$

Proof. (i) It is plain that $c_\alpha(t)$ is invariant under conjugacy. If $\{U_t : t \geqslant 0\}$ is a strongly continuous α cocycle in $\mathcal{M}$ and $\beta_t = \mathrm{Ad}\ U_t \circ \alpha_t$ then it is straightforward that

$$(\beta_t(\mathcal{M})' \cap \mathcal{M}) \vee \beta_t(\mathcal{M}) = U_t \left((\alpha_t(\mathcal{M})' \cap \mathcal{M}) \vee \alpha_t(\mathcal{M}) \right) U_t^*$$

from which $c_\alpha(t) = c_\beta(t)$ follows. Invariance under stable conjugacy follows from the proof of part (ii) below.

(ii) We have, for all $t > 0$,

$$\begin{aligned}
&[(\alpha_t \otimes \beta_t)(\mathcal{M}\overline{\otimes}\mathcal{N})' \cap \mathcal{M}\overline{\otimes}\mathcal{N}] \vee (\alpha_t \otimes \beta_t)(\mathcal{M}\overline{\otimes}\mathcal{N}) \\
&\quad = [(\alpha_t(\mathcal{M})\overline{\otimes}\beta_t(\mathcal{N}))' \cap \mathcal{M}\overline{\otimes}\mathcal{N}] \vee (\alpha_t(\mathcal{M})\overline{\otimes}\beta_t(\mathcal{N})) \\
&\quad = [(\alpha_t(\mathcal{M})'\overline{\otimes}\beta_t(\mathcal{N})') \cap \mathcal{M}\overline{\otimes}\mathcal{N}] \vee (\alpha_t(\mathcal{M})\overline{\otimes}\beta_t(\mathcal{N})) \\
&\quad = [(\alpha_t(\mathcal{M})' \cap \mathcal{M})\overline{\otimes}(\beta_t(\mathcal{N})' \cap \mathcal{N})] \vee (\alpha_t(\mathcal{M})\overline{\otimes}\beta_t(\mathcal{N})) \\
&\quad = [(\alpha_t(\mathcal{M})' \cap \mathcal{M}) \vee \alpha_t(\mathcal{M})]\overline{\otimes}[(\beta_t(\mathcal{N})' \cap \mathcal{N}) \vee \beta_t(\mathcal{N})],
\end{aligned}$$

where we used the equality $(A\overline{\otimes}B) \vee (P\overline{\otimes}Q) = (A \vee P)\overline{\otimes}(B \vee Q)$ (and its dual), which holds for any von Neumann algebras A, B, P and Q. The multiplicativity of the Jones index completes the proof. ∎

We use this to show that the tensor powers of a Clifford flow are pairwise not cocycle conjugate (in fact, not stably conjugate).

PROPOSITION 2.9. *Let α be the Clifford flow of rank n on the hyperfinite II_1 factor $\mathcal{R}$. Then, for every $t > 0$ we have $c_\alpha(t) = 2$.*

Proof. If we choose orthonormal bases $(f_i)_{i\in\mathcal{I}}$ of $\mathcal{L}^2((0,t),\mathfrak{M})$ and $(g_j)_{j\in\mathcal{J}}$ of $\mathcal{L}^2((t,\infty),\mathfrak{M})$ (with $\mathcal{I}$ and $\mathcal{J}$ ordered), so that $(f_i)_{i\in\mathcal{I}} \cup (g_j)_{j\in\mathcal{J}}$ is an orthonormal basis for $\mathcal{L}^2((0,\infty),\mathfrak{M})$, we see that every element w of $\mathcal{R}$ has a unique representation as a sum (in the $\|\cdot\|_2$ norm)

$$w = \sum \lambda_{I,J} Q(I)Q(J), \quad \lambda_{I,J} \in \mathbb{C},$$

where $I = \{i_1 < i_2 < \cdots < i_l\} \subseteq \mathcal{I}$, $J = \{j_1 < j_2 < \cdots < j_k\} \subseteq \mathcal{J}$, and $Q(I) = u(f_{i_1}) \cdots u(f_{i_k})$, $Q(J) = u(g_{j_1}) \cdots u(g_{j_l})$, $Q(\emptyset) = \mathbb{1}$. Note that for $J' \subseteq \mathcal{J}$ we have

$$Q(J')^* \Big(\sum \lambda_{I,J} Q(I)Q(J) \Big) Q(J') = \sum \mu_{I,J}^{J'} \lambda_{I,J} Q(I)Q(J),$$

where $\mu_{I,J}^{J'} = (-1)^{\sigma_{I,J}(J')}$ with $\sigma_{I,J}(J') = |I|\,|J'| + |J|\,|J'| - |J' \cap J|$. Now, if $w \in \alpha_t(\mathcal{R})' \cap \mathcal{R}$, w commutes with $Q(J')$; we must therefore have that $\sigma_{I,J}(J')$ is even for all $J' \subseteq \mathcal{J}$. It follows that $J = \emptyset$ and $|I|$ is even. The conclusion is that $\alpha_t(\mathcal{R})' \cap \mathcal{R}$ is the von Neumann algebra generated by even polynomials in the $u(f)$, with $\operatorname{supp} f \subseteq (0,t)$.

Since $\alpha_t(\mathcal{R})$ is the von Neumann algebra generated by all polynomials in the $u(g)$, with $\operatorname{supp} g \subseteq (t,\infty)$, choosing $f_0 \in \mathcal{L}^2((0,t),\mathfrak{M})$, $\|f_0\| = 1$, we see that $\mathcal{R}$ is generated by $(\alpha_t(\mathcal{R})' \cap \mathcal{R}) \vee \alpha_t(\mathcal{R})$ and the unitary $u(f_0)$. It follows that $[\mathcal{R} : (\alpha_t(\mathcal{R})' \cap \mathcal{R}) \vee \alpha_t(\mathcal{R})] = 2$, i.e. $c_\alpha(t) = 2\ \forall t > 0$. ∎

Corollary 2.10. *If α is the Clifford flow of rank n on $\mathcal{R}$ then, for $k \neq l$, $\alpha^{\otimes k}$ and $\alpha^{\otimes l}$ are not stably conjugate. (Of course $\alpha^{\otimes k}$ denotes the k-fold tensor power of α, an E_0-semigroup of $\mathcal{R}$.)*

Proof. This follows immediately from the previous two propositions. ∎

Remark 2.11. (i) By the previous two propositions, if α is the Clifford flow of rank 1 and β the Clifford flow of rank 2, then $\alpha \otimes \alpha$ and β are not stably conjugate (hence not cocycle conjugate), while they are both of index 2.

