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C^*-ALGEBRAS OF QUADRATURES

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Abstract. Quadrature operators are the $q_\theta = (\mathrm{e}^{-\mathrm{i}\theta}a + \mathrm{e}^{\mathrm{i}\theta}a^*)/\sqrt{2}$ where a and a^* are the annihilation and creation operators on $L^2(\mathbb{R})$. The structure of the C^*-algebra generated by operators $f(q_\theta)$ for f continuous function vanishing at infinity and θ in any subset Θ of $]-\pi,\pi[$ with $\mathrm{Card}(\Theta) \geqslant 2$ is studied. It is shown that it contains all compact operators and it is a C^*-algebra of type I. Its atomic representation and the structure of its spectrum is explicitely given. A trace formula for the operators $f(q_{\theta_1})g(q_{\theta_2})$ is proved.

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1. INTRODUCTION

Let $\mathcal{H}$ be the complex Hilbert space $L^2(\mathbb{R})$, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$, $\mathcal{K}$ the ideal of compact operators and $C_0(\mathbb{R})$ the set of complex continuous functions on $\mathbb{R}$ vanishing at infinity. Let q and p be the position and momentum operators in $\mathcal{H}$ defined by $qu(t) = tu(t)$ and $pu(t) = -\mathrm{i}u'(t)$ on their respective domains $\mathrm{Dom}(q) = \{u \in \mathcal{H} : qu \in \mathcal{H}\}$ and $\mathrm{Dom}(p) = \{u \in \mathcal{H} : u \text{ absolutely continuous}, pu \in \mathcal{H}\}$, and consider the C^*-algebras $\mathcal{Q}_0 = \{f(q) : f \in C_0(\mathbb{R})\}$ and $\mathcal{Q}_{\pi/2} = \{f(p) : f \in C_0(\mathbb{R})\}$. The unitary equivalence between q and p given by the Fourier transform allows us to show that $\mathcal{Q}_0\mathcal{Q}_{\pi/2} \subset \mathcal{K} \subset \mathcal{Q}_{\{0,\pi/2\}}$ where $\mathcal{Q}_{\{0,\pi/2\}}$ is the C^*-algebra generated by $\mathcal{Q}_0 \cup \mathcal{Q}_{\pi/2}$. Moreover, since $f(p)$ and $g(q)$ have continuous spectra for f and g real-valued in $C_0(\mathbb{R})$, it turns out that $\mathcal{Q}_0 \cap \mathcal{K} = \{0\}$ and $\mathcal{Q}_{\pi/2} \cap \mathcal{K} = \{0\}$. It is easy to see that these facts imply

$$\mathcal{Q}_{\{0,\pi/2\}} = \mathcal{Q}_0 + \mathcal{Q}_{\pi/2} + \mathcal{K}.$$

There is a natural generalization of q and p which appears in the context of quantum optics (see [9] and the references therein), the so-called *quadrature* operators q_θ defined for each $\theta \in]-\pi,\pi[$ by

$$q_\theta = \frac{\mathrm{e}^{-\mathrm{i}\theta}a + \mathrm{e}^{\mathrm{i}\theta}a^*}{\sqrt{2}}$$

where a and a^*, the annihilation and creation operators, are respectively the closures of $(q + \mathrm{i}p)/\sqrt{2}$ and $(q - \mathrm{i}p)/\sqrt{2}$ (notice that $q = q_0$, $p = q_{\pi/2}$ and $[q_{\theta_1}, q_{\theta_2}] \neq 0$ whenever $\theta_1 - \theta_2 \notin \pi\mathbb{Z}$). The C^*-algebra $\mathcal{Q}_\Theta$ generated by $\bigcup_{\theta\in\Theta} \mathcal{Q}_\theta$ where $\mathcal{Q}_\theta = \{\, f(q_\theta) : f \in C_0(\mathbb{R})\,\}$ and Θ is any subset of $]-\pi, \pi[$ then arises in a natural way as "good" algebra for studying the irreversible dynamics of the observables of a quantum open system. Indeed, it was shown in [3] that quantum Ornstein-Ulhenbeck semigroups on $\mathcal{B}(\mathcal{H})$ enjoy the Feller property with respect to $\mathcal{Q}_{\{0,\pi/2\}}$.

