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LIE IDEALS IN OPERATOR ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a Banach algebra for which the group of invertible elements is connected. A subspace $\mathcal{L} \subseteq \mathcal{A}$ is a Lie ideal in $\mathcal{A}$ if and only if it is invariant under inner automorphisms. This applies, in particular, to any canonical subalgebra of an AF C^*-algebra. The same theorem is also proven for strongly closed subspaces of a totally atomic nest algebra whose atoms are ordered as a subset of the integers and for CSL subalgebras of such nest algebras.

We also give a detailed description of the structure of a Lie ideal in any canonical triangular subalgebra of an AF C^*-algebra.

KEYWORDS: *Lie ideals, Banach algebras, digraph algebras, nest algebras, triangular AF algebras.*

MSC (2000): 47L40.

1. INTRODUCTION

In view of the close relationship between derivations and automorphisms, it is not surprising that in many settings a subspace of an algebra is a Lie ideal if and only if it is invariant under similarity transformations. We prove this equivalence for closed subspaces of any Banach algebra for which the group of invertible elements is connected. This includes all canonical subalgebras of an AF C^*-algebra. The proof of this result is short and direct. Initially, we proved the equivalence in the context of triangular subalgebras of AF C^*-algebras via a detailed analysis of the structure of Lie ideals in triangular subalgebras. While the structure theorem is no longer needed to prove that Lie ideals are similarity invariant, it remains of independent interest and is described in Section 4 of this paper.

In the process of investigating triangular subalgebras of AF C^*-algebras, it is appropriate to look at triangular subalgebras of finite dimensional C^*-algebras. In fact, with only a moderate additional effort, we can obtain a description of Lie ideals in an arbitrary digraph algebra. In all likelihood these finite dimensional results are not new, but the authors know of no suitable reference (except in more specialized contexts). These results appear in Section 3.

The impetus for this note comes from a similar result by Marcoux and Sourour ([8]) in a much more limited context: direct limits of full upper triangular matrix algebras (T_n's); i.e., subalgebras of UHF C^*-algebras which are strongly maximal triangular in factors. The direct limit algebra context constitutes only a small portion of [8]; most of that paper is devoted to weakly closed Lie ideals in nest algebras and to Lie ideals in algebras of infinite multiplicity. In Section 2, where we present the main theorem, we also prove that strongly closed Lie ideals are similarity invariant in the context of totally atomic nest algebras whose atoms are ordered as a subset of the integers. Since weak and strong closure are identical for subspaces, this result is contained in [8]. Our proof is much shorter than the one in [8], at the price of omitting a considerable amount of information about the structure of Lie ideals in nest algebras. On the other hand, our method also works for CSL-subalgebras of these "integer-ordered" nest algebras, so the domain of validity of the equivalence is extended.

If $\mathcal{A}$ is an algebra, a subspace $\mathcal{L}$ is a *Lie ideal* if $[x, a] = xa - ax \in \mathcal{L}$ whenever $x \in \mathcal{L}$ and $a \in \mathcal{A}$. The subspace $\mathcal{L}$ is said to be *similarity invariant* if $t^{-1}xt \in \mathcal{L}$ whenever $x \in \mathcal{L}$ and t is an invertible element of $\mathcal{A}$. David Pitts has pointed out to the authors an attractive reformulation of the equivalence of these two concepts (when valid): the family of inner derivations of $\mathcal{A}$ and the family of inner automorphisms of $\mathcal{A}$ have the same invariant subspaces.

In order to avoid any ambiguity in the sequel, we shall use the term "associative ideal", rather than the usual term "ideal", for an ordinary (2-sided) ideal. Thus, all associative ideals are also Lie ideals, but not conversely.

For closed subspaces of a Banach algebra, similarity invariance for a subspace implies that the subspace is a Lie ideal. This is an unpublished result of Topping; a brief proof is contained in Theorem 2.1 of this paper. The description of Lie ideals goes back a long way; in a purely algebraic context Herstein ([3]) studied Lie ideals (and their relationship with associative ideals) in 1955. An extensive treatment of the algebraic theory appears in his book ([4]). Lie ideals in the algebra of all linear transformations on an infinite dimensional vector space were studied by Stewart in [14]. Murphy ([10]) investigated Lie ideals and their relationship with associative ideals in algebras with a set of 2×2 matrix units. Fong, Meiers and Sourour ([1]) and Fong and Murphy ([2]) have written about these ideas in the $\mathcal{B}(\mathcal{H})$ context. Marcoux ([6]) identified all Lie ideals in a UHF C^*-algebra and proved that they are similarity invariant (as well as invariant under unitary conjugation). He also described the Lie ideals in algebras of the form $\mathcal{A} \otimes C(X)$, where $\mathcal{A}$ is either a full matrix algebra or a UHF C^*-algebra. Further relevant information in the C^*-algebra setting can be found in [12] and in [7]. In C^*-algebra contexts, invariance under unitary conjugation is generally equivalent to invariance under inner derivations. Moving to the non-self-adjoint operator algebra literature, see Hudson, Marcoux and Sourour ([5]) for a description of the form of Lie ideals in nest algebras and in direct limit algebras which are strongly maximal triangular in factors. And, as mentioned above, [8] shows that a weakly closed subspace in a nest algebra is a Lie ideal if, and only if, it is similarity invariant. The assumption of weak closure can be dropped if the nest has no finite dimensional atoms.

2. LIE SPACES AND SIMILARITY

We begin with a result that refines the relationship between Lie ideals and similarity invariant subspaces given by Topping.

If $\mathcal{A}$ is a unital Banach algebra and $\mathcal{X}$ is a Banach space, then we shall call $\mathcal{X}$ a *bounded, Banach $\mathcal{A}$-bimodule* provided that $\mathcal{X}$ is an $\mathcal{A}$-bimodule such that the identity element of $\mathcal{A}$ acts as the identity on $\mathcal{X}$ and provided that the module action is bounded; that is, there exists a constant K such that $\|axb\| \leqslant K\|a\|\,\|x\|\,\|b\|$ for all a, b in $\mathcal{A}$ and x in $\mathcal{X}$. A linear subspace (not necessarily a submodule!) $\mathcal{L}$ of $\mathcal{X}$ is called a *Lie subspace* over $\mathcal{A}$ provided that $ax - xa \in \mathcal{L}$ for every $x \in \mathcal{L}$ and every $a \in \mathcal{A}$. Thus, a Lie ideal is just a Lie subspace of $\mathcal{A}$. We call a subspace $\mathcal{L}$ of $\mathcal{X}$ *similarity invariant* provided that $a^{-1}xa \in \mathcal{L}$ for every $x \in \mathcal{L}$ and every invertible element $a \in \mathcal{A}$.

Theorem 2.1. *Let $\mathcal{A}$ be a unital Banach algebra, let $\mathcal{X}$ be a bounded, Banach $\mathcal{A}$-bimodule, let $\mathcal{G}$ denote the connected component of the identity in the group of invertible elements of $\mathcal{A}$ and let $\mathcal{L}$ be a closed subspace of $\mathcal{X}$. Then $\mathcal{L}$ is a Lie subspace if and only if $b^{-1}\mathcal{L}b \subseteq \mathcal{L}$ for every $b \in \mathcal{G}$.*

Proof. Assume that $\mathcal{L}$ is a closed Lie subspace and that b is in $\mathcal{G}$. Since b is in the connected component of the identity, b is a finite product of exponentials. Therefore, to prove that $b^{-1}\mathcal{L}b \subseteq \mathcal{L}$, it suffices to prove that $\mathrm{e}^{-a}\mathcal{L}\mathrm{e}^{a} \subseteq \mathcal{L}$, for any $a \in \mathcal{A}$.

Fix $x \in \mathcal{L}$ and set $x(t) = \mathrm{e}^{-ta}x\mathrm{e}^{ta}$. This is an analytic function. An easy induction argument shows that, for all $n \geqslant 0$, the derivatives satisfy the relation $x^{(n+1)}(t) = x^{(n)}(t)a - ax^{(n)}(t)$. Since $x(0) = x \in \mathcal{L}$, it follows that $x^{(n)}(0) \in \mathcal{L}$ for all $n \geqslant 0$. Therefore, all the terms in the power series for $x(t)$ lie in $\mathcal{L}$. Since $\mathcal{L}$ is closed, it follows that $x(t) \in \mathcal{L}$ for all t. In particular, $\mathrm{e}^{-a}x\mathrm{e}^{a} \in \mathcal{L}$.

Conversely, assume that $b^{-1}\mathcal{L}b \subseteq \mathcal{L}$ for every $b \in \mathcal{G}$. Given any $a \in \mathcal{A}$, form $x(t)$ as above. By assumption, $x(t) \in \mathcal{L}$ for all t and hence the derivative $x'(t) \in \mathcal{L}$ for all t. Evaluating at $t = 0$, we find that $xa - ax \in \mathcal{L}$ and our proof is complete. ∎

As mentioned in the introduction, Topping has proven that any closed subspace of $\mathcal{A}$ which is similarity invariant is a Lie ideal. His proof is essentially reproduced in the proof of the converse in Theorem 2.1.

