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CHARACTERIZING ISOMORPHISMS BETWEEN STANDARD OPERATOR ALGEBRAS BY SPECTRAL FUNCTIONS

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ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on an infinite dimensional complex Banach space X, and let Φ be a map from $\mathcal{A}$ onto $\mathcal{B}$. We introduce thirteen parts of spectrum for elements in $\mathcal{A}$ and $\mathcal{B}$ and let $\triangle^{\mathcal{A}}(T)$ denote any one of these thirteen parts of the spectrum of T in $\mathcal{A}$. We show that if Φ satisfies that $\triangle^{\mathcal{A}}(T+S) = \triangle^{\mathcal{B}}(\Phi(T)+\Phi(S))$ and $\triangle^{\mathcal{A}}(T+2S) = \triangle^{\mathcal{B}}(\Phi(T)+2\Phi(S))$ for all $T, S \in \mathcal{A}$, then Φ is either an isomorphism or an anti-isomorphism.

KEYWORDS: *Spectral function, isomorphism, standard operator algebra.*

MSC (2000): Primary 47B49; Secondary 47A10.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The study of linear maps on operator algebras that preserve certain properties of operators has attracted the attention of many mathematicians in recent decades. In particular, surjective linear maps between Banach algebras which preserve the spectrum are extensively studied in connection with a longstanding open problem sometimes called Kaplansky's problem on invertibility preserving linear maps (for a recent survey, see [9]). Let X and Y be two complex Banach spaces, and $\mathcal{B}(X,Y)$ ($\mathcal{B}(X)$ if $X = Y$) be the Banach space (Banach algebra) of all bounded linear operators from X into Y (from X into itself). In [10], Jafarian and Sourour proved that a spectrum-preserving linear map from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ is either an isomorphism or an anti-isomorphism. Aupetit and Mouton [2] extended the result of Jafarian and Sourour to primitive Banach algebras with minimal ideals. Later, Aupetit [1] proved that every spectrum-preserving linear surjection between von Neumann algebras is a Jordan isomorphism. A characterization of spectrum preserving additive maps on $\mathcal{B}(X)$ was obtained by Omladič and Šemrl in [15].

Instead of linear maps preserving spectrum, one can discuss the linear (or additive) maps which preserve various parts of the spectrum. In this direction, [16] by Šemrl is the first one to deal with such linear maps. In [16] it was shown that every linear surjection preserving point spectrum on $\mathcal{B}(X)$ is an automorphism and when X is a Hilbert space, every linear surjection preserving surjectivity spectrum is an automorphism. More generally, the additive surjective maps on standard operator algebras preserving one of several parts of the spectrum were characterized in [4], and the linear surjective maps on some operator algebras compressing one of several parts of the spectrum were characterized in [5].

For an operator $T \in \mathcal{B}(X)$, the symbols $\sigma(T)$, $\sigma_{\mathrm{l}}(T)$, $\sigma_{\mathrm{r}}(T)$, $\partial\sigma(T)$, $\eta\sigma(T)$, $\sigma_{\mathrm{ap}}(T)$, $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{c}}(T)$, and $\sigma_{\mathrm{s}}(T)$, as usual, denote the spectrum, the left spectrum, the right spectrum, the boundary of the spectrum, the full spectrum, the approximate point spectrum, the point spectrum, the compression spectrum, and the surjectivity spectrum of T, respectively. Recall that the full spectrum $\eta\sigma(T)$ of T is the polynomial convex hull of $\sigma(T)$; the compression spectrum $\sigma_{\mathrm{c}}(T)$ of T is the set $\{\lambda \in \mathbb{C} :$ the range of $T - \lambda I$ is not dense in $X\}$; and the surjectivity spectrum $\sigma_{\mathrm{s}}(T)$ of T is the set $\{\lambda \in \mathbb{C} : T - \lambda I$ is not surjective$\}$. Let $\triangle$ denote any one of the symbols σ, σ_{l}, σ_{r}, $\sigma_{\mathrm{l}} \cap \sigma_{\mathrm{r}}$, $\partial\sigma$, $\eta\sigma$, σ_{p}, σ_{c}, σ_{ap}, σ_{s}, $\sigma_{\mathrm{ap}} \cap \sigma_{\mathrm{s}}$, $\sigma_{\mathrm{p}} \cap \sigma_{\mathrm{c}}$, and $\sigma_{\mathrm{p}} \cup \sigma_{\mathrm{c}}$. Then $\triangle(\cdot)$ is a map from $\mathcal{B}(X)$ into $2^{\mathbb{C}}$, which is called a spectral function on $\mathcal{B}(X)$. A map $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is said to be $\triangle(\cdot)$ preserving if $\triangle(\Phi(T)) = \triangle(T)$ for every $T \in \mathcal{B}(X)$. Note that, for a finite rank operator, all parts of the spectrum listed above are the same.

Similarly, we can introduce various spectral functions on the standard operator algebras. Recall that a standard operator algebra $\mathcal{R}$ on a Banach space X is a Banach subalgebra of $\mathcal{B}(X)$ which contains the identity and the ideal of all finite rank operators. Let $\mathcal{R}$ be a standard operator algebra on a complex Banach space X. An element $T \in \mathcal{R}$ is called a left (respectively, right) zero divisor if there exists a nonzero element $S \in \mathcal{R}$ such that $TS = 0$ (respectively, $ST = 0$). We call T a left (respectively, right) topological divisor of zero if there exists a sequence $\{S_n\}_{n=1}^{\infty} \subset \mathcal{R}$ satisfying $\|S_n\| = 1$ such that $TS_n \to 0$ (respectively, $S_nT \to 0$). For $T \in \mathcal{R}$, let us define some spectral functions of T relative to $\mathcal{R}$. As usual, $\sigma_{\mathrm{l}}^{\mathcal{R}}(T)$, $\sigma_{\mathrm{r}}^{\mathcal{R}}(T)$, $\partial\sigma^{\mathcal{R}}(T)$ and $\eta\sigma^{\mathcal{R}}(T)$ stand for the left spectrum, the right spectrum, the boundary of the spectrum and the full spectrum (i.e., the polynomial convex hull of $\sigma^{\mathcal{R}}(T)$) of T relative to $\mathcal{R}$, respectively. Let $\sigma_{\mathrm{p}}^{\mathcal{R}}(T)$ (respectively, $\sigma_{\mathrm{c}}^{\mathcal{R}}(T)$) be the set of all complex numbers λ such that $\lambda I - T$ is a left (respectively, right) zero divisor of $\mathcal{R}$, and let $\sigma_{\mathrm{ap}}^{\mathcal{R}}(T)$ (respectively, $\sigma_{\mathrm{s}}^{\mathcal{R}}(T)$) be the set of all complex numbers λ such that $\lambda I - T$ is a left (respectively, right) topological divisor of zero of $\mathcal{R}$. Obviously, we have $\sigma_{\mathrm{p}}^{\mathcal{R}}(T) \subseteq \sigma_{\mathrm{ap}}^{\mathcal{R}}(T) \subseteq \sigma_{\mathrm{l}}^{\mathcal{R}}(T) \subseteq \sigma^{\mathcal{R}}(T) \subseteq \eta\sigma^{\mathcal{R}}(T)$ ($\sigma_{\mathrm{c}}^{\mathcal{R}}(T) \subseteq \sigma_{\mathrm{s}}^{\mathcal{R}}(T) \subseteq \sigma_{\mathrm{r}}^{\mathcal{R}}(T) \subseteq \sigma^{\mathcal{R}}(T)$) and $\partial\sigma^{\mathcal{R}}(T) \subseteq \sigma_{\mathrm{ap}}^{\mathcal{R}}(T) \cap \sigma_{\mathrm{s}}^{\mathcal{R}}(T)$ for every $T \in \mathcal{R}$. If $\mathcal{R} = \mathcal{B}(X)$, we will simply write $\Delta^{\mathcal{B}(X)}(\cdot)$ as $\Delta(\cdot)$, where Δ is any one of the symbols σ, σ_{l}, σ_{r}, $\partial\sigma$, $\eta\sigma$, σ_{p}, σ_{c}, σ_{ap} and σ_{s}. Then $\Delta(T)$ coincides with the corresponding spectral function of T as an operator on X. It is also obvious that the following relations are true for every $T \in \mathcal{R}$: $\sigma(T) \subseteq \sigma^{\mathcal{R}}(T)$, $\sigma_{\mathrm{l}}(T) \subseteq \sigma_{\mathrm{l}}^{\mathcal{R}}(T)$, $\sigma_{\mathrm{r}}(T) \subseteq \sigma_{\mathrm{r}}^{\mathcal{R}}(T)$, $\partial\sigma^{\mathcal{R}}(T) \subseteq \partial\sigma(T)$, $\eta\sigma^{\mathcal{R}}(T) = \eta\sigma(T)$, and $\Delta^{\mathcal{R}}(T) = \Delta(T)$ whenever Δ takes any one of the symbols σ_{p}, σ_{c}, σ_{ap} and σ_{s}, since $\mathcal{F}(X) \subset \mathcal{R}$, where $\mathcal{F}(X)$ denotes the set of all finite rank operators which is an ideal (not closed) of $\mathcal{B}(X)$.

