J. OPERATOR THEORY
52(2004), 371–384

ON SUBSPACE LATTICES. II. CONTINUITY OF LAT

V.S. SHULMAN and I.G. TODOROV

Communicated by William B. Arveson

ABSTRACT. We study the continuity of the map Lat sending an ultraweakly closed operator algebra to its invariant subspace lattice. We provide an example showing that Lat is in general discontinuous and give sufficient conditions for the restricted continuity of this map. As consequences we obtain that Lat is continuous on the classes of von Neumann and Arveson algebras and give a general approximative criterion for reflexivity, which extends Arveson's theorem on the reflexivity of commutative subspace lattices.

KEYWORDS: *Subspace lattices, continuity, reflexivity, operator algebras.*

MSC (2000): Primary 47D25; Secondary 47D15.

1. INTRODUCTION

The invariant subspace theory explores the map Lat, which sends any collection M of operators on a Banach space $\mathcal{X}$ to the set $\operatorname{lat} M$ of all (closed) subspaces of $\mathcal{X}$ invariant under all operators $T \in M$. The set $S(\mathcal{X})$ of all closed subspaces of $\mathcal{X}$ is a lattice with respect to the operations of the intersection and the closed linear span of the union, and it is evident that $\operatorname{lat} M$ is a sublattice of $S(\mathcal{X})$.

If $\mathcal{X} = \mathcal{H}$ is a Hilbert space then denoting by $[E]$ the orthogonal projection on the closure of a linear subspace $E \subset \mathcal{H}$, we obtain a bijection between $S(\mathcal{H})$ and the set $\operatorname{Proj}(\mathcal{H})$ of all orthogonal projections in $\mathcal{B}(\mathcal{H})$. This allows us to transfer to $S(\mathcal{H})$ the standard operator topologies from $\mathcal{B}(\mathcal{H})$. Clearly weak and strong (as well as σ-weak and σ-strong) operator topologies coincide on $\operatorname{Proj}(\mathcal{H})$, but we prefer to consider the strong operator topology (s), because $\operatorname{Proj}(\mathcal{H})$ is s-closed in the algebra $\mathcal{B}(\mathcal{H})$ of all operators on $\mathcal{H}$. It is easy to see that $\operatorname{lat} M$ is s-closed, for any $M \subseteq \mathcal{B}(\mathcal{H})$.

It is always possible (though not always convenient) to replace M by the weakly (or, equivalently, strongly) closed unital subalgebra $\mathcal{A}(M)$ of $\mathcal{B}(\mathcal{H})$, generated by M (since $\operatorname{lat} M = \operatorname{lat} \mathcal{A}(M)$). Thus one can consider Lat as a map from the set of ultraweakly closed unital operator algebras to the set of strongly closed subspace lattices.

We are mainly interested in criteria for the continuity of Lat and conditions under which a lattice belongs to the image of Lat (or is *reflexive*, in now standard terminology, introduced by Halmos ([7])).

To explain what is meant by "continuity", recall the general notion of the limit space structure in the set 2^X of all subsets of a topological space X. For a net $\{A_\lambda\}$ of subsets of X, denote by $\liminf A_\lambda$ the set of all points $x \in X$ which are limits of nets $\{x_\lambda\}$ with $x_\lambda \in A_\lambda$ and by $\limsup A_\lambda$ the set of all points $x \in X$ which are cluster points of such nets. We say that a net of subsets $\{A_\lambda\}$ of X tends to a set $A \in 2^X$, and write $A = \lim_\lambda A_\lambda$, if $\liminf A_\lambda = \limsup A_\lambda = A$.

Now the (partial) continuity of Lat means the validity of the equality

$$\operatorname{lat}\left(\lim_\lambda \mathcal{A}_\lambda\right) = \lim_\lambda(\operatorname{lat}\mathcal{A}_\lambda) \tag{1.1}$$

for all (some) converging nets $\{\mathcal{A}_\lambda\}_\lambda$ of ultraweakly closed unital operator algebras. Important special classes of converging nets consist of nets which are downward or upward directed (the limits are the intersection and the closed hull of the union, respectively). For upward directed nets $\{\mathcal{A}_\lambda\}$ the equality (1.1) trivially holds. It will be shown in Section 3 that there is a descending sequence of weakly closed algebras $\mathcal{A}_n$ such that $\overline{\bigcup_n \operatorname{lat}\mathcal{A}_n}$ differs from $\operatorname{lat}\left(\bigcap_n \mathcal{A}_n\right)$. Thus, to have "continuity", one must impose restrictions. In Section 3 we prove that continuity holds if all $\mathcal{A}_\lambda$ are von Neumann algebras or Arveson algebras (ultraweakly closed algebras, containing masa's) or are contained in a Bercovici algebra (an algebra, whose commutant contains two isometries with orthogonal ranges). Note that Davidson ([4]) proved the "norm-continuity" of Lat on (and even in) reflexive Arveson algebras.

In Section 4 we apply the results of Section 3 and prove a general "approximative" criterion for reflexivity (Theorem 4.4) which implies immediately the celebrated Arveson's theorem ([1]) on the reflexivity of commutative subspace lattices (CSL's). Establishing a general framework for Arveson's result was one of our most stimulating aims.

2. A COUNTEREXAMPLE

We begin by presenting an example which shows that the map sending an operator algebra $\mathcal{A}$ to its invariant subspace lattice $\operatorname{lat}\mathcal{A}$ is in general discontinuous. We fix a complex separable Hilbert space $\mathcal{H}$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, we denote by $\mathcal{C}_1(\mathcal{H}_1, \mathcal{H}_2)$ the space of nuclear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and set $\mathcal{C}_1(\mathcal{H}) = \mathcal{C}_1(\mathcal{H}, \mathcal{H})$.

When considering the relation (1.1) (see Section 1) one should point out which topology $\operatorname{Proj}(\mathcal{H})$ is being endowed with. Note that a hierarchy of topologies on a set X does not imply the same hierarchy of the corresponding limit structures in 2^X. Indeed if the topology τ_2 is stronger than τ_1 then τ_2-$\liminf \mathcal{A}_\lambda \subset \tau_1$-$\liminf \mathcal{A}_\lambda$ and τ_2-$\limsup \mathcal{A}_\lambda \subset \tau_1$-$\limsup \mathcal{A}_\lambda$. But the following result disproves (1.1) in all possible versions.

Theorem 2.1. *There exists a descending sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of weakly closed algebras acting on a separable Hilbert space and containing the identity operator such that*

$$\operatorname{lat}\Big(\bigcap_n \mathcal{A}_n\Big) \neq \overline{\bigcup_n \operatorname{lat}\mathcal{A}_n}.$$

The proof will require several steps.

Lemma 2.2. *There exists a sequence $x_n \oplus y_n \in \mathcal{H}\oplus\mathcal{H}$ dense in $\mathcal{H}\oplus\mathcal{H}$ such that the set $\{x_n, y_n : n \in \mathbb{N}\}$ is linearly independent.*

Proof. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be (non-closed) subspaces of $\mathcal{H}$ such that $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ and $\mathcal{H}_i$ is dense in $\mathcal{H}$, $i = 1, 2$. There exist $x'_n \in \mathcal{H}_1$ and $y'_n \in \mathcal{H}_2$ such that $\{x'_n \oplus y'_n : n \in \mathbb{N}\}$ is dense in $\mathcal{H}\oplus\mathcal{H}$. For each $n \in \mathbb{N}$ choose $x_n \in \mathcal{H}_1$ in such a way that $\|x_n - x'_n\| < 1/n$ and $\{x_1, \ldots, x_n\}$ are linearly independent. Choose a sequence $\{y_n\}_{n=1}^{\infty}$ in a similar way. It is obvious that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ have the desired properties. ∎