Our aim here is to study the structure of the C^*-algebra $\mathcal{Q}_\Theta$ where Θ is any subset of $]-\pi, \pi[$ with $\mathrm{Card}(\Theta) \geqslant 2$. We show that, if Θ is finite, then

$$\mathcal{Q}_\Theta = \sum_{\theta\in\Theta} \mathcal{Q}_\theta + \mathcal{K},$$

and so in general

$$\mathcal{Q}_\Theta = \varinjlim_{F\in\mathcal{F}} \mathcal{Q}_F$$

where $\mathcal{F}$ is the set of finite subsets of Θ directed by inclusion. We give the atomic representation of $\mathcal{Q}_\Theta$ and deduce that $\mathcal{Q}_\Theta$ is a C^*-algebra of type I; an essential composition serie is given when Θ is countable (Proposition 3.1 and Theorem 3.2). Its spectrum $\widehat{\mathcal{Q}}_\Theta$ is the topological free union $\sum_{\theta\in\Theta} \widehat{\mathcal{Q}}_\theta + \{t_{\mathrm{Id}}\}$ where t_{Id} is the inclusion $\mathcal{Q}_\Theta \subset \mathcal{B}(\mathcal{H})$ (Theorem 3.3); in particular, $\widehat{\mathcal{Q}}_\Theta$ is Hausdorff.

2. STRUCTURE OF $\mathcal{Q}_\Theta$

Let $(h_n)_{n\in\mathbb{N}}$ be the orthonormal basis of $\mathcal{H}$ given by

$$h_n(t) = \pi^{-\frac{1}{4}}(2^n n!)^{-\frac{1}{2}} \mathrm{e}^{-\frac{t^2}{2}} H_n(t)$$

where $H_n(t) = (-1)^n \mathrm{e}^{t^2} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \mathrm{e}^{-t^2}$ is the Hermite polynomial of order n. For all $\theta \in \mathbb{R}$ let F_θ be the unitary operator on $\mathcal{H}$ defined by

$$F_\theta h_n = \mathrm{e}^{\mathrm{i}n\theta} h_n \quad \text{for all } n \in \mathbb{N}.$$

Clearly $\{F_\theta : \theta \in \mathbb{R}\}$ is a group of unitary operators with $F_{2n\pi} = \mathrm{Id}$ and $F_\theta^* = F_{-\theta}$. It is well-known (see e.g. Chapter 4, Section 2, Example 4.18 and 4.22 in [6]) that each F_θ coincides with the operator $P_{\mathrm{i}\theta}$ of the analytic continuation of the Ornstein-Uhlenbeck semigroup $\{P_t : t \geqslant 0\}$ and, by the Mehler's formula, it has an integral representation

$$(F_\theta u)(t) = c_\theta \mathrm{e}^{-\frac{\mathrm{i}t^2}{2}\cot\theta} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{its}{\sin\theta} - \frac{\mathrm{i}s^2}{2}\cot\theta} u(s)\mathrm{d}s$$

for all $u \in \mathcal{H}$ where

$$c_\theta = (2\pi|\sin\theta|)^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}(\frac{\pi \mathrm{sgn}(\theta)}{4} - \frac{\theta}{2})}.$$

Note that $F_{(2n+1)\pi/2}$, $n \in \mathbb{N}$ coincides with the Fourier transform on $\mathcal{H}$

$$(F_{(2n+1)\pi/2}u)(t) = \widehat{u}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}ts} u(s)\mathrm{d}s.$$

For this reason the operators F_θ are also called fractional Fourier transforms (see [8] and the references therein). The following lemma generalises the well-known property of the Fourier transform $F_{\pi/2} f(q) F_{-\pi/2} = f(p)$.

Lemma 2.1. *For each f in $C_0(\mathbb{R})$ and θ in $]-\pi,\pi[$ we have $f(q_\theta) = F_\theta f(q) F_{-\theta}$.*

Proof. Using the well known relations $H_n''(t) - 2tH_n'(t) + 2nH_n(t) = 0$, $H_n'(t) = 2nH_{n-1}(t)$ and $H_{n+1}(t) - 2tH_n(t) + 2nH_{n-1}(t) = 0$ (see e.g. Section A.5 in [5]) we have

$$(q+\mathrm{i}p)h_n = \sqrt{2n}\, h_{n-1} \quad \text{for } n > 0, \quad (q+\mathrm{i}p)h_0 = 0,$$
$$(q-\mathrm{i}p)h_n = \sqrt{2(n+1)}\, h_{n+1} \quad \text{for } n \in \mathbb{N}.$$

Then a short calculation leads to $F_\theta q F_\theta^* = F_\theta q F_{-\theta} = q_\theta$. Therefore we have also $F_\theta f(q) F_{-\theta} = f(q_\theta)$ for all $\theta \in]-\pi,\pi[$. ∎

Let $C_{\mathrm{c}}(\mathbb{R})$ denote the vector space of complex-valued continuous functions on $\mathbb{R}$ with compact support.

