

THE LOCAL TRACE FUNCTION OF SHIFT INVARIANT SUBSPACES

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ABSTRACT. We define the local trace function for subspaces of $L^2(\mathbb{R}^n)$ which are invariant under integer translation. Our trace function contains the dimension function and the spectral function defined in [6] and completely characterizes the given translation invariant subspace. It has properties such as positivity, additivity, monotony and some form of continuity. It behaves nicely under dilations and modulations. We use the local trace function to deduce, using short and simple arguments, some fundamental facts about wavelets such as the characterizing equations, the equality between the dimension function and the multiplicity function and some new relations between scaling functions and wavelets.

KEYWORDS: *Wavelet, scaling function, dimension function, spectral function, multiplicity, shift invariant, trace.*

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1. INTRODUCTION

Shift invariant spaces are closed subspaces of $L^2(\mathbb{R}^n)$ that are invariant under all integer translations (also called shifts). They lie at the very heart of several areas such as the theory of wavelets, spline systems, Gabor systems or approximation theory. Thus, a good understanding of the shift invariant spaces can prove itself fruitful and give results in each of these areas. The local trace function is a new instrument for the analysis of shift invariant spaces. It is associated to a shift invariant subspace of $L^2(\mathbb{R}^n)$ and a positive operator on $l^2(\mathbb{Z}^n)$. And it confirms our expectations: when applied to a specific case it gives results. Several fundamental facts about wavelets can be deduced quite easily with the aid of the local trace function (we mention some of these facts: the equations that characterize wavelets (Remark 6.1), the equality between the dimension function and the multiplicity function (Remark 6.3) and some new equations that relate multiscaling functions to multiwavelets (Theorem 6.4 and Corollary 6.5)). Another nice thing about the

local trace function is that it includes, as special cases, the dimension function and the spectral function introduced by M. Bownik and Z. Rzeszotnik in [6] (their paper was the main source of inspiration for us, several of the theorems and proofs presented here are just extensions of the theorems and proofs from [6]). To get the dimension function, just compute the local trace function using the operator I -the identity on $l^2(\mathbb{Z}^n)$. The spectral function is just the local trace function associated to the projection P_{δ_0} onto the 0-th component (see Proposition 4.14).

The local trace function is based on two main concepts: the range function introduced by Helson ([9]) for shift invariant subspaces of $L^2(\mathbb{R}^n)$ and the trace function for positive operators on $l^2(\mathbb{Z}^n)$. The definition of the local trace functions combines these concepts (see Definition 4.1). Another key fact is a simple observation regarding the link between the trace function and normalized tight frames, namely that the trace function can be computed as in its definition but using a normalized tight frame instead of an orthonormal basis (see Proposition 3.2). Moreover, the normalized tight frames are the only families of vectors that can be used to compute the trace (see Proposition 3.5 and Theorem 3.6).

We recall that a subset $\{e_i : i \in I\}$ of a Hilbert space H is called a *frame* with positive constants A and B if

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f | e_i \rangle|^2 \leq B\|f\|^2, \quad f \in H.$$

If $A = B = 1$ then it is called a *normalized tight frame* (or shortly NTF).

In this paper we define the local trace function, investigate some of its properties and apply it to wavelets to obtain, using only short arguments, some known and some new results. In Section 1 and Section 3 we recall some facts about the range function and the trace function, respectively. We establish the main properties of these functions, properties that we need for Section 4 where we define and study the main character of our paper: the local trace function. Many properties shared by the range function and the trace function are inherited by the local trace function: additivity (Propositions 4.9 and 4.10), monotony (Proposition 4.12), nice behavior with respect to modulation and dilation (Propositions 4.12 and 4.13).

Also, there is a strong connection between our trace function and the Gramian introduced and effectively used by A. Ron and Z. Shen in [10],[11], [12] (see the Remarks 3.8 and 4.7).

Another fact that should be noticed is that the local trace function completely determines the shift invariant subspace: two shift invariant subspaces are equal if and only if their local trace functions are equal (Proposition 4.11).

Section 5 contains some results about the behavior of the local trace function with respect to limits. That is, when a sequence of shift invariant spaces has a limit (which is also shift invariant), in some cases the local trace function of the limit space is the limit of the sequence of local trace functions. One of the conditions is that the convergence is in the strong operator topology, but we put restrictions on the operator (Theorem 5.2); the other theorem is a monotone convergence theorem (Theorem 5.3) and it works for any positive operator on $l^2(\mathbb{Z}^n)$.

In Section 6, we apply the local trace function to wavelets. Some very simple arguments lead to important results: the equations that characterize wavelets

(6.1), (6.2), the equality between the multiplicity function and the dimension function (Remark 6.3) and the relation between scaling functions and wavelets (relations (6.9)–(6.11)) are obtained just by writing some local trace function in two ways.

Before we engage in the analysis of the local trace function, we have to recall some definitions and theorems.

The Fourier transform is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i(x|\xi)} dx, \quad \xi \in \mathbb{R}^n.$$

If V is closed subspace of a Hilbert space H and $f \in H$ we denote by P_V the projection onto V and by P_f the operator defined by:

$$P_f(v) = \langle v | f \rangle f, \quad v \in H.$$

DEFINITION 1.1. A closed subspace V of $L^2(\mathbb{R}^n)$ is called *shift invariant* (or shortly SI) if

$$T_k V = V, \quad k \in \mathbb{Z}^n,$$

where T_k is the translation by k on $L^2(\mathbb{R}^n)$:

$$(T_k f)(\xi) = f(\xi - k), \quad \xi \in \mathbb{R}^n, f \in L^2(\mathbb{R}^n).$$

If \mathcal{A} is a subset of $L^2(\mathbb{R}^n)$ then we denote by $S(\mathcal{A})$ the shift invariant space generated by \mathcal{A} ,

$$S(\mathcal{A}) = \overline{\text{span}}\{T_k \varphi : k \in \mathbb{Z}^n, \varphi \in \mathcal{A}\}.$$

DEFINITION 1.2. Let V be a shift invariant subspace of $L^2(\mathbb{R}^n)$. A subset ϕ of V is called a *normalized tight frame generator* (or NTF generator) for V if

$$\{T_k \varphi : k \in \mathbb{Z}^n, \varphi \in \phi\}$$

is a NTF for V .

We use also the notation $S(\varphi) := S(\{\varphi\})$. φ is called a *quasi-orthogonal generator* for $S(\varphi)$ if

$$\{T_k \varphi : k \in \mathbb{Z}^n\}$$

is a NTF for $S(\varphi)$ and for all $\xi \in \mathbb{R}^n$,

$$\text{Per} |\widehat{\varphi}|^2(\xi) := \sum_{k \in \mathbb{Z}^n} |\widehat{\varphi}|^2(\xi + 2k\pi) \in \{0, 1\}.$$

(actually, the second condition is a consequence of the first but we include it anyway).

Several proofs are available in the literature for the next theorem which guarantees the existence of NTF generators (see [4], [1], [3]).

THEOREM 1.3. *Suppose V is a SI subspace of $L^2(\mathbb{R}^n)$. Then V can be decomposed as an orthogonal sum*

$$V = \bigoplus_{i \in \mathbb{N}} S(\varphi_i),$$

where φ_i is a quasi-orthogonal generator of $S(\varphi_i)$. Moreover $\{\varphi_i : i \in \mathbb{N}\}$ is a NTF generator for V .

In fact, it is possible to derive a choice for the generators φ_i in Theorem 1.3 directly from the Stone's spectral multiplicity theorem, as explained in [3].

2. THE RANGE FUNCTION

In this section we define the range function and state some of its properties. Some preliminary notations are needed.

The Hilbert space of square integrable vector functions $L^2([-\pi, \pi]^n, l^2(\mathbb{Z}^n))$ consists of all vector valued measurable functions $\phi : [-\pi, \pi]^n \rightarrow l^2(\mathbb{Z}^n)$ with the norm

$$\|\phi\| = \left(\int_{[-\pi, \pi]^n} \|\phi(\xi)\|_{l^2(\mathbb{Z}^n)}^2 d\xi \right)^{1/2} < \infty.$$

The scalar product is given by

$$\langle \phi | \psi \rangle := \int_{[-\pi, \pi]^n} \langle \phi(\xi) | \psi(\xi) \rangle_{l^2(\mathbb{Z}^n)} d\xi.$$

The map $\mathcal{T} : L^2(\mathbb{R}^n) \rightarrow L^2([-\pi, \pi]^n, l^2(\mathbb{Z}^n))$ defined for $f \in L^2(\mathbb{R}^n)$ by

$$\mathcal{T}f : [-\pi, \pi]^n \rightarrow l^2(\mathbb{Z}^n), \quad \mathcal{T}f(\xi) = (\widehat{f}(\xi + 2k\pi))_{k \in \mathbb{Z}^n}, \quad \xi \in [-\pi, \pi]^n$$

is an isometric isomorphism (up to multiplication by $1/(2\pi)^{n/2}$) between $L^2(\mathbb{R}^n)$ and $L^2([-\pi, \pi]^n, l^2(\mathbb{Z}^n))$.

We will need some variations of these maps because they will make the formulas nicer. We define $L_{\text{per}}^2(\mathbb{R}^n, l^2(\mathbb{Z}^n))$ to be the space of measurable vector valued functions $\phi : \mathbb{R}^n \rightarrow l^2(\mathbb{Z}^n)$ with the property that $\phi|_{[-\pi, \pi]^n}$ belongs to $L^2([-\pi, \pi]^n, l^2(\mathbb{Z}^n))$ and they are periodic in the following sense:

$$\phi(\xi + 2k\pi) = \lambda(k)^*(\phi(\xi)), \quad \xi \in \mathbb{R}^n,$$

where, for every $k \in \mathbb{Z}^n$, $\lambda(k)$ is the shift by k on $l^2(\mathbb{Z}^n)$ that is

$$(\lambda(k)\alpha)(l) = \alpha(l - k), \quad l \in \mathbb{Z}^n, \alpha \in l^2(\mathbb{Z}^n).$$

The scalar product is defined by the same formula, the integral being taken over $[-\pi, \pi]^n$.

