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GROWTH CONDITIONS AND INVERSE PRODUCING EXTENSIONS

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ABSTRACT. We study the invertibility of Banach algebras elements in their extensions, and invertible extensions of Banach and Hilbert space operators with prescribed growth conditions for the norm of inverses. As applications, the solutions of two open problems are obtained. In the first one we give a characterization of $\mathcal{E}(\mathbb{T})$-subscalar operators in terms of growth conditions. In the second one we show that operators satisfying a Beurling-type growth condition possess Bishop's property (β). Other applications are also given.

KEYWORDS: *Invertible extensions, growth conditions, subscalar operators.*

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1. INTRODUCTION

1.1. PREAMBLE. A bounded linear operator can be made "nicer" by an extension or a dilation to a larger space. One example [31] is the celebrated Sz.-Nagy Dilation Theorem (every Hilbert space contraction has a unitary dilation), or its extension variant (every Hilbert space contraction has a coisometric extension). A Banach space example is a result due to R.G. Douglas [8] stating that a Banach space isometry has an extension to a surjective isometry. Douglas' construction is Hilbertian, in the sense that if the given operator acts on a Hilbert space, then its extension, a unitary operator, acts also on a Hilbert space. In the framework of Banach algebras, a classical result of R.F. Arens [1] states that if an element u of a commutative unital Banach algebra $\mathcal{A}$ is not a topological divisor of zero, then u is invertible in a commutative unital Banach algebra containing $\mathcal{A}$. Other such examples, related to the topic of the present paper, can be found in [29], [26], [24], [27], [28], [6], [5].

1.2. MOTIVATION. The aim of this paper is to study the invertibility of Banach algebras elements in their extensions, and invertible extensions of Banach or Hilbert space operators with prescribed growth conditions for the norm of inverses. We obtain, among other things, generalizations of the above mentioned results of Douglas and Arens.

Our investigations were also motivated by two open problems, which will be solved positively in this paper. The first one is due to K.B. Laursen and M.M. Neumann ([17], Problem 6.1.15) and M. Didas [9] and asks for a characterization in terms of growth conditions of $\mathcal{E}(\mathbb{T})$-subscalar operators, i.e., of operators which are similar to restrictions of $\mathcal{E}(\mathbb{T})$-scalar operators to closed invariant subspaces.

The second open problem asks [20] if operators $T \in B(X)$ satisfying the Beurling-type condition

$$\sum_{n=1}^{\infty} \frac{\log \max(\|T^n\|, m(T^n)^{-1})}{n^2} < \infty \tag{1.1}$$

possess Bishop's property (β) ; see (1.2) for the definition of the minimum modulus $m(T^n)$ and Section 4 for the definition of property (β).

1.3. ORGANIZATION OF THE PAPER. Our first result in the second section is a refinement of the Arens construction. We consider the invertibility of an element u of a Banach algebra $\mathcal{A}$ in an extension of $\mathcal{A}$ with prescribed growth conditions for $\|u^{-k}\|$, $k \geqslant 1$. We then consider extensions of Banach space operators. We use a method due to one of the authors [24] to pass from the Banach algebra case to the case of $B(X)$.

In Section 3 we use an idea of Batty and Yeates [5] to show that, given a real number $p \geqslant 1$ and $T \in B(X)$, there is an isomorphic embedding $\pi : X \mapsto Y$ and an invertible operator $S \in B(Y)$ with prescribed growth conditions for $\|S^{-k}\|$, $k \geqslant 1$, such that T is similar to the restriction of S to $\pi(X)$. Moreover, the space Y may be obtained from X as a quotient of a subspace of an ultraproduct of spaces of the form $L_p(X)$ (i.e., a $SQ_p(X)$-space). In particular, if $p = 2$ and X is a Hilbert space, then so is Y.

In the last section we consider several applications. A characterization for $\mathcal{E}(\mathbb{T})$-subscalar operators is given in Theorem 4.1. The question from [20] concerning operators satisfying the Beurling-type condition (1.1) is positively answered in Theorem 4.5. We then consider operators satisfying some exponential growth conditions. Other applications concerning operators with countable spectrum and Hilbert space contractions with spectrum a Carleson set are given.

1.4. NOTATION AND TERMINOLOGY. We recall now some known facts and introduce some notation. All other undefined terms are classical or will be defined in Section 4.