Theorem 2.1 shows that in order to determine whether or not a closed Lie subspace $\mathcal{L}$ of a bounded, Banach $\mathcal{A}$-bimodule is similarity invariant, it is sufficient to check whether or not $b^{-1}\mathcal{L}b \subseteq \mathcal{L}$ for any collection of elements b that contains at least one representative from each coset in $\mathcal{A}^{-1}/\mathcal{G}$.

Corollary 2.2. *Let $\mathcal{B}$ be an* AF *C^*-algebra with canonical masa $\mathcal{D}$ and let $\mathcal{A}$ be a canonical subalgebra of $\mathcal{B}$, i.e., a subalgebra such that $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{B}$. A closed subspace of $\mathcal{B}$ is a Lie subspace over $\mathcal{A}$ if and only if it is invariant under similarities.*

Proof. Clearly, $\mathcal{B}$ is a bounded, Banach $\mathcal{A}$-bimodule. The invertibles in a canonical subalgebra are connected. (Any invertible t can be closely approximated by — and hence path connected to — an invertible in a finite dimensional approximant of $\mathcal{A}$. Each invertible in a (finite dimensional) digraph algebra is path connected to the identity element.) ∎

We now turn attention to some atomic nest algebras. It is not known whether the invertibles in a nest algebra are connected, not even when the nest is atomic. Therefore Theorem 2.1 does not apply. However, in the "integer ordered" cases we can still obtain the similarity invariance of Lie ideals without using the structure of Lie ideals. Thus we provide, albeit only in a special case, a shortcut to the argument in [8]. This method also works for certain CSL subalgebras of such a nest algebra.

Let N be a subset of $\mathbb{Z}$ and, for each $n \in N$, let $\mathcal{H}_n$ be a Hilbert space. When N is a finite set, the following discussion is valid with some minor modification. It is, however, easy to provide an even simpler proof of Theorem 2.3 for finite nests. Accordingly, we assume that N is an infinite set. Without any loss of generality, we may assume that N is one of $\mathbb{Z}$, $\mathbb{N}$ or $-\mathbb{N}$. Let $\mathcal{H} = \sum_{n\in N}^{\oplus}\mathcal{H}_n$. For each n, let E_n denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$. If $P_n = \bigvee_{k\leqslant n} E_n$, then $\mathcal{N} = \{P_n : n \in N\} \cup \{0, I\}$ is a totally atomic nest in $\mathcal{H}$ whose atoms, $\{E_n\}_{n\in N}$, are order isomorphic to N. ($E_n \ll E_m$ if and only if $E_n\mathcal{H}E_m \subseteq \operatorname{Alg}\mathcal{N}$.)

Let $\mathcal{A}$ be a reflexive subalgebra of $\operatorname{Alg}\mathcal{N}$ such that $\operatorname{Lat}\mathcal{A}$ is a totally atomic lattice whose atoms are exactly the atoms of $\mathcal{N}$. The elements of $\operatorname{Alg}\mathcal{N}$ consist of all upper triangular matrices with respect to the decomposition $\mathcal{H} = \sum_{n\in N}^{\oplus}\mathcal{H}_n$. The elements of $\mathcal{A}$ consist of those matrices in $\operatorname{Alg}\mathcal{N}$ whose entries are 0 in certain specified locations.

Let $A = (A_{i,j})$ be an operator in $\mathcal{A}$. Each entry $A_{i,j}$ is an operator in $\mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ and $A_{ij} = 0$ when $i > j$ and when (i, j) is one of the specified locations mentioned above. For each $n \in \mathbb{N} \cup \{0\}$, define $D_n = \sum_{k\in N} E_kAE_{k+n}$. The matrix for D_n is $(C_{i,j})$, where $C_{i,i+n} = A_{i,i+n}$ for all i and $C_{i,j} = 0$ for all other values of i and j. Now define

$$A(z) = \sum_{n=0}^{\infty} D_n z^n = \left(A_{i,j} z^{j-i}\right).$$

Note that $\|D_n\| \leqslant \sup_{k\in N} \|E_kAE_{k+n}\| \leqslant \|A\|$, for each $n \geqslant 0$. Consequently, the series $\sum_{n=0}^{\infty} D_n z^n$ converges uniformly on any disk $|z| < r < 1$ and $A(z)$ is analytic on the open disk $|z| < 1$.

If $|z| = 1$, then $A(z)$ is unitarily equivalent to A. Indeed, write $z = \mathrm{e}^{\mathrm{i}\theta}$ and let $U(\theta)$ be the diagonal unitary matrix whose n^{th}-diagonal entry is $\mathrm{e}^{\mathrm{i}n\theta}E_n$. Then $U(\theta) \in \mathcal{A}$ and $U(\theta)^*AU(\theta) = A(\mathrm{e}^{\mathrm{i}\theta})$. Thus $\|A(\mathrm{e}^{\mathrm{i}\theta})\| = \|A\|$ for all θ; by the maximum modulus principle, $\|A(z)\| \leqslant \|A\|$ for all $|z| \leqslant 1$. Although the series $\sum_{n=0}^{\infty} D_n z^n$ need not converge uniformly on the whole unit disk, it does converge strongly. This follows from the fact that for any fixed vector $h \in \mathcal{H}$, $\|D_nh\| \to 0$. The function $A(z)$ is continuous with respect to the strong operator topology on the closed unit disk. For any vectors h_1 and h_2 in $\mathcal{H}$, the function $z \to \langle A(z)h_1, h_2\rangle$ is a complex valued analytic function in the open unit disk with continuous boundary values.

Finally, observe that if $A \in \mathcal{A}$ is invertible with inverse B in $\mathcal{A}$, then $A(z)B(z) = I$ for all $|z| \leqslant 1$. Indeed, if $z = e^{i\theta}$ then

$$\begin{aligned} A(z)B(z) &= U(\theta)^* A U(\theta) U(\theta)^* B U(\theta) \\ &= U(\theta)^* AB U(\theta) = I. \end{aligned}$$

Since this identity holds on the boundary of the unit disk and $A(z)B(z)$ is analytic, it holds throughout the unit disk.

Theorem 2.3. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a* CSL *subalgebra of a nest algebra* $\operatorname{Alg}\mathcal{N}$ *whose atoms have order type isomorphic to a subset of the integers. Assume that* $\operatorname{Lat}\mathcal{A}$ *is totally atomic and that the atoms for* $\operatorname{Lat}\mathcal{A}$ *are precisely the atoms for* $\mathcal{N}$*. Let $\mathcal{L} \subseteq B(\mathcal{H})$ be a strongly closed Lie subspace over $\mathcal{A}$. Then $\mathcal{L}$ is invariant under similarities from $\mathcal{A}$.*

Proof. Let $X \in \mathcal{L}$ and let A be an invertible element of $\mathcal{A}$ with inverse B. For $|z| < 1$, it is easy to see that $A(z)$ is in the connected component of the identity in the group of invertibles for $\mathcal{A}$. Since $B(z)$ is the inverse of $A(z)$, Theorem 2.1 implies that $A(z)XB(z) \in \mathcal{L}$ for all $|z| < 1$. But $A(z) \to A$ and $B(z) \to B$ strongly as $z \to 1$ and both $A(z)$ and $B(z)$ are uniformly bounded on the unit disk, so $A(z)XB(z) \to AXB$ strongly. Since $\mathcal{L}$ is strongly closed, $AXB = AXA^{-1} \in \mathcal{L}$. ∎

3. DIGRAPH ALGEBRAS

In this section, we shall describe all the Lie ideals in a family of operator algebras known variously as "digraph algebras," "incidence algebras" and "finite dimensional CSL-algebras." In addition, we shall give an alternate proof that every Lie ideal is similarity invariant. (Since the invertibles are connected in a digraph algebra, this result is a special case of Theorem 2.1.)

