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CHARACTERIZING LIMINAL AND TYPE I GRAPH C^*-ALGEBRAS

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ABSTRACT. We prove that the C^*-algebra of a directed graph E is liminal if and only if the graph satisfies the finiteness condition: if p is an infinite path or a path ending with a sink or an infinite emitter, and if v is any vertex, then there are only finitely many paths starting with v and ending with a vertex in p. Moreover, $C^*(E)$ is type I precisely when the circuits of E are either terminal or transitory, i.e., E has no vertex which is on multiple circuits, and E satisfies the weaker condition: for any infinite path λ, there are only finitely many vertices of λ that get back to λ in an infinite number of ways.

KEYWORDS: *Directed graph, Cuntz-Krieger algebra, graph algebra.*

MSC (2000): 46L05, 46L35, 46L55.

1. INTRODUCTION

A directed graph $E = (E^0, E^1, o, t)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $o, t : E^1 \to E^0$ identifying the origin (source) and the terminus (range) of each edge. The graph is row-finite if each vertex emits at most finitely many edges. A vertex is a sink if it is not an origin of any edge. A vertex v is called singular if it is either a sink or emits infinitely many edges. A path is a sequence of edges $e_1e_2 \cdots e_n$ with $t(e_i) = o(e_{i+1})$ for each $i = 1, 2, \ldots, n-1$. An infinite path is a sequence $e_1e_2 \cdots$ of edges with $t(e_i) = o(e_{i+1})$ for each i.

For a finite path $p = e_1e_2 \cdots e_n$, we define $o(p) := o(e_1)$ and $t(p) := t(e_n)$. For an infinite path $p = e_1e_2 \cdots$, we define $o(p) := o(e_1)$. We regard vertices as paths of length zero, and hence if $v \in E^0, o(v) = v = t(v)$. Define:

$E^* = \bigcup\limits_{n=0}^{\infty} E^n$, where $E^n := \{p : p \text{ is a path of length } n\}$;

$E^{**} := E^* \cup E^\infty$, where E^∞ is the set of infinite paths.

A Cuntz-Krieger E-family consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ satisfying:

(1) $p_{t(e)} = s_e^* s_e \ \forall e \in E^1$;

(2) $\sum_{e \in F} s_e s_e^* \leqslant p_v \ \forall v \in E^0$ and for any finite subset F of $\{e \in E^1 : o(e) = v\}$;

(3) $\sum_{o(e)=v} s_e s_e^* = p_v$ for each non-singular vertex $v \in E^0$.

The graph C^*-algebra $C^*(E)$ is the universal C^*-algebra generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$.

For a finite path $\mu = e_1 e_2 \cdots e_n$, we write s_μ for $s_{e_1} s_{e_2} \cdots s_{e_n}$.

Since the family $\{s_\mu s_\nu^* : \mu, \nu \in E^*\}$ is closed under multiplication, we have:

$$C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^* \text{ and } t(\mu) = t(\nu)\}.$$

The outline of the paper is as follows. In Section 2 we introduce the basic notation and conventions we will use throughout the paper. Section 3 deals with row-finite graphs with no sinks. We begin the section by defining a property of a graph which we later prove to characterize liminal graph C^*-algebras when the graph has no singular vertices. Section 4 provides us with a proposition that gives a method on how to obtain the largest liminal ideal of a C^*-algebra of a row-finite graph with no sinks. In Sections 5, respectively 6, using 'desingularizing graphs' of [4], we generalize the results of Sections 3, respectively 4, to arbitrary graphs. In Section 7 we give a characterization for type I graph C^*-algebras. We finish the section with a proposition on how to obtain the largest type I ideal of a graph C^*-algebra.

2. PRELIMINARIES

Given a directed graph E, we write $v \geqslant w$ if there is a directed path from v to w.

For a directed graph E, we say that $H \subseteq E^0$ is hereditary if $v \in H$ and $v \geqslant w$ imply that $w \in H$. We say that H is saturated if v is not singular and $\{w \in E^0 : v \geqslant w\} \subseteq H$ imply that $v \in H$.

If $z \in \mathbb{T}$, then the family $\{zs_e, p_v\}$ is another Cuntz-Krieger E-family which generates $C^*(E)$, and the universal property gives a homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. This γ_z is a strongly continuous action, called *gauge action*, on $C^*(E)$. See [1] for details.

Let E be a row-finite directed graph, let I be an ideal of $C^*(E)$, and let $H = \{v : p_v \in I\}$. In Lemma 4.2 in [1] they proved that H is a hereditary saturated subset of E^0. Moreover, if $I_H := \overline{\operatorname{span}}\{S_\alpha S_\beta^* : \alpha, \beta \in E^* \text{ and } t(\alpha) = t(\beta) \in H\}$, the map $H \mapsto I_H$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of closed gauge-invariant ideals of $C^*(E)$ ([1], Theorem 4.1 (a)). Letting $F := F(E \setminus H) =$ the subgraph of E that is gotten by removing H and all edges that point into H, it is proven in Theorem 4.1 (b) of [1] that $C^*(F) \cong C^*(E)/I_H$. In case I is not a gauge-invariant ideal, we only get $I_H \subsetneqq I$.

We will use the following notation and conventions.

(i) Every path we take is a directed path.

(ii) A circuit in a graph E is a finite path p with $o(p) = t(p)$. We save the term loop for a circuit of length 1.

(iii) We say that a circuit is terminal if it has no exits, and a circuit is transitory if it has an exit and no exit of the circuit gets back to the circuit.

(iv) $\Lambda_E := \{v \in E^0 : v \text{ is a singular vertex}\}$.

(v) $\Lambda_E^* := t^{-1}(\Lambda_E) \cap E^*$ i.e., Λ_E^* is the set of paths ending with a singular vertex. When there are no ambiguities, we will just use Λ^*.

(vi) We say v gets to w (or reaches w) if there is a path from v to w.

(vii) We say v gets to a path p if v gets to a vertex in p.

(viii) For a subset S of E^0, we write $S \geqslant v$ if $w \geqslant v,\ \forall\, w \in S$.

(ix) For a subset H of E^0, we write Graph(H) to refer to the subgraph of E whose set of vertices is H and whose edges are those edges of E that begin and end in H.

(x) $V(v) := \{w \in E^0 : v \geqslant w\}$.

(xi) $E(v) := \mathrm{Graph}(V(v))$, i.e., $E(v)$ is the subgraph of E that the vertex v can 'see'. Accordingly we use $F(v)$, etc. when the graph is F, etc.

(xii) For $v \in E^0$ let $\Delta(v) := \{e \in E^1 : o(e) = v\}$.

(xiii) For a finite subset F of $\Delta(v)$, we write $V(v;F) := \{v\} \cup \bigcup_{e \in \Delta(v)\setminus F} V(t(e))$.

(xiv) $E(v;F) := \mathrm{Graph}(V(v;F))$.

(xv) For a hereditary subset H of E^0, we write $\overline{H}$ to refer to the saturation of H, i.e. the smallest saturated set containing H. Notice that $\overline{H}$ is hereditary and saturated.

(xvi) For any path λ, λ^0 will denote the vertices of λ.

(xvii) As was used above, $F(E \setminus H)$ will denote the subgraph of E that is obtained by removing H and all edges that point into H.

(xviii) We use $\mathcal{K}$ to denote the space of compact operators on an (unspecified) separable Hilbert space.

3. LIMINAL C^*-ALGEBRAS OF GRAPHS WITH NO SINGULAR VERTICES

We begin this section by a definition.

DEFINITION 3.1. A subset γ of E^0 is called a *maximal tail* if it satisfies the following three conditions:

(i) for any $v_1, v_2 \in \gamma$ there exists $z \in \gamma$ such that $v_1 \geqslant z$ and $v_2 \geqslant z$;

(ii) for any $v \in \gamma$, $\exists e \in E^1$ such that $o(e) = v$ and $t(e) \in \gamma$;

(iii) $v \geqslant w$ and $w \in \gamma$ imply that $v \in \gamma$.

We will prove a result similar to (one direction of) Proposition 6.1 in [1], Proposition 6.1 with a weaker assumption on the graph E and a weaker assumption on the ideal.

LEMMA 3.2. *Let E be a row-finite graph with no sinks. If I is a primitive ideal of $C^*(E)$ and $H = \{v \in E^0 : p_v \in I\}$, then $\gamma = E^0 \setminus H$ is a maximal tail.*

Proof. By Lemma 4.2 in [1], H is hereditary and saturated. The complement of a hereditary set satisfies (iii) in Definition 3.1. Since E has no sinks, and H is saturated, γ satisfies (ii). We prove now (i). Let $v_1, v_2 \in \gamma$ and let $H_i = \{v \in \gamma : v_i \geqslant v\}$. We will first show that $\overline{H}_1 \cap \overline{H}_2 \neq \emptyset$. Let $F = F(E \setminus H)$. For each i, $I_{\overline{H}_i}$ is a non-zero ideal of $C^*(F) \cong C^*(E)/I_H$, hence is of the form I_i/I_H, and $p_{v_i} + I_H \in I_{\overline{H}_i}$. Since each $I_{\overline{H}_i}$ is gauge-invariant, so is $I_{\overline{H}_1} \cap I_{\overline{H}_2}$. Therefore $I_{\overline{H}_1} \cap I_{\overline{H}_2} = I_{\overline{H}_1 \cap \overline{H}_2}$. If $\overline{H}_1 \cap \overline{H}_2 = \emptyset$, then $I_{\overline{H}_1} \cap I_{\overline{H}_2} = \{0\} \subseteq I/I_H$. But

I/I_H is a primitive ideal of $C^*(E)/I_H$, therefore $I_1/I_H \subseteq I/I_H$ or $I_2/I_H \subseteq I/I_H$. Without loss of generality, let $I_1/I_H \subseteq I/I_H$ hence $p_{v_1} + I_H \in I/I_H$ implying that $p_{v_1} \in I_H$ or $p_{v_1} \in I \setminus I_H$. But $p_{v_1} \in I \setminus I_H$ is a contradiction to the construction of H, and $p_{v_1} \in I_H$, which implies that $v_1 \in H$, which is again a contradiction to $v_1 \in \gamma = E^0 \setminus H$. Therefore $\overline{H}_1 \cap \overline{H}_2 \neq \emptyset$. Let $y \in \overline{H}_1 \cap \overline{H}_2$. Applying Lemma 6.2 in [1] to F and v_1 shows that $\exists x \in E^0 \setminus H$ such that $y \geqslant x$ and $v_1 \geqslant x$. Since $y \in \overline{H}_2$ and $\overline{H}_2$ is hereditary, $x \in \overline{H}_2$. Applying Lemma 6.2 in [1] to F and v_2 shows that $\exists z \in E^0 \setminus H$ such that $y \geqslant z$ and $v_2 \geqslant z$. Thus $v_1 \geqslant z$, and $v_2 \geqslant z$ as needed. ∎

Now, we prove that for a row-finite graph E with no sinks, $C^*(E)$ is liminal precisely when the following finiteness condition is satisfied: for any vertex v and any infinite path λ, there is only a finite number of ways to get from v to λ.