Lemma 2.3. *There exists a dense subspace of $\mathcal{C}_1(\mathbb{C}^2, \mathcal{H})$ which does not contain a rank one-operator.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of vectors in $\mathcal{H}$ constructed in Lemma 2.2. Define operators $T_n \in \mathcal{C}_1(\mathbb{C}^2, \mathcal{H})$, $n \in \mathbb{N}$ by letting $T_n e_1 = x_n$, $T_n e_2 = y_n$, where $\{e_1, e_2\}$ is the usual basis of $\mathbb{C}^2$. Let $\mathcal{U}_0$ be the linear span of the operators T_n, $n \in \mathbb{N}$. It is immediate that $\mathcal{U}_0$ is dense in $\mathcal{C}_1(\mathbb{C}^2, \mathcal{H}) = \mathcal{B}(\mathbb{C}^2, \mathcal{H})$. Suppose that $A \in \mathcal{U}_0$ is a rank one operator. This means that, for some coefficients $\alpha_n \in \mathbb{C}$, $n \in \mathbb{N}$, the vectors $\sum_n \alpha_n x_n$ and $\sum_n \alpha_n y_n$ are proportional. But this is impossible since the family $\{x_n, y_n : n \in \mathbb{N}\}$ is linearly independent. ∎

Lemma 2.4. *There exists a dense subspace of $\mathcal{C}_1(\mathcal{H})$ which consists of operators of finite rank and does not contain a rank one operator.*

Proof. Write $\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k$, where each $\mathcal{H}_k$ is isomorphic to $\mathbb{C}^2$ and let $\pi_k : \mathcal{H} \to \mathcal{H}_k$ be the corresponding projections. Let $T_n : \mathbb{C}^2 \to \mathcal{H}$ be the operators from the proof of Lemma 2.3, $T_{k,n} = T_n \pi_k$, $k, n \in \mathbb{N}$, and $\mathcal{U}$ be the linear span of the operators $T_{k,n}$, $k, n \in \mathbb{N}$. If $T = \sum \alpha_{k,n} T_{k,n}$ and $\alpha_{k_0,n_0} \neq 0$, then the compression of T to $\mathcal{H}_{k_0}$ is a non-zero operator of rank strictly greater than 1. It follows that $\operatorname{rank} T > 1$. Suppose, on the other hand, that $A \in \mathcal{B}(\mathcal{H})$ and $\langle A, T_{k,n}\rangle = \operatorname{tr}(AT_{k,n}) = 0$ for each $k, n \in \mathbb{N}$. Since $T_{k,n} = T_n \pi_k$ it follows that, for a fixed k, $\operatorname{tr}((\pi_k A)T_n) = 0$ for each $n \in \mathbb{N}$. Thus $\pi_k A = 0$ for each $k \in \mathbb{N}$ and so $A = 0$. ∎

Lemma 2.5. *There exists a descending sequence $\{\mathcal{M}_n\}_{n=1}^{\infty}$ of weakly closed transitive subspaces of $\mathcal{B}(\mathcal{H})$ such that $\bigcap_n \mathcal{M}_n = \{0\}$.*

Proof. Let $\mathcal{U} \subset \mathcal{C}_1(\mathcal{H})$ be a subspace which satisfies the conditions of Lemma 2.4, $\{T_n\}_{n=1}^{\infty}$ a sequence dense in $\mathcal{U}$ in the trace norm and $\mathcal{M}_n = \{A \in \mathcal{B}(\mathcal{H}) : \langle T_k, A\rangle = 0, k = 1, 2, \ldots, n\}$. Since the operators T_k have finite rank, the space $\mathcal{M}_n$ is weakly closed for each $n \in \mathbb{N}$. Since there are no rank one operators in the preannihilator of $\mathcal{M}_n$, we have that $\mathcal{M}_n$ is transitive for each $n \in \mathbb{N}$. Since $\mathcal{U}$ is dense in $\mathcal{C}_1(\mathcal{H})$, it follows that $\bigcap_n \mathcal{M}_n = \{0\}$. ∎

Proof of Theorem 2.1. Let $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, $\mathcal{M}_n \subset \mathcal{B}(\mathcal{H})$ be the spaces constructed in Lemma 2.5 and

$$\mathcal{A}_n = \left\{ \begin{pmatrix} \lambda\mathbf{1} & X \\ 0 & \mu\mathbf{1} \end{pmatrix} : X \in \mathcal{M}_n, \lambda, \mu \in \mathbb{C} \right\}.$$

Then

$$\bigcap_n \mathcal{A}_n = \left\{ \begin{pmatrix} \lambda\mathbf{1} & 0 \\ 0 & \mu\mathbf{1} \end{pmatrix} : \lambda, \mu \in \mathbb{C} \right\}$$

and $\operatorname{lat}\left(\bigcap_n \mathcal{A}_n\right) = \{L \oplus M : L, M$ closed subspaces of $\mathcal{H}\}$. On the other hand,

$$\begin{aligned} \operatorname{lat}\mathcal{A}_n = &\{L \oplus 0 : L \text{ a closed subspace of } \mathcal{H}\} \\ &\cup \{I \oplus L : L \text{ a closed subspace of } \mathcal{H}\}, \end{aligned}$$

for each $n \in \mathbb{N}$, thus $\operatorname{lat}\left(\bigcap_{n=1}^{\infty} \mathcal{A}_n\right) \neq \bigcap_{n=1}^{\infty} \operatorname{lat}\mathcal{A}_n$. ∎

3. SUFFICIENT CONDITIONS FOR CONTINUITY

We are now going to obtain some partial positive results on the continuity of Lat. The set $\operatorname{Proj}(\mathcal{H})$ will always be endowed with the strong operator topology (s). In $\mathcal{B}(\mathcal{H})$ we consider the ultraweak operator topology (uw). But in order to work with general nets instead of sequences we should consider the bounded ultraweak convergence. Thus, if $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is a net of uw-closed algebras, then $\liminf\limits_{\lambda} \mathcal{A}_\lambda$ consists of all operators A for which there exists a bounded net $\{A_\lambda\}_{\lambda\in\Lambda}$ with $A_\lambda \in \mathcal{A}_\lambda$ such that $A_\lambda \to_\lambda A$ ultraweakly and $\limsup \mathcal{A}_\lambda$ is defined similarly but with cofinal subnets. Since weak and ultraweak topologies coincide on bounded sets, our convergence space is suitable for work with weakly closed algebras as well. In what follows $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ denotes a net of ultraweakly closed algebras containing the identity operator.

Some of the results that follow rely on a certain condition on representability of functionals in a "vector" form. Let $x, y \in \mathcal{H}$; by $\omega_{x,y}|\mathcal{A}$ we denote the functional on an algebra $\mathcal{A}$ given by $A \to (Ax, y)$, $A \in \mathcal{A}$. The functionals on $\mathcal{A}$ of the form $\omega_{x,y}|\mathcal{A}$ are called *vector functionals.*

Definition 3.1. Let $\mathcal{M}$ be an ultraweakly closed algebra of operators on a Hilbert space $\mathcal{H}$. We say that $\mathcal{M}$ possesses *property (CR)* if for each net $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ of uw-continuous functionals on $\mathcal{M}$ with $\varphi_\lambda \to_\lambda \omega_{x,y}|\mathcal{M}$ in norm, there exist vectors x_λ, y_λ, $\lambda \in \Lambda$, such that $\|x_\lambda - x\| \to_\lambda 0$, $\|y_\lambda - y\| \to_\lambda 0$ and $\varphi_\lambda = \omega_{x_\lambda,y_\lambda}|\mathcal{M}$.

It is clear that (CR) is equivalent to the following two conditions:

(CR1) each uw-continuous functional on $\mathcal{M}$ is a vector functional;

(CR2) for any $x, y \in \mathcal{H}$ and $\varepsilon > 0$, there exists $\delta = \delta(x, y, \varepsilon) > 0$ such that if f is an uw-continuous functional on $\mathcal{M}$ and $\|f - \omega_{x,y}|\mathcal{M}\| < \delta$ then there exist x', y' in $\mathcal{H}$ with $f = \omega_{x',y'}|\mathcal{M}$ and $\|x - x'\| < \varepsilon$, $\|y - y'\| < \varepsilon$.

Lemma 3.2. ([12]) *Every von Neumann algebra with properly infinite commutant possesses property* (CR)*.*

Question 1. *Does any von Neumann algebra with cyclic commutant possess property* (CR)*?*

Katsoulis and Trent ([10]) gave a proof of Lemma 3.2 that actually establishes (CR) for any von Neumann algebra $\mathcal{M}$ such that $\mathcal{M}\otimes M_2(\mathbb{C})$ has cyclic commutant.