The map $\mathcal{T}_{\text{per}} : L^2(\mathbb{R}^n) \rightarrow L_{\text{per}}^2(\mathbb{R}^n, l^2(\mathbb{Z}^n))$ is defined by the same formula as \mathcal{T} ,

$$\mathcal{T}_{\text{per}}f : \mathbb{R}^n \rightarrow l^2(\mathbb{Z}^n), \quad \mathcal{T}_{\text{per}}f(\xi) = (\widehat{f}(\xi + 2k\pi))_{k \in \mathbb{Z}^n}, \quad \xi \in \mathbb{R}^n,$$

but notice that ξ is now in \mathbb{R}^n . Also, observe the periodicity property of \mathcal{T}_{per} :

$$\mathcal{T}_{\text{per}}f(\xi + 2s\pi) = \lambda(s)^*(\mathcal{T}_{\text{per}}f(\xi)), \quad \xi \in \mathbb{R}^n, s \in \mathbb{Z}^n.$$

DEFINITION 2.1. A *range function* is a measurable mapping

$$J : [-\pi, \pi]^n \rightarrow \{\text{closed subspaces of } l^2(\mathbb{Z}^n)\}.$$

Measurable means weakly operator measurable, i.e., $\xi \mapsto \langle P_{J(\xi)}a | b \rangle$ is measurable for any choice of vectors $a, b \in l^2(\mathbb{Z}^n)$.

A periodic range function is a measurable function

$$J_{\text{per}} : \mathbb{R}^n \rightarrow \{\text{closed subspaces of } l^2(\mathbb{Z}^n)\},$$

with the periodicity property

$$J_{\text{per}}(\xi + 2k\pi) = \lambda(k)^* (J_{\text{per}}(\xi)), \quad k \in \mathbb{Z}^n, \xi \in \mathbb{R}^n.$$

Sometimes we will use the same letter to denote the subspace $J_{\text{per}}(\xi)$ and the projection onto $J_{\text{per}}(\xi)$. In terms of projections, the periodicity can be written as

$$J_{\text{per}}(\xi + 2k\pi) = \lambda(k)^* J_{\text{per}}(\xi) \lambda(k), \quad k \in \mathbb{Z}^n, \xi \in \mathbb{R}^n.$$

The next theorem due to Helson ([9]) establishes the fundamental connection between shift invariant spaces and the range function. The theorem appears in this form in Proposition 1.5 of [4], the only modification needed is to work with the whole \mathbb{R}^n instead of $[-\pi, \pi]^n$ and use the periodicity.

THEOREM 2.2. *A closed subspace V of $L^2(\mathbb{R}^n)$ is shift invariant if and only if*

$$V = \{f \in L^2(\mathbb{R}^n) : \mathcal{T}_{\text{per}} f(\xi) \in J_{\text{per}}(\xi) \text{ for a.e. } \xi \in \mathbb{R}^n\},$$

for some measurable periodic range function J_{per} . The correspondence between V and J_{per} is bijective under the convention that range functions are identified if they are equal a.e. Furthermore, if $V = S(\mathcal{A})$ for some countable $\mathcal{A} \subset L^2(\mathbb{R}^n)$, then

$$J_{\text{per}}(\xi) = \overline{\text{span}}\{\mathcal{T}_{\text{per}}\varphi(\xi) \mid \varphi \in \mathcal{A}\}, \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

DEFINITION 2.3. If V is a SI subspace of $L^2(\mathbb{R}^n)$ then J_{per} associated to V as in Theorem 2.2 is called the *periodic range function of V* .

The first elementary property of the range function that we will need is additivity. The range function is also unitary in the sense that it preserves orthogonality of subspaces. The precise formulation of these properties is given below.

PROPOSITION 2.4. *Let $(V_i)_{i \in I}$ be a countable family of mutually orthogonal SI subspaces of $L^2(\mathbb{R}^n)$ and denote by J_{V_i} the periodic range function of V_i . If*

$$V = \bigoplus_{i \in I} V_i$$

and J_V is its periodic range function then

$$J_V(\xi) = \bigoplus_{i \in I} J_{V_i}(\xi), \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

where the sum is an orthogonal one.

Proof. Pick some countable $\phi_i \subset V_i$ such that $\{T_k \varphi : k \in \mathbb{Z}^n, \varphi \in \phi_i\}$ spans V_i . Then it is clear that $\{T_k \varphi : k \in \mathbb{Z}^n, \varphi \in \phi\}$ spans V , with the notation $\phi = \bigcup_{i \in I} \phi_i$. Using Helson's Theorem 2.2 we can determine the periodic range functions

$$\begin{aligned} J_{V_i}(\xi) &= \overline{\text{span}}\{\mathcal{T}_{\text{per}}\varphi(\xi) : \varphi \in \phi_i\}, \quad i \in I, \\ J_V(\xi) &= \overline{\text{span}}\{\mathcal{T}_{\text{per}}\varphi(\xi) : \varphi \in \phi\}, \end{aligned}$$

for almost every point $\xi \in \mathbb{R}^n$. This shows that, if we check the orthogonality of the subspaces $J_{V_i}(\xi)$, then we are done.

To check that these subspaces of $L^2(\mathbb{Z}^n)$ are mutually orthogonal, take two arbitrary $i \neq j \in I$ and $\varphi_1 \in \phi_i, \varphi_2 \in \phi_j$. Then, since we are dealing with SI spaces, $T_k\varphi_1$ is perpendicular to φ_2 for any choice of $k \in \mathbb{Z}^n$. Rewriting this in terms of the Fourier transform, we obtain

$$0 = \int_{\mathbb{R}^n} e^{-i\langle k|\xi\rangle} \widehat{\varphi}_1(\xi) \overline{\widehat{\varphi}_2(\xi)} \, d\xi = \int_{[-\pi,\pi]^n} e^{-i\langle k|\xi\rangle} \langle \mathcal{T}\varphi_1(\xi) | \mathcal{T}\varphi_2(\xi) \rangle \, d\xi.$$

For the second equality we applied a periodization. But this shows that all Fourier coefficients of the map $\xi \mapsto \langle \mathcal{T}\varphi_1(\xi) | \mathcal{T}\varphi_2(\xi) \rangle$ are 0 so the map itself is 0 which implies that $\mathcal{T}_{\text{per}}\varphi_1(\xi)$ is perpendicular to $\mathcal{T}_{\text{per}}\varphi_2(\xi)$ for a.e. $\xi \in [-\pi, \pi]^n$ and therefore, because of the periodicity, for a.e. $\xi \in \mathbb{R}^n$. Consequently, the subspaces are mutually orthogonal and the proposition is proved. ■

An easy consequence of the additivity of the range function is the following monotony property:

PROPOSITION 2.5. *Let $(V_j)_{j \in \mathbb{N}}$ be an increasing sequence of SI subspaces of $L^2(\mathbb{R}^n)$. Denote by*

$$V := \overline{\bigcup_{j \in \mathbb{N}} V_j}.$$

Then for a.e. $\xi \in \mathbb{R}^n$, $(J_{V_j}(\xi))_{j \in \mathbb{N}}$ is increasing and

$$J_V(\xi) = \overline{\bigcup_{j \in \mathbb{N}} J_{V_j}(\xi)}.$$

Proof. Let W_j be the orthogonal complement of V_j in V_{j+1} , $j \in \mathbb{N}$. Then W_j are shift invariant too, $V_{j+1} = V_j \oplus W_j$ for all $j \in \mathbb{N}$ and

$$V = V_0 \oplus \bigoplus_{j \in \mathbb{N}} W_j, \quad V_l = V_0 \oplus \bigoplus_{j=0}^{l-1} W_j, \quad l \in \mathbb{N}.$$

The proposition then follows from the additivity property stated in Proposition 2.4. ■

The next proposition can also be found in [9] and [4]. Again the only modification is that we use the periodic extension of the range function.

PROPOSITION 2.6. *Let V be an SI subspace of $L^2(\mathbb{R}^n)$ and J_{per} its periodic range function. Then*

$$\mathcal{T}_{\text{per}}(P_V f)(\xi) = J_{\text{per}}(\xi)(\mathcal{T}_{\text{per}} f(\xi)), \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

DEFINITION 2.7. A subset $\{e_i : i \in I\}$ of a Hilbert space H is called a *Bessel sequence with constant $B > 0$* if

$$\sum_{i \in I} |\langle e_i | f \rangle|^2 \leq B \|f\|^2, \quad f \in H.$$

The next theorem will be extensively used in this paper. Its proof can be found in [4].

THEOREM 2.8. *Let V be an SI subspace of $L^2(\mathbb{R}^n)$, J_{per} its periodic range function and ϕ a countable subset of V . $\{T_k\varphi : k \in \mathbb{Z}^n, \varphi \in \phi\}$ is a frame with constants A and B for V (Bessel family with constant B) if and only if $\{\mathcal{T}_{\text{per}}\varphi(\xi) : \varphi \in \phi\}$ is a frame with constants A and B for $J_{\text{per}}(\xi)$ (Bessel sequence with constant B) for almost every $\xi \in \mathbb{R}^n$.*

3. THE TRACE FUNCTION

In this section we recall the definition of the trace function and gather some of its properties, the main ones and also some others that we will need in the sequel.

DEFINITION 3.1. Let H be a Hilbert space and T a positive operator on H . Then the trace of the operator T is the positive number (can be also ∞) defined by

$$\text{Trace}(T) = \sum_{i \in I} \langle Te_i | e_i \rangle,$$

where $\{e_i : i \in I\}$ is an orthonormal basis for H .

The next proposition shows that the trace is well defined, that does not depend on the choice of the orthonormal basis, and moreover, it can be computed with the same formula using a normalized tight frame.