Banach algebras. All Banach algebras are considered to be complex and with unit. Let u be an element of a Banach algebra $\mathcal{A}$. We write

$$d^{\mathcal{A}}(u) = \inf\{\|ux\| : x \in \mathcal{A}, \|x\| = 1\}.$$

If no confusion can arise then we omit the upper index and write simply $d(u)$ instead of $d^{\mathcal{A}}(u)$.

Let $\mathcal{A}, \mathcal{B}$ be commutative Banach algebras. We say that $\mathcal{B}$ is an extension of $\mathcal{A}$ if there exists an isometrical unit preserving homomorphism $\rho : \mathcal{A} \to \mathcal{B}$. If we identify $\mathcal{A}$ with the image $\rho(\mathcal{A})$ we can consider $\mathcal{A}$ as a closed subalgebra of $\mathcal{B}$ and write simply $\mathcal{A} \subset \mathcal{B}$.

Operators. In this paper X (and Y) will denote complex Banach spaces and H (and K) will denote Hilbert spaces. Denote by $B(X)$ the algebra of all bounded linear operators on the Banach space X. By an operator we always mean a bounded linear operator. Note that for an operator $T \in B(X)$ we can express the quantity $d^{B(X)}(T)$ in a more convenient way by

$$m(T) := d^{B(X)}(T) = \inf\{\|Tx\| : x \in X, \|x\| = 1\}. \tag{1.2}$$

This quantity is called the *minimum modulus* of T [12] or the *lower bound* of T [17].

We denote by $\sigma(T)$ and $\sigma_{\mathrm{ap}}(T)$ the spectrum and the approximate point spectrum of a bounded linear operator $T \in B(X)$, respectively. The latter is given by

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} : \inf\{\|(T-\lambda)x\| : \|x\| = 1\} = 0\}.$$

Note that $m(T) > 0$ if and only if $T \in B(X)$ is one-to-one and of closed range. If T is a Hilbert space operator, then $\sigma_{\mathrm{ap}}(T)$ coincides with the left spectrum and $m(T) > 0$ if and only if T is left invertible.

We say that $S \in B(Y)$ is an extension of $T \in B(X)$ if there is an isometry $\pi : X \to Y$ such that $S\pi = \pi T$. We can also consider X as a subspace of Y and write $T = S_{|X}$.

Banach spaces of class SQ_p. Let $p \geqslant 1$ be a real number. A Banach space E is said to be a SQ_p-space if it is a quotient of a subspace of an L_p-space.

Let X be a Banach space. A Banach space E is said to be a $SQ_p(X)$-space if it is (isometric to) a quotient of a subspace of an ultraproduct of spaces of the form $L_p(\Omega, \mu, X)$, for some measure spaces (Ω, μ). Since ultraproducts of L_p-spaces are L_p-spaces, the latter definition is consistent with the former one. Note that any Banach space is isometric to a subspace (a quotient) of an L_∞-space (respectively an L_1-space). Also, if H is a Hilbert space, then each $SQ_2(H)$-space is a Hilbert space too.

$SQ_p(X)$-spaces are characterized by a theorem of R. Hernandez [13] (for $X = \mathbb{C}$ this goes back to [16]). See also [25] (and Theorem 3.2 of [18]) for a different proof using p-completely bounded maps. Namely, E is a $SQ_p(X)$-space if and only if

$$\|a\|_{p,E} \leqslant \|a\|_{p,X}$$

for each $n \geqslant 1$ and each matrix $a = [a_{ij}] \in M_n(\mathbb{C})$. Here

$$\|[a_{ij}]\|_{p,Y} = \sup \Big[\Big(\sum_i \Big\| \sum_j a_{ij} y_j \Big\|^p \Big)^{1/p} \Big],$$

where the supremum runs over all n-tuples $(y_1, \dots, y_n) \in Y$ satisfying $\sum \|y_j\|^p \leqslant 1$.

Nearness. Let $p \geqslant 1$ and $\beta : \mathbb{N} \to (0, \infty)$. Let X be a subspace of Y. Two operators T and C in $B(Y)$ are said to be *(β, p)-near modulo X* if for every $N \in \mathbb{N}$ and for all $x_1, \dots, x_N \in X$ we have

$$\Big\| \sum_{n=1}^{N} (T^n - C^n) x_n \Big\| \leqslant \Big(\sum_{n=1}^{N} \beta(n)^p \|x_n\|^p \Big)^{1/p}. \tag{1.3}$$

For a constant weight function $\beta(n) \equiv s$ and for $p = 2$ this definition was introduced and studied in [2], [3] under the name of quadratic nearness.

