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UNIFORM CLOSURE AND DUAL BANACH ALGEBRAS

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Abstract. We give a characterization of the "uniform closure" of the conjugate of a C^*-algebra. Some applications in harmonic analysis are given.

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0. INTRODUCTION

Dunkl and Ramirez have studied the following problem in [8]: Let G be a locally compact Abelian group, which bounded continuous functions on G could be uniformly approximated by Fourier-Stieltjes transforms of bounded Borel measures on the Pontryagin dual $\widehat{G}$ of G? Their characterization is based on a comparison of four different topologies on the unit ball of $M(\widehat{G})$ ([8], 3.12). M.L. Bami has used the same idea in [12] to get the same characterization for commutative foundation $*$-semigroups (Bami's result for the discrete case is proved in 5.1.6 of [9]). The crucial role in both proofs is played by the *duality theory* between some algebras of functions on the underlying algebraic objects (Abelian groups or commutative foundation $*$-semigroups). The duality theory in the group case is quite well known, the group algebra $L^1(G)$ is "dual" to the Fourier algebra $A(G)$, which in the Abelian case is simply the set of all Fourier transforms of elements of $L^1(G)$. In the semigroup case, we do not have a natural candidate for the semigroup algebra in general. However, if S is a foundation semigroup, an analogue of the group algebra is introduced and studied by A.C. Baker and J.W. Baker (see [10]).

The main objective of this paper is to put these examples in a general framework. Fortunately, there is such a framework for duality of Banach algebras, introduced by M.E. Walter ([19]). The examples in [19] suggest that the author was motivated by the duality theory of topological groups and wanted to set up a framework which could accommodate some more general algebraic structures, such as (a class of) topological groupoids. We want to use his setup to prove the analogue of the Dunkl-Ramirez theorem in general. The main result of this paper is the following: If two Banach algebras A and B are dual and $C^*(A)$ and $C^*(B)$ are the corresponding C^*-envelopes, then under some moderate conditions (satisfied by many interesting examples) one can characterize the closure of the image of $C^*(A)^*$ in the multiplier norm of $M(C^*(B))$. When G is an Abelian topological group and $A = L^1(G)$ and $B = A(G)$, then

$$C^*(A)^* = C^*(G)^* = B(G) = \{\widehat{\mu} : \mu \in M(\widehat{G})\},$$

and $M(C^*(B)) = M(C_0(G)) = C_{\mathrm{b}}(G)$, and we get the Dunkl-Ramirez theorem. Our proof follows that of [8], and we only have to make the right interpretation of that technique in our general setup. We then apply our theorem to prove versions of Dunkl-Ramirez theorem for the cases where G is not Abelian, or it is another algebraic structure, like a semigroup.

The paper is organized as follows. In the first section we introduce the four topologies considered by Dunkl and Ramirez on the unit ball of an arbitrary C^*-algebra. Then we introduce dual algebras of Martin Walter in Section 2, and prove our theorem and apply it to some algebraic structures in Section 3.

1. FOUR TOPOLOGIES ON THE UNIT BALL OF A C^*-ALGEBRA

Let A be a C^*-algebra and $P(A)$, $S(A)$, A_1^*, and A^* denote the pure state space, state space, closed conjugate unit ball, and the conjugate of A, respectively, all equipped with the w*-topology (the Banach space A^* is usually called the (linear) dual of A, but we prefer to call it the conjugate space, so that we save the term dual for Walter's definition). Let A_1 denote the closed unit ball of A. Following [12], we consider the following four topologies on A_1:

(1) (w) $a_i \to 0$ if and only if $\langle a_i, f\rangle \to 0$, $f \in A^*$,

(2) (wo) $a_i \to 0$ if and only if $\langle a_i x, f\rangle \to 0$, $x \in A, f \in A^*$,

(3) (so) $a_i \to 0$ if and only if $\|a_i x\| \to 0$, $x \in A$,

(4) (uc) $a_i \to 0$ if and only if $a_i \to 0$, uniformly on w*-compact subsets of $S(A)$.

Note that (so) is just the restriction of the strict topology of the multiplier algebra $M(A)$ to A_1. Also note that if in (4) one requires the uniform convergence on w*-compact subsets of A_1^* (instead of $S(A)$), one gets nothing but the norm topology (Banach-Alaoglu).

