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COWEN SETS FOR TOEPLITZ OPERATORS WITH FINITE RANK SELFCOMMUTATORS

WOO YOUNG LEE

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ABSTRACT. Cowen's theorem states that if $\varphi \in L^\infty(\mathbb{T})$ then the Toeplitz operator T_φ is hyponormal if and only if the following "Cowen" set

$$\mathcal{E}(\varphi) = \{k \in H^\infty(\mathbb{T}) : \|k\|_\infty \leqslant 1 \quad \text{and} \quad \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\}$$

is nonempty.

In this paper we give a complete description on the Cowen set $\mathcal{E}(\varphi)$ if the selfcommutator $[T_\varphi^*, T_\varphi]$ is of finite rank. In particular, it is shown that the solution for the cases where φ is of bounded type has a connection with a H^∞ optimization problem.

KEYWORDS: *Cowen sets, H^∞ optimization problem, Toeplitz operators, Hankel operators, hyponormal, bounded type, Carathéodory-Schur interpolation problem.*

MSC (2000): 47B20, 47B35.

1. INTRODUCTION

A bounded linear operator A on a Hilbert space $\mathcal{H}$ is said to be hyponormal if its selfcommutator $[A^*, A] = A^*A - AA^*$ is positive semidefinite. Recall that given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$ in the complex plane $\mathbb{C}$ defined by

$$T_\varphi f = P(\varphi \cdot f),$$

where $f \in H^2(\mathbb{T})$ and P denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Relationships between hyponormal operators and Toeplitz-like operators were discovered in papers [14] and [2]. More recently, the problem of determining which symbols induce hyponormal Toeplitz operators was completely solved by C. Cowen [3] in 1988. Here we shall employ an equivalent variant of Cowen's theorem that was proposed by T. Nakazi and K. Takahashi in [11].

THEOREM 1.1 ([3], [11]). *Suppose that $\varphi \in L^\infty(\mathbb{T})$ is arbitrary and put*

$$\mathcal{E}(\varphi) := \{k \in H^\infty(\mathbb{T}) : \|k\|_\infty \leqslant 1 \quad \text{and} \quad \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\}.$$

Then T_φ is hyponormal if and only if the set $\mathcal{E}(\varphi)$ is nonempty.

Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution of a certain functional equation involving its symbol. This approach has been put to use in the works [4], [5], [6], [8], [9], [10], [11], [15] to study Toeplitz operators on the Hardy space of the unit circle.

Now the set $\mathcal{E}(\varphi)$ will be called the *Cowen set* for the function $\varphi \in L^\infty(\mathbb{T})$. The question about the Cowen set $\mathcal{E}(\varphi)$ is of great interest. Indeed, $\mathcal{E}(\varphi)$ has been studied intensively in recent literature because when φ is of bounded type (i.e., quotient of two bounded analytic functions), it has a connection with the following H^∞ optimization problem which naturally arise in robust control theory (cf. [7]):

H^∞ OPTIMIZATION PROBLEM. *Let $k_0 \in L^\infty(\mathbb{T})$ and θ a fixed inner function in $H^\infty(\mathbb{T})$. Find μ where*

$$\mu = \operatorname{dist}(k_0, \theta H^\infty) \equiv \inf_{h \in H^\infty} \|k_0 - \theta h\|_\infty.$$

In this paper it is shown that via Nehari's Theorem and Adamyan-Arov-Krein Theorem, a solution of a H^∞ optimization problem provides information on $\mathcal{E}(\varphi)$ when φ is of bounded type and T_φ has finite rank selfcommutator.