PROPOSITION 3.2. *Let H be a Hilbert space, T a positive operator on H and $\{f_j : j \in J\}$ a normalized tight frame for H . Then*

$$\text{Trace}(T) = \sum_{j \in J} \langle Tf_j | f_j \rangle.$$

Proof. Let $\{e_i : i \in I\}$ be an orthonormal basis for H .

$$\begin{aligned} \text{Trace}(T) &= \sum_{i \in I} \langle Te_i | e_i \rangle = \sum_{i \in I} \langle T^{1/2}e_i | T^{1/2}e_i \rangle = \sum_{i \in I} \|T^{1/2}e_i\|^2 \\ &= \sum_{i \in I} \sum_{j \in J} |\langle T^{1/2}e_i | f_j \rangle|^2 = \sum_{j \in J} \sum_{i \in I} |\langle T^{1/2}e_i | f_j \rangle|^2 \\ &= \sum_{j \in J} \sum_{i \in I} |\langle e_i | T^{1/2}f_j \rangle|^2 = \sum_{j \in J} \|T^{1/2}f_j\|^2 \\ &= \sum_{j \in J} \langle Tf_j | f_j \rangle. \quad \blacksquare \end{aligned}$$

The following proposition enumerates some of the elementary properties of the trace. For a proof look in any basic book on operator theory (e.g. [14]).

PROPOSITION 3.3. *The trace has the following properties for all a, b positive operators and $\lambda \geq 0$:*

- (i) $\text{Trace}(a + b) = \text{Trace}(a) + \text{Trace}(b)$;
- (ii) $\text{Trace}(\lambda a) = \lambda \text{Trace}(a)$;
- (iii) *if $a \leq b$ then $\text{Trace}(a) \leq \text{Trace}(b)$;*
- (iv) $\text{Trace}(v^*av) \leq \text{Trace}(a)$ *whenever v is a partial isometry;*

- (v) $\text{Trace}(u^*au) = \text{Trace}(a)$ for all unitary u ;
- (vi) $\text{Trace}(x^*x) = \text{Trace}(xx^*)$ for every operator x on H ;
- (vii) $\text{Trace}(|x|) = \text{Trace}(|x^*|)$ for every operator x on H ($|x| := (x^*x)^{1/2}$);
- (viii) $\text{Trace}(P) = \dim PH$ for any projection P in H .

PROPOSITION 3.4. Let P be a projection and $\{e_i : i \in I\}$ a NTF for its range. Then, for any positive operator T , and any vector $f \in H$

$$\begin{aligned}\text{Trace}(TP) &= \sum_{i \in I} \langle Te_i | e_i \rangle, \\ \text{Trace}(P_f P) &= \sum_{i \in I} |\langle f | e_i \rangle|^2 = \|Pf\|^2.\end{aligned}$$

Proof. Let $\{f_j : j \in J\}$ be an orthonormal basis for the orthogonal complement of the range of P . Then $\{e_i : i \in I\} \cup \{f_j : j \in J\}$ is a NTF for the entire Hilbert space. Using Proposition 3.2, the formula for $\text{Trace}(TP)$ follows.

Particularize the formula for $T = P_f$

$$\text{Trace}(P_f P) = \sum_{i \in I} \langle \langle e_i | f \rangle f | e_i \rangle = \sum_{i \in I} |\langle f | e_i \rangle|^2 = \|Pf\|^2. \quad \blacksquare$$

In fact even more is true: the equations of Proposition 3.4 characterize the normalized tight frames for the range of P . This is described in the next statement.

PROPOSITION 3.5. Let H be a Hilbert space, H_0 a closed subspace and $\{e_i : i \in I\}$ a family of vectors from H . The following affirmations are equivalent:

- (i) $\{e_i : i \in I\}$ is a NTF for H_0 ;
- (ii) for every positive operator T on H ,

$$\text{Trace}(TP_{H_0}) = \sum_{i \in I} \langle Te_i | e_i \rangle;$$

- (iii) for every vector $v \in H$,

$$\text{Trace}(P_v P_{H_0}) = \sum_{i \in I} |\langle v | e_i \rangle|^2.$$

(Note that we do not require in (ii) and (iii) that the e_i 's be in H_0 . This will follow from the formulas).

Proof. (i) implies (ii) and (ii) implies (iii) according to Proposition 3.5 and its proof. So we only have to worry about the implication from (iii) to (i). The hypotheses imply that, if $v \perp H_0$, then $P_v P_{H_0} = 0$ so v is perpendicular to e_i for every $i \in I$.

Also, for $v \in H_0$,

$$\sum_{i \in I} |\langle v | e_i \rangle|^2 = \|v\|^2.$$

This shows that all e_i 's are in H_0 and they form a NTF for it. \blacksquare

We can weaken the condition (iii) in Proposition 3.5. The family of vectors is a NTF for the subspace if it satisfies the NTF condition just for some special vectors as shown below:

THEOREM 3.6. *Let H be a Hilbert space and $\{\delta_k : k \in K\}$ a total subset of H (i.e. its closed linear span is H). Let H_0 be a closed subspace for H and $\{e_i : i \in I\}$ a family of vectors in H . The following affirmations are equivalent:*

- (i) $\{e_i : i \in I\}$ is a NTF for H_0 ;
- (ii) for every $r \neq s \in K$ and $\lambda \in \{0, 1, i = \sqrt{-1}\}$

$$(3.1) \quad \sum_{i \in I} |\langle \delta_r + \lambda \delta_s | e_i \rangle|^2 = \|P_{H_0}(\delta_r + \lambda \delta_s)\|^2.$$

Proof. (i) implies (ii) clearly.

Assume (ii). Take $\{f_j : j \in J\}$ a NTF for H_0 . Then

$$(3.2) \quad \|P_{H_0}(v)\|^2 = \sum_{j \in J} |\langle v | f_j \rangle|^2, \quad v \in H.$$

Take $\lambda = 0$ and use (3.1) and (3.2):

$$(3.3) \quad \sum_{i \in I} |\langle \delta_r | e_i \rangle|^2 = \sum_{j \in J} |\langle \delta_r | f_j \rangle|^2, \quad v \in H.$$

Now take $r \neq s$ and $\lambda \in \{1, i\}$. From (3.1) and (3.2):

$$\begin{aligned} & \sum_{i \in I} (|\langle \delta_r | e_i \rangle|^2 + \langle \delta_r | e_i \rangle \overline{\lambda \langle \delta_s | e_i \rangle} + \overline{\langle \delta_r | e_i \rangle} \lambda \langle \delta_s | e_i \rangle + |\lambda|^2 |\langle \delta_s | e_i \rangle|^2) \\ &= \sum_{j \in J} (|\langle \delta_r | f_j \rangle|^2 + \langle \delta_r | f_j \rangle \overline{\lambda \langle \delta_s | f_j \rangle} + \overline{\langle \delta_r | f_j \rangle} \lambda \langle \delta_s | f_j \rangle + |\lambda|^2 |\langle \delta_s | f_j \rangle|^2). \end{aligned}$$

With (3.3) we can reduce this to

$$\begin{aligned} & \overline{\lambda} \sum_{i \in I} \langle \delta_r | e_i \rangle \overline{\langle \delta_s | e_i \rangle} + \lambda \sum_{i \in I} \overline{\langle \delta_r | e_i \rangle} \langle \delta_s | e_i \rangle \\ &= \overline{\lambda} \sum_{j \in J} \langle \delta_r | f_j \rangle \overline{\langle \delta_s | f_j \rangle} + \lambda \sum_{j \in J} \overline{\langle \delta_r | f_j \rangle} \langle \delta_s | f_j \rangle. \end{aligned}$$

Now take $\lambda = 1$ and $\lambda = i$ and the two resulting equation will yield:

$$(3.4) \quad \sum_{i \in I} \langle \delta_r | e_i \rangle \overline{\langle \delta_s | e_i \rangle} = \sum_{j \in J} \langle \delta_r | f_j \rangle \overline{\langle \delta_s | f_j \rangle}.$$

Next we prove that

$$\sum_{i \in I} |\langle v | e_i \rangle|^2 = \sum_{j \in J} |\langle v | f_j \rangle|^2$$

holds for all $v \in S$ where

$$S := \left\{ \sum_{k \in K_0} v_k \delta_k : K_0 \subset K \text{ finite} \right\}.$$

For this take an arbitrary $v = \sum_{k \in K_0} v_k \delta_k$, with K_0 finite. Then

$$\begin{aligned} \sum_{i \in I} |\langle v | e_i \rangle|^2 &= \sum_{i \in I} \left| \sum_{k \in K_0} v_k \langle \delta_k | e_i \rangle \right|^2 \\ &= \sum_{i \in I} \sum_{k, k' \in K_0} v_k \bar{v}_{k'} \langle \delta_k | e_i \rangle \overline{\langle \delta_{k'} | e_i \rangle} \\ &= \sum_{k, k' \in K_0} v_k \bar{v}_{k'} \sum_{i \in I} \langle \delta_k | e_i \rangle \overline{\langle \delta_{k'} | e_i \rangle} \\ &= \sum_{k, k' \in K_0} v_k \bar{v}_{k'} \sum_{j \in J} \langle \delta_k | f_j \rangle \overline{\langle \delta_{k'} | f_j \rangle} \quad (\text{with (3.4)}) \\ &= \sum_{j \in J} |\langle v | f_j \rangle|^2. \end{aligned}$$

To prove the relation for arbitrary $v \in H$, define $\tilde{T}_1 : S \rightarrow l^2(I)$, $T_2 : H \rightarrow l^2(J)$ by

$$\tilde{T}_1 v = (\langle v | e_i \rangle)_{i \in I}, \quad v \in S, \quad T_2 v = (\langle v | f_j \rangle)_{j \in J}.$$

Then \tilde{T}_1 is a well defined linear operator, $\|T_2 v\|^2 = \|P_{H_0} v\|^2$ for $v \in H$ and $\|\tilde{T}_1 v\| = \|T_2 v\|$ for $v \in S$. This shows that $\|\tilde{T}_1 v\|^2 \leq \|v\|^2$ for $v \in S$. Hence, as S is a dense subspace of H , we can extend \tilde{T}_1 to a linear operator T_1 on H with

$$\|T_1 v\|^2 \leq \|v\|^2, \quad \text{for all } v \in H.$$

We claim that $T_1 v = (\langle v | e_i \rangle)_{i \in I}$ for any $v \in H$. Indeed, v can be approximated by vectors v_n in S . Then, for each $i \in I$,

$$(T_1 v)_i = \lim_{n \rightarrow \infty} (T_1 v_n)_i = \lim_{n \rightarrow \infty} \langle v_n | e_i \rangle = \langle v | e_i \rangle.$$

Also, taking the limit,

$$\|T_1 v\|^2 = \|T_2 v\|^2 = \|P_{H_0} v\|^2, \quad v \in H,$$

and this implies (i). ■

THEOREM 3.7. *Let H be a Hilbert space H_0 a closed subspace, $\{\delta_k : k \in K\}$ a total set for H , and $\{f_j : j \in J\}$ a NTF for H_0 . Let $\{e_i : i \in I\}$ be a family of vectors in H . The following affirmations are equivalent:*

- (i) $\{e_i : i \in I\}$ is a NTF for H_0 ;
- (ii) for all $r, s \in H_0$,

$$\sum_{i \in I} \langle \delta_r | e_i \rangle \overline{\langle \delta_s | e_i \rangle} = \sum_{j \in J} \langle \delta_r | f_j \rangle \overline{\langle \delta_s | f_j \rangle}.$$

Proof. Just examine the proof of Theorem 3.6. ■

REMARK 3.8. Theorem 3.7 shows that $\{e_i : i \in I\}$ is a NTF for H_0 if and only if it has the same Gramian as $\{f_j : j \in J\}$. We recall briefly this notion. The Gramian was introduced by A. Ron and Z. Shen in a series of papers and they used it to analyze the structure of shift invariant spaces ([10], [11], [12]).