(ii) If α is the Clifford flow of rank 1 on $\mathcal{R}$ and β is its restriction to the subfactor (isomorphic to $\mathcal{R}$) generated by the even polynomials in the $u(f)$, we have that $\operatorname{ind}(\alpha) = \operatorname{ind}(\beta) = 1$, and $c_\alpha(t) = c_\beta(t) = 2$, $\forall t > 0$. The boundary birepresentation however distinguishes between the two. The boundary birepresentation of α is equivalent to the birepresentation $(x \to R_x, x \to L_{\theta(x)})$, $x \in \mathcal{R}$, of $\mathcal{R}$ on $L^2(\mathcal{R})$, where R_x and L_x denote the standard left and right actions, respectively, of $\mathcal{R}$ on $L^2(\mathcal{R})$ and θ is the automorphism of period two satisfying $\theta(u(f)) = -u(f)$ ([20]). For the restriction β however, the boundary birepresentation is equivalent to the standard birepresentation on $L^2(\mathcal{R})$. We believe that the boundary birepresentation is a cocycle conjugacy invariant and that therefore α and β are not cocycle conjugate. We do not have a proof of this at the moment however.

We next turn to examples of E_0-semigroups on the free group factors.

Example 2.12. We describe a class of examples of E_0-semigroups on $L(F_\infty)$, the von Neumann algebra of the free group on countably many generators. Suppose that $\mathcal{H}$ is an infinite dimensional real Hilbert space, $\mathcal{H}_\mathbb{C}$ is its complexification, and $\mathcal{T}(\mathcal{H}_\mathbb{C}) = \mathbb{C}1 \oplus \bigoplus_{n\geqslant 1} \mathcal{H}_\mathbb{C}^{\otimes n}$ is the full Fock space of $\mathcal{H}_\mathbb{C}$. For every $h \in \mathcal{H}_\mathbb{C}$ define the left creation operator $l(h) \in \mathcal{B}(\mathcal{T}(\mathcal{H}_\mathbb{C}))$ by

$$l(h)\zeta = \begin{cases} h & \text{if } \zeta = 1, \\ h \otimes \zeta & \text{if } \zeta \in \mathcal{T}(\mathcal{H}_\mathbb{C}) \ominus \mathbb{C}1. \end{cases}$$

For $h \in \mathcal{H}$ let $s(h) = \frac{l(h)+l(h)^*}{2}$, and let $\Phi(\mathcal{H})$ be the von Neumann algebra generated by $\{s(h) : h \in \mathcal{H}\}$. By a theorem of Voiculescu, (cf. [28]), $\Phi(\mathcal{H})$ is isomorphic to $L(F_\infty)$ acting standardly on $\mathcal{T}(\mathcal{H}_\mathbb{C})$ with the vacuum vector 1 as a cyclic trace vector.

PROPOSITION 2.13. *Suppose that $\mathcal{U} = \{U_t : t \geqslant 0\}$ is a strongly continuous semigroup of isometries on the real Hilbert space $\mathcal{H}$. Then:*

(i) *there is an E_0-semigroup $\alpha^{\mathcal{U}}$ of $L(F_\infty)$ such that*

$$\alpha_t^{\mathcal{U}}(s(h)) = s(U_t h) \quad \forall t \geqslant 0, \forall h \in \mathcal{H};$$

(ii) *$\alpha^{\mathcal{U}}$ is a shift, i.e. $\bigcap_{t\geqslant 0} \alpha_t^{\mathcal{U}}(L(F_\infty)) = \mathbb{C}\mathbb{1}$, if $\bigcap_{t\geqslant 0} U_t\mathcal{H} = 0$;*

(iii) *there is a free product decomposition $\alpha^{\mathcal{U}} = \alpha * \beta$ where α is a (semi)group of automorphisms on $L(F_n)$ for some $n \in \mathbb{N} \cup \{\infty\}$, and β is a shift on $L(F_\infty)$;*

(iv) $\operatorname{ind}(\alpha^{\mathcal{U}}) = +\infty$.

Proof. (i) Letting $\{V_t : t \in \mathbb{R}\}$ be a unitary dilation of $\mathcal{U}$ on a larger Hilbert space $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}_1$ we see that the formula $\widetilde{\alpha}_t(s(k)) = s(V_t k)$, $k \in \mathcal{K}$, defines an automorphism $\widetilde{\alpha}_t = \operatorname{Ad} \widetilde{V}_t$ of $\Phi(\mathcal{K})$ where $\widetilde{V}_t$ is the second quantization of V_t, so that $\widetilde{\alpha} = \{\widetilde{\alpha}_t : t \in \mathbb{R}\}$ is a one parameter group of automorphisms. Using the free product decomposition $\Phi(\mathcal{K}) \simeq \Phi(\mathcal{H}) * \Phi(\mathcal{H}_1)$ ([29], Lemma 2.6.6), it is easy to see that $\widetilde{\alpha}_t$ leaves $\Phi(\mathcal{H})$ invariant for $t \geqslant 0$, whence, by restriction, we obtain a continuous one-parameter semigroup of endomorphisms of $\Phi(\mathcal{H})$.

(ii) Without loss of generality we can assume that $\mathcal{H}$ is the Hilbert space $L_r^2((0,\infty), \mathcal{C})$ of all square integrable functions form $(0,\infty)$ into an n-dimensional real Hilbert space $\mathcal{C}$, and that $\mathcal{U}$ is the translation semigroup. Suppose that $x \in \bigcap_{n=1}^{\infty} \alpha_{nt} L(F_\infty))$. Since for every n, there is $y_n \in L(F_\infty)$ such that $x = \alpha_{nt}(y_n)$, x is the strong limit of polynomials in $s(h)$, with $h \in L_r^2((nt,\infty),\mathcal{C})$. Since the spaces $L_r^2((nt,\infty),\mathcal{C})$ and $L_r^2((0,nt),\mathcal{C})$ are orthogonal, the families of random variables $\{s(h) : h \in L_r^2((nt,\infty),\mathcal{C})\}$ and $\{s(h) : h \in L_r^2((0,nt),\mathcal{C})\}$ are free, (cf. [29], Theorem 2.6.2), and therefore, $\tau(xp) = \tau(x)\tau(p)$ for every polynomial in the $s(h)$ with $h \in L_r^2((0,nt),\mathcal{C})$. Since n is arbitrary, we get that $\tau(xy) = \tau(x)\tau(y)$ for all $y \in L(F_\infty)$. This implies that $x = \tau(x)1$.