2. MAIN RESULTS

We begin with the connection between Hankel and Toeplitz operators. For φ in $L^\infty(\mathbb{T})$, the *Hankel operator* $H_\varphi : H^2 \to H^2$ is defined by

$$H_\varphi f = J(I-P)(\varphi f),$$

where $J : (H^2)^\perp \to H^2$ is given by $Jz^{-n} = z^{n-1}$ for $n \geqslant 1$. For $\zeta \in L^\infty(\mathbb{T})$, we define

$$\widetilde{\zeta} = \overline{\zeta(\overline{z})}.$$

The following is a basic connection between Hankel and Toeplitz operators:

$$T_{\varphi\psi} - T_\varphi T_\psi = H^*_{\overline{\varphi}} H_\psi \;\; (\varphi, \psi \in L^\infty) \quad \text{and} \quad H_\varphi T_h = H_{\varphi h} = T^*_{\widetilde{h}} H_\varphi \;\; (h \in H^\infty).$$

From this we can see that if $k \in \mathcal{E}(\varphi)$ then

$$[T^*_\varphi, T_\varphi] = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_\varphi H_\varphi = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_{k\,\overline{\varphi}} H_{k\,\overline{\varphi}} = H^*_{\overline{\varphi}}(1 - T_{\widetilde{k}} T^*_{\widetilde{k}}) H_{\overline{\varphi}}. \tag{2.1}$$

For an inner function θ, we write

$$\mathcal{H}(\theta) \equiv H^2 \ominus \theta H^2.$$

If $\varphi \in L^\infty$, write

$$\varphi_+ \equiv P(\varphi) \in H^2 \quad \text{and} \quad \varphi_- \equiv \overline{(I-P)(\varphi)} \in zH^2.$$

Thus we can write $\varphi = \varphi_+ + \overline{\varphi}_-$. Assume that φ is of bounded type, i.e., there are functions ψ_1, ψ_2 in $H^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$$

for almost all $z \in \mathbb{T}$. Since $T_{\overline{z}} H_\varphi = H_\varphi T_z$ it follows from Beurling's theorem that $\ker H_{\overline{\varphi}_-} = \theta H^2$ and $\ker H_{\overline{\varphi}_+} = \theta_+ H^2$ for some inner functions θ, θ_+. If T_φ is hyponormal then, by (2.1), $\|H_{\overline{\varphi}_+} f\| \geqslant \|H_{\overline{\varphi}_-} f\|$ for all $f \in H^2$, so that

$$\theta_+ H^2 = \ker H_{\overline{\varphi}_+} \subseteq \ker H_{\overline{\varphi}_-} = \theta H^2,$$

which implies that θ divides θ_+, i.e., $\theta_+ = \theta_0 \theta$ for some inner function θ_0. Thus if $\varphi = \varphi_+ + \overline{\varphi}_-$ is of bounded type and T_φ is hyponormal then we can write (cf. [8])

$$\varphi_+ = \theta_0 \theta \bar{a} \quad \text{and} \quad \varphi_- = \theta \bar{b},$$

where $a \in \mathcal{H}(\theta_0 \theta)$ and $b \in \mathcal{H}(\theta)$. If $k_0 \in H^\infty$ is a solution of equation

$$b - k_0 a = \theta h \quad \text{for some } h \in H^2 \tag{2.2}$$

then $\mathcal{E}(\varphi)$ can be written as

$$\mathcal{E}(\varphi) = \{\theta_0(k_0 + \theta f) : \ f \in H^\infty \quad \text{and} \quad \|k_0 + \theta f\|_\infty \leqslant 1\}.$$

By Nehari's Theorem [12], we have

$$\operatorname{dist}(k_0,\ \theta H^\infty) = \inf_{f \in H^\infty} \|\bar{\theta} k_0 + f\|_\infty = \|H_{\bar{\theta} k_0}\|. \tag{2.3}$$

Thus we have (see Theorem 8 of [8])

$$T_\varphi \text{ is hyponormal} \iff \|H_{\bar{\theta} k_0}\| \leqslant 1.$$

The following theorem is our main result, which gives a description on the Cowen set $\mathcal{E}(\varphi)$ when the selfcommutator $[T_\varphi^*, T_\varphi]$ is of finite rank. In fact we can prove more:

THEOREM 2.1. *If φ is of bounded type then we have that:*
(i) *If* $\ker H_{\overline{\varphi}} \nsubseteq \ker [T_\varphi^*, T_\varphi]$ *then $\mathcal{E}(\varphi)$ is empty.*
(ii) *If* $\ker H_{\overline{\varphi}} = \ker [T_\varphi^*, T_\varphi]$ *and* $\operatorname{rank} [T_\varphi^*, T_\varphi] < \infty$ *then $\mathcal{E}(\varphi)$ contains infinitely many inner functions.*
(iii) *If* $\ker H_{\overline{\varphi}} \subsetneq \ker [T_\varphi^*, T_\varphi]$ *then $\mathcal{E}(\varphi)$ contains a unique function which is inner.*
If instead φ is not of bounded type such that T_φ is hyponormal then $\mathcal{E}(\varphi)$ contains a unique function.