To state the finiteness condition more precisely, we will use the equivalence relation defined in Definition 1.8 in [8].

If $p = e_1e_2\cdots$ and $q = f_1f_2\cdots \in E^\infty$, we say that $p \sim q$ if and only if $\exists j, k$ so that $e_{j+r} = f_{k+r}$ for all $r \geqslant 0$, i.e., if and only if p and q (eventually) share the same tail.

We use $[p]$ to denote the equivalence class containing p.

DEFINITION 3.3. A row-finite directed graph E that has no sinks is said *to satisfy Condition* (M) if for any $v \in E^0$ and any $[p] \in E^\infty/\sim$ there is only a finite number of representatives of $[p]$ that begin with v.

We note that if E satisfies Condition (M) then every circuit in E is terminal.

LEMMA 3.4. *Let E be a row-finite directed graph with no sinks that satisfies Condition* (M)*. Let F be a subgraph of E so that F^0 is a maximal tail. If F has a circuit, say α, then the saturation of α^0, $\overline{\alpha}^0$, is equal to F^0.*

Proof. Let v_α be a vertex of α. Since α is terminal, $v_\alpha \geqslant z$ implies that z is in α^0. Also, for each $w \in F^0$, by (i) of Definition 3.1, there exists $z \in F^0$ such that $w \geqslant z$ and $v_\alpha \geqslant z$, but z is in α^0 which implies that $z \geqslant v_\alpha$. Therefore $w \geqslant v_\alpha$, i.e., each vertex in F^0 connects to v_α (via a directed path).

Now, assuming the contrary, let $v_1 \notin \overline{\alpha}^0$. Suppose v_1 is in a circuit, say β. Then, by the previous paragraph, $v_1 \geqslant v_\alpha$ hence either $\beta = \alpha$ or β has an exit. But $v_1 \notin \overline{\alpha}^0$, therefore $\beta = \alpha$ is not possible, and since F satisfies Condition (M), β can not have an exit. Thus v_1 is not in a circuit. Therefore $\exists e_1 \in F^1$ such that $o(e_1) = v_1, t(e_1) \notin \overline{\alpha}^0$. Let $v_2 = t(e_1)$. Inductively, $\exists e_n \in F^1$ such that $v_n = o(e_n), t(e_n) = v_{n+1} \notin \overline{\alpha}^0$. Look at the infinite path $e_1e_2\cdots$.

Notice that the v_i's are distinct and each $v_i \geqslant v_\alpha$. Therefore there are infinitely many ways to get to α from v_1, i.e., there are infinitely many representatives of $[\alpha]$ that begin with v_1, which contradicts to the assumption that E satisfies Condition (M). Therefore $F^0 = \overline{\alpha}^0$. ∎

LEMMA 3.5. *Let E be a row-finite directed graph with no sinks that satisfies Condition* (M). *Let F be a subgraph of E so that F^0 is a maximal tail. If F has no circuits then F has a hereditary infinite path, say λ, such that $F^0 = \overline{\lambda}^0$.*

Proof. Since F has no sinks, it must have an infinite path, say λ. Let v_λ be a vertex in λ. By Condition (M), there are only a finite number of infinite paths that begin with v_λ and share a tail with λ. By going far enough on λ, there exists $w \in \lambda^0$ such that $v_\lambda \geqslant w$ and $[\lambda]$ has only one representative that begins with w. By re-selecting v_λ (to be w, for instance) we can assume that there is only one representative of $[\lambda]$ that begins with v_λ. We might, as well, assume that $o(\lambda) = v_\lambda$.

We will now prove that λ^0 is hereditary. Suppose $u \in F^0$ such that $v_\lambda \geqslant u$ and $u \notin \lambda^0$. Since F^0 is a maximal tail and since F has no circuits, by (ii) of Definition 3.1 we can choose $w_1 \in F^0$ such that $v_\lambda \geqslant w_1$ and $v_\lambda \neq w_1$. By (i) of Definition 3.1 there exists $z_1 \in F^0$ such that $u \geqslant z_1$, and $w_1 \geqslant z_1$. If $z_1 \in \lambda^0$ then we have two ways to get to λ from v_λ (through u and through w_1) which contradicts to the choice of v_λ, hence $z_1 \notin \lambda^0$.

Let $w_2 \in \lambda^0$ (far enough) so that $w_2 \ngeqslant z_1$. If such a choice was not possible, we would be able to get to z_1 and hence to any path that begins with z_1 from v_λ in an infinite number of ways, contradicting Condition (M).

Again since F^0 is a maximal tail, there exists $z_2 \in F^0$ such that $w_2 \geqslant z_2$ and $z_1 \geqslant z_2$. Notice that there are (at least) two ways to get to z_2 from v_λ. By inductively choosing a $w_n \in \lambda^0$ and a $z_n \in F^0$ such that $w_n \ngeqslant z_{n-1}, w_n \geqslant z_n$ and $z_{n-1} \geqslant z_n$, there are at least n ways to get to z_n from v_λ (one through w_n and $n-1$ through z_{n-1}).

We now form an infinite path α that contains $z_1, z_2, \dots$ as (some of) its vertices that we can reach to, from v_λ, in an infinite number of ways, which is again a contradiction. Hence no such u can exist. Thus λ is hereditary.

We will now prove that $F^0 = \overline{\lambda}^0$. Assuming the contrary, let $v_1 \notin \overline{\lambda}^0$. Then $\exists e_1 \in F^1$ such that $o(e_1) = v_1, t(e_1) \notin \overline{\lambda}^0$. Inductively, let $v_n = t(e_{n-1})$, then $\exists e_n \in F^1$ such that $v_n = o(e_n), t(e_n) = v_{n+1} \notin \overline{\lambda}^0$. Consider the infinite path $e_1 e_2 \cdots$.

Notice that since F^0 is a maximal tail, for each v_i, $\exists x_i$ such that $v_i \geqslant x_i$ and $v_\lambda \geqslant x_i$. But λ^0 is hereditary hence $x_i \in \lambda^0$, implying that each v_i reaches λ. Therefore there are infinitely many ways to get to λ from v_1, i.e., there are infinitely many representatives of $[\lambda]$ that begin with v_1 which contradicts to Condition (M). Therefore $F^0 = \overline{\lambda}^0$. ∎

LEMMA 3.6. *Let I_H be a primitive ideal of $C^*(E)$, where H is a hereditary saturated subset of E^0. Let $F = F(E \setminus H)$. Then F has no circuits.*

Proof. Note that F^0 is a maximal tail and $C^*(F) \cong C^*(E)/I_H$. Since I_H is a primitive ideal of $C^*(E), \{0\}$ is a primitive ideal of $C^*(F)$.

Suppose that F has a circuit, say α. By Lemma 3.4, $F^0 = \overline{\alpha}^0$. Hence $C^*(F) \cong I_{\alpha^0}$ = the ideal of $C^*(F)$ generated by $\{\alpha^0\}$. Since α has no exits (is hereditary), by Proposition 2.1 in [5], I_{α^0} is Morita equivalent to $C^*(\alpha)$ which is Morita equivalent to $C(\mathbb{T})$. But $\{0\}$ is not a primitive ideal of $C(\mathbb{T})$ implying that $\{0\}$ is not a primitive ideal of $C^*(F)$ which is a contradiction. Hence F has no circuits. ∎

Hidden in the proofs of Lemma 3.5 and Lemma 3.6 we have proven a (less relevant) fact: if a directed graph E with no singular vertices satisfies Condition (M) and F^0 is a maximal tail then F has (essentially) one infinite tail, i.e., $F^\infty / \sim$ contains a single element.

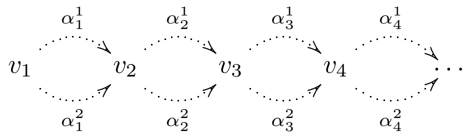
REMARK 3.7. Let E_1 be a subgraph of a directed graph E_2. Applying Theorem 2.34 of [8] and Corollary 2.33 of [8], we observe that $C^*(E_1)$ is a quotient of a C^*-subalgebra of $C^*(E_2)$. (Letting $S_2 = E_2^0 \setminus \{v \in E_2^0 : v \text{ is a singular vertex}\}$.)

LEMMA 3.8. *Let E be a directed graph. Suppose all the circuits of E are transitory and suppose $\exists \lambda \in E^\infty$ such that the number of vertices of λ that emit multiple edges that get back to λ is infinite. Then $C^*(E)$ is not type* I.

Proof. Let $v_1 \in \lambda^0$ such that v_1 emits (at least) two edges that get back to λ. Choose a path $\alpha_1^1 = e_1 e_2 \cdots e_{n_1}$, such that e_1 is not in $\lambda, o(e_1) = v_1$ and $t(\alpha_1^1) = v_2 \in \lambda^0$. If $t(e_{n_1}) = v_1$, i.e. $e_1 e_2 \cdots e_{n_1}$ is a circuit, we extend $e_1 e_2 \cdots e_{n_1}$ so that v_2 is further along λ than v_1 is.

We might again extend α_1^1 along λ, if needed, and assume that v_2 emits (at least) two edges that get back to λ.

Let α_1^2 be the path along λ such that $o(\alpha_1^2) = v_1$ and $t(\alpha_1^2) = v_2$. Inductively, choose $\alpha_k^1 = e_1 e_2 \cdots e_{n_k}$ such that e_1 is not in λ, $o(e_1) = v_k, t(\alpha_k^1) = v_{k+1} \in \lambda^0$; by extending α_k^1, if needed, we can assume that v_{k+1} is further along λ than v_k and emits multiple edges that get back to λ. Let α_k^2 be the path along λ such that $o(\alpha_k^2) = v_k$ and $t(\alpha_k^2) = v_{k+1}$. Now look at the following subgraph of E, call it F:



Now let $\{s_e, p_v : e \in F^1, v \in F^0\}$ be a Cuntz-Krieger F-family.

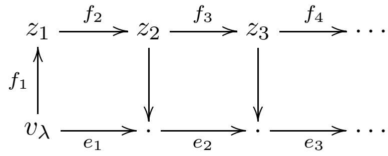
Thus $C^*(F) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in F^* \text{ and } t(\mu) = t(\nu)\}$.

Let $F_k := \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \text{ are paths made up of } \alpha_i^r\text{'s (or just } v_k) \text{ such that } t(\mu) = v_k = t(\nu)\}$.