Fix a finite dimensional Hilbert space $\mathcal{H}$. A digraph algebra is a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ which contains a maximal abelian self-adjoint subalgebra $\mathcal{D}$ of $\mathcal{B}(\mathcal{H})$. Since $\mathcal{D}$ is maximal abelian, the invariant projections for $\mathcal{A}$, $\operatorname{Lat}\mathcal{A}$, are elements of $\mathcal{D}$ and so are mutually commuting. Thus $\mathcal{A}$ is a CSL-algebra. Obviously, $\mathcal{A}$ is finite dimensional; on the other hand, every finite dimensional CSL-algebra acts on a finite dimensional Hilbert space and contains a masa. For another description of $\mathcal{A}$, let n be the dimension of $\mathcal{H}$. Then $\mathcal{A}$ is isomorphic to a subalgebra of M_n which contains all the diagonal matrices, $\mathcal{D}_n$. An $n \times n$ pattern matrix, whose entries consist of 0's and $*$'s, is associated with $\mathcal{A}$. After identifying $\mathcal{A}$ with the matrix algebra to which it is isomorphic, $\mathcal{A}$ consists of all those matrices with arbitrary entries where there are $*$'s in the pattern matrix and 0's in the remaining locations. Not every pattern gives rise to an algebra, but those that do yield all the digraph algebras. This description is the one which gives rise to the term "incidence algebra." The term digraph algebra refers to the fact that associated with $\mathcal{A}$ there is a directed graph on the set of vertices $\{1, 2, \ldots, n\}$. This graph contains all the self loops. Then $\mathcal{A}$ contains the matrix unit e_{ij} if and only if there is a (directed) edge from j to i in the digraph. The matrix units in $\mathcal{A}$ generate $\mathcal{A}$ as an algebra.

Of these three descriptions, we shall primarily use the incidence algebra pattern. Furthermore, for a suitable choice of matrix units, we may assume that $\mathcal{A}$ has a block upper triangular format. One way to see this is to look at the set $\{f_1, \dots, f_p\}$ of minimal central projections in $\mathcal{A} \cap \mathcal{A}^*$. It is then easy to show that for each i, $f_i \mathcal{A} f_i$ is isomorphic to a full matrix algebra and that if $i \neq j$, then at least one of $f_i \mathcal{A} f_j$ and $f_j \mathcal{A} f_i$ is $\{0\}$. Furthermore, if $f_i \mathcal{A} f_j$ contains non-zero elements, then it contains all elements of $f_i \mathcal{B}(\mathcal{H}) f_j$. After a possible reindexing, we may assume that $i > j$ implies $f_i \mathcal{A} f_j = \{0\}$. A selection of matrix units for $\mathcal{B}(\mathcal{H})$ compatible with the minimal central projections puts $\mathcal{A}$ into block upper triangular form. An alternate way to achieve the same form is to select a maximal nest from within $\operatorname{Lat} \mathcal{A}$ and then choose matrix units compatible with the nest. It is then routine to show that $\mathcal{A}$ has a block upper triangular form in which each non-zero block is full.

We shall refer to $\mathcal{E} = \sum_{i=1}^{p} f_i \mathcal{A} f_i$ as the *diagonal* part of $\mathcal{A}$ and $\mathcal{S} = \sum_{i<j} f_i \mathcal{A} f_j$ as the *off-diagonal* part of $\mathcal{A}$. The diagonal part of $\mathcal{A}$ contains, but is in general larger than, the masa $\mathcal{D}$. Since $\mathcal{E} = \mathcal{A} \cap \mathcal{A}^*$, the diagonal part is intrinsically determined. In contrast, $\mathcal{D}$ is determined only up to an inner automorphism from $\mathcal{E}$.

Any associative ideal in $\mathcal{A}$ is, of course, a Lie ideal; we shall be concerned with associative ideals which are subsets of $\mathcal{S}$. One reason for this is that, as we shall see later, if $\mathcal{L}$ is a Lie ideal, then $\mathcal{L} \cap \mathcal{S}$ is an associative ideal.

DEFINITION 3.1. An *off-diagonal associative ideal* is an associative ideal $\mathcal{K}$ which is a subset of $\mathcal{S}$. This is equivalent to requiring that $\mathcal{K} \cap \mathcal{D} = \{0\}$.

Fix, for the moment, an off-diagonal associative ideal $\mathcal{K}$. Then $\mathcal{K}$ is the smallest Lie ideal $\mathcal{L}$ with the property that $\mathcal{L} \cap \mathcal{S} = \mathcal{K}$. It is a simple matter to check that if $f_i \mathcal{K} f_j \neq \{0\}$, then $f_i \mathcal{K} f_j = f_i \mathcal{A} f_j$ $(= f_i \mathcal{B}(\mathcal{H}) f_j)$. In addition, for $t \leqslant i$ and $s \geqslant j$, $f_t \mathcal{K} f_s = f_t \mathcal{A} f_s$. Thus, $\mathcal{K}$ consists only of full blocks and, when the pattern for $\mathcal{K}$ contains a $*$, there is also a $*$ in all locations above and to the right (based on the pattern for $\mathcal{A}$). Note: we use the non-strict interpretations for "above" and "to the right". "Above" permits entries in the same row and "to the right" permits entries in the same column. What we have just described for off-diagonal ideals is, of course, true for all associative ideals.

There are Lie ideals larger than $\mathcal{K}$ whose off-diagonal part (intersection with $\mathcal{S}$) is $\mathcal{K}$. Each of these can be obtained by adding an appropriate subspace of $\mathcal{E}$ to $\mathcal{K}$. Accordingly, we make the following definition:

DEFINITION 3.2. Let $\mathcal{K}$ be an off-diagonal associative ideal in $\mathcal{A}$. A *Lie addend* for $\mathcal{K}$ is a subspace $\mathcal{G}$ of $\mathcal{E}$ with the property that $\mathcal{G} + \mathcal{K}$ is a Lie ideal.

EXAMPLE 3.3. We describe an example of a Lie addend $\mathcal{F}$ for $\mathcal{K}$. Later on, we shall see that this is the largest Lie addend for $\mathcal{K}$; equivalently, $\mathcal{F} + \mathcal{K}$ is the largest Lie ideal which satisfies the property that $\mathcal{L} \cap \mathcal{S} = \mathcal{K}$. Let $\widetilde{S} = \{(i,j) : f_i \mathcal{A} f_j \neq \{0\} \text{ and } i < j\}$ and $\widetilde{K} = \{(i,j) : f_i \mathcal{K} f_j \neq \{0\}\}$. Each $f_i \mathcal{E} f_i$ is isomorphic to a full matrix algebra. Suppose that $(i,j) \in \widetilde{S} \setminus \widetilde{K}$. Then for each $x \in \mathcal{F}$, both $f_i x f_i$ and $f_j x f_j$ must be scalar matrices with equal scalars. On the other hand, if i is such that no ordered pair (i,j) or (j,i) lies in $\widetilde{S} \setminus \widetilde{K}$, then $f_i x f_i$ is arbitrary.

In this case, $f_i\mathcal{F}f_i = f_i\mathcal{B}(\mathcal{H})f_i$ is a subset of $\mathcal{F}$. Thus, we see that each $f_i\mathcal{F}f_i$ is either the scalars or the full matrix algebra $f_i\mathcal{B}(\mathcal{H})f_i$. There are constraints relating some of the scalar blocks, as indicated above; there are no constraints involving the full matrix algebra blocks.

To show that $\mathcal{F}+\mathcal{K}$ is a Lie ideal, it is sufficient to show that for each $x \in \mathcal{F}+\mathcal{K}$ and each matrix unit $e_{ts} \in \mathcal{A}$, $[x, e_{ts}] \in \mathcal{F}+\mathcal{K}$. Since $\mathcal{K}$ is a Lie ideal in $\mathcal{A}$, we may restrict attention to the case in which $x \in \mathcal{F}$. The matrix unit e_{ts} is either in $\mathcal{S}$ or in $\mathcal{E}$. First consider the case in which e_{ts} is in $\mathcal{S}$. Then there is a pair $(i,j) \in \widetilde{S}$ such that $e_{ts} = f_i e_{ts} f_j$. (Note: $i<j$). Now there are two subcases to consider. One is when $(i,j) \in \widetilde{K}$. Write $x = \sum_k f_k x f_k$. If $k \neq i,j$ then $[f_k x f_k, e_{ts}] = 0$, an element of $\mathcal{F}+\mathcal{K}$. If $k=i$ or $k=j$, then $[f_k x f_k, e_{ts}] \in f_i\mathcal{A}f_j = f_i\mathcal{K}f_j$ and so is an element of $\mathcal{K}$. It now follows that $[x, e_{ts}] \in \mathcal{K}$. The remaining subcase occurs when $(i,j) \in \widetilde{S}\setminus\widetilde{K}$. Then, $[x, e_{ts}] = f_i x f_i e_{ts} - e_{ts} f_j x f_j$. But now there is a scalar λ such that $f_i x f_i = \lambda f_i$ and $f_j x f_j = \lambda f_j$; consequently, $[x, e_{ts}] = 0$. The second case to consider is when $e_{ts} \in \mathcal{E}$. Then for some i, $e_{ts} = f_i e_{ts} f_i$. It follows that $[x, e_{ts}]$ is an element of $f_i\mathcal{A}f_i$ and $[x, e_{ts}] = [f_i x f_i, e_{ts}]$. Either $f_i\mathcal{A}f_i$ is a subset of $\mathcal{F}$ (when there is no j such that either (i,j) or (j,i) lies in $\widetilde{S}\setminus\widetilde{K}$) or $f_i x f_i$ is scalar and $[x, e_{ts}] = 0$; either way, $[x, e_{ts}] \in \mathcal{F}+\mathcal{K}$.