To prove Theorem 2.1 we need auxiliary lemmas.

The following lemma is another version of Cowen's theorem.

LEMMA 2.2 ([4], [5], Lemma 1). *If $\varphi \equiv \varphi_+ + \overline{\varphi}_- \in L^\infty$, then $\mathcal{E}(\varphi) \neq \emptyset$ if and only if the equation $H_{\overline{\varphi}_+} k = \bar{z}\widetilde{\varphi}_-$ admits a solution k satisfying $\|k\|_\infty \leqslant 1$.*

T. Nakazi and K. Takahashi [11] noticed that if T_φ is a hyponormal operator such that its selfcommutator is of finite rank then $\mathcal{E}(\varphi)$ contains a finite Blaschke product.

LEMMA 2.3 ([11], Nakazi-Takahashi Theorem). *A Toeplitz operator T_φ is hyponormal and the rank of the selfcommutator $[T^*_\varphi, T_\varphi]$ is finite if and only if there exists a finite Blaschke product k in $\mathcal{E}(\varphi)$ of the form*

$$k(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z - \beta_j}{1 - \overline{\beta}_j z} \qquad (|\beta_j| < 1 \text{ for } j = 1, \dots, n).$$

such that $\deg(k) = \operatorname{rank}[T^*_\varphi, T_\varphi]$, *where* $\deg(k)$ *denotes the degree of k — meaning the number of zeros of k in the open unit disk $\mathbb{D}$.*

The following lemma is a solution of an H^∞ optimization problem.

LEMMA 2.4. *If b and q are finite Blaschke products then*

$$\deg(b) \geqslant \deg(q) \iff \operatorname{dist}(b, qH^\infty) < 1. \tag{2.4}$$

Proof. In general, for a continuous function u on $\mathbb{T}$ with $|u| \equiv 1$,

$$\operatorname{dist}(u,\ H^\infty) < 1 \iff \operatorname{wind}(u) \geqslant 0, \tag{2.5}$$

where wind$(\cdot)$ denotes the winding number with respect to the origin: indeed, this follows from the fact that (see Appendix 4, Theorem 41 of [13])

$$\operatorname{dist}(u,\ H^\infty) < 1 \iff T_u \text{ is left invertible} \iff \operatorname{wind}(u) \geqslant 0, \tag{2.6}$$

where the second implication comes from the observation that T_u is Fredholm and hence, by Coburn's Theorem, T_u is left or right invertible and the Fredholm index of T_u is equal to $-\operatorname{wind}(u)$. Applying (2.5) to $u = \frac{b}{q}$ gives that

$$\operatorname{dist}(b,\ q H^\infty) < 1 \iff \operatorname{wind}\left(\frac{b}{q}\right) \geqslant 0 \iff \deg(b) \geqslant \deg(q). \qquad \blacksquare$$

We are ready for:

Proof of Theorem 2.1. From (2.1) we can see that if T_φ is hyponormal then

$$\ker H_{\overline{\varphi}} \subseteq \ker[T^*_\varphi, T_\varphi],$$

which proves statement (i).

Towards statement (ii), suppose φ is of bounded type. So we can write $\varphi = \theta_0\theta\bar{a} + \bar{\theta}b$ for $a \in \mathcal{H}(\theta_0\theta)$ and $b \in \mathcal{H}(\theta)$. Now suppose $\ker H_{\overline{\varphi}} = \ker[T^*_\varphi, T_\varphi]$ and $\operatorname{rank}[T^*_\varphi, T_\varphi] < \infty$. Since $\ker H_{\overline{\varphi}} = \theta_0\theta H^2$ it follows that

$$\operatorname{ran}[T^*_\varphi, T_\varphi] = (\ker[T^*_\varphi, T_\varphi])^\perp = (\ker H_{\overline{\varphi}})^\perp = H^2 \ominus \theta_0\theta H^2,$$

which implies that $\theta_0\theta$ is a finite Blaschke product since $\operatorname{ran}[T^*_\varphi, T_\varphi]$ is finite dimensional. Also, by Lemma 2.3 there exists a finite Blaschke product $\theta_0 k_0$ in