Note that if $p = 1$, and if the operators $T, C \in B(Y)$ verify $\|T^n - C^n\| \leqslant \beta(n)$ for all $n \geqslant 1$, then (1.3) holds for every $x_n \in Y$.

2. A REFINEMENT OF THE ARENS CONSTRUCTION

The result of R.F. Arens [1] implies that if $\mathcal{A}$ is a commutative Banach algebra and $d^{\mathcal{A}}(u) > 0$, then there exists a commutative extension $\mathcal{B} \supset \mathcal{A}$ such that u is invertible in $\mathcal{B}$. It follows from the Arens construction that $\|u^{-k}\| \leqslant (d^{\mathcal{A}}(u))^{-k}$ $(k \geqslant 1)$. The following theorem gives a necessary and sufficient condition for having invertible extensions of Banach algebra elements with prescribed growth conditions for the norm of inverses.

THEOREM 2.1. *Let u be an element of a commutative Banach algebra $\mathcal{A}$. Let $(c_j)_{j=1}^{\infty}$ be a sequence of positive numbers which is submultiplicative, i.e., $c_{i+j} \leqslant c_i c_j$ for all $i, j \geqslant 1$. Then there is a commutative extension $\mathcal{B} \supset \mathcal{A}$ such that u is invertible in $\mathcal{B}$ and $\|u^{-j}\| \leqslant c_j$ $(j \geqslant 1)$ if and only if we have*

$$\|a_0\| \leqslant \sum_{j=1}^{\infty} c_j \|a_j - a_{j-1} u\|$$

for every sequence $(a_j)_{j=0}^{\infty}$ in $\mathcal{A}$ of finite support.

Proof. Suppose that $\mathcal{B} \supset \mathcal{A}$ is a commutative extension with all the required properties. Let $(a_j)_{j=0}^{\infty}$ be a sequence in $\mathcal{A}$ such that $a_j = 0$ for $j \geqslant n$. Write $f_j = a_j - a_{j-1} u$. Then

$$
\begin{aligned}
\|a_0\|_{\mathcal{A}} = \|a_0\|_{\mathcal{B}} = \|u^{-n}u^n a_0\| \\
&= \Big\| - u^{-n}\Big(\sum_{j=1}^{n} u^{n-j} f_j \Big) \Big\| \\
&= \Big\| \sum_{j=1}^{n} u^{-j} f_j \Big\| \leqslant \sum_{j=1}^{n} c_j \|f_j\|.
\end{aligned}
$$

For the converse, set formally $c_0 = 1$. Consider the algebra $\mathcal{C}$ of all power series $\sum_{i=0}^{\infty} a_i x^i$ in one variable x with coefficients $a_i \in \mathcal{A}$ such that

$$\Big\| \sum_{i=0}^{\infty} a_i x^i \Big\| = \sum_{i=0}^{\infty} \|a_i\| c_i < \infty.$$

With the multiplication given by

$$\Big(\sum_{i=0}^{\infty} a_i x^i \Big) \cdot \Big(\sum_{j=0}^{\infty} a'_j x^j \Big) = \sum_{k=0}^{\infty} x^k \Big(\sum_{i+j=k} a_i a'_j \Big),$$

$\mathcal{C}$ is a commutative Banach algebra containing $\mathcal{A}$ as subalgebra of constants. Let J be the closed ideal generated by the element $1 - ux$ and set $\mathcal{B} = \mathcal{C}/J$. Let $\rho : \mathcal{A} \to \mathcal{B}$ be the composition of the embedding $\mathcal{A} \to \mathcal{C}$ and the canonical homomorphism $\mathcal{C} \to \mathcal{B} = \mathcal{C}/J$. Then

$$\rho(u) \cdot (x + J) = (u + J)(x + J) = 1_{\mathcal{A}} + J = 1_{\mathcal{B}},$$

and so $\rho(u)$ is invertible in $\mathcal{B}$ with the inverse $x + J$. We have $\|(x + J)^n\|_{\mathcal{B}} \leqslant \|x^n\|_{\mathcal{C}} = c_n$ for all $n \geqslant 1$.

It is sufficient to show that ρ is an isometry, i.e., that for each $a \in \mathcal{A}$ we have $\|a\|_{\mathcal{A}} = \|\rho(a)\|_{\mathcal{B}}$.