By Corollary 2.3 in [5], $F_k \cong M_{N_k}(\mathbb{C})$ where N_k is the number of paths made up of α_i^r's (or just v_k) ending with v_k, which is finite.

Also, if $s_\mu s_\nu^* \in F_k$ then $s_\mu s_\nu^* = s_\mu p_{v_k} s_\nu^* = s_\mu s_{\alpha_k^1} s_{\alpha_k^1}^* s_\nu^* + s_\mu s_{\alpha_k^2} s_{\alpha_k^2}^* s_\nu^* \in F_{k+1}$. Hence $F_k \subsetneq F_{k+1}$. Let $\mathcal{A} = \overline{\bigcup_{k=1}^{\infty} F_k}$. Then $\mathcal{A}$ is a C^*-subalgebra of $C^*(F)$. Since $\mathcal{A}$ is a UHF algebra, it is not type I. Therefore $C^*(F)$ has a C^*-subalgebra that is not type I and can not be type I. Since F is a subgraph of E, by Remark 3.7 $C^*(E)$ has a C^*-subalgebra whose quotient is not type I. Therefore $C^*(E)$ is not type I. ∎

It might be useful to keep following graph in mind when reading Lemma 3.9; it can be viewed as a prototype of a graph that satisfies the assumption of the lemma:



Lemma 3.9. *Let E be a directed graph, let $\lambda = e_1e_2\cdots$ be an infinite path in E, and let $o(\lambda) = v_\lambda$. Suppose:*

(i) *E has no circuits;*
(ii) *the number of representatives of* $[\lambda]$ *that begin with v_λ is infinite;*
(iii) *v_λ is the only such vertex in λ^0;*
(iv) *$E = E(v_\lambda)$;*
(v) $\forall v \in E^0\ \exists w \in \lambda^0$ *such that $v \geqslant w$ (i.e., $E^0 \geqslant \lambda^0$).*

Then

(a) $\{0\}$ *is a primitive ideal of $C^*(E)$;*
(b) *$C^*(E)$ is not simple.*

Proof. We prove (a). First note that E satisfies Condition (K) of [1]: every vertex lies on either no circuits or at least two circuits. This is because E has no circuits. We will show that E^0 is a maximal tail. Since E has no sinks, E^0 satisfies (ii) of Definition 3.1, and clearly E^0 satisfies (iii). We will show that E satisfies (i). Let $v_1, v_2 \in E^0$. By (iv) above, $\exists w_1, w_2 \in \lambda^0$ such that $v_i \geqslant w_i$. Since λ is is an infinite path, either $w_1 \geqslant w_2$ or $w_2 \geqslant w_1$. Without loss of generality let $w_2 \geqslant w_1$. We have $v_1 \geqslant w_1$ and $v_2 \geqslant w_2 \geqslant w_1$, hence (i) is satisfied. Therefore E^0 is a maximal tail and, by Proposition 6.1 in [1], $I_\emptyset = \{0\}$ is a primitive ideal of $C^*(E)$.

We will prove (b). Since E is row finite and since v_λ gets to λ infinitely often, $\exists f_1 \in E^1$ such that $o(f_1) = v_\lambda$ and $z_1 := t(f_1)$ gets to λ infinitely often. Moreover, there is no vertex in λ^0 that gets to λ infinitely often except v_λ and E has no circuits, therefore $z_1 \notin \lambda^0$. Inductively, $\exists f_{n+1} \in E^1$ such that $o(f_{n+1}) = z_n$, $z_{n+1} := t(f_{n+1})$ gets to λ infinitely often, and $z_{n+1} \notin \lambda^0$. Notice that the number of representatives of $[\lambda]$ that begin with $t(e_1)$, by (ii) above, is finite. Therefore $t(e_1)$ does not get to any of the z_i's, that is, $t(e_1)$ does not reach the infinite path $f_1f_2\cdots$. Thus E is not co-final. Therefore $C^*(E)$ is not simple. ∎

We are now ready to prove the first of the measure results.

Theorem 3.10. *Let E be a row-finite directed graph with no sinks. $C^*(E)$ is liminal if and only if E satisfies Condition* (M).

Proof. Suppose E satisfies (M). Let I be a primitive ideal of $C^*(E)$, let $H = \{v : p_v \in I\}$, and let $F = F(E \setminus H)$. By Lemma 3.2, F^0 is a maximal tail, and Theorem 4.1 (b) of [1] implies that $C^*(F) \cong C^*(E)/I_H$.

Case 1. $I = I_H$.

Then I_H is a primitive ideal, hence Lemma 3.6 implies that F has no circuits. Using Lemma 3.5, let λ be a hereditary infinite path such that $F^0 = \overline{\lambda}^0$.

$C^*(E)/I_H \cong C^*(F) = I_\lambda = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^*$, such that $t(\alpha) = t(\beta) \in \lambda^0\}$.

By Proposition 2.1 in [5], I_λ is Morita equivalent to $C^*(\lambda) \cong \mathcal{K}(\ell^2(\alpha^0))$. Therefore $C^*(E)$ is liminal.

Case II. $I_H \subsetneq I$.

We will first prove that $F(E \setminus H)$ has a circuit. If $F = F(E \setminus H)$ has no circuits, then by Lemma 3.5 $F^0 = \overline{\lambda}^0$ for some hereditary infinite path λ. Therefore $C^*(F)$ is simple, implying that $C^*(E)/I_H$ is simple. But I/I_H is a (proper) ideal of $C^*(E)/I_H$ therefore $I/I_H = 0$ implying that $I = I_H$. A contradiction.

Hence F must have a circuit, say α. Lemma 3.4 implies that $F^0 = \overline{\alpha}^0$. Using Proposition 2.1 in [5], $C^*(F)$ is Morita equivalent to $C^*(\alpha)$ which is Morita equivalent to $C(\mathbb{T})$ which is liminal. Therefore $C^*(E)/I_H$ is liminal. Since I/I_H is a primitive ideal of $C^*(E)/I_H$ we get $C^*(E)/I \cong (C^*(E)/I_H)/(I/I_H) \cong \mathcal{K}$. Hence $C^*(E)$ is liminal.

To prove the converse, suppose E does not satisfy Condition (M), i.e., there exist an infinite path λ and a $v_\lambda \in E^0$ such that the number of representatives of $[\lambda]$ that begin with v_λ is infinite.

Suppose that E has a non-terminal circuit, say α. Let v be a vertex of α such that $\exists e \in E^1$ which is not an edge of α and $o(e) = v$. Then $p_v = s_\alpha^* s_\alpha \sim s_\alpha s_\alpha^* < s_\alpha s_\alpha^* + s_e s_e^* \leqslant p_v$. Therefore p_v is an infinite projection. Hence $C^*(E)$ cannot be liminal.

Suppose now that all circuits of E are terminal and that the number of representatives of $[\lambda]$ that begin with v_λ is infinite. We might assume that $v_\lambda = o(\lambda)$. We want to prove that $C^*(E)$ is not liminal. If v is a vertex such that $V(v)$ does not intersect λ^0, we can factor $C^*(E)$ by the ideal generated by $\{v\}$. Hence we might assume that $\forall v \in E^0, v \geqslant \lambda^0$. Moreover, this process gets rid of any terminal circuits, and hence we may assume that E has no circuits.

Also, since $V(v_\lambda)$ is hereditary, by Proposition 2.1 in [5], $I_{V(v_\lambda)}$ is Morita equivalent to $C^*(E(v_\lambda))$. Therefore it suffices to show that $C^*(E(v_\lambda))$ is not liminal. Hence we might assume that $E = E(v_\lambda)$.

If for every $v \in \lambda^0$ there exists $w \in \lambda^0$ such that $v \geqslant w$ and $|\{e \in E^1 : o(e) = w\}| \geqslant 2$, then by Lemma 3.8 $C^*(E)$ is not type I, therefore it is not liminal.

Suppose $\exists u \in \lambda^0$ such that $\forall w \in \lambda^0$ with $u \geqslant w, |\{e \in E^1 : o(e) = w\}| = 1$. Notice that there is exactly one representative of $[\lambda]$ that begins with u.

By re-selecting v_λ further along on λ, we might assume that $\forall w \in \lambda^0 \setminus \{v_\lambda\}$ the number of representatives of $[\lambda]$ that begin with w is finite.

Thus E satisfies the following conditions:

(i) E has no circuits;
(ii) the number of representatives of $[\lambda]$ that begin with v_λ is infinite;
(iii) v_λ is the only such vertex in λ^0;
(iv) $E = E(v_\lambda)$; and
(v) $\forall v \in E^0\ \exists w \in \lambda^0$ such that $v \geqslant w$ (i.e., $E^0 \geqslant \lambda^0$).

Therefore by Lemma 3.9 we get:

(a) $\{0\}$ is a primitive ideal of $C^*(E)$;
(b) $C^*(E)$ is not simple.

If $C^*(E)$ is liminal, by (a), since $\{0\}$ is a primitive ideal of $C^*(E)$, $C^*(E) \cong C^*(E)/\{0\}$ is $*$-isomorphic to $\mathcal{K}$. But from (b) $C^*(E)$ cannot be $*$-isomorphic to $\mathcal{K}$ because $\mathcal{K}$ is a simple C^*-algebra. Therefore $C^*(E)$ cannot be liminal. This concludes the proof of the theorem. ∎

4. THE LARGEST LIMINAL IDEAL OF C^*-ALGEBRAS OF GRAPHS WITH NO SINGULAR VERTICES

In this section we will investigate a method of extracting the largest liminal ideal of the C^*-algebra of a row finite graph E with no sinks.

Before we state the proposition, we will extend the definition of the equivalence $\sim$ from E^∞ to $E^{**} = E^\infty \cup E^*$, as it is done in Remark 1.10 in [8]. For $p, q \in E^*$, we say $p \sim q$ if $t(p) = t(q)$.

The proposition gives a method of extracting the largest liminal ideal of $C^*(E)$ of a graph E with no singular vertices by giving a characterization of the set of vertices that generate the ideal. The first part of the proposition, which will eventually be needed, can be proven for a general graph without much complication. Therefore we state that part of the proposition for a general graph.

PROPOSITION 4.1. *Let E be a directed graph and $H = \{v \in E^0 : \forall[\lambda] \in (E^\infty \cup \Lambda^*)/\sim$, the number of representatives of $[\lambda]$ that begin with v is finite$\}$. Then:*

(i) *H is hereditary and saturated.*
(ii) *If E is row-finite with no sinks then I_H is the largest liminal ideal of $C^*(E)$.*

Proof. Suppose $v \in H$ and $v \geqslant w$. Let p be a path from v to w and let $\lambda \in E^\infty \cup \Lambda^*$. If $\beta \sim \lambda$ and $o(\beta) = w$ then $p\beta \sim \lambda$ and $o(p\beta) = v$. Therefore the number of representatives of $[\lambda]$ that begin with w is less than or equal to the number of representatives of $[\lambda]$ that begin with v. Therefore $w \in H$. Thus H is hereditary.