LEMMA 1.1. (Akemann-Glimm) *Let H be a Hilbert space and $S, T \in B(H)$. Let $g : \mathbb{R} \to \mathbb{R}$ be a non negative Borel function and $0 < \theta < 1$. Assume moreover that:*

(i) $0 \leqslant T \leqslant 1$, $S = S^*$, *and* $S \geqslant T$;

(ii) $g \geqslant 1$ *on* $[\theta, +\infty)$;

(iii) $\langle T\zeta, \zeta\rangle \geqslant 1 - \theta$, *for some* $\zeta \in H$.

Then $\langle g(S)\zeta, \zeta\rangle \geqslant 1 - 4\sqrt{\theta}$.

Proof. See Lemma 11.4.4 of [5]. ∎

PROPOSITION 1.2. *Topologies* w, wo, *and* so *coincide on* A_1 *and they are stronger than* uc.

Proof. (wo⊂w). Given $f \in A^*$ and $x \in A$, consider the Arens product $x \cdot f \in A^*$ defined by $x \cdot f(a) = f(ax)$, $a \in A$. If $\{a_i\} \subset A_1$ and $a_i \to 0$ (w) then $\langle a_i x, f\rangle = \langle a_i, x \cdot f\rangle \to 0$, i.e. $a_i \to 0$ (wo).

(w⊂wo). By the Cohen Factorization Theorem [6], we have $A^* = A \cdot A^* = \{x \cdot f : x \in A, f \in A^*\}$. Now if $\{a_i\} \subset A_1$ and $a_i \to 0$ (wo), then given $g \in A^*$, choose $x \in A$ and $f \in A^*$ such that $g = x \cdot f$. Then $\langle a_i, g\rangle = \langle a_i x, f\rangle \to 0$, i.e. $a_i \to 0$ (w).

(wo⊂so). Trivial.

(so⊂wo). We adapt the proof of Theorem 1 of [12]. Given $f \in A^*$, we need only to show that if f restricted to A_1 is so-continuous, then it is also wo-continuous. To this end, assume the so-continuity and note that each $a \in A$ can be associated with the (bounded) linear operator on A, taking $x \in A$ to ax. If $N = \ker(f)$, then by convexity of A_1, we have $(N \cap A_1)^{-\mathrm{so}} \cap A_1 = N \cap A_1$ (Theorem 13.5 of [11]). Hence $(N \cap A_1)^{-\mathrm{wo}} \cap A_1 = N \cap A_1$ (Corollary 5 of [7]), and so f is wo-continuous.

(uc⊂wo). We use an idea of [1]. Take a w*-compact subset K of $S(A)$ and let $f \in K$. Then, given $\theta > 0$ there exists $a_f \in A$ such that $0 \leqslant a_f \leqslant 1$ and $f(a_f) > 1 - \frac{\theta}{2}$ (see the proof of Lemma 4.5 in [1]). Take the w*-neighbourhood $V_f = N(f, a_f) = \{g \in S(A) : |g(a_f) - f(a_f)| < \frac{\theta}{2}\}$ of f in $S(A)$. Then given $g \in V_f$ we have $g(a_f) \geqslant f(a_f) - \frac{\theta}{2} > 1 - \theta$. Cover K by $\{V_f\}_{f \in K}$ and use w*-compactness of K to get $n \geqslant 1$ and $f_1, f_2, \ldots, f_n \in K$ such that $K \subset V_{f_1} \cup \cdots \cup V_{f_n}$. Put $a_i = a_{f_i}$, $i = 1, \ldots, n$ and $a = a_1 + \cdots + a_n$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function which is 0 on $(-\infty, 0]$, 1 on $[\theta, +\infty)$, and linear on $[0, \theta]$. Put $b = g(a)$, then $0 \leqslant b \leqslant 1$. Now given $f \in K$ we have $f \in V_{f_i}$, for some i, say $i = 1$. Then $f(a_1) \geqslant 1 - \theta$. On the other hand, there is a cyclic representation $\{\pi, H, \zeta\}$ of A such that $f(x) = \langle \pi(x)\zeta, \zeta\rangle$, $x \in A$. Take $T = \pi(a_1)$ and $S = \pi(a)$, then clearly $0 \leqslant T \leqslant 1$ and $S \geqslant T$. Hence by above lemma,

$$\begin{aligned} f(b) = \langle \pi(b)\zeta, \zeta\rangle &= \langle \pi(g(a))\zeta, \zeta\rangle \\ &= \langle g(\pi(a))\zeta, \zeta\rangle = \langle g(S)\zeta, \zeta\rangle \\ &\geqslant 1 - 4\sqrt{\theta}. \end{aligned}$$