(iii) This is straightforward, using the Wold decomposition of $\mathcal{U}$.

(iv) There is an obvious extension of $\alpha^{\mathcal{U}}$ to an E_0-semigroup of $\mathcal{B}(\mathcal{T}(\mathcal{H}_{\mathbb{C}}))$ given by $\alpha_t(l(h)) = l(U_t h)$, where $h \in \mathcal{H}_{\mathbb{C}}$ (and U_t is extended to the complexification of $\mathcal{H}$). This E_0-semigroup is completely spatial, of infinite index ([13]). Therefore, its boundary representation is normal, of infinite multiplicity; and its restriction to the domain of the generator of $\alpha^{\mathcal{U}}$ is the left boundary representation of $\alpha^{\mathcal{U}}$. ∎

REMARK 2.14. One may obtain a plethora of examples of E_0-semigroups on other factors using tensor and free product constructions. In particular, if for $i = 1,2$, $\mathcal{M}_i$ is a II_1-factor acting standardly on $\mathcal{H}_i = L^2(\mathcal{M}_i)$, and that $\alpha^i = \{\alpha_t^i : t > 0\}$ is an E_0-semigroup of $\mathcal{M}_i$. Then there is an E_0-semigroup $\alpha = \{\alpha_t : t > 0\}$ of the von Neumann algebra free product $\mathcal{M} = \mathcal{M}_1 * \mathcal{M}_2$ with respect to the canonical traces, such that

$$\alpha_t\left(\lambda_{i_1}(x_{i_1}) \cdots \lambda_{i_n}(x_{i_n})\right) = \lambda_{i_1}(\alpha_t^{i_1}(x_{i_1})) \cdots \lambda_{i_n}(\alpha_t^{i_n}(x_{i_n})),$$

where $x_{i_j} \in \mathcal{M}_{i_j}$, $i_j = 1,2$, $t > 0$, and λ_i is the representation of $\mathcal{M}_i$ on $l^2(\mathcal{M}_1) * L^2(\mathcal{M}_2)$ constructed in [28] (see [29] for definitions and notation concerning the free product of von Neumann algebras). Using the result, proven in [12], that $\mathcal{R} * \mathcal{R} \simeq L(F_2)$ and also that $\mathcal{R} * L(F_n) \simeq L(F_{n+1})$ we then obtain examples of E_0-semigroups on all free group factors. An interesting question is whether every type II_1 factor admits an E_0-semigroup.

3. PRODUCT SYSTEMS

Let $\mathcal{M}$ be a type II_1-factor acting standardly on $L^2(\mathcal{M})$ with cyclic trace vector ξ_0, and $\alpha = \{\alpha_t : t \geqslant 0\}$ an E_0-semigroup of $\mathcal{M}$. For every $t > 0$ consider the set of all intertwining operators for α_t:

$$E_t = \left\{T \in \mathcal{B}(L^2(\mathcal{M})) : \alpha_t(x)T = Tx, \, \forall x \in \mathcal{M}\right\}.$$

The next three propositions describe the structure of E_t.

PROPOSITION 3.1. (i) *For every $T \in E_t$ and $A \in \mathcal{M}'$, $TA \in E_t$.*
(ii) *For every $T \in E_t$ and $A \in \alpha_t(\mathcal{M})'$, $AT \in E_t$.*
(iii) *$\mathcal{M}' = \{S^*T : S, T \in E_t\}$.*
(iv) *The linear span of the set $\{TS^* : S, T \in E_t\}$ is a (σ-weakly dense) two sided ideal of $\alpha_t(\mathcal{M})'$.*

Proof. For $T \in E_t$, $A \in \mathcal{M}'$ and $x \in \mathcal{M}$ we have $\alpha_t(x)TA = TxA = TAx$ and thus $TA \in E_t$. This proves (i) and (ii) is proven similarly. To prove (iii), notice first that if S, T are elements in E_t and $x \in \mathcal{M}$ then $S^*Tx = S^*\alpha_t(x)T = (\alpha_t(x^*)S)^*T = (Sx^*)^*T = xS^*T$, i.e. $S^*T \in \mathcal{M}'$. Therefore $\{S^*T : S, T \in E_t\} \subseteq \mathcal{M}'$. Next, observe that, since the endomorphism α_t is unital, it is trace preserving, and therefore the map $x\xi_0 \to \alpha_t(x)\xi_0$, defined on the dense subspace $\mathcal{M}\xi_0$ of $L^2(\mathcal{M})$, extends to an isometry U_t. Moreover, for $x, y \in \mathcal{M}$, we have:

$$\alpha_t(x)U_t(y\xi_0) = \alpha_t(x)\alpha_t(y)\xi_0 = \alpha_t(xy)\xi_0 = U_t(xy\xi_0) = U_t x(y\xi_0)$$

and therefore $\alpha_t(x)U_t = U_t x$. This means that $U_t \in E_t$. Now if A is an element in $\mathcal{M}'$, then $A = U_t^* U_t A$ with $U_t \in E_t$ and $U_t A \in E_t$ by (i). Thus $A \in \{S^*T : S, T \in E_t\}$, and the proof of (iii) is complete. Finally, if T, S are elements of E_t, $x \in \mathcal{M}'$ and $A, B \in \alpha_t(\mathcal{M})'$ then,

$$TS^*\alpha_t(x) = T(\alpha_t(x^*)S)^* = T(Sx^*)^* = TxS^* = \alpha_t(x)TS^*$$

i.e. $TS^* \in \alpha_t(\mathcal{M})'$ and moreover, $A(TS^*)B = (AT)(B^*S)^*$, with $AT, B^*S \in E_t$ by (ii). This proves that the linear span of the set $\{TS^* : T, S \in E_t\}$ is a two sided ideal in $\alpha_t(\mathcal{M})'$. ∎

It follows easily from (iv) of the previous proposition that every positive element of the linear span of $\{TS^* : S, T \in E_t\}$ is of the form $\sum_{i=1}^{n} T_i T_i^*$, $T_i \in E_t$, and therefore that there is a family $(T_i) \subset E_t$ such that $\sum T_i T_i^* = 1$ in the σ-weak topology of $\alpha_t(\mathcal{M})'$. In fact the T_i's can be chosen to be isometries:

PROPOSITION 3.2. *For every $t > 0$ there is a family $\{U_n(t) : n \in \mathbb{N}\}$ of isometries in E_t such that:*