Before describing the structure of Lie addends, we show that any Lie ideal has the form $\mathcal{L} = \mathcal{G}+\mathcal{K}$, where $\mathcal{K}$ is an off-diagonal associative ideal and $\mathcal{G}$ is a Lie addend for $\mathcal{K}$.

PROPOSITION 3.4. *Let $\mathcal{L}$ be a Lie ideal in $\mathcal{A}$. Let $\mathcal{K} = \mathcal{L}\cap\mathcal{S}$ and $\mathcal{G} = \mathcal{L}\cap\mathcal{E}$. Then $\mathcal{L} = \mathcal{G}+\mathcal{K}$ and $\mathcal{K}$ is an associative ideal in $\mathcal{A}$.*

Proof. As before, $f_1, \dots, f_p$ are the minimal central projections of $\mathcal{E}$ in an order which renders $\mathcal{A}$ block upper triangular after the selection of a system of matrix units compatible with the f_i. Define $\pi : \mathcal{A} \to \mathcal{E}$ by $\pi(x) = \sum_i f_i x f_i$. Note that π is a conditional expectation onto $\mathcal{E}$.

Suppose that $x \in \mathcal{L}$. We claim that $x - \pi(x)$ is an element of $\mathcal{L}$. Since $x - \pi(x) = \sum_{i<j} f_i x f_j$, it will suffice to show that each $f_i x f_j \in \mathcal{L}$. But this follows from the fact that $f_i x f_j = [f_i, [x, f_j]]$ (since $f_j x f_i = 0$ when $i<j$). Since both x and $x-\pi(x)$ are in $\mathcal{L}$, so is $\pi(x)$. We now have

$$\mathcal{K} = \{x \in \mathcal{L} : \pi(x) = 0\} = \{x - \pi(x) : x \in \mathcal{L}\}$$

and

$$\mathcal{G} = \pi(\mathcal{L}) \subseteq \mathcal{L}.$$

For any $x \in \mathcal{L}$, $x = \pi(x) + (x - \pi(x))$, so $\mathcal{L} = \mathcal{G}+\mathcal{K}$. Since $\mathcal{S}$ and $\mathcal{L}$ are Lie ideals in $\mathcal{A}$, $\mathcal{K}$ is also a Lie ideal. It remains to show that $\mathcal{K}$ is an associative ideal.

Let $x \in \mathcal{K}$. We have just seen that $f_i x f_j \in \mathcal{L}$ whenever $i<j$. Since $f_i x f_j$ is clearly in $\mathcal{S}$, it is an element of $\mathcal{K}$. If $f_i\mathcal{K}f_j \neq \{0\}$, then we can find $x \in \mathcal{K}$ such that $f_i x f_j \neq 0$. If y is any rank one element of $\mathcal{A}$ such that $f_i y f_i = y$, then $y f_i x f_j = [y, f_i x f_j]$ is also in $\mathcal{K}$. Similarly, if z is any rank one element of $\mathcal{A}$ such that $f_j z f_j = z$, then $y f_i x f_j z = [y f_i x f_j, z]$ is in $\mathcal{K}$. But suitable choices of

y and z produce any rank one element of $f_i\mathcal{B}(\mathcal{H})f_j$. Thus, $f_i\mathcal{K}f_j \neq \{0\}$ implies that $f_i\mathcal{B}(\mathcal{H})f_j \subseteq \mathcal{K}$; in other words, $\mathcal{K}$ consists of certain of the strictly upper triangular blocks from $\mathcal{A}$.

To show that $\mathcal{K}$ is an associative ideal in $\mathcal{A}$, we need to show that when a block appears in $\mathcal{K}$, so does each block (from the pattern for $\mathcal{A}$) which lies to the right and above. So, assume that $f_i\mathcal{B}(\mathcal{H})f_j \subseteq \mathcal{K}$ and (with $j < k$) that $f_j\mathcal{A}f_k \neq \{0\}$. Let $y \in f_j\mathcal{A}f_k$ be such that $y \neq 0$ and let $x \in f_i\mathcal{B}(\mathcal{H})f_j$ be such that $xy \neq 0$. Now $yx = yf_kf_ix = 0$ (since $i < j < k$), so $xy = [x, y]$ is a non-zero element of $\mathcal{K}$. But this implies that $f_i\mathcal{B}(\mathcal{H})f_k \subseteq \mathcal{K}$. In a similar way, if $k < i < j$ and $f_k\mathcal{A}f_i \neq \{0\}$, then $f_i\mathcal{B}(\mathcal{H})f_j \subseteq \mathcal{K}$. Thus $\mathcal{K}$ is an associative ideal in $\mathcal{A}$. ∎

To complete the description of Lie ideals in $\mathcal{A}$, it remains to describe the structure of an arbitrary Lie addend $\mathcal{G}$ for an off-diagonal associative ideal $\mathcal{K}$. We continue to identify $f_i\mathcal{B}(\mathcal{H})f_i$ with a full matrix algebra acting on $f_i\mathcal{H}$.

PROPOSITION 3.5. *Let $\mathcal{K}$ be a an off-diagonal associative ideal and let $\mathcal{G}$ be a Lie addend for $\mathcal{K}$. Let $\widetilde{S} = \{(i,j) : f_i\mathcal{A}f_j \neq \{0\}$ and $i < j\}$ and $\widetilde{K} = \{(i,j) : f_i\mathcal{K}f_j \neq \{0\}\}$. If $(i,j) \in \widetilde{S} \setminus \widetilde{K}$, then for each $x \in \mathcal{G}$, f_ixf_i and f_jxf_j are scalar matrices with equal scalars. For each i, $f_i\mathcal{G}f_i$ is one of the following four subspaces of $f_i\mathcal{B}(\mathcal{H})f_i$: $\{0\}$, the scalar matrices, the set of all matrices with trace zero, or the full matrix algebra. Each element of $f_i\mathcal{G}f_i$ with trace zero is an element of $\mathcal{G}$.*

In particular, $\mathcal{F}$, the Lie addend in Example 3.3 is a maximal Lie addend. Any subspace $\mathcal{G}$ of $\mathcal{F}$ which satisfies these properties is a Lie addend.

REMARK 3.6. It is not true that $f_i\mathcal{G}f_i \subseteq \mathcal{G}$, when $\mathcal{G}$ is a Lie addend. (This would violate the scalar constraints when $(i,j) \in \widetilde{S} \setminus \widetilde{K}$.) The condition that trace zero elements in $f_i\mathcal{G}f_i$ are in $\mathcal{G}$ essentially says that $\mathcal{G}$ satisfies no non-scalar constraints. (Additional scalar constraints beyond the ones required for membership in $\mathcal{F}$ may be satisfied.)

Proof. If $\mathcal{G}$ is a subspace of $\mathcal{F}$ satisfying the conditions in Proposition 3.5, then a slight variation of the argument in Example 3.3 showing that $\mathcal{F}+\mathcal{K}$ is a Lie ideal shows that $\mathcal{G}+\mathcal{K}$ is a Lie ideal. The assumption that trace zero elements of $f_i\mathcal{G}f_i$ are in $\mathcal{G}$ is used when $x \in \mathcal{G}$ and f_ixf_i is non-scalar. In this situation, there are matrix units supported in the block associated with f_i such that $[x, e] \neq 0$. But when $[x, e] \neq 0$ and $e = f_ief_i$, $[x, e]$ is a non-zero element of $f_i\mathcal{G}f_i$ with trace zero. The assumption implies that $[x, e] \in \mathcal{G} \subseteq \mathcal{G}+\mathcal{K}$, as required for $\mathcal{G}+\mathcal{K}$ to be a Lie ideal.

Now suppose that $\mathcal{G}$ is a Lie addend for $\mathcal{K}$. Let $\mathcal{L}$ denote $\mathcal{G}+\mathcal{K}$. While $f_i\mathcal{L}f_i = f_i\mathcal{G}f_i$ need not be a Lie ideal in $\mathcal{L}$ (indeed, need not even be a subset of $\mathcal{L}$), it is easy to see that $f_i\mathcal{L}f_i$ is a Lie ideal in the full matrix algebra $f_i\mathcal{B}(\mathcal{H})f_i$. Since there are only four Lie ideals in a full matrix algebra, this shows that $f_i\mathcal{G}f_i$ is one of the four subspaces cited in Proposition 3.5.