Now consider a net $\{a_i\} \subset A_1$ such that $a_i \to 0$ (so), then inside $M(A)$ we can write $a_i = a_i b + a_i(1 - b)$, $(1 - b)^2 \leqslant (1 - b)$, and $f(1) = \|f\| = 1$. Hence, for each i

$$\begin{aligned} |f(a_i)| \leqslant |f(a_i b)| + |f(a_i(1 - b))| &\leqslant \|a_i b\| + f(a_i a_i^*)^{1/2} f((1 - b)^2)^{1/2} \\ &\leqslant \|a_i b\| + \|a_i a_i^*\|^{1/2} f(1 - b)^{1/2} \\ &\leqslant \|a_i b\| + (1 - (1 - 4\sqrt{\theta}))^{1/2} \\ &= \|a_i b\| + 2\sqrt[4]{\theta}. \end{aligned}$$

Hence $\sup_{f \in K} |f(a_i)| \leqslant \|a_i b\| + 2\sqrt[4]{\theta}$, and so $a_i \to 0$ uniformly on K, as required. ∎

COROLLARY 1.3. *If A is a C^*-algebra, A_1 is the unit ball of A, and $f : A \to \mathbb{C}$ is continuous with respect to* (uc)*, then f is continuous with respect to* (w).

2. DUAL ALGEBRAS

Notation 2.1. ([19]) If A is a C^*-algebra, $\mathfrak{L}(A)$, $\mathfrak{P}(A)$, and $\mathfrak{D}(A)$ denote the collection of all bounded, completely positive, and completely bounded linear maps of A into A, respectively. $\mathfrak{D}(A)$ is called the *dual algebra* of A.

It can be shown that $\mathfrak{D}(A)$ is a Banach algebra with conjugation (this is the same as involution, except that it preserves the order of multiplication), and if $\mathfrak{B}(A)$ is the closed linear span of $\mathfrak{P}(A)$ in $\mathfrak{D}(A)$ (with respect to the completely bounded norm) then $\mathfrak{B}(A) \subset \mathfrak{D}(A) \subset \mathfrak{L}(A)$ ([19]).

Definition 2.2. ([19]) Let A and B be Banach algebras with involution and conjugation such that there are C^*-algebras $C^*(A)$ and $C^*(B)$ satisfying the following conditions:

(i) There are Banach algebra homomorphisms $i_A : A \to C^*(A)$ and $i_B : B \to C^*(B)$ which are one-one, onto a dense subalgebra, and preserve involution.

(ii) There are norm decreasing Banach algebra isomorphisms $j_A : A \to \mathfrak{D}(C^*(B))$ and $j_B : B \to \mathfrak{D}(C^*(A))$ which preserve conjugation.

Then A and B are called *dual algebras.* If the involutions and conjugations of both algebras are isometric, the duality is called *semirigid.* If moreover both j_A and j_B are isometric, the duality is called *rigid.*

Definition 2.3. Consider the dual algebras A and B. The duality is called *complete* if there are norm decreasing linear injections $k_A : C^*(A)^* \to M(C^*(B))$ and $k_B : C^*(B)^* \to M(C^*(A))$. Here M stands for the multiplier algebra. The duality is called *strongly complete* if moreover there are norm decreasing linear injections $m_A : M(C^*(B)) \to A^*$ and $m_B : M(C^*(A)) \to B^*$ such that $m_A \circ k_A = i_A^*$ and $m_B \circ k_B = i_B^*$.

Example 2.4. If G is a locally compact group then the Fourier algebra $A(G)$ and the group algebra $L^1(G)$ are dual. Here we take $C^*(A(G)) = C_0(G)$ and $C^*(L^1(G)) = C^*(G)$. The duality is rigid ([19]) and strongly complete ([16]).

Example 2.5. If A is $M_n(\mathbb{C})$ with Schur product and trace norm and B is $M_n(\mathbb{C})$ with usual matrix product and L^1 norm, then A and B are dual and duality is rigid ([19]).

Example 2.6. If A is the C^*-algebra of trace class operators on ℓ^2 and B is the subalgebra of $M_\infty(\mathbb{C})$ consisting of countably infinite matrices with finite L^1 norm, then A and B are dual and duality is semirigid ([19]).

Definition 2.7. Consider the dual algebras A and B. The duality is called *amenable* if there are isometric isomorphisms $l_A : C^*(A)^* \to M(B)$ and $l_B : C^*(B)^* \to M(A)$.

Example 2.8. The duality of Example 2.4 is amenable if and only if the locally compact group G is amenable ([14]).