(i) $\sum_{n=1}^{\infty} U_n(t)U_n(t)^* = 1$, *where the convergence is understood with respect to the σ-weak topology of $\alpha_t(\mathcal{M})'$;*

(ii) $\alpha_t(x) = \sum_{n=1}^{\infty} U_n(t)xU_n(t)^*$ *(σ-weakly), $\forall x \in \mathcal{M}$.*

Proof. We note that for $0 \leqslant s < t$, the inclusion $\alpha_t(\mathcal{M}) \subset \alpha_s(\mathcal{M})$ has infinite Jones index. Indeed, for every $n \in \mathbb{N}$ one can choose real numbers $t = t_0 > t_1 > t_2 > \cdots > t_n = s$, and obtain the corresponding (proper) inclusions of II_1-factors, $\alpha_{t_0}(\mathcal{M}) \subset \alpha_{t_1}(\mathcal{M}) \subset \cdots \subset \alpha_{t_n}(\mathcal{M})$ each of index at least 2. The multiplicativity of the Jones index ([16], Proposition 2.1.8.), implies that $[\alpha_s(\mathcal{M}) : \alpha_t(\mathcal{M})] = +\infty$ In particular α_t is a representation of infinite coupling constant and hence equivalent to an infinite direct sum of copies of the standard representation. To each such copy corresponds an isometry $U_n(t)$ in E_t and the family $\{U_n(t) : n \in \mathbb{N}\}$ evidently has the required properties. ∎

We will show later on that the isometries may be chosen in a measurable way.

Proposition 3.1 (i), implies that E_t is a right $\mathcal{M}'$-module, and (iii) of the same proposition shows that the map $(S,T) \to ST^*$ is an $\mathcal{M}'$-valued inner product on E_t, which we will denote by $\langle S,T\rangle$, conjugate linear in the first variable, with respect to which E_t is a Hilbert $\mathcal{M}'$-module (we refer to [8], [14], [19], [24] and [25] for definitions and facts regarding Hilbert modules over von Neumann algebras). The following theorem shows that E_t belongs to the best behaved class of Hilbert w*-modules.

THEOREM 3.3. *E_t is a full, self-dual Hilbert $\mathcal{M}'$-module. Its natural* w*-*topology coincides with the relative σ-weak topology.*

Proof. The norm on E_t defined by the $\mathcal{M}'$-valued inner product, coincides with the operator norm, with respect to which, E_t is complete. Also, (iii) of Proposition 3.1 shows that $\mathcal{M}' = \{\langle S,T\rangle : S,T \in E_t\}$, i.e. E_t is full. In order to prove that E_t is self-dual, we need to show that if $\varphi : E_t \to \mathcal{M}'$ is a bounded module map (where of course $\mathcal{M}'$ is considered as a Hilbert $\mathcal{M}'$-module with inner product $\langle A,B\rangle = A^*B$), then φ is induced by an element in E_t, i.e. that there is an element S in E_t, such that

$$\varphi(T) = \langle S,T\rangle \quad \forall T \in E_t.$$

Without loss of generality we may assume that $\|\varphi\| \leqslant 1$. Then, by [19], Theorem 2.8, $\langle\varphi(T),\varphi(T)\rangle \leqslant \langle T,T\rangle$, $\forall T \in E_t$. We define an operator R on the linear span of vectors of the form $T\xi$, $T \in E_t$, $\xi \in L^2(\mathcal{M})$, by the formula

$$R\Big(\sum_{i=1}^{n} T_i\xi_i\Big) = \sum_{i=1}^{n} \varphi(T_i)\xi_i, \quad T_i \in E_t,\ \xi_i \in L^2(\mathcal{M}),\ i = 1,2,\ldots,n,\ n \in \mathbb{N}.$$

We show below that R is well defined and that it extends to a bounded operator on $L^2(\mathcal{M})$. Notice first that the $n \times n$ matrix $[T_i^*T_j - \varphi(T_i)^*\varphi(T_j)]_{i,j=1}^n$ is a positive element of the von Neumann algebra $M_n(\mathcal{M}')$ of all $n \times n$ matrices with entries in

$\mathcal{M}'$. Indeed this is equivalent to the statement that for any elements $A_1, \dots, A_n$ of $\mathcal{M}'$, the operator $\sum\limits_{i,j=1}^{n} A_i^*(T_i^*T_j - \varphi(T_i)^*\varphi(T_j))A_j$ is positive in $\mathcal{M}'$. And this latter statement is immediate, since

$$\begin{aligned}\sum_{i,j=1}^{n} A_i^*(T_i^*T_j - \varphi(T_i^*)\varphi(T_j))A_j \\ &= \Big\langle \sum_{i=1}^{n} T_iA_i, \sum_{i=1}^{n} T_iA_i \Big\rangle - \Big\langle \sum_{i=1}^{n} \varphi(T_i)A_i, \sum_{i=1}^{n} \varphi(T_i)A_i \Big\rangle \\ &= \Big\langle \sum_{i=1}^{n} T_iA_i, \sum_{i=1}^{n} T_iA_i \Big\rangle - \Big\langle \varphi\Big(\sum_{i=1}^{n} T_iA_i\Big), \varphi\Big(\sum_{i=1}^{n} T_iA_i\Big) \Big\rangle \geqslant 0.\end{aligned}$$

Now if $T_1, \dots, T_n \in E_t$, $\xi_1, \dots, \xi_n \in L^2(\mathcal{M})$ we have

$$\Big\| \sum_{i=1}^{n} \varphi(T_i)\xi_i \Big\|^2 = \sum_{i,j=1}^{n} \langle \varphi(T_i)^*\varphi(T_j)\xi_j, \xi_i \rangle \leqslant \sum_{i,j=1}^{n} \langle T_i^*T_j\xi_j, \xi_i \rangle = \Big\| \sum_{i=1}^{n} T_i\xi_i \Big\|^2.$$