Let $\triangle^{\mathcal{R}}(\cdot)$ denote any one of the spectral functions $\sigma^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{l}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{r}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{l}}^{\mathcal{R}}(\cdot) \cap \sigma_{\mathrm{r}}^{\mathcal{R}}(\cdot)$, $\partial\sigma^{\mathcal{R}}(\cdot)$, $\eta\sigma^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{ap}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{s}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{ap}}^{\mathcal{R}}(\cdot) \cap \sigma_{\mathrm{s}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{p}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{c}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{p}}^{\mathcal{R}}(\cdot) \cap \sigma_{\mathrm{c}}^{\mathcal{R}}(\cdot)$, and $\sigma_{\mathrm{p}}^{\mathcal{R}}(\cdot) \cup \sigma_{\mathrm{c}}^{\mathcal{R}}(\cdot)$, where $\triangle^{\mathcal{R}}(T)$ denotes the spectral function of T relative to the algebra $\mathcal{R}$.

It is clear that if Φ on $\mathcal{R}$ is an additive map preserving the spectral function $\triangle^{\mathcal{R}}(\cdot)$, then we have

$$\triangle^{\mathcal{R}}(T + \lambda S) = \triangle^{\mathcal{R}}(\Phi(T) + \lambda\Phi(S)) \tag{1.1}$$

holds for all $T, S \in \mathcal{R}$ and all $\lambda \in \mathbb{Q}$ (the field of rational numbers). The aim of this paper is to show that, for a surjective map Φ (no linearity or even additivity is assumed) on a standard operator algebra $\mathcal{R}$, the spectral property (1.1) alone is enough to determine the structures of the map Φ. In fact, (1.1) gives a characterization of a map Φ to be an automorphism or an anti-automorphism on $\mathcal{R}$. Moreover, it is enough assuming that (1.1) holds only for two distinct nonzero points in $\mathbb{Q}$, that is, we have the following

MAIN THEOREM. *Let $\mathcal{A}$ and $\mathcal{B}$ be two standard operator algebras on a complex infinite dimensional Banach space X and let $\triangle^{\mathcal{R}}(\cdot)$ denote any one of the spectral functions $\sigma^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{l}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{r}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{l}}^{\mathcal{R}}(\cdot) \cap \sigma_{\mathrm{r}}^{\mathcal{R}}(\cdot)$, $\partial\sigma^{\mathcal{R}}(\cdot)$, $\eta\sigma^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{ap}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{s}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{ap}}^{\mathcal{R}}(\cdot) \cap \sigma_{\mathrm{s}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{p}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{c}}^{\mathcal{R}}(\cdot)$, $\sigma_{\mathrm{p}}^{\mathcal{R}}(\cdot) \cap \sigma_{\mathrm{c}}^{\mathcal{R}}(\cdot)$ and $\sigma_{\mathrm{p}}^{\mathcal{R}}(\cdot) \cup \sigma_{\mathrm{c}}^{\mathcal{R}}(\cdot)$, where $\mathcal{R} = \mathcal{A}$ or $\mathcal{B}$. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective map satisfying $\triangle^{\mathcal{A}}(T + S) = \triangle^{\mathcal{B}}(\Phi(T) + \Phi(S))$ and $\triangle^{\mathcal{A}}(T + 2S) = \triangle^{\mathcal{B}}(\Phi(T) + 2\Phi(S))$ for all $T, S \in \mathcal{A}$, then either*

(i) *there is an invertible operator $A \in \mathcal{B}(X)$ such that $\Phi(T) = ATA^{-1}$ for all $T \in \mathcal{A}$, or*

(ii) *there is an invertible operator $A \in \mathcal{B}(X', X)$ such that $\Phi(T) = AT'A^{-1}$ for all $T \in \mathcal{A}$, where X' is the dual space of X and T' is the adjoint operator of T; in this case, X is reflexive.*

The proof of the main theorem is quite different from the additive case in [4]. Our approach mainly depends on a result concerning the local combination of operators in [7] (also see [8]). The key step is to show that, for Φ stated in the main theorem and for each pair of rank-1 operators T and S, $\Phi(T + S)$ is a local linear combination of $\Phi(T)$ and $\Phi(S)$ with uniformly bounded coefficients. Thus a result in [7] ensures that $\Phi(T + S)$ is a linear combination of $\Phi(T)$ and $\Phi(S)$. This fact enables us to prove that Φ preserves rank-1 idempotency and orthogonality of rank-1 idempotent operators. Then a remarkable result of [12] can be applied to arrive the desired conclusion.

We mention here that the problem of characterizing the structures of maps between operator algebras taking certain spectral properties as invariants was firstly studied in [14] due to Mrčun, where the Lipschitz maps Φ satisfying $\Phi(0) = 0$ and $\sigma(\Phi(T) - \Phi(S)) = \sigma(T - S)$ for all $T, S \in \mathcal{M}_n(\mathbb{C})$ (complex matrix algebra) were discussed. While in [3], the maps Φ satisfying $r(\Phi(T) - \Phi(S)) = r(T - S)$ for all $T, S \in \mathcal{M}_n(\mathbb{C})$ were characterized, where $r(T)$ denotes the spectral radius of T. For the infinite dimensional cases, there is no corresponding known result. However, the surjective maps Φ on $\mathcal{B}(H)$ satisfying $\sigma(\Phi(T)\Phi(S)) = \sigma(TS)$ for all $T, S \in \mathcal{B}(H)$ were described in [13], where H is a complex Hilbert space.