Lemma 2.2. *For each f, g in $C_{\mathrm{c}}(\mathbb{R})$ and θ in $]-\pi,\pi[$ with $\theta \neq 0$ the operator $f(q_\theta)g(q)$ is Hilbert-Schmidt and has kernel*

$$(t,s) \mapsto 2\pi^{-\frac{1}{2}} |\sin\theta|^{-1} \mathrm{e}^{-\frac{\mathrm{i}(t^2-s^2)}{2}\cot\theta} \widehat{f}((t-s)/\sin\theta) g(s). \tag{2.1}$$

Proof. By Lemma 2.1, the operator $f(q_\theta)g(q)$ can be written in the form $F_\theta f(q) F_{-\theta} g(q)$. Moreover, for all $u \in \mathcal{H}$ we have

$$\begin{aligned}(F_\theta f(q) F_{-\theta} u)(t) &= c_\theta c_{-\theta} \mathrm{e}^{-\frac{\mathrm{i}t^2}{2}\cot\theta} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{\mathrm{i}tr}{\sin\theta}} f(r) \Big(\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}rs}{\sin\theta} + \frac{\mathrm{i}s^2}{2}\cot\theta} u(s)\mathrm{d}s \Big) \mathrm{d}r \\ &= c_\theta c_{-\theta} \mathrm{e}^{-\frac{\mathrm{i}t^2}{2}\cot\theta} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{\mathrm{i}s^2\cot\theta}{2}} u(s) \int_{-\infty}^{\infty} \mathrm{e}^{\frac{\mathrm{i}r(t-s)}{\sin\theta}} f(r)\mathrm{d}r\, \mathrm{d}s \\ &= \sqrt{2\pi} c_\theta c_{-\theta} \mathrm{e}^{-\frac{\mathrm{i}t^2}{2}\cot\theta} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{\mathrm{i}s^2\cot\theta}{2}} u(s) \widehat{f}((t-s)/\sin\theta)\mathrm{d}s.\end{aligned}$$

Thus $f(q_\theta)g(q)$ has kernel (2.1). ∎

PROPOSITION 2.3. *For each f, g in $C_0(\mathbb{R})$ and $\theta_1 \neq \theta_2$ in $]-\pi, \pi[$, the operator $f(q_{\theta_1})g(q_{\theta_2})$ is compact.*

Proof. By Lemma 2.1 the operator $f(q_{\theta_1})g(q_{\theta_2})$ can be written in the form

$$\begin{aligned} F_{\theta_1} f(q) F_{-\theta_1} F_{\theta_2} g(q) F_{-\theta_2} &= F_{\theta_2}\left(F_{\theta_1-\theta_2} f(q) F_{-(\theta_1-\theta_2)}\right) g(q) F_{-\theta_2} \\ &= F_{\theta_2} f(q_{\theta_1-\theta_2}) g(q) F_{-\theta_2}. \end{aligned}$$

Therefore it suffices to show that $f(q_\theta)g(q)$ is compact for all non-zero $\theta \in]-\pi, \pi[$. This is clear when f and g have compact support by Lemma 2.2. In the general case when both f and g belong to $C_0(\mathbb{R})$, let (f_n) and (g_n) be sequences in $C_{\mathrm{c}}(\mathbb{R})$ converging uniformly to f and g respectively. Since, for all n, m in $\mathbb{N}$, the norm $\|f_n(q_\theta)g_n(q) - f_m(q_\theta)g_m(q)\|$ is not bigger than

$$\begin{aligned} &\|f_n(q_\theta)(g_n(q) - g_m(q))\| + \|(f_n(q_\theta) - f_m(q_\theta))g_m(q)\| \\ &\leqslant \max\left\{\sup_{n\in\mathbb{N}} \|f_n\|, \sup_{n\in\mathbb{N}} \|g_n\|\right\}(\|g_n - g_m\| + \|f_n - f_m\|), \end{aligned}$$

it follows that the sequence $(f_n(q_\theta)g_n(q))$ is Cauchy in $\mathcal{B}(\mathcal{H})$. Therefore $f(q_\theta)g(q)$ is compact as a norm limit of Hilbert-Schmidt operators. ∎

As a corollary we deduce a generalization of a trace formula due to Accardi (see [1]) for the special case $\theta_1 - \theta_2 = \pi/2$.