Thus R is well defined and bounded and therefore extends to a bounded operator from the closure of $\{T\xi : T \in E_t, \xi \in L^2(\mathcal{M})\}$ to $L^2(\mathcal{M})$. By Proposition 3.2 this closure is $L^2(\mathcal{M})$ and thus $R \in \mathcal{B}(L^2(\mathcal{M}))$. We claim that $R = S^*$ for some $S \in E_t$. To see this, notice that R satisfies the relation $RA\xi = \varphi(A)\xi$, $\forall A \in E_t$, $\forall \xi \in L^2(\mathcal{M})$, i.e. $RA = \varphi(A)$, for $A \in E_t$. In particular $R \cdot E_t \subset \mathcal{M}'$. If $(U_n(t))$ is a sequence of isometries in E_t such that $\sum\limits_{n=1}^{\infty} U_n(t)U_n(t)^* = 1$ (cf. Proposition 3.2), then $R = \sum\limits_{n=1}^{\infty} RU_n(t)U_n(t)^*$ and since $RU_n(t) \in \mathcal{M}'$, R belongs to the σ-weak closure of the set $\{AT^* : A \in \mathcal{M}', T \in E_t\}$. It follows from Proposition 3.1 that this set is $E_t^* = \{T^* : t \in E_t\}$, which is a σ-weakly closed subspace of $\mathcal{B}(L^2(\mathcal{M}))$. Thus $R \in E_t^*$, and $S = R^* \in E_t$. Moreover for $T \in E_t$,

$$\varphi(T) = RT = S^*T = \langle S, T \rangle$$

and this proves that E_t is self dual.

Recall that a self dual Hilbert module over a von Neumann algebra is naturally a dual Banach space and therefore equipped with a w*-topology, (cf. [19] Proposition 3.8). A bounded net (T_α) in E_t converges with respect to the w*-topology to $T \in E_t$, if and only if $T_\alpha^*S \to T^*S$ σ-weakly, for all $S \in E_t$. Since $\{S\xi : S \in E_t, \xi \in L^2(\mathcal{M})\}$ is dense in $L^2(\mathcal{M})$, this happens if and only if $T_\alpha^* \to T^*$ σ-weakly, if and only if $T_\alpha \to T$ σ-weakly. This completes the proof of the theorem. ∎

Let $\mathcal{B}(E_t)$ be the set of all bounded module maps (i.e respecting the right action of $\mathcal{M}'$) from E_t to itself. $\mathcal{B}(E_t)$ is a C^*-algebra, and since E_t is a self-dual module over the von Neumann algebra $\mathcal{M}'$, $\mathcal{B}(E_t)$ is a dual Banach space and therefore a von Neumann algebra, ([19], Proposition 3.10). Note that if A is an operator in $\alpha_t(\mathcal{M})'$, then the map $\varphi_A : E_t \to E_t$, $\varphi_A(T) = AT$ is an element of $\mathcal{B}(E_t)$. We prove below that every element of $\mathcal{B}(E_t)$ arises in this way.

PROPOSITION 3.4. *The map $A \to \varphi_A$ is a $*$-isomorphism between the von Neumann algebras $\alpha_t(\mathcal{M})'$ and $\mathcal{B}(E_t)$.*

Proof. It is evident that $A \to \varphi_A$ is a homomorphism, and since

$$\langle \varphi_A(S), T\rangle = \langle AS, T\rangle = (AS)^*T = S^*A^*T = \langle S, A^*T\rangle = \langle S, \varphi_{A^*}(T)\rangle$$

for all $S, T \in E_t$, we get that $(\varphi_A)^* = \varphi_{A^*}$. If $\varphi_A = 0$, then by considering a sequence $(U_n(t))$ of isometries as in Proposition 3.2, we get that $A = \sum AU_n(t)U_n(t)^* = 0$, since $AU_n(t) = \varphi_A(U_n(t)) = 0$ for all n. Thus $A \to \varphi_A$ is injective. Suppose finally that $\varphi \in \mathcal{B}(E_t)$, and assume without loss of generality that $\|\varphi\| \leqslant 1$. Then $\langle \varphi(T), \varphi(T)\rangle \leqslant \langle T, T\rangle$, $\forall T \in E_t$, and as in the proof of Theorem 3.3, the formula

$$A\Big(\sum_{i=1}^{n} T_i\xi_i\Big) = \sum_{i=1}^{n} \varphi(T_i)\xi_i, \quad T_i \in E_t,\ \xi_i \in L^2(\mathcal{M}),\ i = 1, \ldots, n,\ n \in \mathbb{N}$$

defines a bounded operator on $L^2(\mathcal{M})$, which satisfies $AT = \varphi(T)$, $\forall T \in E_t$. This means, in particular, that $A \cdot E_t \subset E_t$. If $(U_n(t))$ is a sequence of isometries as in the previous paragraph, we have that $A = \sum AU_n(t)U_n(t)^*$ and since $AU_n(t) \in E_t$, A belongs to the σ-weak closure of the linear span of the set $\{ST^* : S, T \in E_t\}$, which is equal to $\alpha_t(\mathcal{M})'$ by Proposition 3.1. Hence $\varphi = \varphi_A$ with $A \in \alpha_t(\mathcal{M})'$, and the proof is complete. ∎

Through Proposition 3.4 we obtain a faithful $*$-homomorphism of $\mathcal{M}'$ into $\mathcal{B}(E_t)$ which we shall denote by φ_t. One can use this homomorphism to define the tensor product $E_s \otimes_{\varphi_t} E_t$ for $s, t > 0$. This is the Hilbert $\mathcal{M}'$-module obtained by completing the algebraic tensor product $E_s \otimes_{\mathcal{M}'} E_t$ with respect to the $\mathcal{M}'$-valued positive semidefinite inner product which is linear in the second variable, conjugate linear in the first, and satisfies

$$\langle S_1 \otimes T_1, S_2 \otimes T_2\rangle = \langle T_1, \varphi_t\left(\langle S_1, S_2\rangle\right)(T_2)\rangle = \langle T_1, \langle S_1, S_2\rangle T_2\rangle,$$

$S_1, S_2 \in E_s$, $T_1, T_2 \in E_t$, where the second equality follows from the fact that the left action of $\mathcal{M}'$ on E_t induced by φ_t is simply left multiplication. For more details about the definition of the tensor product we refer to [25]. The semigroup property of α has as a consequence the following:

THEOREM 3.5. *The map $m : E_s \times E_t \to E_{s+t}$ defined by $m(S, T) = ST$ induces an isometry of the Hilbert $\mathcal{M}'$-module $E_s \otimes_{\varphi_t} E_t$ onto a* w*-dense Hilbert submodule of E_{s+t}.*