Suppose that f_igf_i is an element of $f_i\mathcal{G}f_i$ with trace zero. If $f_igf_i = 0$, then it is certainly an element of $\mathcal{G}$. If $f_igf_i \neq 0$, then there is an element h in $f_i\mathcal{B}(\mathcal{H})f_i$ which does not commute with f_igf_i. But $h \in \mathcal{A}$ and $[g, h] = [f_igf_i, h]$ is then a non-zero element of $\mathcal{L}$. Let x denote this element. Observe that x is a non-zero element of $f_i\mathcal{G}f_i$ which has trace zero. But x is also in $\mathcal{L}$; therefore, the smallest Lie ideal containing x in the algebra $f_i\mathcal{B}(\mathcal{H})f_i$ is also contained in $\mathcal{L}$. This is the

Lie algebra of trace zero matrices in $f_i\mathcal{B}(\mathcal{H})f_i$. Since this is also a subset of $\mathcal{E}$, it is a subset of $\mathcal{G}$. Thus, all elements of $f_i\mathcal{G}f_i$ with trace zero are elements of $\mathcal{G}$.

All that remains is to prove the constraint conditions when $(i,j) \in \widetilde{S} \setminus \widetilde{K}$. The following lemma decreases the need for cumbersome notation.

Lemma 3.7. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be a direct sum of Hilbert spaces and let f_1 and f_2 be the orthogonal projections on $\mathcal{H}_1$ and $\mathcal{H}_2$. Let $x \in f_1\mathcal{B}(\mathcal{H})f_1 + f_2\mathcal{B}(\mathcal{H})f_2$. Assume that $[x,d] = 0$ for all $d \in \mathcal{B}(\mathcal{H})$ for which $d = f_1df_2$. Then there is a scalar λ such that $x = \lambda f_1 + \lambda f_2$.*

Remark 3.8. Of course, the conclusion simply says that x is a scalar operator. But in the application of the lemma, there will be additional summands present, so the $\lambda f_1 + \lambda f_2$ format is more suitable.

Proof. Let $\alpha_1, \alpha_2, \dots$ be an orthonormal basis for $\mathcal{H}_1$ and $\beta_1, \beta_2, \dots$ an orthonormal basis for $\mathcal{H}_2$. Let $d = \alpha_i\beta_j^*$ be the rank one partial isometry in $\mathcal{B}(\mathcal{H})$ with initial space $\mathbb{C}\beta_j$ and final space $\mathbb{C}\alpha_i$. Since $d = f_1df_2$,

$$f_1xf_1d - df_2xf_2 = [f_1xf_1 + f_2xf_2, d] = [x,d] = 0.$$

For any p, $\langle f_1xf_1d\beta_j, \alpha_p\rangle = \langle f_1xf_1\alpha_i, \alpha_p\rangle = \langle x\alpha_i, \alpha_p\rangle$. If $p \neq i$ then $\langle df_2xf_2\beta_j, \alpha_p\rangle = 0$. (The range of d is $\mathbb{C}\alpha_i$, which is orthogonal to α_p.) It follows that

$$\langle x\alpha_i, \alpha_p\rangle = \langle (f_1xf_1d - df_2xf_2)\beta_j, \alpha_p\rangle = 0.$$

This is valid for all pairs of indices p and i with $p \neq i$.

Similarly, for any q, $\langle df_2xf_2\beta_q, \alpha_i\rangle = \langle f_2xf_2\beta_q, \beta_j\rangle = \langle x\beta_q, \beta_j\rangle$. And if $q \neq j$, then $f_1xf_1d\beta_q = 0$. Hence

$$\langle x\beta_q, \beta_j\rangle = -\langle (f_1xf_1d - df_2xf_2)\beta_q, \alpha_i\rangle = 0.$$

This too is valid for all pairs q and j with $q \neq j$.

We also have

$$\langle f_1xf_1d\beta_j, \alpha_i\rangle = \langle x\alpha_i, \alpha_i\rangle, \quad \text{and} \quad \langle df_2xf_2\beta_j, \alpha_i\rangle = \langle x\beta_j, \beta_j\rangle.$$

Therefore,

$$\langle x\alpha_i, \alpha_i\rangle - \langle x\beta_j, \beta_j\rangle = \langle (f_1xf_1d - df_2xf_2)\beta_j, \alpha_i\rangle = 0$$

and $\langle x\alpha_i, \alpha_i\rangle = \langle x\beta_j, \beta_j\rangle$. This holds for any pair i and j. Letting λ be this common value, we now have $x = \lambda f_1 + \lambda f_2$. ∎

It remains to show that if $f_i\mathcal{A}f_j \neq \{0\}$ and $f_i\mathcal{L}f_j = \{0\}$, then for each $x \in \mathcal{G}$ there is a scalar λ such that $f_ixf_i = \lambda f_i$ and $f_jxf_j = \lambda f_j$. Recall that when $f_i\mathcal{A}f_j \neq \{0\}$, then $f_i\mathcal{A}f_j = f_i\mathcal{B}(\mathcal{H})f_j$. Let $d \in \mathcal{B}(\mathcal{H})$ be such that $d = f_idf_j$. For $x \in \mathcal{G}$, $[x,d] = f_i[x,d]f_j \in f_i\mathcal{L}f_j$, so $[x,d] = 0$. An application of Lemma 3.7 completes the proof of the Proposition 3.5. ∎

We now have a complete description of the structure of a Lie ideal in a digraph algebra. Using this description, we give an alternate proof of Theorem 2.1 in the digraph algebra context:

Proof. (Alternate proof of Theorem 2.1 for digraph algebras) We only need to show that if $\mathcal{L}$ is a Lie ideal then it is invariant under all similarity transforms. Write $\mathcal{L}$ in the form $\mathcal{G}+\mathcal{K}$ as above. Since $\mathcal{K}$ is an associative ideal, it is invariant under any similarity transform. This reduces the proof to showing that if $x \in \mathcal{G}$ and if $t \in \mathcal{A}$ is invertible, then $t^{-1}xt \in \mathcal{G}+\mathcal{K}$.

If t is an invertible element of $\mathcal{A}$, then its diagonal part $d = \sum_i f_i t f_i$ is an invertible element of $\mathcal{E}$. (The inverse is $\sum_i f_i t^{-1} f_i$.) Now $d^{-1}t$ is invertible in $\mathcal{A}$ and can be written in the form $d^{-1}t = 1+n$, where 1 is the identity element and n is strictly block upper triangular and therefore nilpotent. So we can split the argument into two cases.

Case 1. Assume that $x \in \mathcal{G}$ and that d is an invertible element of $\mathcal{E}$. For each i, let $x_i = f_i x f_i$ and $d_i = f_i d f_i$. Then $d^{-1}xd = \sum_i d_i^{-1} x_i d_i$. (Here, d_i^{-1} is to be interpreted as the inverse of d_i in $f_i\mathcal{B}(\mathcal{H})f_i$.) If i is such that $f_i\mathcal{G}f_i$ is either $\{0\}$ or the trace zero elements of $f_i\mathcal{B}(\mathcal{H})f_i$, then $d_i^{-1}xd_i$ is a trace zero element of $f_i\mathcal{G}f_i$ and so is in $\mathcal{G}$. If i is such that $f_i\mathcal{G}f_i$ are scalar elements, then $d_i^{-1}x_i d_i = x_i$; in particular, all the scalar constraints are preserved by conjugation by d. This leaves $\sum d_i^{-1}xd_i$ where the sum is taken over those i for which $f_i\mathcal{G}f_i$ is a full matrix algebra. There may be constraints involving these indices, but they are all on $\operatorname{tr} x_i$ only. Since similarity preserves traces, the $d_i^{-1}xd_i$ satisfy the same constraints, so $\sum d_i^{-1}xd_i \in \mathcal{G}$. Thus $d^{-1}xd$ is an element of $\mathcal{G} \subseteq \mathcal{G}+\mathcal{K}$. (We do not need the details, but the constraints which determine $\mathcal{G}$ as a subspace of $\mathcal{F}$ take the form of a set of linear equations in the indeterminates T_i, where the T_i correspond to those f_i for which $f_i\mathcal{G}f_i$ is either the full matrix algebra or the scalars. The elements of $\mathcal{G}$ are those $x \in \mathcal{F}$ such that the numbers $\operatorname{tr}(f_i^{-1}xf_i)$ satisfies all the equations.)