COROLLARY 2.4. *For each f, g in $C_{\mathrm{c}}(\mathbb{R})$ and $\theta_1 \neq \theta_2$ in $]-\pi, \pi[$, the operator $f(q_{\theta_1})g(q_{\theta_2})$ is trace class and*

$$\operatorname{tr}(f(q_{\theta_1})g(q_{\theta_2})) = \frac{1}{2\pi|\sin(\theta_1 - \theta_2)|} \int_{-\infty}^{-\infty} f(t)\mathrm{d}t \int_{-\infty}^{-\infty} g(s)\mathrm{d}s.$$

Proof. Let $\theta \in]-\pi, \pi[\ \setminus\{0\}$, $\delta > 1/2$ put $g(q)f(q_\theta) = xy$ with

$$x = g(q)(1+p^2)^{-\frac{\delta}{2}}(1+q^2)^{\frac{\delta}{2}}, \quad y = (1+q^2)^{-\frac{\delta}{2}}(1+p^2)^{\frac{\delta}{2}} f(q_\theta).$$

A computation as in the proof of Lemma 2.2 shows that both x and y have a continuous square integrable kernel. Therefore x and y have bounded extensions, these are Hilbert-Schmidt and $g(q)f(q_\theta) = xy$ is trace class. It follows that $f(q_{\theta_1})g(q_{\theta_2}) = F_{\theta_2} f(q_{\theta_1-\theta_2})g(q)F_{-\theta_2}$ is trace class for all $\theta_1 \neq \theta_2$ in $]-\pi, \pi[$. The kernel (2.1) being continuous, the trace is given by the integral on the diagonal i.e.

$$\operatorname{tr}(f(q_{\theta_1})g(q_{\theta_2})) = \operatorname{tr}(f(q_{\theta_1-\theta_2})g(q)) = \frac{1}{\sqrt{2\pi}|\sin(\theta_1 - \theta_2)|} \int_{-\infty}^{+\infty} \widehat{f}(0)g(s)\mathrm{d}s.$$

The conclusion follows from the definition of $\widehat{f}(0)$. ∎

LEMMA 2.5. *The C^*-algebra $\mathcal{Q}_\Theta$ contains all compact operators on $\mathcal{H}$.*

Proof. Suppose first that Θ contains 0 and a non-zero θ in $]-\pi,\pi[$. Let $g \in C_c(\mathbb{R})$ and let $g_\theta(t) = e^{(it^2\cot\theta)/2}g(t)$. The Fourier transform of the gaussian distribution with variance ε, $f_\varepsilon(t) = (2\pi\varepsilon)^{-1/2}e^{-t^2/2\varepsilon}$, is $\widehat{f}_\varepsilon(t-s) = (2\pi)^{-1/2}e^{-\varepsilon(t-s)^2/2}$. Therefore by (2.1) the operator $2\pi|\sin\theta|\,\overline{g}(q)f_\varepsilon(q_\theta)g(q) - |g_\theta\rangle\langle g_\theta|$ has the kernel

$$\begin{aligned}k_\varepsilon(t,s) &= \sqrt{2\pi}e^{-\frac{i(t^2-s^2)\cot\theta}{2}}\widehat{f}_\varepsilon((t-s)/\sin\theta)g(s)\overline{g}(t) - g_\theta(s)\overline{g}_\theta(t)\\ &= e^{-\frac{i(t^2-s^2)\cot\theta}{2}}\Big(e^{-\frac{\varepsilon(t-s)^2}{2(\sin\theta)^2}} - 1\Big)g(s)\overline{g}(t).\end{aligned}$$

Notice that k_ε converges to 0 in $L^2(\mathbb{R}^2)$ as ε goes to 0 by dominated convergence. Therefore the operator $2\pi|\sin\theta|\,\overline{g}(q)f_\varepsilon(q_\theta)g(q)$ converges to $|g_\theta\rangle\langle g_\theta|$ (as $\varepsilon \to 0$) in the Hilbert-Schmidt norm. Thus both $|g_\theta\rangle\langle g_\theta|$ and $|g\rangle\langle g|$ belong to $\mathcal{Q}_\Theta$. If $u \in \mathcal{H}$ and (g_n) is a sequence in $C_c(\mathbb{R})$ converging in L^2 norm to u, then $|g_n\rangle\langle g_n|$ converges in norm to $|u\rangle\langle u|$. Thus every finite rank operator belongs to $\mathcal{Q}_\Theta$ and, by norm closure, the same conclusion holds for all compact operators.