Obviously, $\|\rho(a)\|_{\mathcal{B}} = \inf_{c \in \mathcal{C}} \|a + (1 - ux)c\| \leqslant \|a\|_{\mathcal{A}}$.

Suppose on the contrary that there is an $a \in \mathcal{A}$ such that $\|\rho(a)\|_{\mathcal{B}} < \|a\|_{\mathcal{A}}$. Thus there are elements $a_j \in \mathcal{A}$ such that

$$
\begin{aligned}
\|a\|_{\mathcal{A}} > \Big\| a + (1 - ux) \sum_{j=0}^{\infty} a_j x^j \Big\|_{\mathcal{C}} \\
&= \|a - a_0\|_{\mathcal{A}} + \sum_{j=1}^{\infty} c_j \cdot \|a_j - a_{j-1}u\|_{\mathcal{A}} \\
&\geqslant \|a\| - \|a_0\| + \sum_{j=1}^{\infty} c_j \cdot \|a_j - a_{j-1}u\|.
\end{aligned}
$$

Thus $\|a_0\| > \sum_{j=1}^{\infty} c_j \|f_j\|$, where $f_j = a_j - a_{j-1}u$. Moreover, we may assume that only a finite number of elements a_j are non-zero. This contradicts to our assumption. ∎

We introduce the following definition.

DEFINITION 2.2. Let u be an element of a Banach algebra $\mathcal{A}$. Let $(c_j)_{j=1}^{\infty}$ be a sequence of positive numbers which is submultiplicative, i.e., $c_{i+j} \leqslant c_i c_j$ for all $i, j \geqslant 1$. We say that (c_j) *satisfies condition* $(*)$ *for* $u \in \mathcal{A}$ if there exists an increasing sequence (k_n) of integers such that $0 = k_0 < k_1 < k_2 < \cdots$ and

$$(*) \qquad c_j \geqslant (d(u^{k_1})d(u^{k_2-k_1}) \cdots d(u^{k_{n+1}-k_n}))^{-1} \|u^{k_{n+1}-j}\|$$

for all $n \geqslant 0$ and j satisfying $k_n < j \leqslant k_{n+1}$.

THEOREM 2.3. *Let u be an element of a commutative Banach algebra $\mathcal{A}$. Let (c_j) be a sequence of positive numbers satisfying condition $(*)$ for $u \in \mathcal{A}$. Then there is a commutative extension $\mathcal{B} \supset \mathcal{A}$ such that u is invertible in $\mathcal{B}$ and $\|u^{-j}\| \leqslant c_j$ $(j \geqslant 1)$.*

Proof. Set formally $c_0 = 1$. Let $(a_j)_{j=0}^{\infty}$ be a sequence in $\mathcal{A}$ of finite support. Write $f_j = a_j - a_{j-1}u$.

We verify the condition of Theorem 2.1. We have

$$\begin{aligned}
\|a_0\| &\leqslant d(u^{k_1})^{-1}\|a_0 u^{k_1}\| \\
&\leqslant d(u^{k_1})^{-1}(\|a_0 u^{k_1} - a_1 u^{k_1-1}\| + \cdots + \|a_{k_1-1}u - a_{k_1}\| + \|a_{k_1}\|) \\
&\leqslant d(u^{k_1})^{-1}(\|f_1\| \cdot \|u^{k_1-1}\| + \|f_2\| \cdot \|u^{k_1-2}\| + \cdots + \|f_{k_1}\|) \\
&\quad + d(u^{k_1})^{-1} d(u^{k_2-k_1})^{-1}\|a_{k_1}u^{k_2-k_1}\| \\
&\leqslant \sum_{j=1}^{k_1} c_j\|f_j\| + d(u^{k_1})^{-1}d(u^{k_2-k_1})^{-1}(\|a_{k_1}u^{k_2-k_1} - a_{k_1+1}u^{k_2-k_1-1}\| \\
&\quad + \cdots + \|a_{k_2-1}u - a_{k_2}\| + \|a_{k_2}\|) \\
&\leqslant \sum_{j=1}^{k_2} c_j\|f_j\| + d(u^{k_1})^{-1}d(u^{k_2-k_1})^{-1}\|a_{k_2}\| \leqslant \cdots \leqslant \sum_{j=1}^{\infty} c_j\|f_j\|,
\end{aligned}$$

since only a finite number of elements a_j are non-zero. ∎

Using a construction from [24] we obtain a similar result for extensions of Banach space operators.