Several other conditions, related to (and stronger than) property (CR) can be found in [2] and [3]. We mention only a remarkable result by Bercovici ([2]). Let us call an operator algebra $\mathcal{M}$ a *Bercovici algebra* if its commutant contains two isometries with orthogonal ranges. The following lemma is a special case of Theorem 4.3 ([2]).

Lemma 3.3. *Every Bercovici algebra possesses property* (CR)*.*

Since every properly infinite von Neumann algebra contains two isometries with orthogonal ranges, Lemma 3.2 is a partial case of Lemma 3.3.

Question 2. *Suppose that $\mathcal{A}_\lambda \subseteq \mathcal{M}$, where $\mathcal{M}$ is an algebra with the property* (CR) *and $\mathcal{A} = \lim \mathcal{A}_\lambda$. Is it true that* $\operatorname{lat}\mathcal{A} = \lim \operatorname{lat}\mathcal{A}_\lambda$*?*

We do not know the answer to this question even in the case $\mathcal{M}$ is a commutative von Neumann algebra. Some partial results will be presented below. In particular, we will show that the answer is affirmative if $\mathcal{M}$ is a Bercovici algebra.

Lemma 3.4. *Let X and X_λ, $\lambda \in \Lambda$, where Λ is a directed set, be subspaces of a Banach space Y and $X^\perp, X_\lambda^\perp$ be their annihilators in Y^*. Consider the norm topology in Y and the bounded weak* topology in Y^*. If $\limsup\limits_\lambda X_\lambda^\perp \subseteq X^\perp$, then $X \subseteq \liminf\limits_\lambda X_\lambda$.*

Proof. Let $x \in X$, $\|x\| = 1$. Since X_λ are normed spaces, to conclude that $x \in \liminf\limits_\lambda X_\lambda$, it suffices to show that $d(x, X_\lambda) \to 0$. Suppose that this is not true. Then there exists a subnet $\Lambda_0 \subseteq \Lambda$ and $\varepsilon > 0$ such that $d(x, X_\lambda) \geqslant \varepsilon$ for each $\lambda \in \Lambda_0$. For each $\lambda \in \Lambda_0$ choose $f_\lambda \in X_\lambda^\perp$ such that $|f_\lambda(x)| \geqslant \varepsilon$ and $\|f_\lambda\| = 1$. There exists a cluster point, say f, for the net $\{f_\lambda\}_{\lambda\in\Lambda_0}$ in the weak* topology. Since $\limsup\limits_\lambda X_\lambda^\perp \subseteq X^\perp$, we have $f \in X^\perp$. Thus $f(x) = 0$. On the other hand, $|f(x)| = \lim\limits_\lambda |f_\lambda(x)| \geqslant \varepsilon$, a contradiction. ∎

For us, the important case of Lemma 3.4 occurs when $Y = \mathcal{C}_1$, the ideal of nuclear operators on $\mathcal{H}$; then $Y^* = \mathcal{B}(\mathcal{H})$. It will be convenient to formulate it separately. Recall that by $\mathcal{X}_\perp$ we denote the preannihilator of a subspace $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ in $\mathcal{C}_1$.

Corollary 3.5. *If $\limsup\limits_\lambda \mathcal{A}_\lambda \subset \mathcal{A}$ then $\mathcal{A}_\perp \subset \liminf\limits_\lambda \mathcal{A}_{\lambda\perp}$.*

The following lemma settles the easy part of (1.1).

LEMMA 3.6. *If* $\mathcal{A} \subseteq \liminf_\lambda \mathcal{A}_\lambda$ *then* $\limsup_\lambda \operatorname{lat} \mathcal{A}_\lambda \subseteq \operatorname{lat} \mathcal{A}$.

Proof. Let $P \in \limsup_\lambda \operatorname{lat} \mathcal{A}_\lambda$ and $A \in \mathcal{A}$. It suffices to show that $(Ax, y) = 0$ for all unit vectors $x \in P\mathcal{H}$ and $y \in P^\perp \mathcal{H}$. By definition, there exists a bounded net $\{A_\lambda\}_{\lambda \in \Lambda}$, such that $A_\lambda \in \mathcal{A}_\lambda$, $A_\lambda \to_\lambda A$ weakly, and a net $\{P_\mu\}_{\mu \in \Lambda_0}$, where $\Lambda_0 \subseteq \Lambda$ is a cofinal subset, $P_\mu \in \operatorname{lat} \mathcal{A}_\mu$, $P_\mu \to_\mu P$ strongly. Let $C = \sup_{\mu \in \Lambda_0} \|A_\mu\|$, $x_\mu = P_\mu x$, $y_\mu = P_\mu^\perp y$. Then

$$\begin{aligned}
|(Ax, y)| &\leqslant |((A - A_\mu)x, y)| + |(A_\mu x, y)| \\
&\leqslant |((A - A_\mu)x, y)| + |(A_\mu x, y - y_\mu)| + |(A_\mu x, y_\mu)| \\
&\leqslant |((A - A_\mu)x, y)| + C\|y - y_\mu\| + |(A_\mu (x - x_\mu), y_\mu)| \\
&\quad + |(A_\mu x_\mu, y_\mu)| \leqslant |((A - A_\mu)x, y)| \\
&\quad + C\|y - y_\mu\| + C\|x - x_\mu\|.
\end{aligned}$$

Since $A_\mu \to A$ weakly we have that $|((A - A_\mu)x, y)| \to_n 0$. Thus $(Ax, y) = 0$, $P \in \operatorname{lat} \mathcal{A}$. ∎

REMARK 3.7. It is not difficult to see that the conclusion of the previous lemma is valid under a weaker condition: $\mathcal{A} \subseteq$ uw-closure$(\liminf \mathcal{A}_\lambda)$. Indeed, we proved that $(Ax, y) = 0$ for $A \in \liminf \mathcal{A}_\lambda$. Hence this holds for $A \in$ uw-closure$(\liminf \mathcal{A}_\lambda)$.

We do not know if $\liminf \mathcal{A}_\lambda$ is uw-closed if all $\mathcal{A}_\lambda$ are uw-closed. This forces us in what follows to consider the case

$$\limsup \mathcal{A}_\lambda \subset \mathcal{A} \subset \text{uw-closure}(\liminf \mathcal{A}_\lambda), \tag{3.1}$$

which is more general that the condition $\mathcal{A} = \lim_\lambda \mathcal{A}_\lambda$. This will be important in the proof of Theorem 3.14.

Let $\mathcal{L}_\lambda = \operatorname{lat} \mathcal{A}_\lambda$ and $\mathcal{L} = \operatorname{lat} \mathcal{A}$. Given a subset $\mathcal{N} \subseteq \operatorname{Proj}(\mathcal{H})$, we set $\mathcal{E}_\mathcal{N} = \{(x, y) \in \mathcal{H} \times \mathcal{H} : \exists\, P \in \mathcal{N} \text{ with } Px = x \text{ and } Py = 0\}$ ([13]).

LEMMA 3.8. *Suppose that* (3.1) *holds and all* $\mathcal{A}_\lambda$ *are contained in an algebra* $\mathcal{M}$ *with the property* (CR). *Then* $\mathcal{E}_\mathcal{L} = \lim_\lambda \mathcal{E}_{\mathcal{L}_\lambda}$ *with respect to the norm in* $\mathcal{H} \times \mathcal{H}$.