Proof. It is clear that $ST \in E_{s+t}$ when $S \in E_s$ and $T \in E_t$, that m is bilinear, and that $m(SA, T) = m(S, AT)$ for $S \in E_s$, $T \in E_t$ and $A \in \mathcal{M}'$. Thus m induces a map, denoted again by m, from the algebraic tensor product $E_s \otimes_{\mathcal{M}'} E_t$ into E_{s+t}, which respects the right actions of $\mathcal{M}'$. Moreover

$$\begin{aligned}\langle S_1 \otimes T_1, S_2 \otimes T_2\rangle &= \langle T_1, \langle S_1, S_2\rangle T_2\rangle = \langle T_1, S_1^*S_2T_2\rangle = T_1^*S_1^*S_2T_2 \\ &= \langle S_1T_1, S_2T_2\rangle = \langle m(S_1 \otimes T_1), m(S_2 \otimes T_2)\rangle.\end{aligned}$$

Thus m extends to an isometry from $E_s \otimes_{\varphi_t} E_t$ into E_{s+t}. The range of this isometry is a Hilbert submodule of E_{s+t}. To prove that it is w*-dense consider sequences $(U_n(t)) \subset E_t$, $(U_n(s)) \subset E_s$ as in Proposition 3.2. If $A \in E_{s+t}$ then

$$A = \sum_{n,m} U_n(s)U_m(t)U_m(t)^*U_n(s)^*A$$

and because $U_m(t)U_m(t)^*U_n(s)^*A \in E_t \cdot E_{t+s}^* \cdot E_{t+s} = E_t \cdot \mathcal{M}' = E_t$ by Proposition 3.1, we see that A is the σ-weak limit of linear combinations of elements of the form ST, $S \in E_s$, $T \in E_t$. This concludes the proof. ∎

Using Theorems 3.3 and 3.5 we will now show that the set $\mathcal{E}_\alpha = \{(T,t) : T \in E_t, t > 0\}$ has a structure which is analogous to Arveson's product systems ([3]) and which characterizes the E_0-semigroup α, up to cocycle conjugacy.

DEFINITION 3.6. Let $\mathcal{N}$ be a factor of type II_1 with separable predual. A product system of Hilbert modules over $\mathcal{N}$ is a standard Borel space E, together with a measurable map $p : E \to (0,\infty)$, satisfying the following properties:

(i) For every $t > 0$, $E_t = p^{-1}(\{t\})$ is a full self-dual (right) Hilbert $\mathcal{N}$-module. In addition, E_t has the structure of a left $\mathcal{N}$-module, so that

$$\langle ax, y\rangle_t = \langle x, a^*y\rangle_t, \quad \forall x, y \in E_t,\, \forall a \in \mathcal{N}$$

where by $\langle\,\cdot\,,\cdot\,\rangle_t$ we denote the $\mathcal{N}$-valued inner product in E_t.

(ii) There is a measurable map $(x,y) \to xy$ from $E \times E$ into E, called multiplication, with the following properties:

(a) $p(xy) = p(x) + p(y)$

(b) $\langle x_1y_1, x_2y_2\rangle_{s+t} = \langle y_1, \langle x_1, x_2\rangle_s y_2\rangle_t$

whenever $x_1, x_2 \in E_s$, $y_1, y_2 \in E_t$, $s, t > 0$. Moreover the linear span of elements of the form xy, $x \in E_s$, $y \in E_t$, is w*-dense in E_{s+t}.

(iii) There is a full self-dual Hilbert $\mathcal{N}$-module Δ, which is weakly countably generated (in the sense that there is a sequence $(\delta_n) \subset \Delta$ such that $\langle \delta_n, \delta_m\rangle = \delta_{nm}1$ and the set $\left\{ \sum_{n=0}^{k} \delta_n x_n : x_n \in \mathcal{N}, k \in \mathbb{N}\right\}$ is w*-dense in Δ), and a measurable map $\theta : E \to \Delta \times (0,\infty)$ such that θ restricted to each fibre $E_t = p^{-1}(\{t\})$ is an isomorphism of Hilbert modules between E_t and $\Delta \times \{t\}$.

REMARK 3.7. (i) The condition that $\langle ax, y\rangle_t = \langle x, a^*y\rangle_t$ implies that the left action of $\mathcal{N}$ on E_t gives rise to a $*$-homomorphism of $\mathcal{N}$ into the von Neumann algebra $\mathcal{B}(E_t)$ of all bounded module maps from E_t into itself, namely

$$\varphi_t : \mathcal{N} \to \mathcal{B}(E_t), \quad \varphi_t(a)(x) = ax, \quad x \in E_t,\, a \in \mathcal{N}.$$

(ii) Condition (ii) implies that there is an isometric module map from the Hilbert $\mathcal{N}$-module $E_s \otimes_{\varphi_t} E_t$ onto a w*-dense Hilbert submodule of E_{s+t}.

Given an E_0-semigroup α of a II_1-factor $\mathcal{M}$ acting standardly on $L^2(\mathcal{M})$, we associate with it the set

$$\mathcal{E}_\alpha = \{(T,t) : T \in E_t, t > 0\}$$

where $E_t = \{T \in \mathcal{B}(L^2(\mathcal{M})) : \alpha_t(x)T = Tx, \forall x \in \mathcal{M}\}$. We will show that $\mathcal{E}_\alpha$ together with the map $p : \mathcal{E}_\alpha \to (0,\infty)$, $p(T,t) = t$, and with multiplication given by $(T,t)(S,s) = (TS, t+s)$ is a product system of Hilbert modules over the commutant $\mathcal{M}'$ of $\mathcal{M}$ in $\mathcal{B}(L^2(\mathcal{M}))$. Theorem 3.3 and Proposition 3.4, show that condition (i) of Definition 3.1 is satisfied and Theorem 3.3 shows that condition (ii) is satisfied. We need to show that $\mathcal{E}_\alpha$ is a standard Borel space, and that $\mathcal{E}_\alpha$ is locally trivial, in the sense that it satisfies condition (iii) of the definition.

To show that $\mathcal{E}_\alpha$ is a standard Borel space, it is enough to show that it is a Borel subset of $\mathcal{B}(L^2(\mathcal{M})) \times (0,\infty)$, where $\mathcal{B}(L^2(\mathcal{M}))$ has the Borel structure generated by the weak operator topology. Setting $B_n = \{(T,t) \in \mathcal{E}_\alpha : \|T\| \leqslant n\}$ we obviously have $\mathcal{E}_\alpha = \bigcup_{n=1}^{\infty} B_n$, and it is not hard to show that each B_n is closed in $\mathcal{B}(L^2(\mathcal{M})) \times (0,\infty)$ (cf. [3], Proposition 2.2). This shows that $\mathcal{E}_\alpha$ is Borel.