We also remark that in some situations our main result in fact gives a characterization of isomorphisms between standard operator algebras; for instance, in the case that X is not reflexive, or in the case that $\triangle^{\mathcal{R}} = \sigma_{\mathrm{p}}^{\mathcal{R}}$ or $\sigma_{\mathrm{c}}^{\mathcal{R}}$ and the algebra $\mathcal{A}$ is $\mathcal{B}(X)$ or has an element which is left invertible but not invertible. It is obvious however, in many cases, both the form (i) and the form (ii) may occur. In particular, for the Hilbert space case we have the following corollaries.

COROLLARY 1.1. *Let H be a complex infinite dimensional Hilbert space and let $\triangle(\cdot)$ denote any one of the spectral functions $\sigma(\cdot)$, $\sigma_{\mathrm{l}}(\cdot) \cap \sigma_{\mathrm{r}}(\cdot)$, $\partial\sigma(\cdot)$, $\eta\sigma(\cdot)$, $\sigma_{\mathrm{ap}}(\cdot) \cap \sigma_{\mathrm{s}}(\cdot)$, $\sigma_{\mathrm{p}}(\cdot) \cap \sigma_{\mathrm{c}}(\cdot)$ and $\sigma_{\mathrm{p}}(\cdot) \cup \sigma_{\mathrm{c}}(\cdot)$. Then a surjective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies that $\triangle(T + S) = \triangle(\Phi(T) + \Phi(S))$ and $\triangle(T + 2S) = \triangle(\Phi(T) + 2\Phi(S))$ for all T, $S \in \mathcal{B}(H)$ if and only if there is an invertible operator $A \in \mathcal{B}(H)$ such that, either*

(i) $\Phi(T) = ATA^{-1}$ *for all $T \in \mathcal{B}(H)$, or*

(ii) $\Phi(T) = AT^{\mathrm{t}}A^{-1}$ *for all $T \in \mathcal{B}(H)$, where T^{t} denotes the transpose operator of T with respect to an arbitrarily fixed orthonormal basis of H.*

COROLLARY 1.2. *Let H be a complex infinite dimensional Hilbert space and let $\triangle(\cdot)$ denote any one of the spectral functions $\sigma_{\mathrm{l}}(\cdot)$, $\sigma_{\mathrm{r}}(\cdot)$, $\sigma_{\mathrm{ap}}(\cdot)$, $\sigma_{\mathrm{s}}(\cdot)$, $\sigma_{\mathrm{p}}(\cdot)$ and $\sigma_{\mathrm{c}}(\cdot)$. Then a surjective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies that $\triangle(T + S) = \triangle(\Phi(T) + \Phi(S))$ and $\triangle(T + 2S) = \triangle(\Phi(T) + 2\Phi(S))$ for all T, $S \in \mathcal{B}(H)$ if and only if there is an invertible operator $A \in \mathcal{B}(H)$ such that $\Phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(H)$.*

2. PROOF OF THE MAIN THEOREM

Before embarking on the proof, we introduce some more notations. Throughout this section, X will denote a complex infinite dimensional Banach space and X' the dual space of X. $\mathcal{B}(X)$, $\mathcal{F}(X)$, $\mathcal{F}_1(X)$ and $\mathcal{F}_2(X)$ denote the algebra of all bounded linear operators on X, the ideal of all finite rank operators, the set of all operators with rank $\leqslant 1$ and the set of all operators with rank $\leqslant 2$ in $\mathcal{B}(X)$, respectively. If $x \in X$ and $f \in X'$, then $x \otimes f$ stands for the operator of rank at most one defined by

$$(x \otimes f)y = f(y)x \quad (y \in X).$$

Clearly, every operator $T \in \mathcal{F}(X)$ is a finite sum of rank-1 operators. On $\mathcal{F}(X)$, one can define a trace functional tr by

$$\mathrm{tr} T = \sum_n f_n(x_n),$$

where $T = \sum\limits_n x_n \otimes f_n$. Then tr is a well-defined linear functional with the property that $\mathrm{tr}(TS) = \mathrm{tr}(ST)$ for every $T \in \mathcal{F}(X)$ and $S \in \mathcal{B}(X)$. In some places, we also write $f(x)$ as $\langle x, f \rangle$ for $x \in X$ and $f \in X'$.

Before the proof is given we need several lemmas which were proved in [4] and [5]. In the following lemmas, let $\triangle^{\mathcal{A}}(\cdot)$ denote any one of the spectral functions $\sigma^{\mathcal{A}}(\cdot)$, $\sigma_{\mathrm{l}}^{\mathcal{A}}(\cdot)$, $\sigma_{\mathrm{r}}^{\mathcal{A}}(\cdot)$, $\sigma_{\mathrm{l}}^{\mathcal{A}}(\cdot) \cap \sigma_{\mathrm{r}}^{\mathcal{A}}(\cdot)$, $\partial\sigma^{\mathcal{A}}(\cdot)$, $\eta\sigma^{\mathcal{A}}(\cdot)$, $\sigma_{\mathrm{ap}}^{\mathcal{A}}(\cdot)$, $\sigma_{\mathrm{s}}^{\mathcal{A}}(\cdot)$, $\sigma_{\mathrm{ap}}^{\mathcal{A}}(\cdot) \cap \sigma_{\mathrm{s}}^{\mathcal{A}}(\cdot)$, $\sigma_{\mathrm{p}}^{\mathcal{A}}(\cdot)$, $\sigma_{\mathrm{c}}^{\mathcal{A}}(\cdot)$, $\sigma_{\mathrm{p}}^{\mathcal{A}}(\cdot) \cap \sigma_{\mathrm{c}}^{\mathcal{A}}(\cdot)$, and $\sigma_{\mathrm{p}}^{\mathcal{A}}(\cdot) \cup \sigma_{\mathrm{c}}^{\mathcal{A}}(\cdot)$.

LEMMA 2.1 ([4], Lemma 2.2). *Let $\mathcal{A}$ be a standard operator algebra on a complex Banach space X. Then, for an operator $R \in \mathcal{A}$, the following conditions are equivalent:*

(i) *R has rank one;*

(ii) $\triangle^{\mathcal{A}}(T+R) \cap \triangle^{\mathcal{A}}(T+cR) \subseteq \triangle^{\mathcal{A}}(T)$ *for every operator $T \in \mathcal{A}$ with $\sigma^{\mathcal{A}}(T)$ being finite and every scalar $c \neq 1$;*

(iii) $\triangle^{\mathcal{A}}(T+R) \cap \triangle^{\mathcal{A}}(T+cR) \subseteq \triangle^{\mathcal{A}}(T)$ *for every operator $T \in \mathcal{F}_2(X)$ and every scalar $c \neq 1$;*

(iv) $\triangle^{\mathcal{A}}(T+R) \cap \triangle^{\mathcal{A}}(T+2R) \subseteq \triangle^{\mathcal{A}}(T)$ *for every operator $T \in \mathcal{A}$ with $\sigma^{\mathcal{A}}(T)$ being finite;*

(v) $\triangle^{\mathcal{A}}(T+R) \cap \triangle^{\mathcal{A}}(T+2R) \subseteq \triangle^{\mathcal{A}}(T)$ *for every operator $T \in \mathcal{F}_2(X)$.*

LEMMA 2.2 ([4], Lemma 2.3). *Let $\mathcal{A}$ be a standard operator algebra on a complex Banach space X and $T, S \in \mathcal{A}$. If $\triangle^{\mathcal{A}}(T+R) = \triangle^{\mathcal{A}}(S+R)$ for every rank one operator $R \in \mathcal{B}(X)$, then $T = S$.*

The next lemma was obtained in [5], Lemma 4.2, for nine among these thirteen spectral functions. By checking the proof there we see that the lemma is also valid for other four spectral functions.