Proof. Let $(x, y) \in \mathcal{E}_\mathcal{L}$ and $\varepsilon > 0$. By Lemma 2.2 of [13], we have $\omega_{x,y} \in \mathcal{A}_\perp$ whence, by Corollary 3.5, $\omega_{x,y} \in \liminf \mathcal{A}_{\lambda\perp}$. Thus, for any $\delta > 0$ there exists $\lambda_0 \in \Lambda$ such that if $\lambda \succeq \lambda_0$ then there exists $f_\lambda \in \mathcal{A}_{\lambda\perp}$ with $\|\omega_{x,y} - f_\lambda\| < \delta$. Suppose that $\delta = \delta(x, y, \varepsilon)$ of (CR2). Then $f_\lambda|\mathcal{M} = \omega_{x_\lambda, y_\lambda}|\mathcal{M}$ with $\|x_\lambda - x\| < \varepsilon$, $\|y_\lambda - y\| < \varepsilon$. Since $f_\lambda|\mathcal{A}_\lambda = 0$, we have that $(x_\lambda, y_\lambda) \in \mathcal{E}_{\mathcal{L}_\lambda}$. This shows that $(x, y) \in \liminf_\lambda \mathcal{E}_{\mathcal{L}_\lambda}$, $\mathcal{E}_\mathcal{L} \subseteq \liminf \mathcal{E}_{\mathcal{L}_\lambda}$.

The inclusion $\limsup_\lambda \mathcal{E}_{\mathcal{L}_\lambda} \subseteq \mathcal{E}_\mathcal{L}$ can be proved in the same way as Lemma 3.6. ∎

Let $\widetilde{\mathcal{L}} = \liminf \mathcal{L}_\lambda$. It is clear that $\widetilde{\mathcal{L}} \subseteq \operatorname{Proj}(\mathcal{H})$. Note that $\widetilde{\mathcal{L}}$ need not be a lattice.

Proposition 3.9. *Suppose that* (3.1) *holds and all* $\mathcal{A}_\lambda$ *are contained in an algebra* $\mathcal{M}$ *with the property* (CR). *Then the inclusion* $\mathcal{E}_{\widetilde{\mathcal{L}}} \subseteq \mathcal{E}_{\mathcal{L}}$ *holds. If* $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ *is a downward directed net, then*

$$\bigcup_\lambda \mathcal{E}_{\mathcal{L}_\lambda} \subseteq \mathcal{E}_{\widetilde{\mathcal{L}}} \subseteq \overline{\bigcup_\lambda \mathcal{E}_{\mathcal{L}_\lambda}} = \mathcal{E}_{\mathcal{L}}.$$

Proof. Suppose that $(x,y) \in \mathcal{E}_{\widetilde{\mathcal{L}}}$ and $L \in \widetilde{\mathcal{L}}$ is such that $Lx = x$ and $Ly = 0$. Let $L_\lambda \to_\lambda L$ strongly, $L_\lambda \in \mathcal{L}_\lambda$. Then $(x,y) = \lim\limits_\lambda (x_\lambda, y_\lambda)$, where $x_\lambda = L_\lambda x$ and $y_\lambda = L_\lambda^\perp y$. It is clear that $(x_\lambda, y_\lambda) \in \mathcal{E}_{\mathcal{L}_\lambda}$. By Lemma 3.8, $(x,y) \in \mathcal{E}_{\mathcal{L}}$.

If $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is a downward directed net, then $\{\mathcal{L}_\lambda\}_{\lambda\in\Lambda}$ is upward directed and $\widetilde{\mathcal{L}} = \overline{\bigcup_\lambda \mathcal{L}_\lambda}^{s}$. Since $\mathcal{L}_\lambda \subseteq \mathcal{L}$, for each λ, we have that $\bigcup_\lambda \mathcal{E}_{\mathcal{L}_\lambda} \subseteq \mathcal{E}_{\widetilde{\mathcal{L}}}$. The fact that $\overline{\bigcup_\lambda \mathcal{E}_{\mathcal{L}_\lambda}} = \mathcal{E}_{\mathcal{L}}$ follows from Lemma 3.8. ∎

We next state a "one-point" continuity result. Recall that strongly closed sublattices of $\mathrm{Proj}(\mathcal{H})$ are traditionally called subspace lattices. If $\mathcal{L}_\lambda$, $\lambda \in \Lambda$, are subspace lattices on $\mathcal{H}$, we set

$$1\text{-}\liminf_\lambda \mathcal{L}_\lambda = \{P \in \mathrm{Proj}(\mathcal{H}) : \forall x \in \mathcal{H}\ \exists\ P_\lambda \in \mathcal{L}_\lambda, \lambda \in \Lambda, \text{ with } P_\lambda x \to Px\}$$

and

$$\begin{aligned}1\text{-}\limsup_\lambda \mathcal{L}_\lambda = \{P \in \mathcal{P}(\mathcal{H}) : \forall x \in \mathcal{H}\ \exists \text{ a subnet } \Lambda_0 \subseteq \Lambda \text{ and } P_\mu \in \mathcal{L}_\mu,\\ \mu \in \Lambda_0, \text{ with } P_\mu x \to Px\}.\end{aligned}$$

We note that $(1\text{-}\liminf\limits_\lambda \mathcal{L}_\lambda)^\perp = 1\text{-}\liminf\limits_\lambda \mathcal{L}_\lambda^\perp$ and similarly for $1\text{-}\limsup$. If $\mathcal{L}$ is a subspace lattice on $\mathcal{H}$ for which $1\text{-}\liminf\limits_\lambda \mathcal{L}_\lambda = 1\text{-}\limsup\limits_\lambda \mathcal{L}_\lambda = \mathcal{L}$, we say that $\mathcal{L}$ is a one-point limit of the net $\{\mathcal{L}_\lambda\}$ and write $\mathcal{L} = 1\text{-}\lim\limits_\lambda \mathcal{L}_\lambda$.

Theorem 3.10. *Let* $\mathcal{A}, \mathcal{A}_\lambda$, $\lambda \in \Lambda$, *be ultraweakly closed algebras containing the identity operator and contained in an algebra* $\mathcal{M}$ *with the property* (CR). *If* $\lim\limits_\lambda \mathcal{A}_\lambda = \mathcal{A}$ *or, more generally,* (3.1) *holds, then* $1\text{-}\lim\limits_\lambda \operatorname{lat} \mathcal{A}_\lambda = \operatorname{lat} \mathcal{A}$.

Proof. Set $\mathcal{L} = \operatorname{lat}\mathcal{A}$, $\mathcal{L}_\lambda = \operatorname{lat}\mathcal{A}_\lambda$, $\lambda \in \Lambda$. Let $P \in \mathcal{L}$ and $x \in \mathcal{H}$. Then $(Px, P^\perp x) \in \mathcal{E}_{\mathcal{L}}$. By Lemma 3.8, there exist pairs $(x_\lambda, y_\lambda) \in \mathcal{E}_{\mathcal{L}_\lambda}$ such that $(Px, P^\perp x) = \lim (x_\lambda, y_\lambda)$. By the definition of the sets $\mathcal{E}_{\mathcal{L}_\lambda}$, there exist $P_\lambda \in \mathcal{L}_\lambda$ such that $P_\lambda x_\lambda = x_\lambda$ and $P_\lambda^\perp y_\lambda = y_\lambda$. Thus $\|P_\lambda P x - x_\lambda\| = \|P_\lambda(Px - x_\lambda)\| \leqslant \|Px - x_\lambda\| \to_\lambda 0$. It follows that $P_\lambda P x \to_\lambda Px$. Similarly, $P_\lambda^\perp P^\perp x \to_\lambda P^\perp x$. It follows that $P_\lambda x \to_\lambda Px$, so $\mathcal{L} \subseteq 1\text{-}\liminf \mathcal{L}_\lambda$.

Let $P \in 1\text{-}\limsup \mathcal{L}_\lambda$. Fix $x, y \in \mathcal{H}$ with $Px = x$ and $P^\perp y = y$ and let $z = x + y$. There exist $P_\mu \in \mathcal{L}_\mu$, $\mu \in \Lambda_0$, $\Lambda_0 \subseteq \Lambda$, such that $P_\mu z \to_\mu Pz = x$ and therefore ${P_\mu}^\perp z \to_\mu y$. Let $B \in \liminf \mathcal{A}_\lambda$. Then there exist $A_\lambda \in \mathcal{A}_\lambda$, $\lambda \in \Lambda$, such that $\sup\limits_\lambda \|A_\lambda\| < \infty$ and $B = \text{uw-}\lim A_\lambda$. We then have

$$(Bx, y) = \lim_{\mu\in\Lambda_0} (A_\mu P_\mu z, P_\mu^\perp z) = 0.$$

It follows that $(Ax, y) = 0$ for each $A \in \mathcal{A}$, which shows that $P \in \operatorname{lat}\mathcal{A} = \mathcal{L}$. ∎

Recall ([13]) that for a set $\mathcal{M}$ of projections on a Hilbert space $\mathcal{H}$, the 1-closed hull of $\mathcal{M}$ is $\overline{\mathcal{M}}^1 = \{P \in \mathrm{Proj}(\mathcal{H}) : Px \in \overline{\mathcal{M}x}\ \forall x \in \mathcal{H}\}$. The set $\mathcal{M}$ is called *1-closed*, if $\mathcal{M} = \overline{\mathcal{M}}^1$.