Proposition 2.9. *Every amenable duality is complete.*

Proof. The Banach algebra homomorphism $i_B : B \to C^*(B)$ uniquely extends to one from $M(B)$ onto $M(C^*(B))$, still denoted by i_B. Put $k_A = i_B \circ l_A$. k_B is constructed similarly. ∎

Remark 2.10. Example 2.4 shows that the converse of above proposition is not true.

3. UNIFORM CLOSURE OF DUAL ALGEBRAS

Consider the dual algebras A and B. If the duality is strongly complete, then using the norm decreasing linear injection $k_A : C^*(A)^* \to M(C^*(B))$, one can identify $C^*(A)^*$ with a subspace of $M(C^*(B))$, where of course the norm of the latter (which is denoted by $\|\cdot\|_{\mathrm{u}}$) is weaker. In this section we want to calculate the closure of $k_A(C^*(A)^*)$ in $M(C^*(B))$, which we call the *uniform closure* of $C^*(A)^*$.

Theorem 3.1. *Consider the dual algebras A and B. If the duality is rigid and strongly complete, then the closure of $k_A(C^*(A)^*)$ in $M(C^*(B))$ consists exactly of those elements $b \in M(C^*(B))$ which satisfy the following property:*

If $\{a_n\}$ is any sequence in the unit ball A_1 of A such that $\langle a_n, i_A^*(f)\rangle \to 0$ for all $f \in C^*(A)^*$, then $\langle a_n, m_A(b)\rangle \to 0$.

Proof. Assume that b is in the uniform closure of $C^*(A)^*$ and $\{a_n\}$ is any sequence in the unit ball A_1 of A such that $\langle a_n, i_A^*(f)\rangle \to 0$ for all $f \in C^*(A)^*$. Let $\theta > 0$, and take $g \in C^*(A)^*$ such that $\|b - k_A(g)\|_{\mathrm{u}} < \theta$. Then by assumption, $\langle a_n, i_A^*(g)\rangle \to 0$. Therefore

$$\begin{aligned}\limsup_{n\to\infty} |\langle a_n, m_A(b)\rangle| &= \limsup_{n\to\infty} |\langle a_n, m_A(b - k_A(g))\rangle| \\ &\leqslant \limsup_{n\to\infty} \|b - k_A(g)\|_{\mathrm{u}} \cdot \|a_n\| < \theta.\end{aligned}$$

Hence $\langle a_n, m_A(b)\rangle \to 0$.

Conversely, suppose that $b \in M(C^*(B))$ but $b \notin (k_A(C^*(A)^*))^{-\|\cdot\|_{\mathrm{u}}}$. Then by closed graph theorem, $m_A(b)$ is not w-continuous on A_1, where $w = \sigma(A, C^*(A)^*)$. By Corollary 1.3, $m_A(b)$ is not uc-continuous on A_1, hence there is $\theta > 0$ such that for each norm bounded $K \subset C^*(A)^*$ and each $\delta > 0$, there is $a_{K,\delta} \in A_1$ such that

$$|\langle a_{K,\delta}, m_A(b)\rangle| \geqslant \theta, \quad |\langle a_{K,\delta}, i_A^*(f)\rangle| < \delta, \quad f \in K.$$

Fix w*-compact subset $K \subset C^*(A)^*$ and put $a_1 = a_{K,1}$. Then take

$$K_1 = \{f \in C^*(A)^* : |\langle a_1, i_A^*(f)\rangle| \geqslant 1\}$$

and put $a_2 = a_{K_1,1}$. Continuing this way, we put

$$K_n = \{f \in C^*(A)^* : |\langle a_i, i_A^*(f)\rangle| \geqslant 1/n, 1 \leqslant i \leqslant n\}$$

and $a_{n+1} = a_{K_{n,1/n}}$. Then $\langle a_n, f\rangle \to 0$ for all $f \in C^*(A)^*$ (for those f which belong to $\bigcup_{n\geqslant 1} K_n$ use the defining property of $a_{K,\delta}$'s and for others use the defining property of K_n's) but $|\langle a_n, m_A(b)\rangle| \geqslant \theta$, $n \geqslant 1$, and we are done. ∎

It is clear from the proof of the above theorem that we only need to assume a "one way duality" relation between two algebras. More precisely, it is enough that A and B satisfy the following definition.

Definition 3.2. Let A and B be Banach algebras with involution such that there are C^*-algebras $C^*(A)$ and $C^*(B)$ satisfying the following conditions:

(i) There are Banach algebra homomorphisms $i_A : A \to C^*(A)$ and $i_B : B \to C^*(B)$ which are one-one, onto a dense subalgebra, and preserve involution.