LEMMA 2.3. *Let $\mathcal{A}$ be a standard operator algebra on a complex Banach space X. For $T \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ with $\lambda \notin \eta\sigma(T)$, $\lambda \in \triangle^{\mathcal{A}}(T + x \otimes f)$ if and only if $\langle (\lambda I - T)^{-1}x, f \rangle = 1$.*

The following lemma is crucial for our purpose. It is taken from Theorem 2.2 of [7], where it is stated only for Hilbert space case. However, by checking the proof there, it is easily seen that the result holds true also for Banach space case.

LEMMA 2.4. *Let X and Y be complex Banach spaces, and $A, B, C \in \mathcal{B}(X, Y)$. If for every $x \in X$ there are complex numbers $a(x)$ and $b(x)$ such that $Cx = a(x)Ax + b(x)Bx$ and $\sup\{|a(x)|,\ |b(x)| : x \in X\} < \infty$, then C is a linear combination of A and B.*

To prove our main theorem, a result due to Molnă r [12] is also needed. For reader's convenience sake, we list it as a lemma.

LEMMA 2.5. *Let X be an infinite dimensional complex Banach space and $I_1(X)$ the set of all rank-1 idempotent operators in $\mathcal{B}(X)$. Let $\phi : I_1(X) \to I_1(X)$ be a bijective map with the property that $PQ = 0 \Leftrightarrow \phi(P)\phi(Q) = 0$ for all $P, Q \in I_1(X)$. Then there exists an invertible bounded either linear or conjugate linear operator $A : X \to X$ such that $\phi(P) = APA^{-1}$ for all $P \in I_1(X)$.*

Now we are at a position to give the proof of our main result.

Proof of Main Theorem. We complete the proof by checking several claims.

Claim 1. $\Phi(0) = 0$, Φ is injective, $\triangle^{\mathcal{A}}(T) = \triangle^{\mathcal{B}}(\Phi(T))$ for every $T \in \mathcal{A}$ and Φ preserves rank one operators in both directions.

By the assumption, $\triangle^{\mathcal{A}}(T) = \triangle^{\mathcal{A}}(T + 0) = \triangle^{\mathcal{B}}(\Phi(T) + \Phi(0)) = \triangle^{\mathcal{A}}(T + 2 \cdot 0) = \triangle^{\mathcal{B}}(\Phi(T) + 2\Phi(0))$. From the surjectivity of Φ and Lemma 2.2, it follows that $\Phi(0) = 2\Phi(0) = 0$.

Choosing $S = 0$, we have that $\triangle^{\mathcal{A}}(T) = \triangle^{\mathcal{B}}(\Phi(T))$ for every $T \in \mathcal{A}$.

Next we check that Φ is injective. Assume that $\Phi(T) = \Phi(S)$ for some T, $S \in \mathcal{A}$. Then for arbitrary $R \in \mathcal{F}_1(X)$, we have $\triangle^{\mathcal{A}}(T + R) = \triangle^{\mathcal{B}}(\Phi(T) + \Phi(R)) = \triangle^{\mathcal{B}}(\Phi(S) + \Phi(R)) = \triangle^{\mathcal{A}}(S + R)$. By Lemma 2.2, $T = S$.

Since $\triangle^{\mathcal{A}}(T + S) = \triangle^{\mathcal{B}}(\Phi(T) + \Phi(S))$ and $\triangle^{\mathcal{A}}(T + 2S) = \triangle^{\mathcal{B}}(\Phi(T) + 2\Phi(S))$ for all $T, S \in \mathcal{A}$, from Lemma 2.1 and the bijectivity of Φ it follows that Φ preserves rank one operators in both directions.

Claim 2. For every T, $S \in \mathcal{F}_1(X)$, we have that $\mathrm{tr}(TS) = \alpha_{TS}\mathrm{tr}(\Phi(T)\Phi(S))$ for a scalar $\alpha_{TS} \in \{1, \frac{1}{2}, 2, \frac{7}{8}, \frac{8}{7}\}$ depending on T and S. Consequently, $TS = 0$ implies either $\Phi(T)\Phi(S) = 0$ or $\Phi(S)\Phi(T) = 0$.

Note that $\triangle^{\mathcal{A}}(T) = \sigma(T)$ has at most two points, with one being zero, if T is a rank-1 operator. Thus we may write $\triangle^{\mathcal{A}}(T) = \{0, \xi_T\}$ and $\triangle^{\mathcal{A}}(S) = \{0, \xi_S\}$ for some ξ_T and $\xi_S \in \mathbb{C}$ (ξ_T and ξ_S may be zero).

Since $\triangle^{\mathcal{A}}(T + S) = \triangle^{\mathcal{B}}(\Phi(T) + \Phi(S))$, by the spectral mapping theorem, we have that $\triangle^{\mathcal{A}}((T + S)^2 - \xi_S(T + S)) = \triangle^{\mathcal{B}}((\Phi(T) + \Phi(S))^2 - \xi_S(\Phi(T) + \Phi(S)))$. It follows that $\triangle^{\mathcal{A}}(\xi_T T - \xi_S T + TS + ST) = \triangle^{\mathcal{B}}(\xi_T\Phi(T) - \xi_S\Phi(T) + \Phi(T)\Phi(S) + \Phi(S)\Phi(T))$. By Claim 1, both $\xi_T T - \xi_S T + TS + ST$ and $\xi_T\Phi(T) - \xi_S\Phi(T) + \Phi(T)\Phi(S) + \Phi(S)\Phi(T)$ are operators of rank not greater than 2. Thus we have for some scalar $\alpha \in \{1, \frac{1}{2}, 2\}$ that $\mathrm{tr}(\xi_T T - \xi_S T + TS + ST) = \alpha\mathrm{tr}(\xi_T\Phi(T) - \xi_S\Phi(T) + \Phi(T)\Phi(S) + \Phi(S)\Phi(T))$. It follows that

$$\xi_T^2 - \xi_T\xi_S + 2\mathrm{tr}(TS) = \alpha(\xi_T^2 - \xi_T\xi_S) + 2\alpha\mathrm{tr}(\Phi(T)\Phi(S))$$

and so

$$2\mathrm{tr}(TS) - 2\alpha\mathrm{tr}(\Phi(T)\Phi(S)) = (\alpha - 1)(\xi_T^2 - \xi_T\xi_S). \tag{2.1}$$

If $\alpha = 1$ or $\xi_T = 0$ or $\xi_S = \xi_T$, then we already have the desired relation between $\mathrm{tr}(TS)$ and $\mathrm{tr}(\Phi(T)\Phi(S))$. So in the sequel we assume that $\alpha \neq 1$, $\xi_S \neq \xi_T$

and $\xi_T \neq 0$. Similarly, $\triangle^{\mathcal{A}}((T+S)^2 - \xi_T(T+S)) = \triangle^{\mathcal{B}}((\Phi(T)+\Phi(S))^2 - \xi_T(\Phi(T)+\Phi(S)))$ yields that

$$2\mathrm{tr}(TS) - 2\beta\mathrm{tr}(\Phi(T)\Phi(S)) = (\beta-1)(\xi_S^2 - \xi_T\xi_S) \tag{2.2}$$

holds for some $\beta \in \{1, \frac{1}{2}, 2\}$.