Corollary 3.11. *Suppose that* $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ *is a decreasingly directed net of ultraweakly closed algebras containing the identity operator and contained in an operator algebra with the property* (CR) *and let* $\mathcal{A} = \bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$. *Then*

$$\mathrm{lat}\,\mathcal{A} = \overline{\bigcup_{\lambda\in\Lambda} \mathrm{lat}\,\mathcal{A}_\lambda}^1 .$$

In [13], we posed the question whether every subspace lattice is 1-closed. Corollary 3.11 shows that an affirmative answer to this question would imply an affirmative answer to Question 2, for decreasingly directed nets.

Recall ([13]) that a set $\mathcal{M}$ of projections is called *semistrongly closed*, if the set of all ranges of elements of $\mathcal{M}$ is closed as a collection of subsets in $2^{\mathcal{H}}$ (where $\mathcal{H}$ is endowed with its norm topology).

Corollary 3.12. *Under the assumptions of Corollary* 3.11*,* $\mathrm{lat}\,\mathcal{A}$ *is the semistrongly closed subspace lattice generated by* $\mathrm{lat}\,\mathcal{A}_\lambda$*,* $\lambda \in \Lambda$.

Proof. Let $\mathcal{L}_\lambda = \mathrm{lat}\,\mathcal{A}_\lambda$, $\mathcal{L} = \mathrm{lat}\,\mathcal{A}$ and $\mathcal{L}_0$ be the smallest semistrongly closed subspace lattice, containing $\mathrm{lat}\,\mathcal{A}_\lambda$ for each $\lambda \in \Lambda$. By Proposition 3.1 of [13], reflexive lattices are semistrongly closed, and the latters are 1-closed. From Corollary 3.11 we have that

$$\mathcal{L}_0 \subseteq \mathcal{L} = \overline{\bigcup_{\lambda\in\Lambda} \mathcal{L}_\lambda}^1 \subseteq \overline{\mathcal{L}_0}^1 = \mathcal{L}_0.$$

Thus $\mathcal{L}_0 = \mathcal{L}$. ∎

Let C be the algebra of compact operators on H.

Corollary 3.13. *If, under the assumptions of Corollary* 3.11*,* $\mathcal{A} \cap \mathcal{C}$ *is dense in* $\mathcal{A}$ *in the ultraweak topology, then* $\mathrm{lat}\,\mathcal{A}$ *is the subspace lattice generated by* $\mathrm{lat}\,\mathcal{A}_\lambda$*,* $\lambda \in \Lambda$.

Proof. Let $\mathcal{L}_\lambda = \mathrm{lat}\,\mathcal{A}_\lambda$, $\lambda \in \Lambda$ and $\mathcal{L} = \mathrm{lat}\,\mathcal{A}$. Suppose that $P = L_2 - L_1$ where $L_1, L_2 \in \mathcal{L}$ and $L_1 < L_2$. If $(\mathcal{A}\cap\mathcal{C})P\mathcal{H} \subseteq L_1\mathcal{H}$, then $(\mathcal{A}\cap\mathcal{C})L_2\mathcal{H} \subseteq L_1\mathcal{H}$ and $\mathcal{A}L_2\mathcal{H} \subseteq L_1\mathcal{H}$ which is impossible because $\mathcal{A}$ contains the identity operator. Thus $P(\mathcal{A}\cap\mathcal{C})P \neq 0$ for each interval P of $\mathcal{L}$. By Theorem 2.2 of [15] it follows that $\mathcal{L}$ is compact. Let $\mathcal{L}_0$ be the subspace lattice, generated by $\mathcal{L}_\lambda$, $\lambda \in \Lambda$. Clearly $\mathcal{L}_0 \subseteq \mathcal{L}$ and thus $\mathcal{L}_0$ is compact. From [13], $\mathcal{L}_0$ is 1-closed and by Corollary 3.11, $\mathcal{L}_0 = \mathcal{L}$. ∎

Let $\mathcal{K}$ be an infinite dimensional Hilbert space. By $E(\mathcal{B}(\mathcal{K}))$ we will denote the space of all uw-continuous linear functionals on $\mathcal{B}(\mathcal{K})$. For each $\varphi \in E(\mathcal{B}(\mathcal{K}))$, we let $L_\varphi : \mathcal{B}(\mathcal{K}\otimes\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be Tomiyama's right slice map, given on elementary tensors by $L_\varphi(B\otimes A) = \varphi(B)A$, $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ are ultraweakly closed algebras, then $\mathcal{B} \otimes \mathcal{A}$ will denote the ultraweakly closed subalgebra of $\mathcal{B}(\mathcal{K} \otimes \mathcal{H})$, generated by the elementary tensors $B \otimes A$, $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Theorem 3.14. *Suppose that the algebras* $\mathcal{A}_\lambda$, $\lambda \in \Lambda$, *are contained in a Bercovici algebra and* $\mathcal{A} = \lim_\lambda \mathcal{A}_\lambda$. *Then* $\operatorname{lat} \mathcal{A} = \lim_\lambda \operatorname{lat} \mathcal{A}_\lambda$.

Proof. Let $\mathcal{K}$ denote a separable infinite dimensional Hilbert space. It is clear that the algebras $\mathcal{B}(\mathcal{K}) \otimes \mathcal{A}$, $\mathcal{B}(\mathcal{K}) \otimes \mathcal{A}_\lambda$, $\lambda \in \Lambda$, are ultraweakly closed, contain the identity operator and are contained in $\mathcal{B}(\mathcal{K}) \otimes \mathcal{M}$, which is easily seen to be a Bercovici algebra. We claim that $\lim_\lambda (\mathcal{B}(\mathcal{K}) \otimes \mathcal{A}_\lambda) = \mathcal{B}(\mathcal{K}) \otimes \mathcal{A}$. Indeed, it is obvious that all operators $T \otimes S$ where $T \in \mathcal{B}(\mathcal{K})$, $S \in \mathcal{A}$ belong to $\liminf_\lambda \mathcal{B}(\mathcal{K}) \otimes \mathcal{A}_\lambda$ which implies the inclusion $\mathcal{B}(\mathcal{K}) \otimes \mathcal{A} \subseteq \overline{\liminf_\lambda \mathcal{B}(\mathcal{K}) \otimes \mathcal{A}_\lambda}^{\mathrm{uw}}$. Suppose that $T_\mu \in \mathcal{B}(\mathcal{K}) \otimes \mathcal{A}_\mu$, $\mu \in \Lambda_0$ (Λ_0 being a cofinal subset of Λ) and $T_\mu \to_\mu T$ for some operator $T \in \mathcal{B}(\mathcal{K}) \otimes \mathcal{B}(\mathcal{H})$. Then, if $\varphi \in E(\mathcal{B}(\mathcal{K}))$, we have that $L_\varphi(T_\mu) \to_\mu L_\varphi(T)$. Since $L_\varphi(T_\mu) \in \mathcal{A}_\mu$, $\mu \in \Lambda_0$, and $\limsup_\lambda \mathcal{A}_\lambda \subseteq \mathcal{A}$, it follows that $L_\varphi(T) \in \mathcal{A}$. By Theorem 1.9 of [11], $T \in \mathcal{B}(\mathcal{K}) \otimes \mathcal{A}$; hence $\limsup_\lambda \mathcal{B}(\mathcal{K}) \otimes \mathcal{A}_\lambda \subseteq \mathcal{B}(\mathcal{K}) \otimes \mathcal{A}$.