Case 2. Assume that $x \in \mathcal{G}$ and that n is a nilpotent element of $\mathcal{A}$. Let k be the order of nilpotence for n. Observe that

$$\begin{aligned}(1+n)^{-1}x(1+n) &= (1-n+n^2-n^3+\cdots+(-1)^k n^k)x(1+n)\\ &= x+(-nx+xn)+(-nxn+n^2x)+(n^2xn-n^3x)\\ &\quad+(-n^3xn+n^4x)+\cdots+((-1)^{k-1}n^{k-1}xn+(-1)^k n^k x)\\ &\quad+(-1)^k n^k xn\\ &= x-[n,x]+n[n,x]-n^2[n,x]+\cdots+(-1)^{k+1}n^k[n,x].\end{aligned}$$

The last equality uses the fact that $n^{k+1}=0$. Since $x \in \mathcal{G} \subseteq \mathcal{L}$, $[n,x] \in \mathcal{L}$. But n is strictly block upper triangular in $\mathcal{A}$; it follows that $[n,x]$ is also strictly block upper triangular. Hence, $[n,x] \in \mathcal{K}$. Since $\mathcal{K}$ is an associative ideal, $n^j[n,x] \in \mathcal{K}$ for all j. This shows that $(1+n)^{-1}x(1+n) \in \mathcal{G}+\mathcal{K}$, completing the proof of the theorem. ∎

3.1. TRIANGULAR ALGEBRAS (FINITE DIMENSIONAL). Finite dimensional triangular algebras form a subclass of the digraph algebras; consequently the work above on digraph algebras gives a description of the Lie ideals in a finite dimensional triangular algebra. Since our goal is to extend these results to triangular subalgebras of AF C^*-algebras, we pause to describe explicitly the specialization of the digraph algebra results to the triangular algebra context. There is more than one way to phrase this description; the one used below was selected for its compatibility with the use of groupoids in studying subalgebras of AF C^*-algebras.

Any finite dimensional triangular operator algebra is isomorphic to a subalgebra of the upper triangular matrices $\mathcal{T}_n$ which contains the diagonal $\mathcal{D}_n$, for some positive integer n. Accordingly, $\mathcal{B} = M_n(\mathbb{C})$ is the context for the following discussion.

With n fixed, we let $\mathcal{D}$ be the algebra of diagonal matrices; $\mathcal{T}$, the algebra of upper triangular matrices; and $\mathcal{S}$, the algebra of strictly upper triangular matrices in $\mathcal{B}$. If $\mathcal{C} \subseteq \mathcal{B}$ is a bimodule over $\mathcal{D}$, let

$$\operatorname{spec}(\mathcal{C}) = \{(i,j) : c_{ij} \neq 0, \text{ for some } c \in \mathcal{C}\}.$$

It is easy to check that $\mathcal{C} = \{x \in \mathcal{B} : x_{ij} = 0 \text{ whenever } (i,j) \notin \operatorname{spec}(\mathcal{C})\}$.

Let $\mathcal{A}$ be a triangular subalgebra of $\mathcal{T}$ (so $\mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$). Suppose that $\mathcal{K}$ is an associative ideal in $\mathcal{A}$ and that $\mathcal{K} \cap \mathcal{D} = \{0\}$ (i.e., $\mathcal{K} \subseteq \mathcal{A} \cap \mathcal{S}$). Then $\mathcal{K}$ is the smallest Lie ideal in $\mathcal{A}$ with the property that $\mathcal{L} \cap \mathcal{S} = \mathcal{K}$. Let $A = \operatorname{spec}(\mathcal{A})$, $D = \operatorname{spec}(\mathcal{D})$ and $K = \operatorname{spec}(\mathcal{K})$.

Let $\mathcal{E} = \{d \in \mathcal{D} : d_{ii} = d_{jj} \text{ whenever } (i,j) \in A \setminus K\}$. Then $\mathcal{E} + \mathcal{K}$ is a Lie ideal and is the largest Lie ideal such that $\mathcal{L} \cap \mathcal{S} = \mathcal{K}$. Furthermore, if $\mathcal{F}$ is a subspace of $\mathcal{E}$, then $\mathcal{F} + \mathcal{K}$ is a Lie ideal whose off-diagonal part is $\mathcal{K}$. Letting $\operatorname{Lie}(\mathcal{K}) = \{\mathcal{F} + \mathcal{K} : \mathcal{F} \subseteq \mathcal{E}\}$, $\operatorname{Lie}(\mathcal{K})$ is the family of all Lie ideals whose off-diagonal part is $\mathcal{K}$. Finally, if $\mathcal{L}$ is any Lie ideal in $\mathcal{A}$, and if $\mathcal{K} = \mathcal{L} \cap \mathcal{S}$, then $\mathcal{K}$ is an associative ideal and $\mathcal{L} \in \operatorname{Lie}(\mathcal{K})$.

A direct proof of this description of the Lie ideals in a finite dimensional triangular algebra is somewhat simpler than the argument for the digraph algebra case. The primary simplification arises from the fact that all blocks are 1×1 and therefore have no non-zero elements with trace zero. For the full upper triangular matrix algebra case, the description is covered by the literature on Lie ideals in nest algebras.

4. TRIANGULAR SUBALGEBRAS OF AF C^*-ALGEBRAS

In this section, $\mathcal{B}$ will denote an AF C^*-algebra with canonical diagonal $\mathcal{D}$ and $\mathcal{A}$ will be a triangular subalgebra of $\mathcal{B}$ with diagonal $\mathcal{D}$ (i.e., $\mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$). This implies that there is a sequence $\mathcal{B}_n$ of finite dimensional C^*-algebras, each with a maximal abelian self-adjoint subalgebra $\mathcal{D}_n$ such that $\mathcal{B} = \varinjlim \mathcal{B}_n$ and $\mathcal{D} = \varinjlim \mathcal{D}_n$. We can, and do, view the $\mathcal{B}_n$ as a chain of subalgebras of $\mathcal{B}$ in the usual way. Since $\mathcal{A}$ is a bimodule over $\mathcal{D}$, $\mathcal{A}$ is inductive. This means that $\mathcal{A}_n \stackrel{\text{def}}{=} \mathcal{A} \cap \mathcal{B}_n$ is a triangular subalgebra of $\mathcal{B}_n$ with diagonal $\mathcal{D}_n$ and $\mathcal{A} = \varinjlim \mathcal{A}_n$.

In addition to the presentation for $\mathcal{A}$ described above, we shall make use of "coordinitization" for $\mathcal{A}$. Coordinitization, or groupoids, for C^*-algebras is treated in detail in the books of Renault ([13]) and Paterson ([11]). For a good introduction to the use of groupoids in non-self-adjoint algebras, see Muhly and Solel ([9]). For the convenience of the reader, we provide a brief sketch of the most relevant aspects of coordinitization.

Since $\mathcal{B}$ is an AF C^*-algebra, it is a groupoid C^*-algebra. Let G be the groupoid. The groupoid can be realized as a topological equivalence relation on a compact Hausdorff, completely disconnected set X; X will be such that $\mathcal{D} \cong C(X)$. It is possible to pick a system of matrix units $\{e^n_{ij}\}$ for $\mathcal{A}$ such that, for each n, $\{e^n_{ij}\}$ are the matrix units which generate $\mathcal{A}_n$ and each matrix unit in $\mathcal{A}_n$ can be written as a sum of matrix units in $\mathcal{A}_{n+1}$. The matrix units are all normalizing partial isometries for $\mathcal{D}$. (A partial isometry v is normalizing if $v^*\mathcal{D}v \subseteq \mathcal{D}$ and $v\mathcal{D}v^* \subseteq \mathcal{D}$.) The action of a normalizing partial isometry on $\mathcal{D}$ induces a partial homeomorphism on X and the equivalence relation G is exactly the union of the graphs of all the partial homeomorphisms induced by normalizing partial isometries. The multiplication on G is defined for those pairs of elements (x, y) and (w, z) for which $w = y$; the product for a composable pair is given by $(x, y)(y, z) = (x, z)$. Inversion is given by $(x, y)^{-1} = (y, x)$. The topology on G is the one obtained by declaring that each such graph is an open subset of G. It turns out that these sets are all also closed, in fact, compact. This description makes it clear that the groupoid is independent of the presentation, but a very handy fact is that G is the union of the graphs of the matrix units in the presentation.

The elements of $\mathcal{B}$ can be identified with elements of $C_0(G)$ (but not all elements of $C_0(G)$ correspond to elements of $\mathcal{B}$). We won't need all the details of this, but we will need the formula for multiplication: if f and g are elements of $\mathcal{B}$ viewed as functions in $C_0(G)$, then

$$f \cdot g(x, y) = \sum_u f(x, u)g(u, y)$$

where u varies over the equivalence class of x (which is the same as the equivalence class of y). Note, in particular, that if $g \in \mathcal{D}$ then the support of g is in $\{(x, x) : x \in X\}$ and $f \cdot g(x, y) = f(x, y)g(y, y)$ and $g \cdot f(x, y) = g(x, x)f(x, y)$.

If $\mathcal{C} \subseteq \mathcal{B}$ is any bimodule over $\mathcal{D}$, then $C = \{(x, y) \in G : f(x, y) \neq 0 \text{ for some } f \in \mathcal{C}\}$ has the property that $\mathcal{C} = \{f \in \mathcal{B} : \operatorname{supp}(f) \subseteq C\}$. (This is the spectral theorem for bimodules ([9]); we shall refer to C as the spectrum of $\mathcal{C}$ and write $C = \operatorname{spec}(\mathcal{C})$.)