By considering $(T+2S)^2 - \xi_T(T+2S)$ and $(S+2T)^2 - (S+2T)$, it follows from $\triangle^{\mathcal{A}}(T+2S) = \triangle^{\mathcal{B}}(\Phi(T)+2\Phi(S))$ that there exist $\gamma, \delta \in \{1, \frac{1}{2}, 2\}$ such that

$$2\mathrm{tr}(TS) - 2\gamma\mathrm{tr}(\Phi(T)\Phi(S)) = (\gamma-1)(2\xi_S^2 - \xi_T\xi_S) \tag{2.3}$$

and

$$2\mathrm{tr}(TS) - 2\delta\mathrm{tr}(\Phi(T)\Phi(S)) = (\delta-1)(2\xi_T^2 - \xi_T\xi_S). \tag{2.4}$$

We may assume further that $\beta, \gamma, \delta \in \{\frac{1}{2}, 2\}$, $\xi_S \neq 0$, $2\xi_S - \xi_T \neq 0$ and $2\xi_T - \xi_S \neq 0$; otherwise, we have already proved the claim. Solving the equations (2.1) and (2.4) we get

$$\xi_T^2 = \Big(\frac{2}{\delta-1} - \frac{2}{\alpha-1}\Big)\mathrm{tr}(TS) - \Big(\frac{2\delta}{\delta-1} - \frac{2\alpha}{\alpha-1}\Big)\mathrm{tr}(\Phi(T)\Phi(S))$$

and

$$\xi_T\xi_S = \Big(\frac{2}{\delta-1} - \frac{4}{\alpha-1}\Big)\mathrm{tr}(TS) - \Big(\frac{2\delta}{\delta-1} - \frac{4\alpha}{\alpha-1}\Big)\mathrm{tr}(\Phi(T)\Phi(S)),$$

while the equations (2.2) and (2.3) give that

$$\xi_S^2 = \Big(\frac{2}{\gamma-1} - \frac{2}{\beta-1}\Big)\mathrm{tr}(TS) - \Big(\frac{2\gamma}{\gamma-1} - \frac{2\beta}{\beta-1}\Big)\mathrm{tr}(\Phi(T)\Phi(S)).$$

Thus, by (2.2),

$$\begin{aligned}
\frac{2}{\beta-1}&\mathrm{tr}(TS) - \frac{2\beta}{\beta-1}\mathrm{tr}(\Phi(T)\Phi(S)) \\
&= \xi_S^2 - \xi_T\xi_S \\
&= \Big(\frac{2}{\gamma-1} - \frac{2}{\beta-1}\Big)\mathrm{tr}(TS) - \Big(\frac{2\gamma}{\gamma-1} - \frac{2\beta}{\beta-1}\Big)\mathrm{tr}(\Phi(T)\Phi(S)) \\
&\quad - \Big(\frac{2}{\delta-1} - \frac{4}{\alpha-1}\Big)\mathrm{tr}(TS) + \Big(\frac{2\delta}{\delta-1} - \frac{4\alpha}{\alpha-1}\Big)\mathrm{tr}(\Phi(T)\Phi(S)),
\end{aligned}$$

which gives

$$\begin{aligned}
\Big(\frac{2}{\beta-1} &- \frac{2}{\alpha-1} + \frac{1}{\delta-1} - \frac{1}{\gamma-1}\Big)\mathrm{tr}(TS) \\
&= \Big(\frac{2\beta}{\beta-1} - \frac{2\alpha}{\alpha-1} + \frac{\delta}{\delta-1} - \frac{\gamma}{\gamma-1}\Big)\mathrm{tr}(\Phi(T)\Phi(S)) \\
&= \Big(\frac{2}{\beta-1} - \frac{2}{\alpha-1} + \frac{1}{\delta-1} - \frac{1}{\gamma-1}\Big)\mathrm{tr}(\Phi(T)\Phi(S)).
\end{aligned}$$

If $\frac{2}{\beta-1} - \frac{2}{\alpha-1} + \frac{1}{\delta-1} - \frac{1}{\gamma-1} \neq 0$, then $\mathrm{tr}(TS) = \mathrm{tr}(\Phi(T)\Phi(S))$ and the claim is true. Note that $\alpha, \beta, \gamma, \delta \in \{\frac{1}{2}, 2\}$ and there are sixteen possible choices of $(\alpha, \beta, \gamma, \delta)$. It is easily checked that there are four choices for which we have $\frac{2}{\beta-1}-\frac{2}{\alpha-1}+\frac{1}{\delta-1}-\frac{1}{\gamma-1}=0$, that is, $(\alpha,\beta,\gamma,\delta)$ takes one of the following cases:

(1°) $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$;
(2°) $(2,2,2,2)$;
(3°) $(\frac{1}{2},\frac{1}{2},2,2)$;
(4°) $(2,2,\frac{1}{2},\frac{1}{2})$.

It follows from the equations (2.2) and (2.3) that any one of (1°) and (2°) leads to a contradiction, namely $\xi_S=0$. So, cases (1°) and (2°) can not occur.

Assume (3°). Equations (2.1) and (2.2) together imply that $\xi_T=-\xi_S$ and therefore, equation (2.2) and (2.3) become

$$2\mathrm{tr}(TS)-\mathrm{tr}(\Phi(T)\Phi(S))=-\xi_S^2$$

and

$$2\mathrm{tr}(TS)-4\mathrm{tr}(\Phi(T)\Phi(S))=3\xi_S^2,$$

respectively. Thus we have

$$\mathrm{tr}(\Phi(T)\Phi(S))=-\frac{4}{3}\xi_S^2,\quad \mathrm{tr}(TS)=-\frac{7}{6}\xi_S^2$$

and hence,

$$\mathrm{tr}(TS)=\frac{7}{8}\mathrm{tr}(\Phi(T)\Phi(S)).$$

Similarly, the case (4°) implies that

$$\mathrm{tr}(TS)=\frac{8}{7}\mathrm{tr}(\Phi(T)\Phi(S)).$$

This completes the proof of Claim 2.

Claim 3. For $T,\ S\in\mathcal{F}_1(X)$ with $T+S\in\mathcal{F}_1(X)$, we have $\Phi(T+S)=\alpha\Phi(T)+\beta\Phi(S)$ for some nonzero $\alpha,\ \beta\in\mathbb{C}$ depending on T and S. Moreover, $\Phi(\lambda S)=\xi_\lambda\Phi(S)$ for every $\lambda\in\mathbb{C}$.