Applying Theorem 3.10 and Proposition 4.2 (iii) of [13] to the algebras $\mathcal{B}(\mathcal{K}) \otimes \mathcal{A}_\lambda$ and $\mathcal{B}(\mathcal{K}) \otimes \mathcal{A}$, we obtain $1\text{-}\lim_\lambda (\mathbf{1} \otimes \operatorname{lat} \mathcal{A}_\lambda) = \mathbf{1} \otimes \operatorname{lat} \mathcal{A}$. Let $\{e_i\}$ and $\{f_i\}$ be orthonormal bases of $\mathcal{K}$ and $\mathcal{H}$ respectively and let $x = \sum_{i=1}^{\infty} \frac{1}{i} e_i \otimes f_i$. Take $P \in \operatorname{lat} \mathcal{A}$. Then, for each $\lambda \in \Lambda$, there exists $P_\lambda \in \operatorname{lat} \mathcal{A}_\lambda$ such that $(I \otimes P)x = \lim_\lambda (I \otimes P_\lambda)x$. This means that $P_\lambda f_i \to_\lambda P f_i$ for each $i \in \mathbb{N}$, and hence $P_\lambda \to P$ strongly. In other words, $\operatorname{lat} \mathcal{A} \subseteq \liminf_\lambda \operatorname{lat} \mathcal{A}_\lambda$. By Lemma 3.6, $\operatorname{lat} \mathcal{A} = \lim_\lambda \operatorname{lat} \mathcal{A}_\lambda$. ∎

The following evident consequence of Theorem 3.14 was first obtained in [12]. It is important for applications in Section 4, where only the case $\mathcal{M} = \mathbf{1} \otimes \mathcal{B}(\mathcal{H})$ will be considered.

Corollary 3.15. *If* $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ *is a decreasingly directed net of ultraweakly closed subalgebras of a von Neumann algebra* $\mathcal{M}$ *with properly infinite commutant then* $\operatorname{lat} \mathcal{A} = \overline{\bigcup_\lambda \operatorname{lat} \mathcal{A}_\lambda}^{\mathrm{s}}$.

Now we apply some results of [13]. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we will denote by $\mathcal{L}_1 \otimes \mathcal{L}_2$ the subspace lattice on (the Hibert space tensor product) $\mathcal{H}_1 \otimes \mathcal{H}_2$, generated by all projections of the form $L_1 \otimes L_2$, where $L_i \in \mathcal{L}_i$, $i = 1, 2$. For a subspace lattice $\mathcal{L} \subseteq \operatorname{Proj}(\mathcal{H})$, let $\operatorname{conv} \mathcal{L}$ be the weakly closed convex hull of $\mathcal{L}$. If $A \in \mathcal{B}(\mathcal{H})$ is a positive contraction, let $E_s(A)$ be the spectral projection of A, corresponding to the set $[s, 1]$, $0 \leqslant s \leqslant 1$. Let also $\Phi(\mathcal{L})$ be the collection of all positive contractions A on $\mathcal{H}$ such that $E_s(A) \in \mathcal{L}$ for each $s \in [0, 1]$.

We recall some definitions from [13].

Definition 3.16. Let $\mathcal{L}$ be a subspace lattice and $\mathcal{P}$ the lattice of all projections on a separable infinite dimensional Hilbert space. We say that $\mathcal{L}$ possesses *property (p)*, if $\mathcal{P} \otimes \mathcal{L}$ is reflexive. We say that $\mathcal{L}$ possesses *property (c)* (respectively, *(c′)*) if $L_\varphi(\mathcal{P} \otimes \mathcal{L}) \subseteq \operatorname{conv} \mathcal{L}$ (respectively $L_\varphi(\mathcal{P} \otimes \mathcal{L}) \subseteq \Phi(\mathcal{L})$) for each uw-continuous state φ on $\mathcal{B}(\mathcal{H})$.

Theorem 3.17. *Let* $\mathcal{A}_\lambda$, $\lambda \in \Lambda$, *be reflexive algebras such that* $\operatorname{lat}\mathcal{A}_\lambda$ *possesses properties* (c′) *and* (p) *for each* $\lambda \in \Lambda$. *If* $\mathcal{A} = \lim_\lambda \mathcal{A}_\lambda$, *then* $\operatorname{lat}\mathcal{A} = \lim_\lambda \operatorname{lat}\mathcal{A}_\lambda$.

Proof. We first show that, for any $\varphi \in E(\mathcal{B}(\mathcal{K}))$,

$$L_\varphi(\operatorname{lat}(\mathbf{1} \otimes \mathcal{A}_\lambda)) \subseteq \Phi(\operatorname{lat}\mathcal{A}_\lambda). \tag{3.2}$$

Indeed, by formula (3.1), Section 4 of [13],

$$\mathcal{P} \otimes \operatorname{lat}\mathcal{A}_\lambda = \operatorname{lat}(\mathbf{1} \otimes \operatorname{alg}\operatorname{lat}\mathcal{A}_\lambda) = \operatorname{lat}(\mathbf{1} \otimes \mathcal{A}_\lambda),$$

since $\mathcal{A}_\lambda$ is reflexive. Now

$$L_\varphi(\operatorname{lat}(\mathbf{1} \otimes \mathcal{A}_\lambda)) = L_\varphi(\mathcal{P} \otimes \operatorname{lat}\mathcal{A}_\lambda) \subseteq \Phi(\operatorname{lat}\mathcal{A}_\lambda)$$

by (c′). If $P \in \operatorname{lat}\mathcal{A}$, we have $I \otimes P \in \operatorname{lat}(\mathbf{1} \otimes \mathcal{A})$ and so, by Corollary 3.15, there is a net $\{Q_\lambda\}_{\lambda\in\Lambda}$ of projections such that $Q_\lambda \in \operatorname{lat}(\mathbf{1} \otimes \mathcal{A}_\lambda)$, $Q_\lambda \to I \otimes P$. Setting $T_\lambda = L_\varphi(Q_\lambda)$ we have $T_\lambda \to_\lambda P$ and, by the proof of the Theorem of [9], $E_{1/2}(T_\lambda) \to_\lambda P$. By (3.2), $E_{1/2}(T_\lambda) \in \operatorname{lat}\mathcal{A}_\lambda$ and so $P \in \liminf_\lambda \operatorname{lat}\mathcal{A}_\lambda$. Hence $\operatorname{lat}\mathcal{A} \subseteq \liminf \operatorname{lat}\mathcal{A}_\lambda$. The validity of the inclusion $\limsup \mathcal{A}_\lambda \subseteq \operatorname{lat}\mathcal{A}$ follows from Lemma 3.6. ∎

Theorem 3.17 has two immediate corollaries.

Corollary 3.18. *Let* $\mathcal{A}_\lambda$, $\lambda \in \Lambda$, *be von Neumann algebras. If* $\mathcal{A} = \lim_\lambda \mathcal{A}_\lambda$, *then* $\operatorname{lat}\mathcal{A} = \lim_\lambda \operatorname{lat}\mathcal{A}_\lambda$.

Proof. By Corollary 4.10 and Proposition 4.17 of [13], the algebras $\mathcal{A}_\lambda$ possess properties (p) and (c′). The conclusion follows from Theorem 3.17. ∎

Recall that by an *Arveson algebra* we mean an ultraweakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing a maximal abelian selfadjoint algebra of $\mathcal{B}(\mathcal{H})$.

Corollary 3.19. *Let* $\mathcal{A}_\lambda$, $\lambda \in \Lambda$, *be Arveson algebras. If* $\mathcal{A} = \lim_\lambda \mathcal{A}_\lambda$, *then* $\operatorname{lat}\mathcal{A} = \lim_\lambda \operatorname{lat}\mathcal{A}_\lambda$.

Proof. It follows from the proof of Theorem 3.17 that it suffices to prove (3.2). It was done essentially by Arveson ([1], Theorem 2.1.5). The proof was coordinate-free in spirit but not in details. A proof of (3.2), completely released of the separability restriction, can be found in Lemma 22.16 of [5] (there is a difference in the formulations, but it is not difficult to see their equivalence). ∎

Recall that the commutant $\mathcal{A}'$ of a set $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is the algebra of those operators on $\mathcal{H}$ which commute with all operators in $\mathcal{A}$. The above results on the continuity of Lat allow us to establish the continuity of the map $\mathcal{A} \to \mathcal{A}'$ on the class of von Neumann algebras.