THEOREM 2.4. *Let T be an operator acting on a Banach space X. Let (c_j) be a sequence of positive numbers satisfying condition $(*)$ for $T \in B(X)$. Then there exists a Banach space Y containing X as a closed subspace and an invertible operator $S \in B(Y)$ such that $S|X = T$ and $\|S^{-j}\| \leqslant c_j$ $(j \geqslant 1)$. Moreover, we have $\|S^j\| \leqslant \|T^j\|$ $(j \geqslant 1)$ and $\sigma(S) \subset \sigma(T)$.*

Proof. Let $\mathcal{A}$ be a maximal commutative subalgebra of $B(X)$ containing T.

Set $\mathcal{B} = \mathcal{A} \oplus X$. Define the norm and multiplication in $\mathcal{B}$ by

$$\|A \oplus x\| = \|A\| + \|x\|$$

and

$$(A \oplus x)(A' \oplus x') = AA' \oplus (Ax' + A'x) \quad (A, A' \in \mathcal{A},\ x, x' \in X).$$

Then $\mathcal{B}$ is a commutative Banach algebra and $A \mapsto A \oplus 0\ \ (A \in \mathcal{A})$ is an isometrical embedding $\mathcal{A} \to \mathcal{B}$.

Let $n \geqslant 0$. It is easy to show that

$$d^{\mathcal{B}}(T^n \oplus 0) = d^{B(X)}(T^n) = m(T^n).$$

By Theorem 2.3, there exists a commutative Banach algebra $\mathcal{C} \supset \mathcal{B}$ such that $T \oplus 0$ is invertible in $\mathcal{C}$ and

$$\|(T \oplus 0)^{-j}\|_{\mathcal{C}} \leqslant c_j \quad (j \geqslant 1).$$

Consider the operator $S : \mathcal{C} \to \mathcal{C}$ defined by $Sc = (T \oplus 0)c\ (c \in \mathcal{C})$. Then S is invertible and

$$\|S^{-j}\| \leqslant c_j \quad (j \geqslant 1).$$

For $x \in X$ we have

$$S(0 \oplus x) = (T \oplus 0)(0 \oplus x) = 0 \oplus Tx.$$

If we identify $x \in X$ with $0 \oplus x \in \mathcal{B} \subset \mathcal{C}$, then $T = S_{|X}$.

The relation $\|S^j\| \leqslant \|T^j\|\ (j \geqslant 1)$ is easy to verify.

Finally, we have

$$\sigma^{B(X)}(T) = \sigma^{\mathcal{A}}(T) \supset \sigma^{\mathcal{B}}(T \oplus 0) \supset \sigma^{\mathcal{C}}(T \oplus 0) \supset \sigma^{B(\mathcal{C})}(S). \quad \blacksquare$$

3. EXTENSIONS TO $SQ_p(X)$-SPACES

In this section we study the similarity to restrictions of invertible operators acting on $SQ_p(X)$-spaces.

The proof of the following result uses an idea from [5].

THEOREM 3.1. *Let $(c_j)_{j=1}^{\infty}$ be a sequence of positive numbers which is submultiplicative. Let $p \geqslant 1$ be a fixed real number, X a Banach space and $T \in B(X)$.*

(i) *Suppose that there exists a Banach space Y, $M \geqslant 1$, an operator $\pi : X \to Y$ such that $\|x\| \leqslant M\|\pi(x)\|$ for all $x \in X$, and an invertible operator $S \in B(Y)$ such that $S\pi = \pi T$ and S^{-1} is (c, p)-near the null operator modulo $\pi(X)$, that is*

$$\Big\| \sum_{j=1}^{n} S^{-j}\pi(y_j) \Big\| \leqslant \Big(\sum_{j=1}^{n} c_j^p \|y_j\|^p \Big)^{1/p}$$

for every $n \geqslant 1$ and all $y_j \in X$. Then we have

$$\|x\|^p \leqslant M^p (c_n^p \|x_0\|^p + c_{n-1}^p \|x_1\|^p + \cdots + c_1^p \|x_{n-1}\|^p),$$

whenever $T^n x = x_0 + Tx_1 + \cdots + T^{n-1}x_{n-1}$.