Choose nonzero $R\in\mathcal{F}_1(X)$ arbitrarily. Write $\Phi(R)=x\otimes f$. Then, by Claim 2, one gets

$$\mathrm{tr}(TR)=\alpha_{xf}\mathrm{tr}(\Phi(T)\Phi(R)),$$
$$\mathrm{tr}(SR)=\beta_{xf}\mathrm{tr}(\Phi(S)\Phi(R))$$

and

$$\mathrm{tr}((T+S)R)=\gamma_{xf}\mathrm{tr}(\Phi(T+S)\Phi(R)),$$

where $\alpha_{xf},\beta_{xf},\gamma_{xf}\in\{1,\frac{1}{2},2,\frac{7}{8},\frac{8}{7}\}$.

Suppose that T and S are linearly dependent; without loss of generality, say $T=\lambda S$ for some scalar λ. Then $\mathrm{tr}(TR)=\alpha_{xf}\mathrm{tr}(\Phi(T)\Phi(R))=\lambda\mathrm{tr}(SR)=\lambda\beta_{xf}\mathrm{tr}(\Phi(S)\Phi(R))$. That is $\langle\alpha_{xf}\Phi(T)x,f\rangle=\langle\lambda\beta_{xf}\Phi(S)x,f\rangle$. Since $R\in\mathcal{F}_1(X)$ is arbitrary and $\Phi(\mathcal{F}_1(X))=\mathcal{F}_1(X)$ by Claim 1, we see that $\Phi(T)\in\mathbb{C}\Phi(S)$. It is also easily seen that, in this case, we can find nonzero scalars α and β such that $\Phi(T+S)=\alpha\Phi(T)+\beta\Phi(S)$.

Assume that T and S are linearly independent; then

$$\begin{aligned}\mathrm{tr}(\gamma_{xf}\Phi(T+S)\Phi(R)) &= \mathrm{tr}((T+S)R) = \mathrm{tr}(TR+SR) = \mathrm{tr}(TR)+\mathrm{tr}(SR)\\ &= \alpha_{xf}\mathrm{tr}(\Phi(T)\Phi(R)) + \beta_{xf}\mathrm{tr}(\Phi(S)\Phi(R))\\ &= \mathrm{tr}((\alpha_{xf}\Phi(T)+\beta_{xf}\Phi(S))\Phi(R)).\end{aligned}$$

Consequently,

$$\langle(\alpha_{xf}\Phi(T)+\beta_{xf}\Phi(S))x, f\rangle = \langle\gamma_{xf}\Phi(T+S)x, f\rangle \tag{2.5}$$

for every $x \in X$ and $f \in X'$. In particular, for every $f \in \{\Phi(T)x, \Phi(S)x\}^{\perp}$ and $x \in X$, we have $\langle\gamma_{xf}\Phi(T+S)x, f\rangle = 0$, which implies that $\Phi(T+S)x$ lies in the linear span of $\Phi(T)x$ and $\Phi(S)x$. Say $\Phi(T+S)x = \xi_x\Phi(T)x + \eta_x\Phi(S)x$. Assume that there exists a $u \in X$ such that $\Phi(T)u$ and $\Phi(S)u$ are linearly dependent with $\Phi(T)u \neq 0$, $\Phi(S)u \neq 0$ and $\Phi(T+S)u \neq 0$. Then $\Phi(T)$ and $\Phi(S)$ may be written as $\Phi(T) = y \otimes g_1$, $\Phi(S) = y \otimes g_2$. Choose $f \in X'$ with $\langle y, f\rangle \neq 0$. Since $\Phi(T+S)u = \xi_u\Phi(T)u + \eta_u\Phi(S)u$, we have that $\Phi(T+S) = y \otimes g_3$ for some functional g_3. It follows from equation (2.5) and a simple computation that $\langle x, \alpha_{xf}g_1 + \beta_{xf}g_2\rangle = \langle x, \gamma_{xf}g_3\rangle$ holds for every $x \in X$. This implies that g_3 lies in the linear span of g_1 and g_2. Thus $\Phi(T+S) = \alpha\Phi(T) + \beta\Phi(S)$ for some scalars α and β. Now we assume that the above vector u does not exist. By Lemma 2.4, to show that $\Phi(T+S) = \alpha\Phi(T) + \beta\Phi(S)$ for some scalar α and β, we only need to check that $\{\xi_x : x \in X\}$ and $\{\eta_x : x \in X\}$ are bounded sets. If $\Phi(T)x$ and $\Phi(S)x$ are linearly independent, then there is a linear functional $f \in X'$ such that $\langle\Phi(T)x, f\rangle = 0$ and $\langle\Phi(S)x, f\rangle \neq 0$. It follows from the equation (2.5) that $\langle\beta_{xf}\Phi(S)x, f\rangle = \langle\gamma_{xf}\eta_x\Phi(S)x, f\rangle$. Thus we have $\eta_x = \frac{\beta_{xf}}{\gamma_{xf}}$. Note that $\beta_{xf}, \gamma_{xf} \in \{1, \frac{1}{2}, 2, \frac{7}{8}, \frac{8}{7}\}$, so $|\eta_x| \leqslant 4$. Similarly, $|\xi_x| \leqslant 4$, too. Assume that $\Phi(T)x$ and $\Phi(S)x$ are linearly dependent. Then from the discussion above, at least one of $\Phi(T)x$, $\Phi(S)x$ and $\Phi(T+S)x$ is zero. In the case that $\Phi(T)x \neq 0$ and $\Phi(S)x \neq 0$, or in the case that $\Phi(T)x = \Phi(S)x = 0$, we must have $\Phi(T+S)x = 0$. Thus we can choose $\xi_x = \eta_x = 0$. If $\Phi(S)x = 0$ and $\Phi(T)x \neq 0$, then, for every $f \in X'$, $\langle(\alpha_{xf}\Phi(T)+\beta_{xf}\Phi(S))x, f\rangle = \langle\gamma_{xf}\Phi(T+S)x, f\rangle = \langle\alpha_{xf}\Phi(T)x, f\rangle$. Hence $\Phi(T+S)x = \frac{\alpha_{xf}}{\gamma_{xf}}\Phi(T)x$. Thus we can choose $\eta_x = 0$ and $\xi_x = \frac{\alpha_{xf}}{\gamma_{xf}}$, which are bounded by 4. Similarly, if $\Phi(S)x \neq 0$ and $\Phi(T)x = 0$, we have $\Phi(T+S)x = \eta_x\Phi(S)x$, where η_x is bounded by 4. Therefore, there exist $\alpha, \beta \in \mathbb{C}$ such that $\Phi(T+S) = \alpha\Phi(T) + \beta\Phi(S)$. Finally, let us check that $\alpha\beta \neq 0$. If $\alpha = 0$, then $\Phi(T+S) = \beta\Phi(S)$. Thus $T+S$ lies in the linear span of S which contradicts to the fact that T and S are linearly independent. Similarly, we can check that $\beta \neq 0$.

Claim 4. Let $T \in \mathcal{F}_1(X)$. Then either $\Phi(T)\Phi(S) = 0$ for every $S \in \mathcal{F}_1(X)$ with $TS = 0$, or $\Phi(S)\Phi(T) = 0$ for every $S \in \mathcal{F}_1(X)$ with $TS = 0$.