Corollary 3.20. *If $\mathcal{A}, \mathcal{A}_\lambda$ are von Neumann algebras and $\mathcal{A} = \lim_\lambda \mathcal{A}_\lambda$ with respect to the bounded ultraweak convergence then $\mathcal{A}' = \lim_\lambda \mathcal{A}'_\lambda$ with respect to the bounded $*$-strong convergence.*

Proof. Any operator $B \in \mathcal{A}'$ is a norm limit of finite linear combinations $\sum_k \alpha_k P_k$ with $\sum_k |\alpha_k| \leqslant 4\|B\|$, where $P_k \in \mathrm{Proj}(\mathcal{A}')$. Since by Corollary 3.18 each projection in $\mathcal{A}'$ is a strong limit of projections in $\mathcal{A}'_\lambda$, B is a $*$-strong limit of a bounded net $(B_\lambda)_{\lambda\in\Lambda}$ with $B_\lambda \in \mathcal{A}'_\lambda$. In other words $\mathcal{A}' \subseteq *\text{-strong}\liminf_\lambda \mathcal{A}_\lambda$.

Let $B_\mu \in \mathcal{A}'_\mu$ ($\mu \in \Lambda_0 \subseteq \Lambda$) form a bounded net that converges $*$-strongly to an operator B. For $A \in \mathcal{A}$ let $A_\mu \in \mathcal{A}_\mu$ be a bounded net converging weakly to A. Then, for all $x, y \in \mathcal{H}$,

$$((BA - AB)x, y) = (B(A - A_\mu)x, y) - ((A - A_\mu)Bx, y) + (A_\mu x, (B^* - B^*_\mu)y)$$
$$-(A_\mu(B - B_\mu)x, y) \to 0.$$

Thus $B \in \mathcal{A}'$ that is $\limsup_\lambda \mathcal{A}' \subseteq \mathcal{A}'$. ∎

4. APPROXIMATIVITY AND REFLEXIVITY

In this section we look at the equality $\mathrm{lat}\Big(\bigcap_\lambda \mathcal{A}_\lambda\Big) = \overline{\bigcup_\lambda \mathrm{lat}\,\mathcal{A}_\lambda}^{\mathrm{s}}$ from “the reverse side”, that is, we consider the properties of the union of a directed net of lattices. For example, the above equality implies that the strongly closed hull of the union of a directed net of reflexive lattices is (under certain conditions) a reflexive lattice.

One of the obstacles in the work with projection lattices is that the lattice operations are not completely consistent with the topology. In particular, the closure of a projection lattice, even in finite-dimensional spaces, need not be a lattice. Some interesting counterexamples and positive results can be found in the papers of Gilfeather and Larson ([6]) and Symes ([14]).

Thus, the closed hull of the union of an increasing net of subspace lattices need not be a lattice. This forces us to distinguish (at least) two types of “approximativity”. Let us say that a property (Pr) is *approximative*, if each lattice $\mathcal{L}$ with $\mathcal{L} = \overline{\bigcup_\lambda \mathcal{L}_\lambda}^{\mathrm{s}}$, where $\{\mathcal{L}_\lambda\}$ is an upward directed net of lattices possessing (Pr), possesses (Pr) as well. Say that (Pr) is *strictly approximative*, if the fact that $\mathcal{L}_\lambda$ possesses (Pr) for each $\lambda \in \Lambda$ implies that the subspace lattice generated by the union of $\mathcal{L}_\lambda$, $\lambda \in \Lambda$, possesses (Pr) as well. It is clear that if (Pr) is strictly approximative, then (Pr) is approximative.

Note also that if a lattice $\mathcal{L}$ is the (strong) closure of the union of an increasing net $\{\mathcal{L}_\lambda\}$ of subspace lattices, then one cannot conclude that $\mathcal{N} \otimes \mathcal{L}$ is the closure of the union of the net $\{\mathcal{N} \otimes \mathcal{L}_\lambda\}$, but only that

$$\mathcal{N} \otimes \mathcal{L} = \bigvee_\lambda \mathcal{N} \otimes \mathcal{L}_\lambda, \tag{4.1}$$

where by $\bigvee_\lambda \mathcal{N}_\lambda$ is denoted the smallest subspace lattice, containing the union of the lattices $\mathcal{N}_\lambda$. If the lattices $\mathcal{N}$ and $\mathcal{L}_\lambda$, $\lambda \in \Lambda$, act on the same space, then

$$\mathcal{N} \vee \Big(\bigvee_{\lambda\in\Lambda} \mathcal{L}_\lambda \Big) = \bigvee_{\lambda\in\Lambda} (\mathcal{N} \vee \mathcal{L}_\lambda). \tag{4.2}$$

Identity (4.2) is evident while (4.1) follows from (4.2) and the identities $\mathcal{N} \otimes \mathcal{L} = (\mathcal{N} \otimes \mathbf{1}) \vee (\mathbf{1} \otimes \mathcal{L})$ and $\bigvee_\lambda (\mathbf{1} \otimes \mathcal{L}_\lambda) = \mathbf{1} \otimes \Big(\bigvee_\lambda \mathcal{L}_\lambda \Big)$.

Proposition 4.1. *Property (p) is strictly approximative, hence approximative.*

Proof. Let $\{\mathcal{L}_\lambda\}$ be an increasingly directed net of subspace lattices, each of which possesses (p) and let $\mathcal{L}$ be the subspace lattice, generated by their union. Set $\mathcal{A}_\lambda = \operatorname{alg} \mathcal{L}_\lambda$, $\lambda \in \Lambda$, and $\mathcal{A} = \operatorname{alg} \mathcal{L}$. Then $\{\mathcal{A}_\lambda\}$ is a downward directed net of reflexive algebras and $\mathcal{A} = \bigcap_\lambda \mathcal{A}_\lambda$. Indeed, it is clear that $\mathcal{A} \subseteq \mathcal{A}_\lambda$ for each λ. On the other hand, $\mathcal{L}_\lambda \subseteq \operatorname{lat} \Big(\bigcap_\lambda \mathcal{A}_\lambda \Big)$ for each λ, whence $\mathcal{L} \subseteq \operatorname{lat} \Big(\bigcap_\lambda \mathcal{A}_\lambda \Big)$. Thus $\bigcap_\lambda \mathcal{A}_\lambda \subseteq \operatorname{alg} \mathcal{L} = \mathcal{A}$.

We have

$$\mathcal{P} \otimes \mathcal{L} \supseteq \overline{\bigcup_\lambda \mathcal{P} \otimes \mathcal{L}_\lambda}^{s} = \overline{\bigcup_\lambda \operatorname{lat}(\mathbf{1} \otimes \mathcal{A}_\lambda)}^{s} = \operatorname{lat}(\mathbf{1} \otimes \mathcal{A}).$$

The first inclusion in the above chain is obvious. The second equality follows from the fact that the lattices $\mathcal{L}_\lambda$ possess property (p); indeed, a lattice $\mathcal{M}$ possesses (p) if and only if $\mathcal{P} \otimes \mathcal{M} = \operatorname{lat}(\mathbf{1} \otimes \operatorname{alg} \mathcal{M})$ (see identity (2) of [13]). The third equality follows from Corollary 3.15. On the other hand, $\mathcal{P} \otimes \mathcal{L}$ is clearly contained in $\operatorname{lat}(\mathbf{1} \otimes \mathcal{A})$. We thus obtained that $\mathcal{P} \otimes \mathcal{L}$ is reflexive, that is, $\mathcal{L}$ possesses (p). ∎

The next result strengthens Theorem 4.9 of [13].