For a given countable family of vectors $\{e_i : i \in I\}$ define the operator $K : l^2(I) \rightarrow H$, initially on sequences $f = (f_i)_{i \in I}$ with compact support, by

$$(3.5) \quad K(f) = \sum_{i \in I} f_i e_i.$$

If K extends to a bounded operator (this is the case when $\{e_i : i \in I\}$ forms a Bessel family), then its adjoint $K^* : H \rightarrow l^2(I)$ is given by

$$(3.6) \quad K^*(v) = (\langle v | e_i \rangle)_{i \in I}, \quad v \in l^2(\mathbb{Z}^n).$$

Suppose $\{e_i : i \in I\}$ is a subset of H and K is defined as in (3.5). The Gramian of the system $\{e_i : i \in I\}$ is $G : l^2(I) \rightarrow l^2(I)$ defined by $G = K^*K$. The dual Gramian of the system $\{e_i : i \in I\}$ is $\tilde{G} : H \rightarrow H$ defined by $\tilde{G} = KK^*$.

Note that

$$(3.7) \quad \langle \tilde{G}f | g \rangle = \langle K^*f | K^*g \rangle = \sum_{i \in I} \langle f | e_i \rangle \overline{\langle g | e_i \rangle} =: B(f, g).$$

The trace function involves expressions of the form

$$\sum_{i \in I} |\langle f | e_i \rangle|^2 = B(f, f) = \langle \tilde{G}f | f \rangle.$$

We can recuperate the dual Gramian if these expressions are given, because by polarization

$$B(f, g) = \frac{1}{4} \sum_{k=0}^3 i^k B(f + i^k g, f + i^k g).$$

B , as a sesquilinear form gives rise to an operator and this operator has to be the dual Gramian \tilde{G} . Another fact that is worth noticing is that, if $\{e_i : i \in I\}$ is a NTF for some subspace H_0 of $l^2(\mathbb{Z}^n)$, then the dual Gramian \tilde{G} is the projection onto H_0 , P_{H_0} . This is because P_{H_0} verifies the equation (3.7).

PROPOSITION 3.9. *Let $(P_j)_{j \in J}$ be a family of mutually orthogonal projections in a Hilbert space H . Then*

$$\text{Trace} \left(T \sum_{j \in J} P_j \right) = \sum_{j \in J} \text{Trace}(TP_j),$$

for any positive operator T on H .

Proof. Consider $\{e_i : i \in I_j\}$ an orthonormal basis for the range of P_j , ($j \in J$). Then, $\{e_i : i \in \bigcup_{j \in J} I_j\}$ is an orthonormal basis for the range of $\sum_{j \in J} P_j$.

The additivity property is now clear, if we use Proposition 3.4. ■

The next proposition is an easy consequence of the additivity of the trace mentioned in Proposition 3.9.

PROPOSITION 3.10. *Let $(P_j)_{j \in \mathbb{N}}$ be an increasing sequence of projections in some Hilbert space H , $P = \sup_{j \in \mathbb{N}} P_j$ and T a positive operator on H . Then $(\text{Trace}(TP_j))_{j \in \mathbb{N}}$ increases to $\text{Trace}(TP)$.*

PROPOSITION 3.11. *Let T be a positive operator on a Hilbert space H , P a projection on H and U a unitary on H . Then*

$$\text{Trace}(UTPU^*) = \text{Trace}(TP).$$

Proof. Let $\{e_i : i \in I\}$ be an orthonormal basis for H such that $\{e_i : i \in I_0\}$ is an orthonormal basis for UPH , with $I_0 \subset I$. Then $\{U^*e_i : i \in I_0\}$ is an orthonormal basis for PH and $PU^*e_i = 0$ for $i \in I \setminus I_0$; so

$$\text{Trace}(UTPU^*) = \sum_{i \in I_0} \langle UTPU^*e_i | e_i \rangle = \sum_{i \in I_0} \langle TPU^*e_i | U^*e_i \rangle = \text{Trace}(TP). \quad \blacksquare$$

LEMMA 3.12. *If H is a Hilbert space, H_0 a closed subspace and $f \in H$, then $\text{Trace}(P_f P_{H_0}) \leq \|f\|^2$ with equality if and only if $f \in H_0$.*

Proof. Let $\{e_i : i \in I\}$ be an orthonormal basis for H such that $\{e_i : i \in I_0\}$ is an orthonormal basis for H_0 with $I_0 \subset I$. Then

$$\begin{aligned} \|f\|^2 &= \sum_{i \in I_0} |\langle f | e_i \rangle|^2 + \sum_{i \in I \setminus I_0} |\langle f | e_i \rangle|^2 \\ &= \sum_{i \in I_0} \langle P_f e_i | e_i \rangle + \sum_{i \in I \setminus I_0} |\langle f | e_i \rangle|^2 \\ &= \text{Trace}(P_f P_{H_0}) + \sum_{i \in I \setminus I_0} |\langle f | e_i \rangle|^2. \end{aligned}$$

Thus the inequality holds and $\text{Trace}(P_f P_{H_0}) = \|f\|^2$ if and only if $\langle f | e_i \rangle = 0$ for all $i \in I \setminus I_0$ which is equivalent to $f \in H_0$. \blacksquare

LEMMA 3.13. *Let H be a Hilbert space, H_1 and H_2 two closed subspaces. Then $H_1 \subset H_2$ if and only if $\text{Trace}(P_f P_{H_1}) \leq \text{Trace}(P_f P_{H_2})$ for all $f \in H$.*

Proof. If H_1 is contained in H_2 then, using Proposition 3.9, the inequality between the traces is immediate.

For the converse, assume $f \in H_1$. Then, by Lemma 3.12, $\text{Trace}(P_f P_{H_1}) = \|f\|^2$ so $\|f\|^2 \leq \text{Trace}(P_f P_{H_2})$. But then, with the same lemma, $\text{Trace}(P_f P_{H_2}) = \|f\|^2$ which implies that $f \in H_2$. As f was arbitrary, the inclusion is proved. \blacksquare

4. THE LOCAL TRACE FUNCTION

DEFINITION 4.1. Let V be a SI subspace of $L^2(\mathbb{R}^n)$, T a positive operator on $l^2(\mathbb{Z}^n)$ and let J_{per} be the range function associated to V . We define the local trace function associated to V and T as the map from \mathbb{R}^n to $[0, \infty]$ given by the formula

$$\tau_{V,T}(\xi) = \text{Trace}(TJ_{\text{per}}(\xi)), \quad \xi \in \mathbb{R}^n.$$

We define the restricted local trace function associated to V and a vector f in $l^2(\mathbb{Z}^n)$ by

$$\tau_{V,f}(\xi) = \text{Trace}(P_f J_{\text{per}}(\xi)) (= \tau_{V,P_f}(\xi)), \quad \xi \in \mathbb{R}^n,$$

where P_f is the operator on $l^2(\mathbb{Z}^n)$ defined by $P_f(v) = \langle v | f \rangle f$.

PROPOSITION 4.2. For all $f \in l^2(\mathbb{Z}^n)$,

$$\tau_{V,f}(\xi) = \|J_{\text{per}}(\xi)(f)\|^2, \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Proof. Use the Proposition 3.4. ■

THEOREM 4.3. Let V be a SI subspace of $L^2(\mathbb{R}^n)$ and $\phi \subset V$ a NTF generator for V . Then for every positive operator T on $l^2(\mathbb{Z}^n)$ and any $f \in l^2(\mathbb{Z}^n)$,

$$(4.1) \quad \tau_{V,T}(\xi) = \sum_{\varphi \in \phi} \langle T \mathcal{T}_{\text{per}} \varphi(\xi) | \mathcal{T}_{\text{per}} \varphi(\xi) \rangle, \quad \text{for a.e. } \xi \in \mathbb{R}^n;$$

$$(4.2) \quad \tau_{V,f}(\xi) = \sum_{\varphi \in \phi} |\langle f | \mathcal{T}_{\text{per}} \varphi(\xi) \rangle|^2, \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Proof. According to Theorem 2.8, $\{\mathcal{T}_{\text{per}} \varphi(\xi) : \varphi \in \phi\}$ is a NTF for $J_{\text{per}}(\xi)$ for a.e. $\xi \in \mathbb{R}^n$. With the Proposition 3.4 we obtain (4.1). Take $T = P_f$ and (4.1) becomes (4.2). ■

THEOREM 4.4. Let V be a SI subspace of $L^2(\mathbb{R}^n)$, J_{per} its periodic range function and ϕ a countable subset of $L^2(\mathbb{R}^n)$. Then following affirmations are equivalent:

- (i) $\phi \subset V$ and ϕ is a NTF generator for V ;
- (ii) for every $f \in l^2(\mathbb{Z}^n)$

$$(4.3) \quad \sum_{\varphi \in \phi} |\langle f | \mathcal{T}_{\text{per}} \varphi(\xi) \rangle|^2 = \|J_{\text{per}}(\xi)(f)\|^2, \quad \text{for a.e. } \xi \in \mathbb{R}^n;$$

- (iii) for every $0 \neq l \in \mathbb{Z}^n$ and $\alpha \in \{0, 1, i\}$,

$$(4.4) \quad \sum_{\varphi \in \phi} |\widehat{\varphi}(\xi) + \bar{\alpha} \widehat{\varphi}(\xi + 2l\pi)|^2 = \|J_{\text{per}}(\xi)(\delta_0 + \alpha \delta_l)\|^2, \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Proof. (i) implies (ii) just as an application of Theorem 4.3 and Proposition 4.2. (iii) is just a particular case of (ii), namely $f = \delta_0 + \alpha \delta_l$. So assume (iii) holds. Then (4.4) can be rewritten as:

$$\sum_{\varphi \in \phi} \langle \delta_0 + \alpha \delta_l | \mathcal{T}_{\text{per}}(\xi) \rangle|^2 = \|J_{\text{per}}(\delta_0 + \alpha \delta_l)\|^2.$$

Now take $r \neq s \in \mathbb{Z}^n$ and apply this equation to $l = s - r$, $\xi = \xi + 2\pi r$. A short computation, that uses the periodicity of \mathcal{T}_{per} and J_{per} and the fact that $\lambda(r)$ is unitary, will lead to

$$\sum_{\varphi \in \phi} |\langle \delta_r + \alpha \delta_s | \mathcal{T}_{\text{per}}(\xi) \rangle|^2 = \|J_{\text{per}}(\xi)(\delta_r + \alpha \delta_s)\|^2, \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Now we can use Theorem 3.6 to conclude that, for a.e. $\xi \in \mathbb{R}^n$,

$$\{\mathcal{T}_{\text{per}}\varphi(\xi) : \varphi \in \phi\}$$

is a NTF for $J_{\text{per}}(\xi)$. And, with Theorem 2.8, (i) is obtained. ■

Applying this theorem to $V = L^2(\mathbb{R}^n)$ we deduce the next corollary which can be found also in [10]:

COROLLARY 4.5. *A countable subset φ of $L^2(\mathbb{R}^n)$ is a NTF generator for $L^2(\mathbb{R}^n)$ if and only if the following equations hold for a.e. $\xi \in \mathbb{R}^n$:*

$$\begin{aligned} \sum_{\varphi \in \phi} |\widehat{\varphi}(\xi)|^2 &= 1, \\ \sum_{\varphi \in \phi} \widehat{\varphi}(\xi) \overline{\widehat{\varphi}(\xi + 2l\pi)} &= 0, \quad l \in \mathbb{Z}^n, l \neq 0. \end{aligned}$$

THEOREM 4.6. *Let V be a SI subspace of $L^2(\mathbb{R}^n)$, ϕ_1 a NTF generator for V and ϕ_2 a countable family of vectors from $L^2(\mathbb{R}^n)$. The following affirmations are equivalent:*

- (i) $\phi_2 \subset V$ and ϕ_2 is a NTF generator for V ;
- (ii) for every $l \in \mathbb{Z}^n$,

$$\sum_{\varphi \in \phi_2} \widehat{\varphi}(\xi) \overline{\widehat{\varphi}(\xi + 2l\pi)} = \sum_{\varphi \in \phi_1} \widehat{\varphi}(\xi) \overline{\widehat{\varphi}(\xi + 2l\pi)}, \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Proof. By Theorem 2.8, (i) is equivalent to

$$(4.5) \quad \{\mathcal{T}_{\text{per}}\varphi(\xi) : \varphi \in \phi_2\} \text{ is a NTF for } J_{\text{per}}(\xi) \text{ for a.e. } \xi \in \mathbb{R}^n.$$

But we know that $\{\mathcal{T}_{\text{per}}\varphi(\xi) : \varphi \in \phi_1\}$ is a NTF for $J_{\text{per}}(\xi)$ (again, by Theorem 2.8). Therefore, using Theorem 3.7, (4.5) is equivalent to:

For every $r, s \in \mathbb{Z}^n$ and a.e. $\xi \in \mathbb{R}^n$,

$$\sum_{\varphi \in \phi_2} \langle \delta_r | \mathcal{T}_{\text{per}}\varphi(\xi) \rangle \overline{\langle \delta_s | \mathcal{T}_{\text{per}}\varphi(\xi) \rangle} = \sum_{\varphi \in \phi_1} \langle \delta_r | \mathcal{T}_{\text{per}}\varphi(\xi) \rangle \overline{\langle \delta_s | \mathcal{T}_{\text{per}}\varphi(\xi) \rangle},$$

which is exactly

$$\sum_{\varphi \in \phi_2} \widehat{\varphi}(\xi + 2r\pi) \overline{\widehat{\varphi}(\xi + 2s\pi)} = \sum_{\varphi \in \phi_1} \widehat{\varphi}(\xi + 2r\pi) \overline{\widehat{\varphi}(\xi + 2s\pi)}, \quad r, s \in \mathbb{Z}^n, \xi \in \mathbb{R}^n.$$

This implies (ii) and (ii) implies this, because one can take $l = s - r$, $\xi = \xi + 2r\pi$. ■

REMARK 4.7. Theorem 4.6 shows that ϕ_2 is a NTF generator for V if and only if it has the same dual Gramian as ϕ_1 . Recall, that the dual Gramian of a countable subset ϕ of $L^2(\mathbb{R}^n)$ is defined as the function which assigns to each $\xi \in \mathbb{R}^n$ the dual Gramian of the set $\{\mathcal{T}_{\text{per}}\varphi(\xi) : \varphi \in \phi\}$ (see Remark 3.8 and [10], [11], [12], [4] for details). The dual Gramian satisfies the equations:

$$\langle \tilde{G}(\xi)\delta_r | \delta_s \rangle = \sum_{\varphi \in \phi} \widehat{\varphi}(\xi + 2\pi r)\overline{\widehat{\varphi}(\xi + 2\pi s)}.$$

PROPOSITION 4.8. (Periodicity) *Let V be a SI subspace, T a positive operator on $l^2(\mathbb{Z}^n)$, $f \in l^2(\mathbb{Z}^n)$. Then, for $k \in \mathbb{Z}^n$*

$$(4.6) \quad \tau_{V,T}(\xi + 2k\pi) = \tau_{V,\lambda(k)T\lambda(k)^*}(\xi), \quad \text{for a.e. } \xi \in \mathbb{R}^n;$$

$$(4.7) \quad \tau_{V,f}(\xi + 2k\pi) = \tau_{V,\lambda(k)f}(\xi), \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

where $\lambda(k)$ is the unitary operator on $l^2(\mathbb{Z}^n)$ defined by $(\lambda(k)\alpha)(l) = \alpha(l - k)$ for all $l \in \mathbb{Z}^n$, $\alpha \in l^2(\mathbb{Z}^n)$.

Proof. The periodicity of the local trace function is a consequence of the periodicity of the range function. Indeed, by Definition 2.1, we know that for a.e. $\xi \in \mathbb{R}^n$ and every $k \in \mathbb{Z}^n$,

$$J_{\text{per}}(\xi + 2k\pi) = \lambda(k)^* J_{\text{per}}(\xi)\lambda(k).$$

Apply this in the definition of the local trace function and use Proposition 3.11:

$$\begin{aligned} \tau_{V,T}(\xi + 2k\pi) &= \text{Trace}(TJ_{\text{per}}(\xi + 2k\pi)) = \text{Trace}(T\lambda(k)^* J_{\text{per}}(\xi)\lambda(k)) \\ &= \text{Trace}(\lambda(k)T\lambda(k)^* J_{\text{per}}(\xi)) = \tau_{V,\lambda(k)T\lambda(k)^*}(\xi). \end{aligned}$$

Since $\lambda(k)P_f\lambda(k)^* = P_{\lambda(k)f}$, (4.7) follows from (4.8). ■

PROPOSITION 4.9. *Let V be an SI space, T, S positive operators on $l^2(\mathbb{Z}^n)$, $f \in l^2(\mathbb{Z}^n)$:*

- (i) $0 \leq \tau_{V,T}(\xi) \leq \infty$, $0 \leq \tau_{V,f} \leq \|f\|^2$;
- (ii) $\tau_{V,T+S} = \tau_{V,T} + \tau_{V,S}$;
- (iii) $\tau_{V,\lambda T} = \lambda\tau_{V,T}$, $\lambda > 0$;
- (iv) $\tau_{V,\lambda f} = |\lambda|^2\tau_{V,f}$, $\lambda \in \mathbb{C}$.

Proof. Everything follows from Section 3. ■

PROPOSITION 4.10. (Additivity) *Suppose $(V_i)_{i \in I}$ are mutually orthogonal SI subspaces (I countable) and let $V = \bigoplus_{i \in I} V_i$. Then, for every positive operator T on $l^2(\mathbb{Z}^n)$ and every $f \in l^2(\mathbb{Z}^n)$:*

$$(4.8) \quad \tau_{V,T} = \sum_{i \in I} \tau_{V_i,T}, \quad \text{a.e. on } \mathbb{R}^n;$$

$$(4.9) \quad \tau_{V,f} = \sum_{i \in I} \tau_{V_i,f}, \quad \text{a.e. on } \mathbb{R}^n.$$

Proof. Let J_i be the periodic range function of V_i , $i \in I$ and J the periodic range function of V . The range function is additive (proposition 2.4) so

$$J(\xi) = \sum_{i \in I} J_i(\xi), \quad \text{a.e. on } \mathbb{R}^n.$$

Also the trace has additive properties (Proposition 3.9) and these imply the additivity of the local trace function expressed in (4.8). Again (4.9) is just a particular case of (4.8). ■

PROPOSITION 4.11. (Monotony and injectivity) *Let V, W be SI subspaces.*

(i) $V \subset W$ if and only if $\tau_{V,T} \leq \tau_{W,T}$ a.e. for all positive operators T if and only if $\tau_{V,f} \leq \tau_{W,f}$ a.e. for all $f \in l^2(\mathbb{Z}^n)$.