When Θ contains two points $\theta_1,\theta_2 \in\,]-\pi,\pi[$ with $\theta_1 \neq \theta_2$ it suffices to recall the identity $\mathcal{Q}_{\{\theta_1,\theta_2\}} = F_{\theta_1}\mathcal{Q}_{\{0,\theta_2-\theta_1\}}F_{-\theta_1}$. Indeed, $F_{\theta_1}\mathcal{K}F_{-\theta_1} = \mathcal{K}$. ∎

PROPOSITION 2.6. *If Θ is finite, then $\mathcal{Q}_\Theta = \sum_{\theta\in\Theta}\mathcal{Q}_\theta + \mathcal{K}$.*

Proof. Notice that $\mathcal{Q}_{\theta_1}\mathcal{Q}_{\theta_2} \subset \mathcal{K}$ for all $\theta_1 \neq \theta_2$ in Θ by Proposition 2.3, and since for each real-valued $f \in C_0(\mathbb{R})$ and $\theta \in \Theta$, $f(q_\theta)$ has continuous spectrum we have $\mathcal{Q}_\theta \cap \mathcal{K} = \{0\}$. For each $x = \sum_{\theta\in\Theta} x_\theta + z$ and $y = \sum_{\theta\in\Theta} y_\theta + z'$ in $\sum_{\theta\in\Theta}\mathcal{Q}_\theta + \mathcal{K}$, we have $(x-z)(y-z') = \sum_{\theta\in\Theta} x_\theta y_\theta + \sum_{\substack{\theta\in\Theta\\ \theta'\in\Theta\\ \theta'\neq\theta}} x_\theta y_{\theta'} \in \sum_{\theta\in\Theta}\mathcal{Q}_\theta + \mathcal{K}$, and so $\sum_{\theta\in\Theta}\mathcal{Q}_\theta + \mathcal{K}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing $\mathcal{K} \cup \bigcup_{\theta\in\Theta}\mathcal{Q}_\theta$. For each $\theta_0 \in \Theta$ and $\sum_{\theta\in\Theta} x_\theta + z \in \sum_{\theta\in\Theta}\mathcal{Q}_\theta + \mathcal{K}$ we have

$$\|x_{\theta_0}\|\Big\|\sum_{\theta\in\Theta} x_\theta + z\Big\| \geqslant \Big\|x_{\theta_0}\Big(\sum_{\theta\in\Theta} x_\theta + z\Big)\Big\| \geqslant \|x_{\theta_0}^2 + z'\|$$

for some $z' \in \mathcal{K}$, and since $\mathcal{Q}_{\theta_0} + \mathcal{K}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ with $\mathcal{Q}_{\theta_0} \cap \mathcal{K} = \{0\}$, there is a constant $c_{\theta_0} > 0$ such that $c_{\theta_0}\|x_{\theta_0}^2 + z'\| \geqslant \|x_{\theta_0}^2\| = \|x_{\theta_0}\|^2$. Thus $c_{\theta_0}\Big\|\sum_{\theta\in\Theta} x_\theta + z\Big\| \geqslant \|x_{\theta_0}\|$ and $c = \sum_{\theta\in\Theta} c_\theta$ satisfies

$$c\Big\|\sum_{\theta\in\Theta} x_\theta + z\Big\| \geqslant \sum_{\theta\in\theta}\|x_\theta\|. \tag{2.2}$$

Let $\Big(\sum_{\theta\in\Theta} x_{n,\theta} + z_n\Big)$ be a sequence in $\sum_{\theta\in\Theta}\mathcal{Q}_\theta + \mathcal{K}$ converging to $y \in \mathcal{B}(\mathcal{H})$. By (2.2), $(x_{n,\theta})$ is a Cauchy sequence converging to some $x_\theta \in \mathcal{Q}_\theta$ for all $\theta \in \Theta$, and $\Big(\sum_{\theta\in\Theta} x_{n,\theta}\Big)$ is Cauchy converging to $\sum_{\theta\in\Theta} x_\theta$. Thus, (z_n) is a Cauchy sequence with some limit $z \in \mathcal{K}$, and $y = \sum_{\theta\in\Theta} x_\theta + z$. Therefore $\sum_{\theta\in\Theta} \mathcal{Q}_\theta + \mathcal{K}$ is closed and so it is the C^*-algebra generated by $\mathcal{K} \cup \bigcup_{\theta\in\Theta} \mathcal{Q}_\theta$ which is equal to $\mathcal{Q}_\Theta$ by Lemma 2.5. ∎

Let $\mathcal{F}$ denote the set of finite subsets of Θ directed by inclusion; since $\bigcup_{F\in\mathcal{F}} \mathcal{Q}_F$ is a self-adjoint algebra containing $\mathcal{K}$ and all the $\mathcal{Q}_\theta$ with $\theta \in \Theta$, we obtain the following:

COROLLARY 2.7. *The C^*-algebra $\mathcal{Q}_\Theta$ is the inductive limit of the directed system $\{\, \mathcal{Q}_F : F \in \mathcal{F} \,\}$.*

3. SPECTRUM OF $\mathcal{Q}_\Theta$

Let $\mathrm{Irr}(\mathcal{A})$ denote the set of irreducible representations of a C^*-algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ denote its spectrum. Moreover, for all $\pi \in \mathrm{Irr}(\mathcal{A})$ we denote by t_π the image of π in $\widehat{\mathcal{A}}$.