Proposition 4.2. *Let $\mathcal{L}$ be a CSL, $\mathcal{N}_0$ and $\mathcal{N}$ the projection lattices of von Neumann algebras, least one of which is injective and suppose that $\mathcal{N}$ commutes with $\mathcal{L}$. Then the lattice $\mathcal{N}_0 \otimes (\mathcal{N} \vee \mathcal{L})$ possesses property (p).*

Proof. Note first that

$$\mathcal{N}_0 \otimes (\mathcal{N} \vee \mathcal{L}) = (\mathcal{N}_0 \otimes \mathcal{N}) \vee (\mathbf{1} \otimes \mathcal{L}). \tag{4.3}$$

Indeed, we have that

$$\begin{aligned} \mathcal{N}_0 \otimes (\mathcal{N} \vee \mathcal{L}) &= (\mathcal{N}_0 \otimes \mathbf{1}) \vee (\mathbf{1} \otimes (\mathcal{N} \vee \mathcal{L})) \\ &= (\mathcal{N}_0 \otimes \mathbf{1}) \vee (\mathbf{1} \otimes \mathcal{N}) \vee (\mathbf{1} \otimes \mathcal{L}) = (\mathcal{N}_0 \otimes \mathcal{N}) \vee (\mathbf{1} \otimes \mathcal{L}). \end{aligned}$$

Since any CSL is the strongly closed hull of the union of an upward directed net of finite lattices, Proposition 4.1 and identity (4.2) allow us to assume that $\mathcal{L}$ is finite. On the other hand, by Theorem 4.9 of [13], $\mathcal{N}_0 \otimes \mathcal{N}$ is a von Neumann lattice which moreover commutes with $\mathbf{1} \otimes \mathcal{L}$. By Theorem 4 of [10] and identity (4.3), $\mathcal{N}_0 \otimes (\mathcal{N} \vee \mathcal{L})$ is reflexive in this case and the proof is complete. ∎

It will be convenient to formulate separately some immediate consequences of the above result.

Corollary 4.3. (i) *Every CSL possesses property (p).*

(ii) *The tensor product of a CSL and a von Neumann lattice possesses property (p).*

(iii) *If a CSL $\mathcal{L}$ commutes with a von Neumann lattice $\mathcal{N}$, then $\mathcal{N} \vee \mathcal{L}$ possesses property (p).*

The main result in the present section is the following.

Theorem 4.4. *Let $\{\mathcal{L}_\lambda\}_{\lambda\in\Lambda}$ be an increasingly directed net of subspace lattices possessing properties (p) and (c) and $\mathcal{L}$ be the subspace lattice generated by its union. Then $\mathcal{L}$ is reflexive.*

Proof. By Proposition 4.1, we have that $\mathcal{L}$ possesses (p) and moreover, as is seen from its proof,

$$\mathcal{P} \otimes \mathcal{L} = \overline{\bigcup_\lambda \mathcal{P} \otimes \mathcal{L}_\lambda}^{\,s}.$$

If $Q \in \mathcal{P} \otimes \mathcal{L}$, then there exist projections $Q_\lambda \in \mathcal{P} \otimes \mathcal{L}_\lambda$ for each λ, such that $Q_\lambda \to_\lambda Q$ in the strong operator topology. It follows that $L_\varphi(Q_\lambda) \to L_\varphi(Q)$ strongly, for each $\varphi \in E(\mathcal{B}(\mathcal{K}))$. On the other hand, $L_\varphi(Q_\lambda) \in \operatorname{conv} \mathcal{L}_\lambda \subseteq \operatorname{conv} \mathcal{L}$ for each λ and since $\operatorname{conv} \mathcal{L}$ is strongly closed, we conclude that $L_\varphi(Q) \in \operatorname{conv} \mathcal{L}$ for each $\varphi \in E(\mathcal{B}(\mathcal{K}))$. We hence have that $\mathcal{L}$ possesses property (c). By Theorem 4.14 and Proposition 4.16 of [13], $\mathcal{L}$ is reflexive. ∎

Remarks 4.5. (i) From Proposition 4.1 and the proof of Theorem 4.4 it follows that the property "(p) and (c)" is strictly approximative. Is (c) approximative?

(ii) A less general result than Theorem 4.4, namely if we replace (c) by (c′), follows from Theorem 3.17.

(iii) Theorem 4.4 extends Arveson's reflexivity theorem for CSL's ([1]). To see this, we need only to check that finite CSL's possess property (c). But, if $\mathcal{L}$ is a finite CSL, every projection $Q \in \mathcal{P} \otimes \mathcal{L}$ can be written in the form $Q = \sum_{E\in a(\mathcal{L})} P(E) \otimes E$, where $P(\cdot)$ is a decreasing (in the sense that $P(E) \leqslant P(F)$ if $F\mathcal{B}(\mathcal{H})E \subseteq \operatorname{alg} \mathcal{L}$) projection valued function defined on the set $a(\mathcal{L})$ of atoms of $\mathcal{L}$. Hence $L_\varphi(Q) = \sum_{E\in a(\mathcal{L})} t(E)E$, where $t : a(\mathcal{L}) \to [0,1]$ is a decreasing function. Writing each atom of $\mathcal{L}$ as the difference of two projections in $\mathcal{L}$, it is easy to see that such a sum belongs to $\operatorname{conv}(\mathcal{L})$.

Question 3. *Is the tensor product of a CSL and a von Neumann lattice reflexive? In particular, if $\mathcal{P}_n$ is the subspace lattice of all projections in an n-dimensional Hilbert space, is $\mathcal{P}_n \otimes \mathcal{L}$ reflexive for each CSL $\mathcal{L}$?*

For properly infinite von Neumann lattices the affirmative answer follows easily from Corollary 4.10 of [13] and Corollary 4.3 (ii) (see also more general results of Katsoulis ([10]) and Symes ([14]) based on the continuity theorem of [12]).

Acknowledgements. It is a pleasure to express our gratitude to J. Erdos, A. Katavolos and L. Turowska for helpful discussions.

REFERENCES

1. W.B. Arveson, Operator algebras and invariant subspaces, *Ann. Math.* **100**(1974), 433–532.
2. H. Bercovici, Hyper-reflexivity and the factorization of linear functionals, *J. Funct. Anal.* **158**(1998), 242–252.
3. H. Bercovici, C. Foiaş, C.M. Pearcy, *Dual Algebras with Applications to Invariant Subspaces and Dilation Theory*, CBMS Regional Conf. Ser. in Math., vol. 56, Amer. Math. Soc., Providence RI 1985.
4. K.R. Davidson, Perturbations of reflexive operator algebras, *J. Operator Theory* **15**(1986), 289–305.
5. K.R. Davidson, *Nest Algebras*, Longman, 1988
6. F. Gilfeather, D.R. Larson, Structure in reflexive subspace lattices, *J. London Math. Soc.* **26**(1982), 117–131.
7. P.R. Halmos, Reflexive lattices of subspaces, *J. London Math. Soc.* **4**(1971), 257–263.
8. P.R. Halmos, Limsups of Lats, *Indiana Univ. Math. J.* **29**(1980), 293–311.
9. A. Hopenwasser, Tensor products of reflexive subspace lattices, *Michigan Math. J.* **31**(1984), 359–370.
10. E. Katsoulis, Reflexivity for a class of subspace lattices, *Math. Proc. Cambridge Philos. Soc.* **119**(1996), 67–71.
11. J. Kraus, The slice map problem for σ-weakly closed subspaces of von Neumann algebras, *Trans. Amer. Math. Soc.* **279**(1983), 357–376.
12. V.S. Shulman, On subspace lattices in a Hilbert space, *Funktsional. Anal. i Prilozhen.* **23**(1989), 86–87.
13. V.S. Shulman, I.G. Todorov, On subspace lattices I. Closedness type properties and tensor products, *Integral Equations Operator Theory*, to appear.
14. D. Symes, Structure of doubly generated reflexive lattices, *Quart. J. Math.* **49**(1998), 229–235.
15. B.H. Wagner, Weak limits of projections and compactness of subspace lattices, *Trans. Amer. Math. Soc.* **304**(1987), 515–535.

V.S. SHULMAN
School of Communications Technologies and Mathematical Sciences
University of North London
Holloway, London N7 8DB
UNITED KINGDOM
E-mail: shulman_v@yahoo.com

I.G. TODOROV
Department of Pure Mathematics
Queen's University Belfast
Belfast, BT7 1NN Northern Ireland
Northern Ireland
UNITED KINGDOM
E-mail: i.todorov@qub.ac.uk

Received November 18, 2002; revised April 20, 2003.