(ii) $V = W$ if and only if $\tau_{V,T} = \tau_{W,T}$ a.e. for all positive operators T if and only if $\tau_{V,f} = \tau_{W,f}$ a.e. for all $f \in l^2(\mathbb{Z}^n)$.

Proof. (i) It is clear that the first statement implies the second; just use the additivity property for the SI spaces V and $W \ominus V$. The third statement is just a particular case of the second one. So the only interesting implication is from the third statement to the first one.

Let J_V and J_W be the corresponding periodic range functions for V and W . The hypothesis implies, according to Lemma 3.13, that

$$J_V(\xi) \subset J_W(\xi), \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

and this implies in turn that $V \subset W$ (just look at Theorem 2.2).

(ii) is a consequence of (i) by a double inclusion argument. ■

For $a \in \mathbb{R}^n$ we define the modulation of $f \in L^2(\mathbb{R}^n)$ by

$$M_a(f)(x) = e^{i\langle a | x \rangle} f(x), \quad x \in \mathbb{R}^n.$$

The local trace function behaves nicely under modulation. This is expressed in the next proposition.

PROPOSITION 4.12. (Modulation) *Let V be a SI subspace and $a \in \mathbb{R}^n$. Then $M_a V$ is a SI subspace and for all positive operators T on $l^2(\mathbb{Z}^n)$ and all vectors $f \in l^2(\mathbb{Z}^n)$:*

$$(4.10) \quad \tau_{M_a V, T}(\xi) = \tau_{V, T}(\xi - a), \quad \text{for a.e. } \xi \in \mathbb{R}^n;$$

$$(4.11) \quad \tau_{M_a V, f}(\xi) = \tau_{V, f}(\xi - a), \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Proof. The modulation and the translations satisfy a commutation relation: for $k \in \mathbb{Z}^n$ and $\varphi \in L^2(\mathbb{R}^n)$,

$$T_k M_a \varphi(x) = e^{i\langle a | x - k \rangle} \varphi(x - k) = e^{-i\langle a | k \rangle} M_a T_k \varphi(x), \quad x \in \mathbb{R}^n.$$

This relation shows that $M_a V$ is shift invariant.

Now take a NTF generator ϕ for V (it exists by Theorem 1.3). Then, as M_a is unitary,

$$\{M_a T_k \varphi : \varphi \in \phi\}$$

is a NTF for $M_a V$. Using the commutation relation and the fact that $e^{-i\langle a | k \rangle}$ are just constants of modulus 1, we see that

$$\{T_k M_a \varphi : \varphi \in \phi\}$$

is a NTF for $M_a V$. Therefore we can safely use Theorem 4.3 and compute:

$$\tau_{M_a V, T}(\xi) = \sum_{\varphi \in \phi} \langle T \mathcal{T}_{\text{per}}(M_a \varphi)(\xi) | \mathcal{T}_{\text{per}}(M_a \varphi)(\xi) \rangle.$$

But

$$\mathcal{T}_{\text{per}} M_a \varphi(\xi) = (\widehat{M_a \varphi}(\xi + 2k\pi))_{k \in \mathbb{Z}^n} = (\widehat{\varphi}(\xi - a + 2k\pi))_{k \in \mathbb{Z}^n} = \mathcal{T}_{\text{per}} \varphi(\xi - a),$$

and (4.10) follows with (4.11) as its consequence. ■

The dilation by an $n \times n$ non-singular matrix A is the unitary operator on $L^2(\mathbb{R}^n)$ defined by

$$D_A f(x) = |\det A|^{1/2} f(Ax), \quad x \in \mathbb{R}^n, f \in L^2(\mathbb{R}^n).$$

We will consider only matrices A which preserve the lattice \mathbb{Z}^n , because in this case $D_A V$ is shift invariant whenever V is.

PROPOSITION 4.13. (Dilation) *Let V be a SI subspace and A an $n \times n$ integer matrix with $\det A \neq 0$. Then $D_A V$ is shift invariant and, for every positive operator T on $l^2(\mathbb{Z}^n)$ and every vector $f \in l^2(\mathbb{Z}^n)$:*

$$(4.12) \quad \tau_{D_A V, T}(\xi) = \sum_{d \in \mathcal{D}} \tau_{V, D_d^* T D_d}((A^*)^{-1}(\xi + 2d\pi)), \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

$$(4.13) \quad \tau_{D_A V, f}(\xi) = \sum_{d \in \mathcal{D}} \tau_{V, D_d^* f}((A^*)^{-1}(\xi + 2d\pi)), \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

where \mathcal{D} is a complete set of representatives of the cosets $\mathbb{Z}^n / A^* \mathbb{Z}^n$ and D_d is the linear operator on $l^2(\mathbb{Z}^n)$ defined by

$$(D_d \alpha)(k) = \begin{cases} \alpha(l), & \text{if } k = d + A^* l, \\ 0, & \text{otherwise,} \end{cases} \quad k \in \mathbb{Z}^n, \alpha \in l^2(\mathbb{Z}^n).$$

Proof. The dilation and the translation satisfy the following commutation relation which can be easily verified:

$$T_k D_A = D_A T_{Ak}, \quad k \in \mathbb{Z}^n.$$

This shows that $D_A V$ is shift invariant.

We can decompose V as the orthogonal sum $V = \bigoplus_{i \in I} S(\varphi_i)$, where φ_i is a quasi-orthogonal generator of $S(\varphi_i)$ (see Theorem 1.3). Since D_A is unitary, $D_A V = \bigoplus_{i \in I} D_A S(\varphi_i)$ and, using the additivity property of the local trace function (Proposition 4.10), it suffices to prove the formula (4.12) just for the case when $V = S(\varphi)$ with φ quasi-orthogonal generator for $S(\varphi)$. We will assume this is the case.

From the commutation relation we see that, if \mathcal{L} is a complete set of $|\det A|$ representatives of the cosets $\mathbb{Z}^n/A\mathbb{Z}^n$, then

$$\{D_A T_l \varphi : l \in \mathcal{L}\}$$

will span $D_A V$ by translations.

For $l \in \mathcal{L}$, consider

$$\begin{aligned} \phi_l(\xi) &= \mathcal{T}_{\text{per}}(D_A T_l \varphi)(\xi) \\ &= \left(|\det A|^{-(1/2)} \widehat{\varphi}((A^*)^{-1}(\xi + 2k\pi)) e^{-i\langle (A^*)^{-1}(\xi + 2k\pi) | l \rangle} \right)_{k \in \mathbb{Z}^n}. \end{aligned}$$

For $d \in \mathcal{D}$ define $\psi_d \in L^2_{\text{per}}(\mathbb{R}^n, l^2(\mathbb{Z}^n))$ by

$$\begin{aligned} \psi_d(\xi)(k) &= \begin{cases} \widehat{\varphi}((A^*)^{-1}(\xi + 2k\pi)), & \text{if } k \in d + A^*\mathbb{Z}^n, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} \widehat{\varphi}((A^*)^{-1}(\xi + 2d\pi) + 2\pi l), & \text{if } k = d + A^*l, \text{ with } l \in \mathbb{Z}^n, \\ 0, & \text{otherwise,} \end{cases} \\ &= D_d (\mathcal{T}_{\text{per}} \varphi((A^*)^{-1}(\xi + 2\pi d))). \end{aligned}$$

Then $\phi_l(\xi)$ and $\psi_d(\xi)$ are related by the following linear equations:

$$\phi_l(\xi) = e^{-i\langle (A^*)^{-1}\xi | l \rangle} |\det A|^{-1/2} \sum_{d \in \mathcal{D}} e^{-i\langle (A^*)^{-1}2\pi d | l \rangle} \psi_d(\xi).$$

Since the $|\det A| \times |\det A|$ matrix

$$\left(|\det A|^{-1/2} e^{-i\langle (A^*)^{-1}2\pi d | l \rangle} \right)_{d \in \mathcal{D}, l \in \mathcal{L}}$$

is unitary, it follows that $\{\psi_d(\xi) : d \in \mathcal{D}\}$ and $\{\phi_l(\xi) : l \in \mathcal{L}\}$ span the same subspace of $l^2(\mathbb{Z}^n)$, namely $J_{D_A V}(\xi)$ (use Theorem 2.2).

φ is a quasi-orthogonal generator so $\text{Per} |\widehat{\varphi}|^2$ is a characteristic function that is $\|\mathcal{T}_{\text{per}} \varphi(\xi)\|_{l^2(\mathbb{Z}^n)}$ is either 0 or 1, and as D_d is an isometry, $\|\psi_d(\xi)\|_{l^2(\mathbb{Z}^n)} \in \{0, 1\}$. Also $\psi_d(\xi)$ and $\psi_{d'}(\xi)$ are perpendicular when $d \neq d'$ and, in conclusion $\{\psi_d(\xi) : d \in \mathcal{D}\}$ is a NTF for $J_{D_A V}(\xi)$. Therefore we can use these vectors to compute the local trace function (Proposition 3.4):

$$\begin{aligned} \tau_{D_A V, T}(\xi) &= \sum_{d \in \mathcal{D}} \langle T \psi_d(\xi) | \psi_d(\xi) \rangle \\ &= \sum_{d \in \mathcal{D}} \langle T D_d (\mathcal{T}_{\text{per}} \varphi((A^*)^{-1}(\xi + 2\pi d))) | D_d (\mathcal{T}_{\text{per}} \varphi((A^*)^{-1}(\xi + 2\pi d))) \rangle \\ &= \sum_{d \in \mathcal{D}} \langle D_d^* T D_d (\mathcal{T}_{\text{per}} \varphi((A^*)^{-1}(\xi + 2\pi d))) | \mathcal{T}_{\text{per}} \varphi((A^*)^{-1}(\xi + 2\pi d)) \rangle \\ &= \sum_{d \in \mathcal{D}} \tau_{V, D_d^* T D_d}((A^*)^{-1}(\xi + 2\pi d)). \end{aligned}$$

This proves (4.12). (4.13) follows from (4.12) because $D_d^* P_f D_d = P_{D_d^* f}$. ■

As we promised, the local trace function incorporates the dimension function and the spectral function defined by M. Bownik and Z. Rzeszotnik in [6].