PROPOSITION 3.1. *The atomic representation π_{a} of $\mathcal{Q}_\Theta$ is*

$$\pi_{\mathrm{a}} = \pi_{\mathrm{Id}} \oplus \bigoplus_{\substack{\theta\in\Theta \\ \pi_\theta\in\mathrm{Irr}(\mathcal{Q}_\theta)}} \widetilde{\pi}_\theta$$

where π_{Id} is the inclusion $\mathcal{Q}_\Theta \subset B(\mathcal{H})$ and $\widetilde{\pi}_\theta$ the unique element $\pi \in \mathrm{Irr}(\mathcal{Q}_\Theta)\backslash\{\pi_{\mathrm{Id}}\}$ such that $\pi_{|\mathcal{Q}_\theta} = \pi_\theta$.

Proof. Since $\mathcal{Q}_\Theta$ acts irreducibly on $\mathcal{H}$, $(\pi_{\mathrm{Id}}, \mathcal{H})$ is an irreducible representation of $\mathcal{Q}_\Theta$. Let $(\pi, \mathcal{H}_\pi) \in \mathrm{Irr}(\mathcal{Q}_\Theta)\backslash\{(\pi_{\mathrm{Id}}, \mathcal{H})\}$. Since π must vanish on $\mathcal{K}$ ([7], Theorem 10.4.6), there exists $\theta \in \Theta$ such that $\pi(\mathcal{Q}_\theta) \neq \{0\}$ by Corollary 2.7. For each $x_\theta \in \mathcal{Q}_\theta$ and $x \in \mathcal{Q}_\Theta$ we have $\pi(x)\pi(x_\theta) = \lim \pi(x_i)\pi(x_\theta) = \lim \pi(x_i x_\theta)$ for some net (x_i) in $\bigcup_{F\in\mathcal{F}} \mathcal{Q}_F$ by Corollary 2.7. Since $\mathcal{Q}_{\theta'}\mathcal{Q}_\theta \subset K$ for all $\theta' \neq \theta$ in Θ, we have $\pi(x_i x_\theta) \in \pi(\mathcal{Q}_\theta)$ and so

$$\pi(x)\pi(x_\theta) \in \pi(\mathcal{Q}_\theta). \tag{3.1}$$

Thus $\pi(\mathcal{Q}_\theta)$ is a closed ideal in $\pi(\mathcal{Q}_\Theta)$. Since $\pi(\mathcal{Q}_\theta) \neq \{0\}$, we have $(\pi_{|\mathcal{Q}_\theta}, \pi(\mathcal{Q}_\theta)\mathcal{H}_\pi) \in \mathrm{Irr}(\mathcal{Q}_\theta)$ ([10], Lemma 4.1.5), and since $\pi(\mathcal{Q}_\theta)\mathcal{H}_\pi$ is invariant for π by (3.1), we have $\pi(\mathcal{Q}_\theta)\mathcal{H}_\pi = \mathcal{H}_\pi$. Therefore $(\pi_{|\mathcal{Q}_\theta}, \mathcal{H}_\pi) \in \mathrm{Irr}(\mathcal{Q}_\theta)$. For each $\theta' \neq \theta$ in Θ we have $\pi(\mathcal{Q}_{\theta'})\mathcal{H}_\pi = \pi(\mathcal{Q}_{\theta'})\pi(\mathcal{Q}_\theta)\mathcal{H}_\pi = \{0\}$ whence $\pi_{|\mathcal{Q}_{\theta'}} = 0$. Thus for each $(\pi, \mathcal{H}_\pi) \in \mathrm{Irr}(\mathcal{Q}_\Theta)\backslash\{(\pi_{\mathrm{Id}}, \mathcal{H})\}$ there is a unique θ_π such that $\pi(\mathcal{Q}_{\theta_\pi}) \neq \{0\}$; moreover $(\pi_{|\mathcal{Q}_{\theta_\pi}}, \mathcal{H}_\pi)$ is a character on $\mathcal{Q}_{\theta_\pi}$ and $\pi_{|\mathcal{Q}_{\theta_\pi}}$ determines completely π by Corollary 2.7. Therefore there is an injective map

$$\mathrm{Irr}(\mathcal{Q}_\Theta)\backslash\{(\pi_{\mathrm{Id}}, \mathcal{H})\} \to \bigcup_{\theta\in\Theta} \mathrm{Irr}(\mathcal{Q}_\theta), \quad \pi \mapsto \pi_{|\mathcal{Q}_{\theta_\pi}}. \tag{3.2}$$