In order to prove the local triviality condition, we will need the following lemma:

LEMMA 3.8. *Suppose $\mathcal{N}$ is a II_1-factor with separable predual, $\mathcal{K},\mathcal{H}$ are separable Hilbert spaces, π_0 is a representation of $\mathcal{N}$ on $\mathcal{K}$, and for $t>0$, (π_t) is a family of representations of $\mathcal{N}$ on $\mathcal{H}$ with the following properties:*

(i) *for every ξ,η in $\mathcal{H}$ and every x in $\mathcal{N}$, the map $t \to \langle \pi_t(x)\xi,\eta\rangle$ is continuous;*

(ii) *for each $t>0$ there is a unitary operator $V_t : \mathcal{H} \to \mathcal{K}$ such that $V_t\pi_t(x)V_t^* = \pi_0(x)$, $\forall x \in \mathcal{N}$.*

Then there is a family $(U_t)_{t>0}$ of unitary operators from $\mathcal{H}$ to $\mathcal{K}$ such that $U_t\pi_t(x)U_t^* = \pi_0(x)$, $\forall x \in \mathcal{N}$ and for every $\xi,\eta \in \mathcal{H}$, the map $t \to \langle U_t\xi,\eta\rangle$ is Borel measurable.

Proof. Let $\mathcal{U}(\mathcal{H},\mathcal{K})$ be the set of all unitary operators from $\mathcal{H}$ to $\mathcal{K}$, equipped with the strong operator toplology. The separability condition implies that $\mathcal{U}(\mathcal{H},\mathcal{K})$ is a Polish space. There is a continuous map, $U \to \operatorname{Ad} U^* \circ \pi_0$ from $\mathcal{U}(\mathcal{H},\mathcal{K})$ into the set $\mathrm{Rep}_{\mathcal{H}}(\mathcal{N})$ of all representations of $\mathcal{N}$ on $\mathcal{H}$, with the topology of pointwise strong convergence. The range of this map is the set X of all representations of $\mathcal{N}$ on $\mathcal{H}$ which are unitarily equivalent to π_0.

On $\mathcal{U}(\mathcal{H},\mathcal{K})$ there is an action of the unitary group of $\pi_0(\mathcal{N})'$ by homeomorphisms (namely the action by left multiplication). It is clear that the orbits of that action are closed and therefore by a theorem of Dixmier, ([11]), if $\mathcal{U}(\mathcal{H},\mathcal{K})/\sim$ is the quotient Borel space by the equivalence relation defined by the action, then the canonical map $q : \mathcal{U}(\mathcal{H},\mathcal{K}) \to \mathcal{U}(\mathcal{H},\mathcal{K})/\sim$ has a Borel cross section. Note that two elements U,V of $\mathcal{U}(\mathcal{H},\mathcal{K})$ are mapped on the same element of X if and only if $U = WV$ for some unitary W in $\pi_0(\mathcal{N})'$ and, consequently, there is a $1-1$ Borel map from X to $\mathcal{U}(\mathcal{H},\mathcal{K})/\sim$. It follows that the map $U \to \operatorname{Ad} U^* \circ \pi_0$ from $\mathcal{U}(\mathcal{H},\mathcal{K})$ to X, has a Borel cross section. Since the map $t \to \pi_t$ is continuous and hence measurable, there is a measurable map $t \to U_t$ from $(0,\infty)$ to $\mathcal{U}(\mathcal{H},\mathcal{K})$, so that $\operatorname{Ad} U_t \circ \pi_t = \pi_0$. This completes the proof. ∎

COROLLARY 3.9. *Let α be an E_0-semigroup of the II_1-factor $\mathcal{M}$ acting standardly on $\mathcal{H} = L^2(\mathcal{M})$, and fix $t_0 > 0$. There is a measurable family $(U_t)_{t>0}$ of unitary elements in $\mathcal{B}(\mathcal{H})$ such that*

$$\alpha_t(x) = U_t\alpha_{t_0}(x)U_t^*, \quad \forall t>0, \forall x \in \mathcal{M}.$$

Proof. We can view the E_0-semigroup α as a family $(\alpha_t)_{t>0}$ of representations of the II_1-factor $\mathcal{M}$ on $L^2(\mathcal{M})$, continuous with respect to the topology of pointwise strong convergence. Since all these representations have infinite coupling constant, each one of them is unitarily equivalent to the amplification of the standard action of $\mathcal{M}$, on the space $L^2(\mathcal{M}) \otimes \mathcal{K}$, where $\dim \mathcal{K} = \infty$. We denote this representation by π_0, and apply Lemma 3.8 to obtain a measurable family $(V_t)_{t>0}$ of unitary operators from $L^2(\mathcal{M})$ to $L^2(\mathcal{M}) \otimes \mathcal{K}$ such that

$$\alpha_t(x) = V_t^* \pi_0(x) V_t \quad t > 0, x \in \mathcal{M}.$$

Letting $U_t = V_t^* V_{t_0}$ we obtain a measurable family $(U_t)_{t>0}$ of unitaries, such that

$$\alpha_t(x) = U_t \alpha_{t_0}(x) U_t^*. \quad ∎$$

As a consequence of the previous corollary we can define a map $\theta : \mathcal{E}_\alpha \to E_{t_0} \times (0, \infty)$ by $\theta(T, t) = (U_t^* T, t)$. For $x \in \mathcal{M}$ and $T \in E_t$ we have, $\alpha_{t_0}(x) U_t^* T = U_t \alpha_t(x) T = U_t T x$ and thus θ is well defined and a Borel isomorphism by the measurability of $(U_t)_{t>0}$. Moreover the restriction of θ on $p^{-1}(\{t\})$ is an isometric module map onto $E_{t_0} \times \{t\}$. This shows that $\mathcal{E}_\alpha$ satisfies condition (iii) of Definition 3.1 and therefore we have finally completed the proof of the following:

THEOREM 3.10. *If α is an E_0-semigroup of the II_1-factor $\mathcal{M}$ acting standardly on $L^2(\mathcal{M})$, then the set*

$$\mathcal{E}_\alpha = \{(T, t) \in \mathcal{B}(L^2(\mathcal{M})) \times (0, \infty) : \alpha_t(x) T = T x, \forall x \in \mathcal{M}\}$$

is a product system of Hilbert modules over the commutant $\mathcal{M}'$ of $\mathcal{M}$.