Recall that for a shift invariant subspace V of $L^2(\mathbb{R}^n)$, its dimension function is defined as

$$\dim_V(\xi) = \dim J_{\text{per}}(\xi), \quad \xi \in \mathbb{R}^n,$$

where J_{per} is the periodic function associated to V .

The spectral function introduced in [6] is defined by

$$\sigma_V(\xi + 2k\pi) = \|J_{\text{per}}(\xi)\delta_k\|^2, \quad \xi \in [-\pi, \pi]^n, k \in \mathbb{Z}^n,$$

where $\delta_k \in l^2(\mathbb{Z}^n)$, $\delta_k(l) = \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{otherwise,} \end{cases} k \in \mathbb{Z}^n$.

For a similar treatment of the dimension function, using the Gramian the reader can also consult [13].

PROPOSITION 4.14. *Let V be a SI space.*

- (i) $\tau_{V,I} = \dim_V$.
- (ii) $\tau_{V,\delta_0} = \sigma_V$.

Proof. Let J_{per} be the periodic range function of V :

$$\tau_{V,I}(\xi) = \text{Trace}(IJ_{\text{per}}(\xi)) = \dim J_{\text{per}}(\xi) = \dim_V(\xi).$$

For $\xi \in [-\pi, \pi]^n$, by Proposition 4.2,

$$\tau_{V,\delta_0}(\xi) = \|J_{\text{per}}(\xi)\delta_0\|^2 = \sigma_V(\xi).$$

If $k \in \mathbb{Z}^n$ then, using the periodicity of the local trace function stated in Proposition 4.8,

$$\tau_{V,\delta_0}(\xi + 2k\pi) = \tau_{V,\lambda(k)\delta_0}(\xi) = \tau_{V,\delta_k}(\xi) = \|J_{\text{per}}(\xi)\delta_k\|^2 = \sigma_V(\xi). \quad \blacksquare$$

5. CONVERGENCE THEOREMS

In this section we study the behavior of the local trace function with respect to limits. More precisely, we consider the following question: if $(V_j)_{j \in \mathbb{N}}$ is a sequence of SI subspaces of $L^2(\mathbb{R}^n)$ such that P_{V_j} converges in the strong operator topology to P_V for some SI subspace V , what can be said about the convergence of the local trace functions $\tau_{V_i,T}$?

We begin with a lemma and then give some useful partial answers to this question.

LEMMA 5.1. *Suppose $(V_j)_{j \in \mathbb{N}}$ and V are SI subspaces of $L^2(\mathbb{R}^n)$ such that P_{V_j} converges in the strong operator topology to P_V . Then for any $f \in L^2(\mathbb{R}^n)$ and any $A > 0$*

$$\int_{[-A,A]^n} \|J_{V_j}(\xi)(\mathcal{T}_{\text{per}}f(\xi)) - J_V(\xi)(\mathcal{T}_{\text{per}}f(\xi))\|^2 d\xi \quad \text{converges to 0 as } j \rightarrow \infty,$$

where J_{V_j} and J_V are the corresponding periodic range functions.

Proof. We can consider $A = \pi$ because then the result is obtained using the periodicity of the range function. With Proposition 2.6,

$$J_{V_j}(\xi)(\mathcal{T}_{\text{per}}f(\xi)) = \mathcal{T}_{\text{per}}(P_{V_j}f)(\xi), \quad \xi \in \mathbb{R}^n.$$

And then

$$\begin{aligned} & \int_{[-\pi, \pi]^n} \|J_{V_j}(\xi)(\mathcal{T}_{\text{per}}f(\xi)) - J_V(\xi)(\mathcal{T}_{\text{per}}f(\xi))\|^2 d\xi \\ &= \int_{[-\pi, \pi]^n} \|\mathcal{T}_{\text{per}}(P_{V_j}f)(\xi) - \mathcal{T}_{\text{per}}(P_Vf)(\xi)\|^2 d\xi \\ &= (2\pi)^n \|P_{V_j}f - P_Vf\|_{L^2(\mathbb{R}^n)}^2 \rightarrow 0, \quad \text{as } j \rightarrow \infty. \quad \blacksquare \end{aligned}$$

THEOREM 5.2. *Let $(V_j)_{j \in \mathbb{N}}$ and V be SI subspaces such that P_{V_j} converges to P_V in the strong operator topology. Then, for any bounded measurable set E and every $f \in L^2(\mathbb{Z}^n)$*

$$\int_E |\tau_{V_j, f}(\xi) - \tau_{V, f}(\xi)| d\xi \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Proof. Let J_{V_j} and J_V be the corresponding periodic range function. We can assume $E = [-\pi, \pi]^n$ because then the result is a consequence of the periodicity of the local trace function (Proposition 4.8) and we can assume also $\|f\| = 1$. Using Lemma 5.1 for a $g \in L^2(\mathbb{R}^n)$ with $\mathcal{T}g(\xi) = f$ for all $\xi \in [-\pi, \pi]^n$, we have

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \int_{[-\pi, \pi]^n} \|J_{V_j}(\xi)f - J_V(\xi)f\|^2 d\xi \\ &\geq \limsup_{j \rightarrow \infty} \int_{[-\pi, \pi]^n} \left| \|J_{V_j}(\xi)f\| - \|J_V(\xi)f\| \right|^2 d\xi \\ &= \limsup_{j \rightarrow \infty} \int_{[-\pi, \pi]^n} |\tau_{V_j, f}^{1/2}(\xi) - \tau_{V, f}^{1/2}(\xi)|^2 d\xi \\ &\geq \frac{1}{4} \limsup_{j \rightarrow \infty} \int_{[-\pi, \pi]^n} |\tau_{V_j, f}(\xi) - \tau_{V, f}(\xi)|^2 d\xi \\ &\geq \frac{1}{4(2\pi)^n} \limsup_{n \rightarrow \infty} \left(\int_{[-\pi, \pi]^n} |\tau_{V_j, f}(\xi) - \tau_{V, f}(\xi)| d\xi \right)^2, \end{aligned}$$

where for the third line we used Proposition 4.2, for the fourth we used the fact that $\tau_{V, f}, \tau_{V_j, f}$ are less than $\|f\|^2 = 1$ (Proposition 4.9 (i)) and for the last one we used Hölder's inequality. \blacksquare

THEOREM 5.3. *Let $(V_j)_{j \in \mathbb{N}}$ be an increasing sequence of SI subspaces of $L^2(\mathbb{R}^n)$,*

$$V := \overline{\bigcup_{j \in \mathbb{N}} V_j},$$

T a positive operator on $l^2(\mathbb{Z}^n)$ and $f \in l^2(\mathbb{Z}^n)$. Then, for a.e. $\xi \in \mathbb{R}^n$, $\tau_{V_j, T}(\xi)$ increases to $\tau_{V, T}(\xi)$ and $\tau_{V_j, f}(\xi)$ increases to $\tau_{V, f}(\xi)$.

Proof. The monotony is a consequence of Proposition 4.11.

Again, we denote by J_{V_j} and J_V the corresponding periodic range function. By Proposition 2.5, $(J_{V_j}(\xi))_{j \in \mathbb{N}}$ is increasing and

$$J_V(\xi) = \overline{\bigcup_{j \in \mathbb{N}} J_{V_j}(\xi)},$$

for almost every $\xi \in \mathbb{R}^n$.

Apply now Proposition 3.10 to our situation to conclude that

$$\tau_{V, T}(\xi) = \text{Trace}(TJ_V(\xi)) = \lim_{j \rightarrow \infty} \text{Trace}(TJ_{V_j}(\xi)) = \lim_{j \rightarrow \infty} \tau_{V_j, T}(\xi).$$

The second statement is a particular case of the first. ■

6. WAVELETS AND THE LOCAL TRACE FUNCTION

We reserved this section for applications of the local trace function to wavelets. Throughout this section A will be an $n \times n$ dilation matrix (i.e. all eigenvalues λ have $|\lambda| > 1$), and A preserves the lattice \mathbb{Z}^n (i.e. $A\mathbb{Z}^n \subset \mathbb{Z}^n$).

REMARK 6.1. The local trace function can be used to obtain the characterization of wavelets (see [5] or [7]), namely:

A set $\Psi = \{\psi^1, \dots, \psi^L\}$ is a NTF wavelet (i.e. the set

$$X(\Psi) := \{D_A^j T_k \psi^l : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l \in \{1, \dots, L\}\}$$

is a NTF for $L^2(\mathbb{R}^n)$) if and only if the following equations hold for a.e. $\xi \in \mathbb{R}^n$:

$$(6.1) \quad \sum_{\psi \in \Psi} \sum_{j=-\infty}^{\infty} |\widehat{\psi}|^2((A^*)^j \xi) = 1,$$

$$(6.2) \quad \sum_{\psi \in \Psi} \sum_{j \geq 0} \widehat{\psi}((A^*)^j \xi) \overline{\widehat{\psi}((A^*)^j(\xi + 2s\pi))} = 0, \quad s \in \mathbb{Z}^n \setminus A^* \mathbb{Z}^n.$$

The argument used here will be the same as the one used in [5], the only difference is that, instead of the Gramian, we employ the local trace function. Here is a sketch of the proof: it is known (see [8]) that $X(\Psi)$ is an affine NTF for $L^2(\mathbb{R}^n)$ if and only if the quasi-affine system $X^q(\Psi)$ is a NTF for $L^2(\mathbb{R}^n)$. Recall that

$$X^q(\Psi) := \{\widetilde{\psi}_{j,k}^l : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l \in \{1, \dots, L\}\},$$

with the convention

$$\widetilde{\psi}_{j,k}(x) = \begin{cases} D_A^j T_k \psi(x), & \text{if } j \geq 0, k \in \mathbb{Z}^n, \\ |\det A|^{j/2} T_k D_A^j \psi(x), & \text{if } j < 0, k \in \mathbb{Z}^n. \end{cases}$$

This can be reformulated as

$$\{\widetilde{\psi}_{j,0}^l : j < 0\} \cup \{\widetilde{\psi}_{j,r}^l : j \geq 0, r \in \mathcal{L}_j\}$$

is a NTF generator for $L^2(\mathbb{R}^n)$ (here \mathcal{L}_j is a complete set of representatives of $\mathbb{Z}^n/A^j\mathbb{Z}^n$).