Suppose $v \in E^0$ is not singular and $\{w \in E^0 : v \geqslant w\} \subseteq H$. Let $\triangle(v) = \{e \in E^1 : o(e) = v\}$. Note that $\triangle(v)$ is a finite set and $\forall e \in \triangle(v), t(e) \in H$. Let $\lambda \in E^\infty \cup \Lambda^*$ and $\beta \sim \lambda$ where $o(\beta) = v$. Then the first edge of β is in $\triangle(v)$. Therefore the number of representatives of $[\lambda]$ that begin with v is equal to the sum of the number of representatives of $[\lambda]$ that begin with a vertex in $\{t(e) : e \in \triangle(v)\}$, which is a finite sum of finite numbers. Therefore $v \in H$. Hence H is saturated.

To prove (ii), suppose E is row-finite with no sinks. Let $F = \text{Graph}(H)$. Clearly F satisfies Condition (M). Hence Theorem 3.10 implies that $C^*(F)$ is liminal. By Proposition 2.1 of [5], I_H is Morita equivalent to $C^*(F)$. Hence I_H is a liminal ideal. What remains is to prove that I_H is the largest liminal ideal of $C^*(E)$.

Let I be the largest liminal ideal of $C^*(E)$. Thus $I_H \subseteq I$. Since the largest liminal ideal of a C^*-algebra is invariant under automorphisms, I is gauge invariant, therefore $I = I_K$ for some saturated hereditary subset K of E^0 which includes

H. We will prove that $K \subseteq H$. Let $G = \text{Graph}(K)$. Since I_K is Morita equivalent to $C^*(G)$, $C^*(G)$ is liminal hence, by Theorem 3.10, G satisfies Condition (M). Let $v \in K = G^0$. If $\beta \in E^\infty$ with $o(\beta) = v$, $\beta^0 \subseteq K$ because K is hereditary. Therefore $\beta \in G^\infty$. Now let $[\lambda] \in E^\infty/\sim$, and let γ be a representative of $[\lambda]$ that begins with v. (If no such γ exists then the number of representatives of $[\lambda]$ is zero.) Then $\{\beta \in E^\infty : \beta \sim \lambda, o(\beta) = v\} = \{\beta \in G^\infty : \beta \sim \gamma, o(\beta) = v\}$, i.e., the set of representatives of $[\lambda]$ that begin with v is subset of the set of representatives of $[\gamma]$ (as an equivalence class of $G^\infty/\sim$) that begin with v. Since G satisfies Condition (M) the second set is finite. Therefore $v \in H$, implying that $K \subseteq H$. Therefore $I_H = I_K$. ∎

5. Liminal C^*-algebras of general graphs

In this section we will consider a general graph E and give the necessary and sufficient conditions for $C^*(E)$ to be liminal in terms of the properties of the graph.

In [4] the authors gave a recipe on how to "desingularize a graph E", that is, obtain a graph F that has no singular vertices (by adding a tail at every singular vertex of E) so that $C^*(E)$ and $C^*(F)$ are Morita equivalent. Therefore, we will use this idea of desingularizing E and use the results of the previous sections to get the needed results.

We will begin by extending the definition of Condition (M) from row-finite graphs with no sinks to general graphs:

Definition 5.1. A graph E is said to satisfy Condition (M) if $\forall[p] \in (E^\infty \cup \Lambda^*)/\sim$ and any $v \in E^0$, the number of representatives of $[p]$ that begin with v is finite.

Notice that when E is a row-finite graph with no sinks, Definition 3.3 and Definition 5.1 say the same thing.

Since we need to use the results of the previous sections, it is important to check that Condition (M) is preserved by the desingularization process. We will do that in the next two lemmas.

Remark 5.2. Lemma 2.6 (i) of [4] states that if F is a desingularization of a directed graph E then there are bijective maps:

$$\varphi : E^* \to \{\beta \in F^* : o(\beta), t(\beta) \in E^0\} \quad \text{and} \quad \varphi_\infty : E^\infty \cup \Lambda^* \to \{\lambda \in F^\infty : o(\lambda) \in E^0\}.$$

The map φ preserves origin and terminus (and hence preserves circuits). The map φ_∞ preserves origin.

Lemma 5.3. *The map φ_∞ preserves the equivalence, in fact, for $p, q \in E^\infty \cup \Lambda^*$, $p \sim q$ if and only if $\varphi_\infty(p) \sim \varphi_\infty(q)$.*

Proof. Observe that if $\mu\nu \in E^\infty \cup \Lambda^*$ where $\mu \in E^*$ then $\varphi_\infty(\mu\nu) = \varphi(\mu)\varphi_\infty(\nu)$ and $\varphi_\infty(\nu) \in F^\infty$.

Now let $p = e_1e_2\cdots$, $q = f_1f_2\cdots \in E^\infty$ such that $p \sim q$, $\exists i, j$ such that $e_{i+r} = f_{j+r}$ $\forall r \in \mathbb{N}$. Thus $p = \mu_1\nu$ and $q = \mu_2\nu$ where $\mu_1 = e_1e_2\cdots e_i, \mu_2 = f_1f_2\cdots f_j$ and $\nu = e_{i+1}e_{i+2}\cdots = f_{j+1}f_{j+2}\cdots$. Therefore $\varphi_\infty(p) = \varphi(\mu_1)\varphi_\infty(\nu)$ and $\varphi_\infty(q) = \varphi(\mu_2)\varphi_\infty(\nu)$ implying $\varphi_\infty(p) \sim \varphi_\infty(q)$.

If $p, q \in \Lambda^*$ such that $p \sim q$ then $t(p) = t(q)$ is a singular vertex. Hence $\varphi_\infty(t(p)) = \varphi_\infty(t(q))$. Moreover $\varphi_\infty(p) = \varphi(p)\varphi_\infty(t(p))$ and $\varphi_\infty(q) = \varphi(q)\varphi_\infty(t(q))$ implying $\varphi_\infty(p) \sim \varphi_\infty(q)$.

Hence $\varphi_\infty(p) \sim \varphi_\infty(q)$ whenever $p \sim q$.

To prove the converse, suppose $\varphi_\infty(p_1) \sim \varphi_\infty(p_2)$ for $p_1, p_2 \in E^\infty \cup \Lambda^*$.

Claim. If $p_1 \in \Lambda^*$ then $p_2 \in \Lambda^*$. If $p_1 \in E^\infty$ then $p_2 \in E^\infty$.

We prove the claim. Suppose $p_1 \in \Lambda^*$. Thus $\varphi_\infty(p_1) = \varphi(p_1)e_1e_2\cdots$ where $e_1e_2\cdots$ is the tail added to $t(p_1)$ in the construction of F, i.e., $t(p_1) = o(e_1e_2\cdots)$. Therefore, $\varphi_\infty(p_1) \sim e_2e_3\cdots$. Since $\varphi_\infty(p_1) \sim \varphi_\infty(p_2)$ we get $\varphi_\infty(p_2) \sim e_2e_3\cdots$. If $p_2 \in E^\infty$ then $p_2 = f_1f_2\cdots$ for some $f_1, f_2, \ldots \in E^1$. Therefore $\varphi_\infty(p_2) = \varphi(f_1)\varphi(f_2)\cdots$. Implying that, so $\varphi(f_1)\varphi(f_2)\cdots \sim e_2e_3\cdots$. But for each $i \geqslant 1$ we have $o(\varphi(f_i)), t(\varphi(f_i)) \in E^0$ and by the construction of F, for each $i \geqslant 2$, $o(e_i), t(e_i) \notin E^0$. Therefore $\varphi(f_1)\varphi(f_2)\cdots$ cannot be equivalent to the path $e_2e_3\cdots$, which is a contradiction. Therefore $p_2 \in \Lambda^*$. The second statement follows from the contrapositive of the first statement by symmetry.

Now suppose $p_1 \in \Lambda^*$. By the above claim, $p_2 \in \Lambda^*$. Thus $\varphi_\infty(p_2) = \varphi(p_2)g_1g_2\cdots$, where $g_1g_2\cdots$ is the tail added to $t(p_2)$ in the construction of F. Hence $\varphi_\infty(p_2) \sim g_1g_2\cdots$. Since $\varphi_\infty(p_1) \sim e_1e_2\cdots$, we get $e_1e_2\cdots \sim g_1g_2\cdots$. Notice that (by the construction of F) $t(p_1)$ is the only entrance of $e_1e_2\cdots$ and $t(p_2)$ is the only entrance to $g_1g_2\cdots$. Therefore either $t(p_1) = o(g_i)$ for some i or $t(p_2) = o(e_i)$ for some i. Without loss of generality suppose $t(p_1) = o(g_i)$, thus $e_1e_2\cdots = g_ig_{i+1}\cdots$. But $t(p_2) = o(g_1)$ is the only vertex in the path $g_1g_2\cdots$ that belongs to E^0 and $t(p_1) \in E^0$. Hence $t(p_1) = t(p_2)$. Therefore $p_1 \sim p_2$.

If $p_1 \in E^\infty$ then, by the above claim, $p_2 \in E^\infty$. Notice that $\forall v \in \varphi_\infty(p_i)^0$ either $v \in E^0$ (hence in p_i^0) or $\exists w \in p_i^0$ such that $v \geqslant w$. Since $\varphi_\infty(p_1) \sim \varphi_\infty(p_2), \varphi_\infty(p_i) = \mu_i\nu$, for some $\mu_1, \mu_2 \in F^*$ and some $\nu \in F^\infty$, and $t(\mu_1) = t(\mu_2) = o(\nu)$. Extending μ_1 and μ_2 along ν, if needed, we may assume that $t(\mu_i) \in E^0$, i.e., $\mu_1, \mu_2 \in \{\beta \in F^* : o(\beta), t(\beta) \in E^0\}, \nu \in \{\beta \in F^\infty : o(\beta) \in E^0\}$ and $t(\mu_1) = t(\mu_2) = o(\nu)$. Therefore $\mu_i = \varphi(\delta_i), \nu = \varphi_\infty(\gamma)$ for some $\delta_i \in E^*$ and some $\gamma \in E^\infty \cup \Lambda^*$. Implying, so $p_i = \varphi_\infty^{-1}(\mu_i\nu) = \varphi_\infty^{-1}(\varphi(\delta_i)\varphi_\infty(\gamma)) = \varphi_\infty^{-1}(\varphi_\infty(\delta_i\gamma)) = \delta_i\gamma$. Thus $p_1 \sim p_2$. ∎

LEMMA 5.4. *Let F be a desingularization of a directed graph E. Then E satisfies Condition* (M) *if and only if F satisfies Condition* (M).