We need only to check that if there is an $R \in \mathcal{F}_1(X)$ such that $TR = 0$, $\Phi(T)\Phi(R) = 0$ and $\Phi(R)\Phi(T) \neq 0$, then the first assertion is true; and if there is an $R \in \mathcal{F}_1(X)$ such that $TR = 0$, $\Phi(T)\Phi(R) \neq 0$ and $\Phi(R)\Phi(T) = 0$, then the latter assertion is true.

Let $T = x_1 \otimes f_1$, $R = x_2 \otimes f_2$ and $S = x_3 \otimes f_3$ and assume that $TR = TS = 0$, $\Phi(T)\Phi(R) = 0$ and $\Phi(R)\Phi(T) \neq 0$. It is clear that $\langle x_2, f_1\rangle = \langle x_3, f_1\rangle = 0$. Let us first check that $\Phi(T)\Phi(x_2 \otimes f) = 0$ for every $f \in X'$. Suppose there exists an $f \in X'$ such that $\Phi(T)\Phi(x_2 \otimes f) \neq 0$. By Claim 2, $\Phi(x_2 \otimes f)\Phi(T) = 0$ and either $\Phi(x_2 \otimes (f + f_2))\Phi(T) = 0$ or $\Phi(T)\Phi(x_2 \otimes (f + f_2)) = 0$. But on the other hand, by Claim 3, we have $\Phi(x_2 \otimes (f + f_2))\Phi(T) = (\alpha_1\Phi(x_2 \otimes f) + \alpha_2\Phi(x_2 \otimes f_2))\Phi(T) \neq 0$ and $\Phi(T)\Phi(x_2 \otimes (f + f_2)) = \Phi(T)(\alpha_1\Phi(x_2 \otimes f) + \alpha_2\Phi(x_2 \otimes f_2)) \neq 0$, a contradiction.

Now we verify that $\Phi(T)\Phi(S) = 0$. Assume, on the contrary, that $\Phi(T)\Phi(S) = \Phi(x_1 \otimes f_1)\Phi(x_3 \otimes f_3) \neq 0$; then $\Phi(S)\Phi(T) = \Phi(x_3 \otimes f_3)\Phi(x_1 \otimes f_1) = 0$. Just like the argument as in the above paragraph, we can get $\Phi(x_3 \otimes f)\Phi(T) = 0$ for every $f \in X'$. Thus $\Phi(T)\Phi(x_2 \otimes f_3) = \Phi(x_3 \otimes f_2)\Phi(T) = 0$. Note that either $\Phi(T)\Phi((x_2 + x_3) \otimes f_2) = 0$ or $\Phi((x_2 + x_3) \otimes f_2)\Phi(T) = 0$. So we have $\Phi(T)\Phi(x_3 \otimes f_2) = 0$ and $\Phi(x_3 \otimes f_2)\Phi(T) = 0$. Similarly, $\Phi(T)\Phi(x_2 \otimes f_3) = 0$ and $\Phi(x_2 \otimes f_3)\Phi(T) = 0$. Because either $\Phi((x_2 + x_3) \otimes (f_2 + f_3))\Phi(T) = 0$ or $\Phi(T)\Phi((x_2 + x_3) \otimes (f_2 + f_3)) = 0$, one sees that either

$$\begin{aligned}\Phi(T)\alpha_1\Phi(x_2 \otimes f_2) &+ \Phi(T)\alpha_2\Phi(x_3 \otimes f_3)\\ &+ \Phi(T)\alpha_3\Phi(x_2 \otimes f_3) + \Phi(T)\alpha_4\Phi(x_3 \otimes f_2) = 0\end{aligned}$$

for some nonzero numbers $\alpha_i \in \mathbb{C}$ or

$$\begin{aligned}\beta_1\Phi(x_2 \otimes f_2)\Phi(T) &+ \beta_2\Phi(x_3 \otimes f_3)\Phi(T)\\ &+ \beta_3\Phi(x_2 \otimes f_3)\Phi(T) + \beta_4\Phi(x_3 \otimes f_2)\Phi(T) = 0\end{aligned}$$

for some nonzero numbers $\beta_i \in \mathbb{C}$, $i = 1, \ldots, 4$. However, this leads to $\Phi(T)\Phi(x_3 \otimes f_3) = \Phi(T)\Phi(S) = 0$, a contradiction. Hence the first assertion of the claim is true.

The second assertion can be checked similarly.

Claim 5. For $x \in X$ and $f \in X'$, denote $L_x = \{x \otimes h : h \in X'\}$ and $R_f = \{u \otimes f : u \in X\}$. Then exactly one of the following is true:

(a) there exist maps $x \mapsto y(x)$ from X into X and $f \mapsto g(f)$ from X' into X' such that $\Phi(L_x) \subseteq L_{y(x)}$ for every $x \in X$ and $\Phi(R_f) \subseteq R_{g(f)}$ for every $f \in X'$;

(b) there exist maps $x \mapsto g(x)$ from X into X' and $f \mapsto y(f)$ from X' into X such that $\Phi(L_x) \subseteq R_{g(x)}$ for every $x \in X$ and $\Phi(R_f) \subseteq L_{y(f)}$ for every $f \in X'$.

It follows from Claim 3 that both $\Phi(L_x)$ and $\Phi(R_f)$ are additive subgroups consisting of rank-1 operators. So $\Phi(L_x)$ and $\Phi(R_f)$ are contained in L_y or R_g for some y and g depending on x and f, respectively.

Now we show that either

(I) $\Phi(L_x) \subseteq L_{y(x)}$ for every $x \in X$, or

(II) $\Phi(L_x) \subseteq R_{g(x)}$ for every $x \in X$.

Suppose that there exist vectors x_1 and x_2 satisfying

$$\dim(\mathrm{span}\Phi(L_{x_1})) \geqslant 2, \quad \dim(\mathrm{span}\Phi(L_{x_2})) \geqslant 2$$

and

$$\Phi(L_{x_1}) \subseteq L_{y_1}, \quad \Phi(L_{x_2}) \subseteq R_{g_2}.$$

Then there exist f_1 and f_2 in X' such that $\Phi(x_1 \otimes f_1) = y_1 \otimes g_1$ and $\Phi(x_2 \otimes f_2) = y_2 \otimes g_2$ such that y_1 is linearly independent to y_2 and g_1 is linearly independent to g_2. Let $\Phi(x_1 \otimes f_2) = y_1 \otimes g_3$ and $\Phi(x_2 \otimes f_1) = y_3 \otimes g_2$. It follows from Claim 3 that

$$\Phi(x_1 \otimes f_2 + x_2 \otimes f_2) = \alpha_1 \Phi(x_1 \otimes f_2) + \alpha_2 \Phi(x_2 \otimes f_2) = \alpha_1 y_1 \otimes g_3 + \alpha_2 y_2 \otimes g_2.$$

Because Φ preserves rank one operators in both directions, $\alpha_1 y_1 \otimes g_3 + \alpha_2 y_2 \otimes g_2$ is a rank one operator. Since y_1 and y_2 are linearly independent, we see that g_2 and g_3 must be linearly dependent, i.e., $g_3 = \beta_1 g_2$ for some scalar β_1. Similarly, since $\Phi(x_2 \otimes f_1 + x_1 \otimes f_1)$ and $\Phi((x_1 + x_2) \otimes (f_1 + f_2))$ are rank one operators, we can get $y_3 = \beta_2 y_1$ for some scalar β_2 and $g_1 = \beta_3 g_2$. Thus g_1 is linearly dependent to g_2, arriving a contradiction.