Now use Corollary 4.5, and all one has to do is a computation. The computation is done in Lemma 2.3 in [5] and one obtains, for a.e. $\xi \in \mathbb{R}^n$,

$$\sum_{\psi \in \Psi} \sum_{j=-\infty}^{\infty} |\widehat{\psi}|^2((A^*)^j(\xi)) = 1,$$

$$t_{(A^*)^{-m}p}((A^*)^{-m}(\xi)) = 0, \quad p \in \mathbb{Z}^n, p \neq 0,$$

where

$$t_s(\xi) := \sum_{\psi \in \Psi} \sum_{j \geq 0} \widehat{\psi}((A^*)^j\xi) \overline{\widehat{\psi}}((A^*)^j(\xi + 2s\pi)), \quad \xi \in \mathbb{R}^n, s \in \mathbb{Z}^n \setminus A^*\mathbb{Z}^n$$

and $m = \max\{j \in \mathbb{Z} : (A^*)^{-j}p \in \mathbb{Z}^n\}$. This leads immediately to the equivalence to (6.1) and (6.2).

Another observation is that, if we know that the equivalence “ $X(\psi)$ is a NTF for $L^2(\mathbb{R}^n)$ if and only if (6.1) and (6.2) hold”, then the argument above also shows that the affine system $X(\psi)$ is a NTF for $L^2(\mathbb{R}^n)$ if and only if $X^q(\psi)$ is a NTF for $L^2(\mathbb{R}^n)$.

THEOREM 6.2. *Let $\Psi = \{\psi^1, \dots, \psi^L\}$ be a semiorthogonal wavelet, i.e. the affine system*

$$\{D_A^j T_k \psi : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \psi \in \Psi\}$$

is a NTF for $L^2(\mathbb{R}^n)$ and $W_i \perp W_j$ for $i \neq j$, where

$$W_j = \overline{\text{span}}\{D_A^j T_k \psi : k \in \mathbb{Z}^n, \psi \in \Psi\} = D_A^j(S(\Psi)), \quad j \in \mathbb{Z},$$

$$V_j = \bigoplus_{i < j} W_i.$$

Then:

(i) *The set*

$$\{|\det A|^{j/2} T_k D_A^j \psi : j < 0, k \in \mathbb{Z}^n\}$$

is a NTF for V_0 .

(ii) *For $\psi \in \Psi$, $j \geq 1$ and $\xi \in \mathbb{R}^n$, denote by $f_\psi^j(\xi)$ the vector in $l^2(\mathbb{Z}^n)$ defined by*

$$f_\psi^j(\xi)(k) = \widehat{\psi}((A^*)^j(\xi + 2k\pi)), \quad k \in \mathbb{Z}^n.$$

Then, for almost every $\xi \in \mathbb{R}^n$, the set

$$\{f_\psi^j(\xi) : j \geq 1, \psi \in \Psi\}$$

is a NTF for $J_{V_0}(\xi)$, where J_{V_0} is the periodic range function of V_0 .

(iii) *For every positive operator T on $l^2(\mathbb{Z}^n)$, every $f \in l^2(\mathbb{Z}^n)$ and almost every $\xi \in \mathbb{R}^n$,*

$$(6.3) \quad \tau_{V_0, f}(\xi) = \sum_{\psi \in \Psi} \sum_{j=1}^{\infty} \langle T f_\psi^j(\xi) | f_\psi^j(\xi) \rangle,$$

$$(6.4) \quad \tau_{V_0, f}(\xi) = \sum_{\psi \in \Psi} \sum_{j=1}^{\infty} \left| \sum_{k \in \mathbb{Z}^n} \widehat{\psi}((A^*)^j(\xi + 2k\pi)) \overline{f}_k \right|^2.$$

Proof. (i) follows, as we mentioned before, from the equivalence between affine frames and quasi-affine frames (see [8] or Theorem 1.4 in [5]) and the orthogonality relations given in the hypothesis.

(ii) If we compute the Fourier transform of $|\det A|^{-j/2} D_A^{-j} \psi$ for $j > 0$ we get $\widehat{\psi}((A^*)^j \xi)$ and (ii) is obtained from (i) and Theorem 2.8.

(iii) The fact asserted in (ii) shows that we can use the vectors f_{ψ}^j to compute the trace and the resulting formulas are exactly (6.3) and (6.4). ■

REMARK 6.3. Another fundamental fact from the theory of wavelets can be obtained with the aid of the local trace function: the dimension function equals the multiplicity function (see [15]). Let's recall some notions. Consider $\Psi := \{\psi^1, \dots, \psi^L\}$ a semi-orthogonal wavelet (as in Theorem 6.2). We keep the notations for V_j and W_j . The dimension function associated to Ψ is

$$D_{\Psi}(\xi) = \sum_{k \in \mathbb{Z}^n} \sum_{\psi \in \Psi} \sum_{j \geq 1} |\widehat{\psi}|^2((A^*)^j(\xi + 2k\pi)).$$

The definition of the multiplicity function requires a little bit of harmonic analysis (for details look in [3] or [1]). The subspace V_0 is invariant under translations by integers (shift invariant). So one has a unitary representation of the locally compact abelian group \mathbb{Z}^n on V_0 by translations. The Stone-Mackey theory shows that this representation is determined by a projection-valued measure which in turn is determined by a positive measure class on the dual group $\widehat{\mathbb{Z}^n}$ and a measurable multiplicity function $m_{V_0} : \widehat{\mathbb{Z}^n} \rightarrow \{0, 1, \dots, \infty\}$. $\widehat{\mathbb{Z}^n}$ can be identified with $[-\pi, \pi]^n$. In our case, the measure is the Lebesgue measure, so the determinant is the multiplicity function m_{V_0} . The beautiful result is

$$(6.5) \quad D_{\Psi}(\xi) = m_{V_0}(\xi), \quad \xi \in [-\pi, \pi]^n.$$

We use the local trace function to prove it. First, if we use the equation (6.3) with $T = I$, the identity on $l^2(\mathbb{Z}^n)$, we obtain

$$(6.6) \quad \dim_{V_0}(\xi) = \tau_{V_0, I}(\xi) = D_{\Psi}(\xi), \quad \xi \in [-\pi, \pi]^n.$$

To equate the local trace function with the multiplicity function we use a result from [2]: if

$$S_j := \{\xi \in [-\pi, \pi]^n : m_{V_0}(\xi) \geq j\}, \quad j \in \mathbb{N}, j \geq 1$$

then there exists a NTF generator $\{\varphi_j : j \geq 1\}$ for V_0 such that

$$(6.7) \quad \sum_{k \in \mathbb{Z}^n} \widehat{\varphi}_i \overline{\widehat{\varphi}_j}(\xi + 2k\pi) = \begin{cases} \chi_{S_i}(\xi), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

We can use Theorem 4.3 for $T = I$ and we have for a.e. ξ :

$$\tau_{V_0, I}(\xi) = \sum_{j \geq 1} \langle \mathcal{T}_{\text{per}} \varphi_j(\xi) | \mathcal{T}_{\text{per}} \varphi_j(\xi) \rangle = \sum_{j \geq 1} \sum_{k \in \mathbb{Z}^n} |\widehat{\varphi}_j|^2(\xi) = \sum_{j \geq 1} \chi_{S_j}(\xi).$$

Therefore

$$(6.8) \quad \dim_{V_0}(\xi) = \tau_{V_0, I}(\xi) = m_{V_0}(\xi), \quad \xi \in [-\pi, \pi]^n.$$

Consequently the multiplicity function and the dimension function are just two disguises of the local trace function at $T = I$.

The next theorem gives the equations that relates scaling functions (i.e. NTF generators for V_0) wavelets.

THEOREM 6.4. *Let $\Psi = \{\psi^1, \dots, \psi^L\}$ be a semi-orthogonal wavelet as in Theorem 6.2. Let Φ be a countable subset of $L^2(\mathbb{R}^n)$. The following affirmations are equivalent:*

- (i) Φ is contained in V_0 and is a NTF generator for V_0 ;
- (ii) the following equations hold for every $s \in \mathbb{Z}^n$,

$$(6.9) \quad \sum_{\psi \in \Psi} \sum_{j \geq 1} \widehat{\psi}((A^*)^j \xi) \overline{\widehat{\psi}((A^*)^j(\xi + 2s\pi))} = \sum_{\varphi \in \Phi} \widehat{\varphi}(\xi) \overline{\widehat{\varphi}(\xi + 2s\pi)},$$

for a.e. $\xi \in \mathbb{R}^n$.

Proof. Use Theorem 4.6 and Theorem 6.2. ■

COROLLARY 6.5. *Let $\Psi = \{\psi^1, \dots, \psi^L\}$ be a semi-orthogonal wavelet as in Theorem 6.2 and Φ a NTF generator for V_0 . Then*

$$(6.10) \quad \sum_{\psi \in \Psi} |\widehat{\psi}|^2(\xi) = \sum_{\varphi \in \Phi} |\widehat{\varphi}|^2((A^*)^{-1}\xi) - \sum_{\varphi \in \Phi} |\widehat{\varphi}|^2(\xi), \quad \text{a.e. on } \mathbb{R}^n.$$

For all $s \in \mathbb{Z}^n \setminus A^*\mathbb{Z}^n$,

$$(6.11) \quad \sum_{\psi \in \Psi} \widehat{\psi}(\xi) \overline{\widehat{\psi}(\xi + 2s\pi)} = - \sum_{\varphi \in \Phi} \widehat{\varphi}(\xi) \overline{\widehat{\varphi}(\xi + 2s\pi)}, \quad \text{a.e. on } \mathbb{R}^n.$$

Proof. For (6.10), take $s = 0$, write the equation (6.9) for ξ and $(A^*)^{-1}\xi$, and subtract the first from the second. For (6.11), subtract equation (6.9) from equation (6.2). ■

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