Proof. We will prove the only if side. Recall that F has no singular vertices. Suppose F does not satisfy Condition (M). Let $v \in F^0$ and $[\lambda] \in F^\infty/\sim$ such that the number of representatives of $[\lambda]$ that begin with v is infinite.

If $v \notin E^0$ then v is on an added tail to a singular vertex v_0 of E and there is (only one) path from v_0 to v. Then the number of representatives of $[\lambda]$ that begin with v (in the graph F) is equal to the number of representatives of $[\lambda]$ that begin with v_0 (in the graph F). If the latter is finite then the first is finite, hence we might assume that $v \in E^0$. Moreover, every path in F^∞ is equivalent to one whose origin lies in E^0. Therefore we might choose a representative λ with $o(\lambda) \in E^0$.

The set of representatives of $[\lambda]$ that begin with v is $\{\beta \in F^\infty : o(\beta) = v \text{ and } \lambda \sim \beta\}$. Since φ_∞ is bijective, $\varphi_\infty^{-1}\{\beta \in F^\infty : o(\beta) = v \text{ and } \lambda \sim \beta\}$ is an infinite subset of $E^\infty \cup \Lambda^*$. As φ_∞^{-1} preserves origin and the equivalence, $\varphi_\infty^{-1}\{\beta \in F^\infty : o(\beta) = v \text{ and } \beta \sim \lambda\} = \{\varphi_\infty^{-1}(\beta) \in E^\infty \cup \Lambda^* : o(\varphi_\infty^{-1}(\beta)) =$

v and $\varphi_\infty^{-1}(\beta) \sim \varphi_\infty^{-1}(\lambda)\}$. Thus $[\varphi_\infty^{-1}(\lambda)]$ has infinite representatives that begin with v. Therefore E does not satisfy Condition (M).

To prove the converse, suppose E does not satisfy Condition (M). Let $v \in E^0$ and $[p] \in (E^\infty \cup \Lambda^*)/\sim$ such that the number of representatives of $[p]$ that begin with v is infinite. The set of representatives of $[p]$ that begin with v is $\{q \in E^\infty \cup \Lambda^* : o(q) = v \text{ and } q \sim p\}$. Since φ_∞ is bijective, $\varphi_\infty\{q \in E^\infty \cup \Lambda^* : o(q) = v \text{ and } q \sim p\}$ is an infinite subset of F^∞. As φ_∞ preserves origin and the equivalence, $\varphi_\infty\{q \in E^\infty \cup \Lambda^* : o(q) = v \text{ and } q \sim p\} = \{\varphi_\infty(q) \in F^\infty : o(\varphi_\infty(q) = v \text{ and } \varphi_\infty(q) \sim \varphi_\infty(p)\}$. Thus $[\varphi_\infty(p)]$ has infinitely many representatives that begin with v. Therefore F does not satisfy Condition (M). ∎

We can now write the main theorem in its full generalities.

THEOREM 5.5. *Let E be a directed graph. $C^*(E)$ is liminal if and only if E satisfies Condition* (M).

Proof. Let F be a desingularization of E. Then E satisfies Condition (M)

$$\begin{aligned} &\Leftrightarrow F \text{ satisfies Condition (M)} \\ &\Leftrightarrow C^*(F) \text{ is liminal} \\ &\Leftrightarrow C^*(E) \text{ is liminal.} \end{aligned}$$ ∎

6. THE LARGEST LIMINAL IDEAL OF C^*-ALGEBRAS OF GENERAL GRAPHS

In this section we will identify the largest liminal ideal of $C^*(E)$ for a general graph E.

We will, once again, follow the construction in [4]. For a hereditary saturated subset H of E^0, define:

$$B_H := \{v \in \Lambda : 0 < |o^{-1}(v) \cap t^{-1}(E^0 \setminus H)| < \infty\}.$$

Thus B_H is the set of infinite emitters that point into H infinitely often and out of H at least once but finitely often. In [4] it is proven that the set $\{(H, S) : H$ is a hereditary saturated subset of E^0 and $S \subseteq B_H\}$ is a lattice with the lattice structure $(H, S) \leqslant (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$. Observe that, since $B_H \cap H = \emptyset, (H, S) \leqslant (H, S')$ if and only if $S \subseteq S'$.

Let E be a directed graph and F be a desingularization of E, let H be a hereditary saturated subset of E^0, and let $S \subseteq B_H$. Following the construction in [4], define:

$$\widetilde{H} := H \cup \{v_n \in F^0 : v_n \text{ is on a tail added to a vertex in } H\}.$$

Thus $\widetilde{H}$ is the smallest hereditary saturated subset of F^0 containing H.

Let $S \subseteq B_H$, and let $v_0 \in S$. Let $v_i = t(e_i)$, where $e_1 e_2 \cdots$ is the tail added to v_0 in the construction of F. If N_{v_0} is the smallest non-negative integer such that $t(e_j) \in H$, $\forall j \geqslant N_{v_0}$, we have that $\forall j \geqslant N_{v_0}, v_j$ emits exactly two edges: one pointing to v_{j+1} and one pointing to a vertex in H. Define

$$T_{v_0} := \{v_n \in F^0 : v_n \text{ is on a tail added to } v_0 \text{ and } n \geqslant N_{v_0}\}$$

and

$$H_S := \widetilde{H} \cup \bigcup_{v_0 \in S} T_{v_0}.$$

Lemma 3.2 in [4] states that the above construction defines a lattice isomorphism from the lattice $\{(H,S) : H$ is a hereditary saturated subset of E^0 and $S \subseteq B_H\}$ onto the lattice of hereditary saturated subsets of F^0.

Let $\{t_e, q_v\}$ be a generating Cuntz-Krieger F-family and $\{s_e, p_v\}$ be the canonical generating Cuntz-Krieger E-family. Let $p = \sum_{v \in E^0} q_v$. Since $C^*(E)$ and $C^*(F)$ are Morita equivalent via the imprimitivity bimodule $pC^*(F)$, it follows that the Rieffel correspondence between ideals in $C^*(F)$ and ideals in $C^*(E)$ is given by the map $I \mapsto pIp$.

Let H be a hereditary saturated subset of E^0 and $S \subseteq B_H$. For $v_0 \in S$, define

$$p^H_{v_0} := p_{v_0} - \sum_{\substack{o(e)=v_0 \\ t(e)\notin H}} s_e s_e^*$$

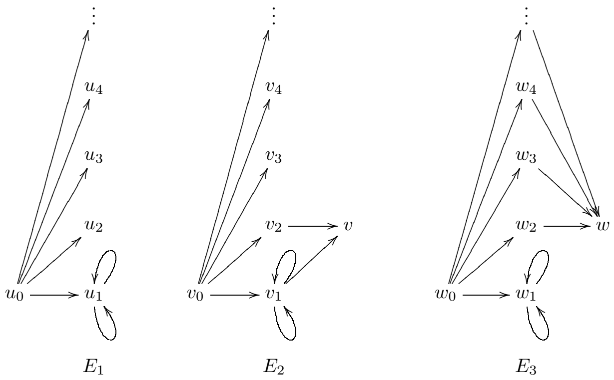
and

$$I_{(H,S)} := \text{ the ideal generated by } \{p_v : v \in H\} \cup \{p^H_v : v_0 \in S\}.$$

Proposition 3.3 in [4] states that if E satisfies Condition (K): every vertex of E lies on either no circuits or at least two circuits, then $pI_{H_S}p = I_{(H,S)}$. The assumption that E satisfies Condition (K) was only used to make sure that all the ideals of F are gauge invariant. Therefore whenever I is a gauge invariant ideal of $C^*(F)$ and $H_S = \{v \in F^0 : p_v \in I\}$, since $I = I_{H_S}$, we have $pIp = pI_{H_S}p = I_{(H,S)}$. Moreover, Theorem 3.5 in [4] states that if E satisfies Condition (K) then the map $(H,S) \mapsto I_{(H,S)}$ is a bijection from the lattice $\{(H,S) : H$ is a hereditary saturated subset of E^0 and $S \subseteq H\}$ onto the lattice of ideals in $C^*(E)$. Without the assumption that E satisfies Condition (K) the bijection will be from the lattice $\{(H,S) : H$ is a hereditary saturated subset of E^0 and $S \subseteq H\}$ onto the lattice of gauge invariant ideals in $C^*(E)$. Hence the gauge invariant ideals of E are of the form $I_{(H,S)}$ for some hereditary saturated subset H of E^0 and for some $S \subseteq B_H$.

To identify the largest liminal ideal of $C^*(E)$, first recall that the largest liminal ideal of a C^*-algebra is invariant under automorphisms. Therefore the largest liminal ideal of $C^*(E)$ has to be of the form $I_{(H,S)}$ for some hereditary saturated subset H of E^0 and a subset S of B_H. We set $H_l = \{v \in E^0 :\ \forall[\lambda] \in (E^\infty \cup \Lambda^*_E)/\sim$, the number of representatives of $[\lambda]$ that begin with v is finite$\}$. Since Graph(H_l) satisfies Condition (M), we see that the ideal $I_{H_l} = I_{(H_l,\emptyset)}$ is a subset of the largest liminal ideal of $C^*(E)$. While it is true that $H = H_l$, as illustrated in the following example, it is not automatically clear what S can be.

Example 6.1. Consider the following graphs:



Let $I_{(H_i,S_i)}$ denote the largest liminal ideal of $C^*(E_i)$. It is not hard to see that $H_1 = \{u_2, u_3, \ldots\}$, $H_2 = \{v_2, v_3, \ldots\} \cup \{v\}$, $H_3 = \{w_2, w_3, \ldots\} \cup \{w\}$, $B_{H_1} = \{u_0\}$, $B_{H_2} = \{v_0\}$, and $B_{H_3} = \{w_0\}$. A careful computation shows that $S_1 = \{u_0\}$, $S_2 = \{v_0\}$, while $S_3 = \emptyset$. Notice that we can reach from v_0 to v in an infinite number of ways, but not through H. We can reach from w_0 to w through H in an infinite number of ways.

For a hereditary and saturated subset H of E^0 and $v \in B_H$, we define $D_{(v,H)} := \{e \in \Delta(v) : t(e) \notin H\}$, that is, $D_{(v,H)}$ is the set of all edges that begin with v and point outside of H. Notice that $D_{(v,H)}$ is a non empty finite set.