(ii) There is norm decreasing linear injection

$$k_A : C^*(A)^* \to M(C^*(B)),$$

where M stands for the multiplier algebra.

Then A is called *semidual* to B. In this case the concepts such as rigidity, (strong) completeness, and amenability are defined similarly.

Example 3.3. If S is a foundation topological $*$-semigroup whose $*$-representations separate the points of S, then the Fourier algebra $A(S)$ ([2]) is semidual to the semigroup algebra $M_a(S)$. The semiduality is rigid and strongly complete. Here we take $C^*(A(S)) = C_0(S)$ and $C^*(M_a(S)) = C^*(S)$ ([2]). This is in particular true for any (discrete) inverse semigroup (with $M_a(S)$ replaced by $\ell^1(S)$, see [3]).

Theorem 3.4. *Consider the involutive Banach (normed) algebras A and B. If A is semidual to B and the semiduality is rigid and strongly complete, then the closure of $k_A(C^*(A)^*)$ in $M(C^*(B))$ consists exactly of those elements $b \in M(C^*(B))$ which satisfy the following property:*

If $\{a_n\}$ is any sequence in the unit ball A_1 of A such that $\langle a_n, i_A^*(f)\rangle \to 0$ for all $f \in C^*(A)^*$, then $\langle a_n, m_A(b)\rangle \to 0$.

Corollary 3.5. *Let S be a foundation topological $*$-semigroup with identity whose $*$-representations separate the points of S, and let $B(S)$ denote the Fourier-Stieltjes algebra of S. Then for a function $f \in C_{\mathrm{b}}(S)$ the following are equivalent:*

(i) *$f \in B(S)^{-\|\cdot\|_\infty}$;*

(ii) *If $\{\mu_n\}$ is any sequence in the unit ball of $M_a(S)$ such that $\int_S g\,\mathrm{d}\mu_n \to 0$ as $n \to \infty$, for all $g \in P(S)$, then $\int_S f\,\mathrm{d}\mu_n \to 0$, as $n \to \infty$.*

Proof. See Example 3.3 and Theorem 3.4. ∎

As far as I know, this result is new even for locally compact groups (although $B(G)^{-\|\cdot\|_\infty}$ has been studied in other directions; see for instance [4]).

Corollary 3.6. *Let G be a topological group and m be a left Haar measure on G, and let $B(G)$ denote the Fourier-Stieltjes algebra of G. Then for a function $f \in C_{\mathrm{b}}(G)$ the following are equivalent:*

(i) *$f \in B(G)^{-\|\cdot\|_\infty}$;*

(ii) *If $\{f_n\}$ is any sequence in the unit ball of $L^1(G)$ such that $\int_G g f_n \mathrm{d}m \to 0$ as $n \to \infty$, for all $g \in P(G)$, then $\int_G f f_n \mathrm{d}m \to 0$, as $n \to \infty$.*

If we compare Corollary 3.5 with the main result of [12] which asserts that

PROPOSITION 3.7. *Let S be a commutative separative foundation semigroup with identity and let $R(S)$ denote the L^∞-representation algebra of S. Then for a function $f \in C_b(S)$ the following are equivalent:*

(i) $f \in R(S)^{-\|\cdot\|_\infty}$;

(ii) *If $\{\mu_n\}$ is any sequence in the unit ball of $M_a(S)$ such that $\widehat{\mu}_n(\chi) \to 0$ as $n \to \infty$, for all $\chi \in \widehat{S}$, then $\int_S f \mathrm{d}\mu_n \to 0$, as $n \to \infty$.*

and use the Remark 3.1(b) of [13], we get

COROLLARY 3.8. *If S is as in above proposition, then $B(S)$ is uniformly dense in $R(S)$.*

If G is a topological (or measured) groupoid then the Fourier algebra $A(G)$ has been studied by several authors ([18], [17], [15]). The definitions in these papers are not exactly the same, but of course they coincide if G is a group. If one can show that $A(G)$ is semidual to the convolution algebra $C_c(G)$ (here $C_c(G)$ is only a normed $*$-algebra, but that does not change anything in our proof), then Theorem 3.4 could be used to characterize the closure of $M(G)$ in $M(C^*(G))$. Here we take $C^*(A(G)) = C_0(G)$ and $C^*(C_c(G)) = C^*(G)$. Note that Proposition 2.3 in [18] provides a norm decreasing injection from $B(G)$ into $\mathfrak{D}(C^*(G))$, but in contrast with group case, $B(G)$ is no longer the same as the conjugate space of $C^*(G)$ (a more sophisticated relation using module Haagerup tensor products is provided in [18]).

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