Each $\pi_{\theta_0} \in \bigcup_{\theta\in\Theta} \mathrm{Irr}(\mathcal{Q}_\theta)$ can be extended to an element $\widetilde{\pi}_{\theta_0} \in \mathrm{Irr}(\mathcal{Q}_\Theta)\backslash\{\pi_{\mathrm{Id}}\}$ by defining $\widetilde{\pi}_{\theta_0}(K) = \widetilde{\pi}_{\theta_0}(\mathcal{Q}_\theta) = \{0\}$ for all $\theta \neq \theta_0$ in Θ; since $\theta_{\widetilde{\pi}_{\theta_0}} = \theta_0$ we have $\widetilde{\pi}_{\theta_0|\mathcal{Q}_{\theta_0}} = \pi_{\theta_0}$ and the map (3.2) is onto. ∎

THEOREM 3.2. *$\mathcal{Q}_\Theta$ is a C^*-algebra of type* I. *If Θ is finite or countable, then the family*

$$\{\, \mathcal{J}_n : 0 \leqslant n < \mathrm{Card}(\Theta) + 2 \,\}$$

where $\mathcal{J}_0 = \{0\}$, $\mathcal{J}_1 = \mathcal{K}$, $\mathcal{J}_2 = \mathcal{Q}_{\theta_1} + \mathcal{K}$, $\mathcal{J}_n = \mathcal{Q}_{\{\theta_1,\ldots,\theta_{n-1}\}}$ for all $3 \leqslant n <$ $\mathrm{Card}(\Theta) + 2$ is an essential composition series for $\mathcal{Q}_\Theta$ such that $\mathcal{J}_{n+1}/\mathcal{J}_n$ has continuous trace for all $0 \leqslant n < \mathrm{Card}(\Theta) + 1$.

Proof. By Proposition 3.1, each $\pi \in \mathrm{Irr}(\mathcal{Q}_\Theta)\backslash\{\pi_{\mathrm{Id}}\}$ is one dimensional and so $\mathcal{Q}_\Theta$ is a C^*-algebra of type I. Assume that Θ is finite or countable. For each $0 \leqslant n < \mathrm{Card}(\Theta)+2$, $\mathcal{J}_n$ is closed in $\mathcal{Q}_\Theta$ by Proposition 2.6, and since $\mathcal{Q}_{\theta_1}\mathcal{Q}_{\theta_2} \subset \mathcal{K}$ for all $\theta_1 \neq \theta_2$ in Θ, $\mathcal{J}_n$ is an ideal in $\bigcup_{0\leqslant k<\mathrm{Card}(\Theta)+2} \mathcal{J}_k$; thus $\mathcal{J}_n$ is a closed ideal in $\mathcal{Q}_\Theta$ by Corollary 2.7. Since $\mathcal{Q}_\Theta$ is the norm closure of $\bigcup_{0\leqslant n<\mathrm{Card}(\Theta)+2} \mathcal{J}_n$, the family $\{\, \mathcal{J}_n : 0 \leqslant n < \mathrm{Card}(\Theta) + 2 \,\}$ is a composition series for $\mathcal{Q}_\Theta$. Let $1 \leqslant n < \mathrm{Card}(\Theta) + 2$ and $\mathcal{J}$ a non-zero closed ideal in $\mathcal{J}_{n+1}$. If $\mathcal{J} \cap \mathcal{J}_n = \{0\}$, then each element $x \in \mathcal{J}$ has a form $x = x_1 + \cdots + x_n + z$ with $x_i \in \mathcal{Q}_{\theta_i}$, $1 \leqslant i \leqslant n$, $z \in \mathcal{K}$ and $x_n \neq 0$. Since $\mathcal{K}x_n x \in \mathcal{K}\mathcal{J} = \{0\}$, we have $x_n x = x_n^2 + z' = 0$ for some $z' \in \mathcal{K}$, i.e., $x_n^2 \in \mathcal{K}$ which is impossible. Thus $\mathcal{J} \cap \mathcal{J}_n \neq \{0\}$ and $\mathcal{J}_n$ is an essential ideal in $\mathcal{J}_{n+1}$. Since $\mathcal{J}_1 = \mathcal{K}$ and $\mathcal{J}_{n+1}/\mathcal{J}_n = \mathcal{Q}_n$ for all $1 \leqslant n < \mathrm{Card}(\Theta) + 1$ the theorem is proved. ∎