Proposition 6.2. *Let E be a directed graph and $H = \{v \in E^0 : \forall[\lambda] \in (E^\infty \cup \Lambda_E^*)/\sim$, the number of representatives of $[\lambda]$ that begin with v is finite$\}$. Let $S = \{v \in B_H : E(v; D_{(v,H)})$ satisfies Condition* (M). *Then $I_{(H,S)}$ is the largest liminal ideal of $C^*(E)$.*

Proof. That H is hereditary and saturated was proved in Proposition 4.1. Let $I_{(H',S')}$ be the largest liminal ideal of $C^*(E)$ and let F be a desingularization of E. In what follows, we will prove that $I_{(H,S)} = I_{(H',S')}$. To do that we will prove: $H \subseteq H'$, $H' \subseteq H$, $S \subseteq S'$ and $S' \subseteq S$, in that order.

We will prove that $H \subseteq H'$. Notice that $I_{H'_{S'}}$ is the largest liminal ideal of $C^*(F)$. Using Proposition 4.1 we get that $H'_{S'} = \{v \in F^0 : \forall[\lambda] \in F^\infty/\sim$, the number of representatives of $[\lambda]$ that begin with v is finite$\}$.

Let $G_H = \mathrm{Graph}(H)$. Notice that by Proposition 2.1 of [5], I_H is Morita equivalent to $C^*(G_H)$. Since G_H satisfies Condition (M), by Theorem 3.10, $C^*(G_H)$ is liminal. Therefore $I_H = I_{(H,\emptyset)}$ is liminal. By the maximality of $I_{(H',S')}, I_{(H,\emptyset)} \subseteq I_{(H',S')}$, implying that $H \subseteq H'$. We will prove that $H' \subseteq H$. Let $G_{H'} = \mathrm{Graph}(H')$. Then $I_{H'} = I_{(H',\emptyset)} \subseteq I_{(H',S')}$. Hence $I_{H'}$ is liminal. By Proposition 2.1 of [5], $I_{H'}$ is Morita equivalent to $C^*(G_{H'})$. Therefore $G_{H'}$ satisfies Condition (M).

Let $v \in H'$. If $\beta \in E^{**}$ with $o(\beta) = v$ then, since H' is hereditary, $\beta \in G_{H'}$. Now let $[\lambda] \in (E^\infty \cup \Lambda^*)/\sim$. If γ is a representative of $[\lambda]$ with $o(\gamma) = v$ then $\gamma \in G_{H'}^\infty \cup \Lambda_{G_{H'}}^*$. Therefore the set of representatives of $[\lambda]$ that begin with v is $\{\beta \in E^\infty \cup \Lambda^* : o(\beta) = v, \beta \sim \gamma\} = \{\beta \in G_{H'}^\infty \cup \Lambda_{G_{H'}}^* : o(\beta) = v, \beta \sim \gamma\}$ which is finite, since $G_{H'}$ satisfies Condition (M). Therefore $v \in H$, hence $H' \subseteq H$.

Next we will prove that $S \subseteq S'$. Let $v_0 \in S$. To show that $v_0 \in S'$ we will show that $v_n \in H_{S'}$ whenever $n \geqslant N_{v_0}$, i.e., $\forall n \geqslant N_{v_0}$, and $\forall[\lambda] \in F^\infty / \sim$, the number of representatives of $[\lambda]$ beginning with v_n is finite.

Let $n \geqslant N_{v_0}$ and let $[\lambda] \in F^\infty / \sim$. If $[\lambda]$ has no representative that begins with v_n then there is nothing to prove. Let γ be a representative of $[\lambda]$ $o(\gamma) = v_n$.

First suppose that $\gamma^0 = \{v_n, v_{n+1}, \ldots\}$, i.e., γ is the part of the tail added to v_0 in the construction of F. Then $\{\beta \in F^\infty : o(\beta) = v_n, \beta \sim \gamma\} = \{\gamma\}$ since γ has no entry other than v_n. Therefore the number of representatives of $[\lambda]$ beginning with v_n is 1.

Now suppose γ^0 contains a vertex not in $\{v_n, v_{n+1}, \ldots\}$. Recalling that $\forall k \geqslant N_{v_0}, v_k$ emits exactly two edges, one pointing to v_{k+1} and one pointing to a vertex in H, let $w \in H$ be the first such vertex, i.e., $w \in H \cap \gamma^0$ is chosen so that whenever $v \geqslant w$ and $v \in \{v_n, v_{n+1}, \ldots\}$ then $v \notin H$. If p is the (only) path from v_0 to v_n and q is the path from v_n to w along γ, then $\gamma = q\mu$ for some $\mu \in F^\infty$ with $o(\mu) = w$. Moreover, $\varphi_\infty^{-1}(p\gamma) = \varphi_\infty^{-1}(pq\mu) = \varphi^{-1}(pq)\varphi_\infty^{-1}(\mu)$ and $\varphi^{-1}(pq)$ is an edge in E^1 with $o(\varphi^{-1}(pq)) = v_0$ and $t(\varphi^{-1}(pq)) = w \in H$. Therefore $\varphi_\infty^{-1}(p\gamma) \in E(v_0; D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})}$. The set of representatives of $[\lambda]$ that begin with v_n is $\{\beta \in F^\infty : o(\beta) = v_n, \beta \sim p\gamma\}$. If $\beta \in F^\infty$ is any representative of $[\lambda]$ that begins with v_n then $\beta \sim p\gamma \sim \mu$. Hence β^0 has to contain a vertex in H. Applying the same argument to β we see that $p\beta$ is a representative of $[\lambda], o(p\beta) = v$ and $\varphi_\infty^{-1}(p\beta) \in E(v_0; D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})}$.

Hence $|\{\beta \in F^\infty : o(\beta) = v_n, \beta \sim p\gamma\}| = |\{p\beta \in F^\infty : p\beta \sim p\gamma\}| = |\{\varphi_\infty^{-1}(p\beta) \in E(v_0; D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})} : \varphi_\infty^{-1}(p\beta) \sim \varphi_\infty^{-1}(p\gamma)\}|$ which is finite, since $E(v_0; D_{(v_0,H)})$ satisfies Condition (M).

In each case, the number of representatives of $[\lambda]$ beginning with v_n is finite, implying that $v_n \in H_{S'}$. Therefore $v_0 \in S'$.

Finally, we will prove that $S' \subseteq S$. Let $v_0 \in S'$. We will show that $E(v_0; D_{(v_0,H)})$ satisfies Condition (M). Let $\lambda \in E(v_0; D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})}$. If a vertex $v \neq v_0$ is in $E(v_0; D_{(v_0,H)})^0$ then it is in H, hence, by the definition of H, the number of representatives of $[\lambda]$ beginning with v is finite. What remains is to show that the number of representatives of $[\lambda]$ beginning with v_0 is finite. Noting that $v_{N_{v_0}} \in H_{S'}$, for any $\gamma \in F^\infty$ the set $\{\mu \in F^\infty : o(\mu) = v_{N_{v_0}}, \mu \sim \gamma\}$ is finite. In particular, the set $\{\mu \in F^\infty : o(\mu) = v_{N_{v_0}}, \mu \sim \varphi_\infty(\lambda)\}$ is finite.

Let $\beta = e_1e_2\cdots \in E(v_0; D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})}$ with $o(\beta) = v_0$. Then $\varphi_\infty(\beta) = \varphi(e_1)\varphi_\infty(e_2e_3\cdots) \in F^\infty$ and $o(\varphi(e_1)) = v_0, t(\varphi(e_1)) = o(\varphi_\infty(e_2e_3\cdots)) \in H$. Let p be the path from v_0 to $v_{N_{v_0}}$.

We will first show that the set $\{\beta = e_1e_2\cdots \in E(v_0; D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})} : e_1e_2\cdots \sim \lambda$ and $v_{N_{v_0}} \in \varphi(e_1)^0\}$ is finite.

If $v_{N_{v_0}} \in \varphi(e_1)^0$ then $\varphi_\infty(\beta) = p\mu$ for some $\mu \in F^\infty$ with $o(\mu) = v_{N_{v_0}}$. Hence $|\{e_1e_2\cdots \in E(v_0; D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})} : e_1e_2\cdots \sim \lambda$ and $v_{N_{v_0}} \in \varphi(e_1)^0\}| = |\{\varphi_\infty(e_1e_2\cdots) \in F^\infty : \varphi_\infty(e_1e_2\cdots) \sim \varphi_\infty(\lambda), o(e_1) = v_0, t(e_1) \in H$ and $v_{N_{v_0}} \in \varphi(e_1)^0\}| = |\{p\mu \in F^\infty : p\mu \sim \varphi_\infty(\lambda)\}| = |\{\mu \in F^\infty : o(\mu) = v_{N_{v_0}}$ and $\mu \sim \varphi_\infty(\lambda)\}|$ which is finite.

We will next show that the set $\{e_1e_2\cdots \in E(v_0;D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})} : e_1e_2\cdots \sim \lambda$ and $v_{N_{v_0}} \notin \varphi(e_1)^0\}$ is finite.

Observe that the set $\mathcal{E} := \{e \in \Delta : t(e) \in H$ and $v_{N_{v_0}} \notin \varphi(e)\}$ is finite. Moreover, $\forall e \in \mathcal{E}$ the set $\{\beta \in E^\infty \cup \Lambda^*_E : o(\beta) = t(e), \beta \sim \lambda\}$ is finite, since $\{t(e) : e \in \mathcal{E}\} \subseteq H$. Hence $|\{e_1e_2\cdots \in E(v_0;D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})} : e_1e_2\cdots \sim \lambda, v_{N_{v_0}} \notin \varphi(e_1)^0\}| = |\{e_1e_2\cdots \in E^\infty \cup \Lambda^*_E : e_1e_2\cdots \sim \lambda, t(e_1) \in K\}| = |\{\beta \in E^\infty \cap \Lambda^*_E : o(\beta) \in K, \beta \sim \lambda\}|$ which is finite, as the set is a finite union of finite sets.

Therefore the set $\{\beta \in E(v_0;D_{(v_0,H)})^\infty \cup \Lambda^*_{E(v_0;D_{(v_0,H)})} : \beta \sim \lambda\}$ is a union of two finite sets, hence is finite. Thus $v_0 \in S$. It follows that $S \subseteq S'$ concluding the proof. ∎

7. TYPE I GRAPH C^*-ALGEBRAS

In this section we will characterize type I graph C^*-algebras.

We say that an edge e reaches a path p if $t(e)$ reaches p, i.e. if there is a path q such that $o(q) = t(e)$ and $q \sim p$.

If v is a sink then we regard $\{v\}$ as a tree.

For an infinite path λ, we use N_λ to denote the number of vertices of λ that emit multiple edges that get back to λ.