(ii) *Let* $M \geqslant 1$ *and* $p \geqslant 1$. *Suppose that the equality*

$$T^n x = x_0 + Tx_1 + \cdots + T^{n-1}x_{n-1} \quad (x_i \in X, 1 \leqslant i \leqslant n)$$

always implies

$$\|x\|^p \leqslant M^p(c_n^p\|x_0\|^p + c_{n-1}^p\|x_1\|^p + \cdots + c_1^p\|x_{n-1}\|^p).$$

Then there exists a Banach space $(Y, |\cdot|)$ *which is a* $SQ_p(X)$*-space, an isomorphic embedding* $\pi : X \to Y$ *satisfying* $\frac{\|x\|}{M2^{(p-1)/p}} \leqslant |\pi(x)| \leqslant \|x\|$ $(x \in X)$*, and an invertible operator* $S \in B(Y)$ *such that* $S\pi = \pi T$ *and* $\|S^{-j}\| \leqslant c_j$ *for every* $j \geqslant 1$. *Moreover,* S^{-1} *is* (c, p)*-near the null operator modulo* $\pi(X)$, $\|S^j\| \leqslant \|T^j\|$ $(j \geqslant 1)$ *and* $\sigma(S) \subset \sigma(T)$.

Proof. (i) Suppose that T has an invertible extension S as in the statement of the theorem and let $\pi : X \to Y$ satisfy $\|x\| \leqslant M\|\pi(x)\|$ for all $x \in X$ and $S\pi = \pi T$. Suppose that $T^n x = x_0 + Tx_1 + \cdots + T^{n-1}x_{n-1}$. Then

$$\begin{aligned}\|x\| &\leqslant M\|\pi(x)\| = M\|S^{-n}S^n\pi(x)\| = M\|S^{-n}\pi(T^n x)\| \\ &= M\Big\|S^{-n}\pi\Big(\sum_{k=0}^{n-1} T^k x_k\Big)\Big\| = M\Big\|\sum_{k=0}^{n-1} S^{-(n-k)}\pi(x_k)\Big\| \\ &\leqslant M\Big(\sum_{k=0}^{n-1} c_{n-k}^p\|x_k\|^p\Big)^{1/p}.\end{aligned}$$

(ii) Suppose now that

$$\|x\|^p \leqslant M^p(c_n^p\|x_0\|^p + c_{n-1}^p\|x_1\|^p + \cdots + c_1^p\|x_{n-1}\|^p),$$

whenever $T^n x = x_0 + Tx_1 + \cdots + T^{n-1}x_{n-1}$. For $x_0 = T^n x$ we get

$$\|T^n x\| \geqslant \frac{1}{Mc_n}\|x\|.$$

In particular, each operator T^n is injective.

The equivalence relation. Let $X_0 = X \times \mathbb{Z}$. We define an equivalence relation on X_0 by $(x, t) \sim (y, s)$ if there exists $m \in \mathbb{N}$ such that $s + m \in \mathbb{N}$, $t + m \in \mathbb{N}$ and $T^{s+m}x = T^{t+m}y$.

Let $X_1 = X_0/\sim$ be the space of equivalence classes. We denote the equivalence class containing (x, t) by $[x, t]$. Each equivalence class contains a member (x, t) with $t \in \mathbb{N}$.

The operations. The operations

$$\begin{aligned}[x, t] + [y, s] &= [T^s x + T^t y, s + t], && s, t \in \mathbb{N},\ x, y \in X, \\ \alpha[x, t] &= [\alpha x, t], && t \in \mathbb{N},\ \alpha \in \mathbb{C},\end{aligned}$$

endow X_1 with a structure of vector space.

The norm. Set $c_0 = 1$. We define the norm on X_1 as follows. For $[x,t] \in X_1$, set

$$|[x,t]|^p = \inf\Big\{ \sum_{i=0}^n \|x_i\|^p c_i^p : n \in \mathbb{N}, \sum_{i=0}^n [x_i, i] = [x,t] \Big\}.$$

We note that the existence of a decomposition $[x,t] = \sum_{i=0}^n [x_i, i]$ with $t \geqslant n$ is equivalent to

$$x = \sum_{i=0}^n T^{t-i} x_i.$$

It is easy to see that $|\cdot|$ is well-defined and $|\lambda[x,t]| = |\lambda|\,|[x,t]|$ $(\lambda \in \mathbb{C})$.