In the same way, we can check that either

(I′) $\Phi(R_f) \subseteq L_{y(f)}$ for every $f \in X'$, or
(II′) $\Phi(R_f) \subseteq R_{g(f)}$ for every $f \in X'$.

Now we only need to show that (I) and (I′), (II) and (II′) do not occur simultaneously. Assume, on the contrary, that both (I) and (I′) hold true. Fix a vector $f_0 \in X'$; then, for every $x \in X$, we have $x \otimes f_0 \in L_x \cap R_{f_0}$. It follows that $\Phi(x \otimes f_0) \in L_{y(x)} \cap L_{y(f_0)}$, which implies that $y(x)$ and $y(f_0)$ are linearly dependent. Thus $\Phi(\mathcal{F}_1(X)) \subseteq L_{y(f_0)}$, this contradicts the property that Φ preserves rank-1 operators in both directions. Applying the same process, we can prove that (II) and (II′) can not occur at the same time.

In Claim 6–Claim 8, we will assume that the case (a) holds true, and then prove that Φ has the form (i) in the theorem, i.e., Φ is an isomorphism.

Claim 6. For any $T, S \in \mathcal{F}_1(X)$, $TS = 0$ implies that $\Phi(T)\Phi(S) = 0$.

Pick a rank one operator $T \in \mathcal{F}_1(X)$. By Claim 4, we already knew that either $\Phi(T)\Phi(S) = 0$ for every $S \in \mathcal{F}_1(X)$ with $TS = 0$; or $\Phi(S)\Phi(T) = 0$ for every $S \in \mathcal{F}_1(X)$ with $TS = 0$. Assume that the latter occurs, we shall induce a contradiction. Write $T = x_0 \otimes f_0$ and $\Phi(T) = y_0 \otimes g_0$. Note that, for every $g \in X'$, there is a vector $f \in X'$ such that $\Phi(R_f) \subseteq R_g$. Take $x \in \ker f_0$; then $Tx \otimes f = 0$ and $\Phi(x \otimes f)\Phi(T) = \Phi(x \otimes f) y_0 \otimes g_0 = 0$, which implies that $\langle y_0, g \rangle = 0$. Thus we get $y_0 = 0$; since g is arbitrary, this is impossible.

Claim 7. There exists an invertible operator $A \in \mathcal{B}(X)$ such that $\Phi(T) = ATA^{-1}$ for every $T \in \mathcal{F}_1(X)$.

Since Φ is bijective and Φ^{-1} has the same properties as Φ, we have $TS = 0 \Leftrightarrow \Phi(T)\Phi(S) = 0$ for any $T, S \in \mathcal{F}_1(X)$. Using Lemma 2.5, we see that there exists an invertible bounded either linear or conjugate linear operator $A : X \to X$ such that $\Phi(P) = APA^{-1}$ for every rank-1 idempotent $P \in \mathcal{B}(X)$.

By Claim 4 and $\triangle^{\mathcal{A}}(T) = \triangle^{\mathcal{B}}(\Phi(T))$ for every $T \in \mathcal{A}$, it follows that $\Phi(\lambda P) = \lambda\Phi(P)$ for every rank-1 idempotent and scalar λ. So the operator A in above paragraph is linear. Moreover $\Phi(x \otimes f) = A(x \otimes f)A^{-1}$ for all $x \in X$ and $f \in X'$ with $\langle x, f\rangle \neq 0$.

For any $x \in X$ and $f \in X'$ with $\langle x, f\rangle = 0$, take $g \in X'$ such that $\langle x, g\rangle = 1$. Then

$$\begin{aligned}\Phi(x \otimes f) &= \Phi(x \otimes (f-g) + x \otimes g) = \xi_1\Phi(x \otimes (f-g)) + \xi_2\Phi(x \otimes g)\\ &= A(\xi_1 x \otimes (f-g) + \xi_2 x \otimes g)A^{-1} = A(x \otimes (\xi_1(f-g) + \xi_2 g)A^{-1}\end{aligned}$$

for some nonzero scalars ξ_1 and ξ_2. Noticing that Φ preserves the spectrum of operators, we must have $\alpha_1 = \alpha_2$. Therefore, $\Phi(x \otimes f) = \xi A(x \otimes f)A^{-1}$ for some scalar ξ. Next we verify that $\xi = 1$. Take a rank-1 idempotent P. It is easy to see that $\sigma(P + x \otimes f) = \sigma(\Phi(P) + \Phi(x \otimes f)) = \sigma(P + \xi x \otimes f)$. So $\langle(\lambda - P)^{-1}x, f\rangle = 1$ if and only if $\xi\langle(\lambda - P)^{-1}x, f\rangle = 1$ for every $\lambda \notin \sigma(P)$. Thus $\xi = 1$.

Claim 8. $\Phi(T) = ATA^{-1}$ for every $T \in \mathcal{A}$.

By Lemma 2.3, for every $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$, we have that

$$\lambda \in \eta\sigma(T + x \otimes f) \Leftrightarrow \langle(\lambda I - T)^{-1}x, f\rangle = 1.$$

Thus, for any $T \in \mathcal{A}$, $x \in X$, $f \in X'$ and any $\lambda \in \mathbb{C}$ with $|\lambda| > \max\{\|T\|, \|\Phi(T)\|\}$, $\lambda \in \triangle^{\mathcal{A}}(T + x \otimes f) = \triangle^{\mathcal{B}}(\Phi(T) + A(x \otimes f)A^{-1})$ if and only if $\langle(\lambda I - T)^{-1}x, f\rangle = \langle(\lambda I - \Phi(T))^{-1}Ax, (A^{-1})'f\rangle = 1$. It follows that

$$\langle(I - \omega T)^{-1}x, f\rangle = \langle(I - \omega\Phi(T))^{-1}Ax, (A^{-1})'f\rangle \tag{2.6}$$

holds for every $\omega \in \mathbb{C}$ with $0 < |\omega| < \min\{\|T\|^{-1}, \|\Phi(T)\|^{-1}\}$ and for every $T \in \mathcal{A}$, $x \in X$, $f \in X'$. Since each side of Equation (2.6) is analytic in $\{\omega : 0 < |\omega| < \min\{\|T\|^{-1}, \|\Phi(T)\|^{-1}\}\}$ with removable singularity at 0, taking the derivative at $\omega = 0$, we get $\langle Tx, f\rangle = \langle\Phi(T)Ax, (A^{-1})'f\rangle = \langle A^{-1}\Phi(T)Ax, f\rangle$. Therefore, we have $\Phi(T) = ATA^{-1}$ for all $T \in \mathcal{A}$.

A similar argument shows that the case (b) in Claim 5 will implies that Φ has the form (ii) in the main theorem, i.e., Φ is an anti-isomorphism. The reflexivity of X in this case is easily checked (for example, see proof of Theorem 1.4 in [6]). We omit the details here. ∎

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