LEMMA 7.1. *Let E be a directed graph with:*
(i) *every circuit in E is either terminal or transitory;*
(ii) *for any $\lambda \in E^\infty, \mathrm{N}_\lambda$ is finite.*
Then there exists $v \in E^0$ such that $E(v)$ is either a terminal circuit or a tree.

Proof. Let $z_1 \in E^0$. If $E(z_1)$ is neither a terminal circuit nor a tree, then there exists $z_2 \neq z_1$ such that z_1 and z_2 do not belong to a common circuit, and there are (at least) two paths from z_1 to z_2.

Notice that $\exists w_1 \in E^0$ such that $z_1 \geqslant w_1 \geqslant z_2$ and w_1 emits multiple edges that reach z_2 (perhaps is z_1 itself). Observe that, by construction, $z_2 \not\geqslant z_1$.

Inductively: if $E(z_i)$ is neither a terminal circuit nor a tree, then there exists $z_{i+1} \neq z_i$ such that z_i and z_{i+1} do not belong to a common circuit, and there are (at least) two paths from z_i to z_{i+1}. Again $\exists w_i \in E^0$ such that $z_i \geqslant w_i \geqslant z_{i+1}$ and w_i emits multiple edges that reach z_{i+1}. Observe also that $z_{i+1} \not\geqslant z_i$ and hence $w_{i+1} \not\geqslant w_i$.

This process has to end, for otherwise, let $\lambda \in E^\infty$ be such that $z_i, w_i \in \lambda^0$, $\forall i$. Then λ has infinite number of vertices that emit multiple edges that reach λ, namely $w_1, w_2, \ldots$ contradicting the assumption. ∎

REMARK 7.2. Let $\lambda, \gamma \in E^\infty$. If $\lambda = p\gamma$, for some $p \in E^*$, then $\mathrm{N}_\gamma \leqslant \mathrm{N}_\lambda \leqslant \mathrm{N}_\gamma + |p^0|$, where $|p^0|$ = the number of vertices in p, which is finite since p is a finite path. Therefore, N_λ is finite if and only if N_γ is finite. Moreover, if $\lambda \sim \mu$ then $\lambda = p\gamma, \mu = q\gamma$ for some $p, q \in E^*$ and some $\gamma \in E^\infty$. Hence N_λ is finite if and only if N_γ is finite if and only if N_μ is finite.

THEOREM 7.3. *Let E be a graph. $C^*(E)$ is type* I *if and only if:*
(i) *every circuit in E is either terminal or transitory;*
(ii) *for any $\lambda \in E^\infty$, N_λ is finite.*

We will first prove the following lemma.

LEMMA 7.4. *Let E be a directed graph and F be a desingularization of E. E satisfies* (i) *and* (ii) *of Theorem* 7.3 *if and only if F satisfies* (i) *and* (ii) *of Theorem* 7.3.

Proof. That E satisfies (i) if and only if F satisfies (i) follows from the fact that the map φ of Remark 5.2 preserves circuits.

Now we suppose that E satisfies (i), equivalently F satisfies (i).

Suppose E fails to satisfy (ii). Let $\lambda \in E^\infty$ such that N_λ is infinite. Suppose $v \in \lambda^0$ and p is a path such that $o(p) = v, t(p) \in \lambda^0$. Let q be the path along λ such that $o(q) = v, t(q) = t(p)$, then $\exists \beta \in E^*$ and $\mu \in E^\infty$ such that $\lambda = \beta q \mu$. Since φ preserves origins and termina, $o(\varphi(p)) = v = o(\varphi(q))$ and $t(\varphi(p)) = t(\varphi(q))$. Moreover, $\varphi_\infty(\lambda) = \varphi_\infty(\beta q \mu) = \varphi(\beta)\varphi(q)\varphi_\infty(\mu)$. Since φ is bijective, $\varphi(p) = \varphi(q)$ if and only if $p = q$. Therefore, if v (as a vertex in E) emits multiple edges that get back to λ then it (as a vertex in F) emits multiple edges that get back to $\varphi(\lambda)$, implying that $\mathrm{N}_{\varphi_\infty(\lambda)}$ is infinite. Hence F does not satisfy (ii).

To prove the converse, suppose E satisfies (ii). Let $\lambda \in F^\infty$. If $o(\lambda) \notin E^0$, then $o(\lambda)$ is on a path extended from a singular vertex. Using Remark 7.2, we may extend λ (backwards) and assume that $o(\lambda) \in E^0$. Let $\gamma = {\varphi_\infty}^{-1}(\lambda) \in E^\infty \cup \Lambda^*$.

First suppose $\gamma \in \Lambda^*$. Then $v_0 := t(\gamma) \sim \gamma$. Hence $\varphi_\infty(v_0) \sim \varphi_\infty(\gamma) = \lambda$. Using Remark 7.2, we may assume that $\lambda = \varphi_\infty(v_0)$, that is, λ is the path added to v_o in the construction of F. Thus each vertex of λ emits exactly two edges: one pointing to a vertex in λ (the next vertex) and one pointing to a vertex in E^0. Since v_0 is the only entry to λ, if a vertex v of λ emits multiple edges that get back to λ then $v \geqslant v_0$. And since F satisfies (i), there could be at most one such vector, for otherwise v_0 would be on multiple circuits. Hence N_λ is at most 1.

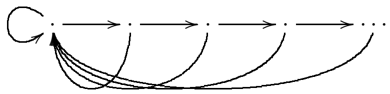
Now suppose $\gamma \in E^\infty$. Since E satisfies (ii), N_γ is finite. Going far enough on γ, let $w \in \gamma^0$ be such that no vertex of γ that w can reach to emits multiple edges that get back to γ. Let $\mu \in E^*, \beta \in E^\infty$ be such that $\gamma = \mu\beta$ and $t(\mu) = w = o(\beta)$, then $\lambda = \varphi(\mu)\varphi_\infty(\beta)$. Hence $\lambda \sim \varphi_\infty(\beta)$. Moreover, each $v \in \beta^0$ emits exactly one edge that gets to β, which, in fact, is an edge of β.

Let $v \in \beta^0$ and $p \in F^*$ be such that $o(p) = v, t(p) \in \lambda^0$. Extending p, if needed, we may assume that $t(p) \in \beta^0$. Let $q \in F^*$ be the path along λ such that $o(q) = v$ and $t(q) = t(p)$. Since φ is bijective, $\varphi^{-1}(p) = \varphi^{-1}(q)$ if and only if $p = q$. But v can get to β in only one way, therefore $\varphi^{-1}(p) = \varphi^{-1}(q)$, implying that $p = q$. Thus v emits (in the graph F) only one edge that gets to λ. Hence, for each vertex $v \in \varphi_\infty(\beta)$, if $v \in E^0$ then v emits only one edge that gets to λ.

Now let $v \in \varphi_\infty(\beta) \setminus E^0$. Then v is on a path extended from a singular vertex, say v_0. Since $w \geqslant v_0$, by the previous paragraph, v_0 emits only one edge that gets to λ. Let p be the (only) path from v_0 to v. Let $\mu, \nu \in F^*$ be such that $t(\mu), t(\nu) \in \lambda^0$ and $o(\mu) = o(\nu) = v$. Extending μ or ν along λ, if needed, we can assume that $t(\mu) = t(\nu)$. Again extending them along λ we can assume that $t(\mu) = t(\nu) \in \beta^0$. Observe that $o(p\mu) = o(p\nu) = v_0$ and $t(p\mu) = t(p\nu) \in \beta^0$. Therefore $o(\varphi^{-1}(p\mu)) = o(\varphi^{-1}(p\nu)) = v_0$ and $t(\varphi^{-1}(p\mu)) = t(\varphi^{-1}(p\nu)) \in \beta^0$. But each vertex in β emits exactly one edge that gets to β, i.e., there is exactly one path from v_0 to $t(\varphi^{-1}(p\mu))$ hence $p\mu = p\nu$. Therefore, $\mu = \nu$. That is, v emits only one edge that gets to λ. Therefore $\mathrm{N}_{\varphi_\infty(\beta)} = 0$. By Remark 7.2, we get that N_λ is finite. ∎

REMARK 7.5. The fact that E satisfies (ii) of Theorem 7.3 does not imply that its desingularization F satisfies (ii) of Theorem 7.3 as illustrated by the following example.

EXAMPLE 7.6. If E is the $\mathcal{O}_\infty$ graph (one vertex with infinitely many loops), which clearly satisfies (ii) of Theorem 7.3, then its desingularization does not satisfy (ii) of Theorem 7.3. The desingularization looks like this:



Proof of Theorem 7.3. We first prove the "if" side. We will first assume that E is a row-finite graph with no sinks. Let $(I_\rho)_{0\leqslant\rho\leqslant\alpha}$ be an increasing family of ideals of $C^*(E)$ such that:

(i) $I_0 = \{0\}, C^*(E)/I_\alpha$ is antiliminal;

(ii) if $\rho \leqslant \alpha$ is a limit ordinal, $I_\rho = \overline{\bigcup_{\beta<\rho} I_\beta}$;

(iii) if $\rho < \alpha, I_{\rho+1}/I_\rho$ is a liminal ideal of $C^*(E)/I_\rho$ and is non zero.

We prove that $I_\alpha = C^*(E)$. Since I_α is the largest type I ideal of $C^*(E)$, it is gauge invariant. Let H be a hereditary saturated subset of E^0 such that $I_\alpha = I_H$. If $H \neq E^0$ then let $F = F(E \setminus H)$. Clearly F satisfies (i) and (ii) of the theorem. Using Lemma 7.1 let $v_0 \in F^0$ be such that $K = \{v \in F^0 : v_0 \geqslant v\}$ is the set of vertices of either a terminal circuit or a tree. Let $G = \mathrm{Graph}(K)$, thus G is either a terminal circuit or a tree. By Proposition 2.1 of [5], I_K is Morita equivalent to $C^*(G)$. Moreover G satisfies Condition (M), hence by Theorem 3.10, $C^*(G)$ is liminal. And, so I_K is an ideal of $C^*(F) \cong C^*(E)/I_\alpha$ contradicting the assumption that $C^*(E)/I_\alpha$ is antiliminal. It follows that $I_\alpha = C^*(E)$. Therefore $C^*(E)$ is type I.

For an arbitrary graph E, let F be a desingularization of E. By Lemma 7.4, F satisfies (i) and (ii) of the theorem. By the above argument, $C^*(F)$ is type I. Therefore $C^*(E)$ is type I.

To prove the converse, suppose E has a non-terminal non-transitory circuit, that is, E has a vertex that is on (at least) two circuits. Let v_0 be a vertex on two circuits, say α and β. Let F be the subgraph containing (only) the edges and vertices of α and β.