Recall that the *free union* $\sum_{i\in I} X_i$ of a family $\{X_i : i \in I\}$ of topological spaces is the set $\bigcup_{i\in I}\{i\} \times X_i$ endowed with the topology $\Big\{G \subset \bigcup_{i\in I}\{i\} \times X_i :$ $G \cap (\{i\} \times X_i)$ is open in $\{i\} \times X_i$ for all $i \in I\Big\}$. If X is a topological space and $\{X_i, i \in I\}$ an open cover of X, then the family of homeomorphisms $\{\psi_i :$ $\{i\} \times X_i \to X_i : i \in I\}$ defined by $\psi_i(i, x_i) = x_i$ induces a homeomorphism $\Big(\sum_{i\in I} X_i\Big)/\mathcal{R} \simeq X$ where $(i, x_i)\mathcal{R}(j, x'_j)$ if $x_i = x'_j$ ([4], Theorem 8.5 and Example 2, p. 131).

THEOREM 3.3. *$\widehat{\mathcal{Q}}_\Theta = \sum_{\theta\in\Theta} \widehat{\mathcal{Q}}_\theta + \{t_{\mathrm{Id}}\}$ where t_{Id} corresponds to π_{Id}. In particular, $\widehat{\mathcal{Q}}_\Theta$ is Hausdorff.*

Proof. Since $\mathcal{Q}_\Theta$ is a C^*-algebra of type I by Theorem 3.2, $\widehat{\mathcal{Q}}_\Theta$ coincides with the primitive spectrum ([2], Proposition 1.5.4). By Proposition 3.1, for each $\pi \in \mathrm{Irr}(\mathcal{Q}_\Theta)\backslash\{\pi_{\mathrm{Id}}\}$ there exists $\theta_\pi \in \Theta$ and a maximal ideal $\mathcal{M}_{\theta_\pi}$ of $\mathcal{Q}_{\theta_\pi}$ such that

$$\bigcup_{\substack{F\in\mathcal{F}\\ \theta_\pi\notin F}} \mathcal{Q}_F + \mathcal{M}_{\theta_\pi} \subset \mathrm{Ker}\,\pi. \tag{3.3}$$

For each $\theta \in \Theta$, $\mathcal{Q}_\theta \subset \bigcap_{t'\in\widehat{\mathcal{Q}}_\Theta\backslash\widehat{\mathcal{Q}}_\theta} t'$ by (3.3), and each $t \in \widehat{\mathcal{Q}}_\theta$ does not contain $\mathcal{Q}_\theta$; it follows that $\forall t \in \widehat{\mathcal{Q}}_\theta, t \not\supset \bigcap_{t'\in\widehat{\mathcal{Q}}_\Theta\backslash\widehat{\mathcal{Q}}_\theta} t'$, which shows that $\widehat{\mathcal{Q}}_\theta$ is open in $\widehat{\mathcal{Q}}_\Theta$. The set $\{t_{\mathrm{Id}}\}$ is open in $\widehat{\mathcal{Q}}_\Theta$ since $\mathcal{K} \subset \bigcap_{t' \in \widehat{\mathcal{Q}}_\Theta \setminus \{t_{\mathrm{Id}}\}} t'$. Therefore $\{\widehat{\mathcal{Q}}_\theta : \theta \in \Theta\} \cup \{t_{\mathrm{Id}}\}$ is an open cover of $\widehat{\mathcal{Q}}_\Theta$. For each θ, θ' in Θ, $t_\theta \in \widehat{\mathcal{Q}}_\theta$ and $t'_{\theta'} \in \widehat{\mathcal{Q}}_{\theta'}$, we have by (3.3) $t_\theta = t'_{\theta'}$ if and only if $\theta = \theta'$ and $t_\theta = t'_\theta$ so that $\widehat{\mathcal{Q}}_\Theta = \sum_{\theta \in \Theta} \widehat{\mathcal{Q}}_\theta + \{t_{\mathrm{Id}}\}$. The last assertion is obvious since $\widehat{\mathcal{Q}}_\theta$ is Hausdorff for all $\theta \in \Theta$. ∎

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