With this terminology, $\operatorname{spec}(\mathcal{D}) = \{(x, x) : x \in X\}$. It is customary to identify $D = \operatorname{spec}(\mathcal{D})$ with X (which is the spectrum of $\mathcal{D}$ in the usual sense for abelian C^*-algebras) by writing x in place of (x, x), but we won't do so in this treatment. If $A = \operatorname{spec}(\mathcal{A})$, then A is a subrelation of G whose intersection with its reversal (the spectrum of $\mathcal{A}^*$) is exactly D. Let $S = A \setminus D$. If $f \in \mathcal{A}$, then we can write $f = f|_D + f|_S$; this gives a decomposition of f into a diagonal part and an off-diagonal part.

There is another way to effect the same decomposition, via a contractive conditional expectation onto the diagonal. To define this conditional expectation we use the presentation $\varinjlim \mathcal{B}_n$ and the matrix unit system $\{e^n_{ij}\}$. For each n, define π_n on $\mathcal{B}$ by $\pi_n(f) = \sum\limits_i e^n_{ii} f e^n_{ii}$. The sequence of maps π_n converges pointwise to a contractive conditional expectation of $\mathcal{B}$ onto $\mathcal{D}$. If $f \in \mathcal{A}$, it can be shown that $\pi(f) = f|_D$ and $f - \pi(f) = f|_S$. In particular, the conditional expectation is independent of the choice of presentation or the choice of matrix units.

If f and g are elements of $\mathcal{A}$, then the index of summation in the formula for the product $f \cdot g(x, y) = \sum\limits_u f(x, u)g(u, y)$, runs over all elements u such that $(x, u) \in A$ and $(u, y) \in A$. (This is always a countable set.) In particular, if $x = y$

then the only possible value for u is x, so $f \cdot g(x,x) = f(x,x)g(x,x)$. From this it immediately follows that all commutators vanish on D.

Let $\mathcal{K}$ be an associative ideal in $\mathcal{A}$ such that $\mathcal{K} \cap \mathcal{D} = \{0\}$. Let $K = \operatorname{spec}(\mathcal{K})$. Thus $K \cap D = \emptyset$ and $K \subseteq S$. Also, $\pi(f) = 0$ for all $f \in \mathcal{K}$. Trivially, $\mathcal{K}$ is a Lie ideal in $\mathcal{A}$. Now let $\mathcal{E} = \{f \in \mathcal{D} : f(x,x) = f(y,y) \text{ whenever } (x,y) \in S \setminus K\}$. Let $\mathcal{F}$ be any subspace of $\mathcal{E}$ and let $\mathcal{L} = \mathcal{F} + \mathcal{K}$. With this notation:

PROPOSITION 4.1. *$\mathcal{L}$ is a Lie ideal in $\mathcal{A}$.*

Proof. It suffices to prove that $[f,e] \in \mathcal{L}$ for any $f \in \mathcal{F}$ and any matrix unit e in a matrix unit system for $\mathcal{A}$. If e is a diagonal matrix unit, then $[f,e] = 0 \in \mathcal{L}$. So assume that e is off-diagonal. Viewed as a function in $C_0(G)$, e is the characteristic function of a G-set which is contained in S. Since the support of $[f,e]$ is disjoint from D, it is contained in S. If we can show that it is contained in K, then $[f,e] \in \mathcal{K} \subseteq \mathcal{L}$. So let $(x,y) \in S \setminus K$. Then

$$\begin{aligned}[f,e](x,y) &= f \cdot e(x,y) - e \cdot f(x,y) = f(x,x)e(x,y) - e(x,y)f(y,y) \\ &= (f(x,x) - f(y,y))e(x,y) = 0.\end{aligned}$$

This shows that $[f,e]$ is supported in K. ∎

The next theorem shows that all closed Lie ideals have this form.

THEOREM 4.2. *If $\mathcal{L}$ is a Lie ideal in $\mathcal{A}$ then $\mathcal{L} = \mathcal{F} + \mathcal{K}$, where $\mathcal{K}$ is a diagonal disjoint associative ideal in $\mathcal{A}$ and $\mathcal{F}$ is a subspace of*

$$\mathcal{E}_{\mathcal{K}} = \{f \in \mathcal{D} : f(x,x) = f(y,y) \text{ whenever } (x,y) \in S \setminus K\}.$$

Proof. Assume that $\mathcal{L}$ is a Lie ideal in $\mathcal{A}$. We may define $\mathcal{K}$ in any of several equivalent ways:

$$\mathcal{K} = \{f \in \mathcal{L} : \pi(f) = 0\} = \{f - \pi(f) : f \in \mathcal{L}\} = \{f \in \mathcal{L} : \operatorname{supp}(f) \subseteq S\}.$$

We must show that $\mathcal{K}$ is an associative ideal in $\mathcal{A}$; the first step in that direction is to show that $\mathcal{K}$ is a bimodule over $\mathcal{D}$. (The inductivity of $\mathcal{K}$ will be helpful in showing that $\mathcal{K}$ is an associative ideal.) The following lemma is useful in showing that $\mathcal{K}$ is a bimodule over $\mathcal{D}$.

LEMMA 4.3. *Let $\mathcal{L}$ be a Lie ideal in $\mathcal{A}$ and let $f \in \mathcal{L}$. Let $d_1, \dots, d_q$ be the minimal diagonal projections in $\mathcal{D}_m$. Then, for each i, $d_i f - d_i f d_i$ is an element of $\mathcal{L}$.*

Proof. Fix i. If $j \neq i$ then $[[d_i, f], d_j] = [d_i f - f d_i, d_j] = d_i f d_j + d_j f d_i \in \mathcal{L}$. Hence, $\sum\limits_{j \neq i} d_i f d_j + \sum\limits_{j \neq i} d_j f d_i \in \mathcal{L}$. We also have

$$[d_i, f] = d_i f - f d_i = \sum_{j=1}^{n} d_i f d_j - \sum_{j=1}^{n} d_j f d_i = \sum_{j \neq i} d_i f d_j - \sum_{j \neq i} d_j f d_i \in \mathcal{L}.$$

Take an average to see that $\displaystyle\sum_{j \neq i} d_i f d_j = d_i f(1 - d_i) = d_i f - d_i f d_i \in \mathcal{L}$. (Or, more succinctly, $d_i f - d_i f d_i = (1/2)\Big([d_i, f] + \displaystyle\sum_{j \neq i} [d_i, [f, d_j]]\Big) \in \mathcal{L}$.) ∎

Let $f \in \mathcal{K}$. Fix n and let $e_1, \dots, e_p$ be the minimal diagonal projections in $\mathcal{D}_n$. As above

$$f - \pi_n(f) = \sum_{i \neq j} e_i f e_j$$

and $e_i f e_j + e_j f e_i \in \mathcal{L}$ for each pair i, j with $i \neq j$. By the way, even when $e_j \mathcal{A}_n e_i = \{0\}$, it is possible that both $e_i f e_j$ and $e_j f e_i$ are non-zero.

In order to show that $\mathcal{K}$ is a left $\mathcal{D}$ module, it suffices to show that $d(e_i f e_j + e_j f e_i) \in \mathcal{K}$ for all $d \in \mathcal{D}$ and all pairs i, j with $i \neq j$. It then follows that $df - d\pi_n(f) \in \mathcal{K}$ and, since $\mathcal{K}$ is closed, $df = \lim_n (df - d\pi_n(f)) \in \mathcal{K}$. To do this for all d, it is enough to show that $d(e_i f e_j + e_j f e_i) \in \mathcal{K}$ when d is a minimal diagonal matrix unit in $\mathcal{D}_m$ and $m \geqslant n$.

Fix $m \geqslant n$ and let d be a minimal diagonal matrix unit in $\mathcal{D}_m$. Since $m \geqslant n$, d is a subprojection of one of the e_k. In particular, since $i \neq j$, either $de_i = 0$ or $de_j = 0$; in either case $d(e_i f e_j + e_j f e_i)d = 0$. But then Lemma 4.3 yields

$$d(e_i f e_j + e_j f e_i) = d(e_i f e_j + e_j f e_i) - d(e_i f e_j + e_j f e_i)d \in \mathcal{L}.$$

But $\pi(d(e_i f e_j + e_j f e_i)) = d\pi(e_i f e_j + e_j f e_i) = 0$, so $d(e_i f e_j + e_j f e_i) \in \mathcal{K}$, as desired.