$\mathcal{E}(\varphi)$ such that $\deg(\theta_0 k_0) = \operatorname{rank}[T_\varphi^*, T_\varphi]$. Thus k_0 is a finite Blaschke product such that $\deg(\theta_0 k_0) = \operatorname{rank} H_{\overline{\varphi}} = \deg(\theta_0\theta)$, and hence $\deg(k_0) = \deg(\theta)$. So by Lemma 2.4, we have that $\operatorname{dist}(k_0,\ \theta H^\infty) < 1$, and hence by (2.3), $\|H_{\bar{\theta}k_0}\| < 1$. Remembering Adamyan-Arov-Krein theorem which states that if $f \in L^\infty$ and $\operatorname{dist}(f,\ H^\infty) < 1$ then $f + H^\infty$ contains a unimodular function, we can see that if $\|H_{\bar{\theta}k_0}\| < 1$, then $k_0 + \theta H^\infty$ contains an inner function. Thus $\theta_0 k_0 + \theta_0\theta H^\infty$ contains an inner function, and in turn, $\mathcal{E}(\varphi)$ contains an inner function. Since

$$\begin{aligned} 1 > \operatorname{dist}(\bar{\theta}\, k_0,\ H^\infty) &= \operatorname{dist}(\bar{z}\bar{\theta}k_0,\ \bar{z}H^\infty) \\ &= \operatorname{dist}(\bar{z}\bar{\theta}k_0 + \bar{z}c,\ H^\infty) \quad \text{for a suitable } c \\ &= \|H_{\bar{z}\bar{\theta}(k_0+\theta c)}\|, \end{aligned}$$

we can choose different constants α_n ($n \in \mathbb{Z}_+$) such that $\|H_{\bar{z}\bar{\theta}(k_0+\theta\alpha_n)}\| < 1$. Applying again Adamyan-Arov-Krein theorem to $H_{\bar{z}\bar{\theta}(k_0+\theta\alpha_n)}$, there exists $q_n \in H^\infty$ such that $k_0 + \theta\alpha_n + z\theta q_n$ are inner functions. Evidently, $\theta_0 k_0 + \theta_0\theta(\alpha_n + z\theta q_n) \in \mathcal{E}(\varphi)$ and are different. This proves statement (ii).

Towards statement (iii), suppose $\ker H_{\overline{\varphi}} \subsetneq \ker[T_\varphi^*, T_\varphi]$. If $\mathcal{E}(\varphi)$ contains a function k which is not inner then $\ker(1 - T_{\widetilde{k}}T_{\widetilde{k}}^*) = \{0\}$: indeed if $g = T_{\widetilde{k}}T_{\widetilde{k}}^* g$ then $\|g\|^2 = \|T_{\widetilde{k}}^* g\|^2$, and hence

$$\int |g|^2 \mathrm{d}\mu = \|g\|^2 = \|T_{\widetilde{k}}^* g\|^2 \leqslant \|\overline{\widetilde{k}} g\|^2 = \int |\widetilde{k}|^2 |g|^2 \mathrm{d}\mu,$$

which implies that $g = 0$ a.e. if $\widetilde{k}$ is not inner. Thus by (2.1) we have that $\ker[T_\varphi^*, T_\varphi]$
$\subseteq \ker H_{\overline{\varphi}}$, which forces that $\ker H_{\overline{\varphi}} = \ker[T_\varphi^*, T_\varphi]$, a contradiction. If instead $\mathcal{E}(\varphi)$ contains two different inner functions then $\mathcal{E}(\varphi)$ has a function which is not inner: for if k_1 and k_2 ($k_1 \neq k_2$) are inner functions in $\mathcal{E}(\varphi)$ then we can easily see that $\frac{k_1+k_2}{2} \in \mathcal{E}(\varphi)$ and $\frac{k_1+k_2}{2}$ is not an inner function since every inner function is an extreme point of the unit ball of H^∞. Thus $\mathcal{E}(\varphi)$ contains a unique inner function. This proves statement (iii).