$\mathcal{A} := \overline{\mathrm{span}}\{s_\mu s_\nu^* : \mu, \nu$ are paths made by α and β or just $v_0\}$ is a C^*-subalgebra of $C^*(F)$. But $\mathcal{A} \cong \mathcal{O}_2$ which is not type I. Hence $C^*(F)$ is not type

I. By Remark 3.7, $C^*(E)$ has a sub-algebra whose quotient is not type I therefore $C^*(E)$ is not type I.

Suppose now that each circuit in E is either terminal or transitory and $\exists\lambda \in E^\infty$ such that N_λ is infinite. Let $v_\lambda = o(\lambda)$. Let $G = E(v)$. If v is a vertex such that $V(v)$ does not intersect λ^0, we can factor $C^*(G)$ by the ideal generated by $\{v\}$. This process gets rid of any terminal circuits of G. By Lemma 3.8, $C^*(G)$ is not type I, implying that $C^*(E)$ is not type I. ∎

Next we will identify the largest type I ideal of the C^*-algebra of a graph E. For a vertex v of E (respectively F), recall that $E(v)$ (respectively $F(v)$) denotes the subgraph of E (respectively F) that v can 'see'.

We begin with the following lemma.

LEMMA 7.7. *Let E be a directed graph, F a desingularization of E and $v \in E^0$. Then $F(v)$ is a desingularization of $E(v)$.*

Proof. Let $u \in E(v)^0 = \{w \in E : v \geqslant w\}$. Let p be a path in E with $o(p) = v$, and $t(p) = u$, then $\varphi(p)$ is a path in F with $o(\varphi(p)) = v$, and $t(\varphi(p)) = u$. Hence $u \in F(v)^0$, implying that $E(v)^0 \subseteq F(v)^0$. Clearly $F(v)$ has no singular vertices. Let $v_0 \in E(v)^0$ be a singular vertex. If v_n is a vertex on the path added to v_0 in the construction of F, since $F(v)^0$ is hereditary and $v_0 \in F(v)^0$, we get $v_n \in F(v)^0$. Therefore the path added to v_0 is in the graph $F(v)$. To show that $F(v)$ has exactly the vertices needed to desingularize $E(v)$, let $w \in F(v)^0$. Let p be a path in $F(v)$ with $o(p) = v$ and $t(p) = w$. If $w \in E^0$ then $\varphi^{-1}(p) \in E^*$ and $o(\varphi^{-1}(p)) = v$ and $t(\varphi^{-1}(p)) = w$. Therefore $v \geqslant w$ in the graph E. Hence $w \in E(v)^0$. If $w \notin E^0$ then there is a singular vertex, say $v_0 \in E^0$, such that w is on the path added to v_0 in the construction of F. Since the path from v_0 to w has no other entry than v_0 and since $v \geqslant w$, we must have $v \geqslant v_0$. Hence w is on the the graph obtained when $E(v)$ is desingularized. Therefore $F(v)$ is a desingularization of $E(v)$. ∎

The following corollary follows from Lemma 7.7 and Lemma 7.4.

COROLLARY 7.8. *Let E be a directed graph, F a desingularization of E and $v \in E^0$. Then $E(v)$ satisfies* (i) *and* (ii) *of Theorem* 7.3 *if and only if $F(v)$ satisfies* (i) *and* (ii) *of Theorem* 7.3.

The next proposition identifies the largest type I ideal of the C^*-algebra of a row-finite graph E with no sinks. The first part of the proposition, which will be needed later, is written for a general graph as it is proven without the need of the property that E is row-finite and has no sinks.

PROPOSITION 7.9. *Let E be a directed graph and*

$$H = \{v \in E^0 : E(v) \textit{ satisfies } (i) \textit{ and } (ii) \textit{ of Theorem 7.3}\}.$$

Then

(i) *H is a hereditary saturated subset of E^0.*

(ii) *If E is a row-finite graph with no sinks then I_H is the largest type* I *ideal of $C^*(E)$.*

Proof. We first prove (i). That H is hereditary follows from $v \geqslant w \Rightarrow E(v) \supseteq E(w)$. We prove now that H is saturated. Suppose $v \in E^0$ and $\{w \in E^0 : v \geqslant w\} \subseteq H$. Let $\triangle(v) = \{e \in E^1 : o(e) = v\}$. Note that $\forall e \in \triangle(v)$, $t(e) \in H$. If there is a circuit at v, i.e., v is a vertex of some circuit, then $v \geqslant v$, implying that $v \in H$. Suppose there are no circuits at v. If there is a vertex $w \in E(v)^0$ on a circuit, then it is in $E(t(e))^0$ for some $e \in \triangle(v)$. But $t(e) \in H$, hence w cannot be on multiple circuits, i.e, $E(v)$ has no non-terminal and non-transitory circuits. Hence $E(v)$ satisfies (i) of Theorem 7.3. Let $\lambda \in E(v)^\infty$, then $\exists e \in \triangle(v)$ and $\beta \in E(t(e))$ such that $\lambda \sim \beta$. Since $t(e) \in H$, N_β is finite. Using Remark 7.2 we get that N_λ is finite. Therefore $v \in H$. Hence H is saturated.

To prove (ii), suppose E is row-finite with no sinks. Let $F = \mathrm{Graph}(H)$. Clearly F satisfies (i) and (ii) of Theorem 7.3, hence by Theorem 7.3, $C^*(F)$ is type I. Moreover, by Proposition 2.1 of [5], I_H is Morita equivalent to $C^*(F)$. Hence I_H is type I. Let I be the largest type I ideal of $C^*(E)$, then $I_H \subseteq I$. Since I is gauge invariant, $I = I_K$ for some hereditary saturated subset K of E^0 that includes H. We will prove that $K \subseteq H$. Let $G = \mathrm{Graph}(K)$. Since I_K is Morita equivalent to $C^*(G)$, $C^*(G)$ is type I, hence G satisfies (i) and (ii) of Theorem 7.3. Let $v \in K$; since $E(v) \subseteq G$, $E(v)$ satisfies (i) and (ii) of Theorem 7.3. Therefore $v \in H$, hence $K \subseteq H$. ∎

The next proposition generalizes Proposition 7.9.

PROPOSITION 7.10. *Let E be a directed graph and*

$$H = \{v \in E^0 : E(v) \textit{ satisfies (i) and (ii) of Theorem 7.3}\}.$$

Then $I_{(H,B_H)}$ is the largest type I *ideal of $C^*(E)$.*

Proof. Let $I_{(H',S')}$ be the largest type I ideal of $C^*(E)$ and let F be a desingularization of E; then $I_{H'_{S'}}$ is the largest type I ideal of $C^*(F)$. From (ii) of Proposition 7.9, we get that $H'_{S'} = \{v \in F^0 : F(v)$ satisfies (i) and (ii) of Theorem 7.3$\}$.

We will prove that $H \subseteq H'$. Let $G_H = \mathrm{Graph}(H)$. Clearly G_H satisfies (i) and (ii) of Theorem 7.3, hence $C^*(G_H)$ is type I. By Proposition 2.1 of [5], I_H is Morita equivalent to $C^*(G_H)$. Therefore $I_H = I_{(H,\emptyset)}$ is type I. By the maximality of $I_{(H',S')}, I_{(H,\emptyset)} \subseteq I_{(H',S')}$, implying that $H \subseteq H'$.

We will prove that $H' \subseteq H$. Let $G_{H'} = \mathrm{Graph}(H')$. Then $I_{H'} = I_{(H',\emptyset)} \subseteq I_{(H',S')}$. Hence $I_{H'}$ is liminal. By Proposition 2.1 of [5], $I_{H'}$ is Morita equivalent to $C^*(G_{H'})$, implying that $C^*(G_{H'})$ is liminal. Hence $G_{H'}$ satisfies (i) and (ii) of Theorem 7.3.

Let $v \in H'$. Since H' is hereditary and $E(v)^0 \subseteq H'$ it follows that $E(v)$ is a subgraph of $G_{H'}$. Thus $E(v)$ satisfies (i) and (ii) of Theorem 7.3. Therefore $v \in H$, hence $H' \subseteq H$.

Since $S' \subseteq B_H$, as $H = H'$, it remains to prove that $B_H \subseteq S'$. Let $v_0 \in B_H$. To show that $v_0 \in S'$ we will show that $\forall n \geqslant N_{v_0}, v_n \in H_{S'}$ i.e., $F(v_n)$ satisfies (i) and (ii) of Theorem 7.3. Let $n \geqslant N_{v_0}$ and suppose $F(v_n)$ does not satisfy (i) of Theorem 7.3. Let α be a non-terminal and non-transitory circuit in $F(v_n)$, and let $v \in \alpha^0$.

If v is on the infinite path added to v_0 in the construction of F then v_0 is in the circuit α. Notice that $v_n \geqslant v \geqslant v_0$. Recall that $\forall k \geqslant N_{v_0}, v_k$ emits exactly two edges, one pointing to v_{k+1} and one pointing to a vertex in H. Following along α, we get that $v \geqslant w$ for some vertex $w \in H$ of α. But H is hereditary, therefore $v_0 \in H$, which contradicts to the fact that $H \cap B_H = \emptyset$.

Suppose now that v is not on the infinite path added to v_0. Let p be a path from v_n to v. Then p must contain a vertex, say w, in H. Notice that $w \geqslant v$ which implies that $v \in F(w)$. Since $F(w)^0$ is hereditary, α is in the graph $F(w)$. Hence $F(w)$ contains a non-terminal and non-transitory circuit. Since $w \in H$, $E(w)$ satisfies (i) and (ii) of Theorem 7.3. But this contradicts to Corollary 7.8. Therefore $F(v_n)$ satisfies (i) of Theorem 7.3.

To prove that $F(v_n)$ satisfies (ii) of Theorem 7.3, let $\lambda \in F(v_n)^\infty$. Either λ is on the tail added to v_0 on the construction of F or λ^0 contains a vertex in H.

If λ is on the tail added to v_0 then $\mathrm{N}_\lambda = 0$. Otherwise let $w \in \lambda^0 \cap H$. Then $\lambda = p\mu$ for some $p \in F(v_n)^*$ and some $\mu \in F(v_n)^\infty$ with $o(p) = v_n, t(p) = w = o(\mu)$, implying that $\lambda \sim \mu$. Since $w \in H$, $E(w)$ satisfies (i) and (ii) of Theorem 7.3. By Corollary 7.8, we get that $F(w)$ satisfies (i) and (ii) of Theorem 7.3. Hence N_μ is finite and Remark 7.2 implies that N_λ is finite. Therefore $F(v_n)$ satisfies (ii) of Theorem 7.3.

We have established that $F(v_n)$ satisfies (i) and (ii) of Theorem 7.3. Therefore $v_n \in H_{S'}$ and hence $B_H \subseteq S'$. This concludes the proof. ∎

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