This shows that $\mathcal{K}$ is a left $\mathcal{D}$-module. A similar argument shows that $\mathcal{K}$ is a right $\mathcal{D}$-module. Or, alternatively, $\mathcal{K}^*$ is the off-diagonal part of a closed Lie ideal in $\mathcal{A}^*$ and so is a left $\mathcal{D}$-module, which immediately implies that $\mathcal{K}$ is a right $\mathcal{D}$-module. One consequence of this fact is that we now know that $\mathcal{K}$ is the closed linear span of all the off-diagonal matrix units in $\mathcal{L}$. (Note: the diagonal part of $\mathcal{L}$ is not inductive, in general.)

Next, we show that $\mathcal{K}$ is a left ideal. Let $f \in \mathcal{K}$. It suffices to show that $ef \in \mathcal{K}$ for all matrix units e in $\mathcal{A}$. Assume that e is an off-diagonal matrix unit in $\mathcal{A}_n$. (We do not need to consider diagonal matrix units, since we know that $\mathcal{K}$ is a $\mathcal{D}$-module.)

Let $\varepsilon > 0$. We claim that there is $g \in \mathcal{K}$ such that $\|g - ef\| < \varepsilon$, i.e., $\operatorname{dist}(ef, \mathcal{K}) < \varepsilon$. Since $\mathcal{K}$ is closed and ε is arbitrary, the claim implies that $ef \in \mathcal{K}$ and $\mathcal{K}$ is a left ideal.

Let p be a positive integer such that $p \geqslant n$ and $\|\pi_q(f)\| < \varepsilon/3$ for all $q \geqslant p$. (We can find such p since $\pi_p(f) \to \pi(f) = 0$ as $p \to \infty$.) With $q \geqslant p$, let $b \in \mathcal{K} \cap \mathcal{B}_q = \mathcal{K} \cap \mathcal{A}_q$ be such that $\|b - f\| < \varepsilon/3$. ($\mathcal{K}$ is inductive.) Let $c = b - \pi_q(b)$. Since $b \in \mathcal{K}$ and $\mathcal{K}$ is a bimodule over $\mathcal{D}$, $\pi_q(b) \in \mathcal{K} \subseteq \mathcal{L}$. Therefore $c \in \mathcal{K} \cap \mathcal{A}_q \subseteq \mathcal{L} \cap \mathcal{A}_q$. Now

$$\begin{aligned} \|c - f\| &= \|b - \pi_q(b) - f\| = \|b - f - \pi_q(f) + \pi_q(f - b)\| \\ &\leqslant \|b - f\| + \|\pi_q(f)\| + \|\pi_q(f - b)\| = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus, $\|ec - ef\| \leqslant \|c - f\| \leqslant \varepsilon$. So we just need to show that $ec \in \mathcal{K}$.

Since e is an off-diagonal matrix unit in $\mathcal{A}_n$ and $q \geqslant n$, e can be written as a sum of off-diagonal matrix units in $\mathcal{A}_q$. So we need to prove that $hc \in \mathcal{K}$ when h is an off-diagonal matrix unit in $\mathcal{A}_q$, and for this it is enough to prove that $hc \in \mathcal{L}$, since $\pi(hc) = 0$. Let $\mathcal{S}_q$ be the subalgebra of $\mathcal{A}_q$ generated by all the off-diagonal matrix units in $\mathcal{A}_q$ and let $\mathcal{L}' = \mathcal{L} \cap \mathcal{S}_q$. If $x \in \mathcal{A}_q$ and $y \in \mathcal{L}' \subseteq \mathcal{L}$, then $[x, y] \in \mathcal{L}$, since $\mathcal{L}$ is a Lie ideal. But $[x, y] \in \mathcal{S}_q$ also, since it is a commutator of two elements in $\mathcal{A}_q$. Thus $\mathcal{L}'$ is a diagonal disjoint Lie ideal in $\mathcal{A}_q$. But this implies that $\mathcal{L}'$ is an associative ideal in $\mathcal{A}_q$ (Proposition 3.4). Since $c \in \mathcal{L} \cap \mathcal{A}_q$ and $\pi_q(c) = 0$, $c \in \mathcal{L}'$. Therefore, $hc \in \mathcal{L}' \subseteq \mathcal{L}$. This completes the argument that $\mathcal{K}$ is a left ideal. It also shows that $\mathcal{K}^*$ is a left ideal in $\mathcal{A}^*$, whence $\mathcal{K}$ is a right ideal in $\mathcal{A}$.

We have now shown that if $\mathcal{L}$ is a Lie ideal in $\mathcal{A}$ then the diagonal disjoint part $\mathcal{K}$ is an associative ideal in $\mathcal{A}$. Let $\mathcal{F} = \mathcal{L} \cap \mathcal{D}$. From the way that $\mathcal{K}$ is defined, it is clear that $\mathcal{L} = \mathcal{F} + \mathcal{K}$. To complete the description of $\mathcal{L}$, we need to show that if $f \in \mathcal{F}$ then $f(x,x) = f(y,y)$ whenever $(x,y) \in S \setminus K$. Let $(x,y) \in S \setminus K$. Let e be a matrix unit in $\mathcal{A}$ such that $(x,y) \in \operatorname{supp}(e)$. Since $f \in \mathcal{L}$, the commutator $[f,e]$ is also in $\mathcal{L}$. Since commutators vanish on the diagonal, $[f,e] \in \mathcal{K}$. Hence

$$\begin{aligned} 0 = [f,e](x,y) &= f \cdot e(x,y) - e \cdot f(x,y) \\ &= f(x,x)e(x,y) - e(x,y)f(y,y) = f(x,x) - f(y,y). \end{aligned}$$

Thus, $f(x,x) = f(y,y)$ whenever $(x,y) \in S \setminus K$. ∎

With $\mathcal{L}$ and $\mathcal{K}$ as above, if we let

$$\mathcal{E}_{\mathcal{K}} = \{f \in \mathcal{D} : f(x,x) = f(y,y) \text{ whenever } (x,y) \in S \setminus K\},$$

then we have shown that $\mathcal{K}$ is the smallest Lie ideal and $\mathcal{E}_{\mathcal{K}} + \mathcal{K}$ is the largest Lie ideal which has $\mathcal{K}$ as its off-diagonal part. Once again, the structure of Lie ideals in triangular AF algebras permits an alternative proof of Theorem 2.1 for this context.

Proof. (Alternate proof of Theorem 2.1 for triangular subalgebras of AF C^*-algebras). As usual, we need only prove that closed Lie ideals are invariant under similarities. Let $\mathcal{L}$ be a closed Lie ideal and write $\mathcal{L} = \mathcal{F} + \mathcal{K}$ as above. Let t be an invertible element of $\mathcal{A}$. Since the invertible elements of an operator algebra form an open set, there is a sequence $t_p \in \mathcal{A}_p$ of invertible elements in $\mathcal{A}_p$ such that $t_p \to t$ and $t_p^{-1} \to t^{-1}$. If we can show that $t_p^{-1}\mathcal{L}t_p \subseteq \mathcal{L}$ for all p, then $t^{-1}\mathcal{L}t \subseteq \mathcal{L}$.

So we have reduced the proof to the case where $t \in \mathcal{A}_p$ for some p. Since $\mathcal{A}_p$ is isomorphic to a triangular matrix algebra, we can write $t = d(1+n)$, where d is a diagonal invertible element of $\mathcal{A}_p$ and n is a nilpotent element of $\mathcal{A}_p$. Let k be the order of nilpotence of n.

Since $\mathcal{K}$ is an associative ideal, $t^{-1}\mathcal{K}t \subseteq \mathcal{K}$. So we need prove that $t^{-1}\mathcal{F}t \subseteq \mathcal{F} + \mathcal{K} = \mathcal{L}$. Since d is diagonal, $d^{-1}\mathcal{F}d = \mathcal{F}$. This leaves invariance under conjugation by $1+n$. For any $f \in \mathcal{F}$,

$$\begin{aligned} (1+n)^{-1}f(1+n) &= (1 - n + n^2 - n^3 + \cdots + (-1)^k n^k) f (1+n) \\ &= f + (-nf + fn) + (-nfn + n^2 f) + (n^2 fn - n^3 f) \\ &\quad + (-n^3 fn + n^4 f) + \cdots + ((-1)^{k-l} n^{k-1} fn + (-1)^k n^k f) \\ &\quad + (-1)^k n^k fn \\ &= f - [n,f] + n[n,f] - n^2[n,f] + \cdots + (-1)^{k+1} n^k [n,f]. \end{aligned}$$

Now $[n,f] \in \mathcal{L}$ and has off-diagonal support, so $[n,f] \in \mathcal{K}$. Since $\mathcal{K}$ is an associative ideal, $(1+n)^{-1}f(1+n) \in \mathcal{F} + \mathcal{K} = \mathcal{L}$. Thus $t^{-1}\mathcal{L}t \subseteq \mathcal{L}$; $\mathcal{L}$ is invariant under inner automorphisms. ∎

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