We next prove that $\mathcal{E}_\alpha$ is invariant under cocycle conjugacy. First, we make a few comments concerning sequences of isometries $(U_n(t)) \subset E_\alpha(t)$, with the property $\sum_{n=1}^{\infty} U_n(t) U_n(t)^* = 1$. We call such a sequence, a weak orthonormal basis for the Hilbert $\mathcal{M}'$-module $E_\alpha(t)$. The existence of such a basis is the content of Proposition 3.2. Moreover, one can choose for $t > 0$ orthonormal bases $(U_n(t))$ in $E_\alpha(t)$, in such a way that for each $n \in \mathbb{N}$ and for all $\xi, \eta \in L^2(\mathcal{M})$, the function $t \to \langle U_n(t)\xi, \eta \rangle$ is measurable. This follows immediately from the local triviality condition that $\mathcal{E}_\alpha$ satisfies, since one can choose a weak orthonormal basis for $E_\alpha(t_0)$ (for some fixed $t_0 > 0$), and use the isomorphism θ to obtain bases in each $E_\alpha(t)$, in a measurable way (cf. the paragraph before Theorem 3.10). Note finally that, if $(U_n(t))$ is a weak orthonormal basis for $E_\alpha(t)$ and V is a unitary element in $\mathcal{M}'$ then the family $(W_n(t))$ where $W_n(t) = V U_n(t)$ is also a weak orthonormal basis. Indeed, $W_n(t) \in E_t$, $W_n(t)^* W_n(t) = U_n(t)^* V^* V U_n(t) = 1$ and $\sum W_n(t) W_n(t)^* = \sum V U_n(t) U_n(t)^* V^* = 1$.

DEFINITION 3.11. Two product systems E and F of Hilbert modules over the II_1 factors $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, if there is a $*$-isomorphism $\theta : \mathcal{M} \to \mathcal{N}$ and a measurable map $\psi : E \to F$ such that:

(i) $\psi(xy) = \psi(x)\psi(y)$, for all $x, y \in E$;

(ii) the restriction of ψ to the fibre E_t is an isometric bimodule map onto the fibre F_t, i.e. it satisfies $\psi(axb) = \theta(a)\psi(x)\theta(b)$ and $\langle \psi(x), \psi(y) \rangle = \theta(\langle x, y \rangle)$ $\forall x, y \in E_t$.

THEOREM 3.12. *Let α, β be two E_0-semigroups of the II_1-factor $\mathcal{M}$. Then α and β are cocycle conjugate, if and only if the product systems of Hilbert modules $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are isomorphic.*

Proof. That the isomorphism class of the product system of α is invariant under conjugacy is straightforward and we can safely ommit the details.

If β is a cocycle perturbation of α and $(U_t)_{t>0}$ is a continuous unitary family in $\mathcal{M}$ such that $U_{t+s} = U_t\alpha_t(U_s)$ and $\beta_t(x) = U_t\alpha_t(x)U_t^*$, $\forall x \in \mathcal{M}$, then we define $\psi : \mathcal{E}_\alpha \to \mathcal{E}_\beta$ by

$$\psi(T,t) = (U_tT,t), \quad T \in E_\alpha(t).$$

It is straightforward that ψ is an isomorphism.

Conversely, suppose that $\psi : \mathcal{E}_\alpha \to \mathcal{E}_\beta$ is an isomorphism. We follow the proof of [5], Theorem 3.18. Because $\mathcal{E}_\alpha$ satisfies the local triviality condition (iii) of Definition 3.6, we can choose families of weak orthonormal bases for the fibres $E_\alpha(t)$, $t > 0$, i.e. families $(V_n(t))_{n\in\mathbb{N}}$ of isometries in $E_\alpha(t)$ such that $\sum_{n=1}^{\infty} V_n(t)V_n(t)^* = 1$, so that for each $n \in \mathbb{N}$ and each $\xi, \eta \in L^2(\mathcal{M})$ the map $t \to \langle V_n(t)\xi, \eta\rangle$ is measurable. We define U_t by the formula

$$U_t = \sum_{n=1}^{\infty} \psi(V_n(t))V_n(t)^*.$$

As in [5], one shows that $\{U_t : t \geqslant 0\}$ is a strongly continuous unitary cocycle and that $\beta_t = \operatorname{Ad} U_t \circ \alpha_t$.

It only remains to show that U_t belongs to $\mathcal{M}$ for $t > 0$. Let U be a unitary element in $\mathcal{M}'$ and define for $n \in \mathbb{N}$, $t > 0$,

$$W_n(t) = UV_n(t).$$

By the remarks preceding Definition 3.11, it follows that for each $t > 0$ the family $(W_n(t))_{n\in\mathbb{N}}$ is a weak orthonormal basis for $E_\alpha(t)$. Moreover we have, just as in the case of $(V_n(t))_{n\in\mathbb{N}}$, that

$$\psi(T) = \Big(\sum_{n=1}^{\infty} \psi(W_n(t))W_n(t)^*\Big)T.$$

For $m \in \mathbb{N}$ we then have

$$\Big(\sum_{n=1}^{\infty} \psi(V_n(t))V_n(t)^*\Big)V_m(t) = \psi(V_m(t)) = \Big(\sum_{n=1}^{\infty} \psi(W_n(t))W_n(t)^*\Big)V_m(t).$$

Multiplying on the right by $V_m(t)^*$ and summing over all m we obtain

$$U_t = \sum_{n=1}^{\infty} \psi(V_n(t))V_n(t)^* = \sum_{n=1}^{\infty} \psi(W_n(t))W_n(t)^*.$$

Using this we show below that U_t commutes with U:

$$\begin{aligned} UU_t &= U\Big(\sum_{n=1}^{\infty} \psi(V_n(t))V_n(t)^*\Big) = \sum_{n=1}^{\infty} \psi(UV_n(t))V_n(t)^* \\ &= \sum_{n=1}^{\infty} \psi(UV_n(t))(UV_n(t))^*U = \sum_{n=1}^{\infty} \psi(W_n(t))W_n(t)^*U = U_tU. \end{aligned}$$

Since U was an arbitrary unitary element of $\mathcal{M}'$, we conclude that $U_t \in \mathcal{M}$. This completes the proof of the theorem. ∎

Among isomorphism invariants of product systems, we specifically mention the automorphism group which we believe can be computed in specific cases. For the case of the Clifford flow of index n we know that the automorphism group is a certain subgroup of the automorphism group $G_{\mathcal{H}}$ of the exponential product system $E_{\mathcal{H}}$ considered in [3], but we have been, so far, unable to compute it explicitly.

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