For the second assertion write $\varphi = \varphi_+ + \overline{\varphi}_-$. If φ is not of bounded type then by an argument of Abrahamse ([1], Lemma 3), we have that $\ker H_{\overline{\varphi}_+} = \ker H_{\overline{\varphi}} = \{0\}$. Thus the solution k of the equation $H_{\overline{\varphi}_+} k = \bar{z}\widetilde{\varphi}_-$ should be unique. Thus the second assertion follows at once from Lemma 2.2. ∎

We would like to remark that if $H_{\bar{\theta}k_0}$ attains its norm (e.g., it is of finite rank) then $\operatorname{dist}(k_0,\ \theta H^\infty) = 1$ implies that $\mathcal{E}(\varphi)$ contains a unique inner function. To see this, recall (cf. p. 202 in [13]) that if $f \in L^\infty$ and H_f attains its norm then $f + H^\infty$ contains a unique element of least norm which is of the form $\lambda \frac{\bar{h}}{h\nu}$, where $\lambda \in \mathbb{C}$, h is an outer function and ν is an inner function. So if $\|H_{\bar{\theta}k_0}\| = 1$ and $H_{\bar{\theta}k_0}$ attains its norm then by (2.3), $\bar{\theta}k_0 + H^\infty$ contains a unique unimodular function. Thus $\mathcal{E}(\varphi)$ contains a unique inner function.

We now turn our attention to the cases of Toeplitz operators with symbols that are trigonometric polynomials. If φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero, then the rank of the selfcommutator $[T_\varphi^*, T_\varphi]$ is finite. Thus if T_φ is hyponormal then by Lemma 2.3, $\mathcal{E}(\varphi)$ contains a finite Blaschke product.

We now have:

COROLLARY 2.5. *Let* $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ *be such that* T_φ *is a hyponormal operator.*

(i) *If* rank $[T_\varphi^*, T_\varphi] < N$ *then* $\mathcal{E}(\varphi)$ *contains a unique finite Blaschke product.*

(ii) *If* rank $[T_\varphi^*, T_\varphi] = N$ *then* $\mathcal{E}(\varphi)$ *contains infinitely many inner functions. Furthermore if* $b \in \mathcal{E}(\varphi)$ *is a finite Blaschke product then* $\deg(b) \geqslant N$.

Proof. Since $\ker H_{\overline{\varphi}} = z^N H^2$, part (i) corresponds to the case where $\ker H_{\overline{\varphi}} \subsetneq \ker [T_\varphi^*, T_\varphi]$ and part (ii) corresponds to the case where $\ker H_{\overline{\varphi}} = \ker [T_\varphi^*, T_\varphi]$. Thus the statement (i) and the first assertion of statement (ii) follow at once from Theorem 2.1 together with Lemma 2.3.

For the second assertion of statement (ii), assume to the contrary that $b \in \mathcal{E}(\varphi)$ is a finite Blaschke product of degree less than N. By Lemma 2.3, there exists a finite Blaschke product $k \in \mathcal{E}(\varphi)$ of degree N. Then we have

$$\widehat{k}(j) = \widehat{b}(j) \quad \text{for } j = 1, \ldots, N-1,$$

where $\widehat{f}(j)$ means the j-th Fourier coefficients of $f \in H^\infty$. Thus by the uniqueness argument of Lemma 1 in [10], we should have that $b = k$, a contradiction. ∎

Corollary 2.5(i) is an extended result of Corollary 4 in [9].

The following is an immediate result from Corollary 2.5.

COROLLARY 2.6. *Suppose that* $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ *and that* k *is a finite Blaschke product in* $\mathcal{E}(\varphi)$.

(i) *If* $\deg(k) < N$ *then* rank $[T_\varphi^*, T_\varphi] = \deg(k)$.

(ii) *If* $\deg(k) \geqslant N$ *then* rank $[T_\varphi^*, T_\varphi] = N$.

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WOO YOUNG LEE, DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151–742, KOREA

E-mail address: wylee@math.snu.ac.kr

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