Let $[x,t]$ and $[y,s]$ be two elements of X_1 decomposed by $[x,t] = \sum_i [x_i, i]$ and $[y,s] = \sum_i [y_i, i]$. Then $[x,t] + [y,s] = \sum_i [x_i + y_i, i]$. By the triangular inequality in ℓ^p, we have

$$\begin{aligned} |[x,t] + [y,s]| &\leqslant \Big(\sum_i \|x_i + y_i\|^p c_i^p \Big)^{1/p} \leqslant \Big(\sum_i (\|x_i\| + \|y_i\|)^p c_i^p \Big)^{1/p} \\ &\leqslant \Big(\sum_i \|x_i\|^p c_i^p \Big)^{1/p} + \Big(\sum_i \|y_i\|^p c_i^p \Big)^{1/p}. \end{aligned}$$

Taking the infimum on the right hand side over all decompositions of $[x,t]$ and $[y,s]$ we get $|[x,t] + [y,s]| \leqslant |[x,t]| + |[y,s]|$.

We show that $|\cdot|$ is a norm. Let $x \in X$ and $t \geqslant 0$. Consider a decomposition

$$[x,t] = \sum_{i=0}^n [x_i, i]$$

with $x_i \in X$. Then

$$[x,t] = \sum_{i=0}^n [x_i, i] = \sum_{i=0}^n [T^{n-i} x_i, n] = \Big[\sum_{i=0}^n T^{n-i} x_i, n \Big].$$

Hence

$$T^n (x - T^t x_0) = T^t \Big(\sum_{i=1}^n T^{n-i} x_i \Big) = \sum_{i=1}^n T^{n-i} (T^t x_i).$$

By hypothesis, we have

$$\|x - T^t x_0\|^p \leqslant M^p \Big(\sum_{i=1}^n c_i^p \|T^t x_i\|^p \Big).$$

Since

$$\frac{1}{2^{p-1}} \|x\|^p - \|T^t x_0\|^p \leqslant \|x - T^t x_0\|^p,$$

we get

$$\frac{1}{2^{p-1}} \|x\|^p \leqslant M^p \sum_{i=0}^n c_i^p \|T^t x_i\|^p \leqslant M^p \|T^t\|^p \sum_{i=0}^n c_i^p \|x_i\|^p.$$

Since this is true for all such decompositions, we obtain

$$|[x,t]| \geqslant \frac{1}{2^{(p-1)/p} M \|T^t\|} \|x\|.$$

In particular, $|[x,t]| \neq 0$ whenever $x \neq 0$.

The isomorphic embedding π. The space X embeds isomorphically into X_1. The embedding is given by $\pi : x \to [x, 0]$ and the trivial decomposition $[x, 0] = [x, 0]$ gives $|\pi(x)| \leqslant \|x\|$. The previous paragraph, for $t = 0$, shows that

$$|\pi(x)| \geqslant \frac{1}{M 2^{(p-1)/p}} \|x\|.$$

The operator S. Define S on X_1 by $S[x,s] = [x, s-1]$, $x \in X$, $s \in \mathbb{Z}$. Clearly the definition of S is correct, S is a linear map and $S\pi = \pi T$.

The inequality

$$|S^j[x,t]| \leqslant \|T^j\| \cdot \Big|[x,t]\Big|$$

can be proved exactly as in [5]. Thus $\|S^j\| \leqslant \|T^j\|$ for all $j \geqslant 0$.

We show now that $|S^{-s}[x,t]| \leqslant c_s |[x,t]|$ for all positive s and all classes $[x,t]$. Consider a decomposition

$$[x,t] = \sum_{i=0}^{n} [x_i, i]$$

with $x_i \in X$. Then

$$[x, t+s] = \sum_{i=0}^{n} [x_i, i+s].$$

Thus

$$|[x, t+s]|^p \leqslant \sum_{i=0}^{n} c_{i+s}^p \|x_i\|^p.$$

Using the submultiplicativity of the sequence $c = (c_j)_{j=1}^{\infty}$ we obtain

$$|S^{-s}[x,t]|^p = |[x, t+s]|^p \leqslant c_s^p \sum_{i=0}^{n} c_i^p \|x_i\|^p.$$

This yields the announced estimate.

We show now that

$$\Big| \sum_{j=1}^{n} S^{-j} \pi(y_j) \Big| \leqslant \Big(\sum_{j=1}^{n} c_j^p \|y_j\|^p \Big)^{1/p}$$

for every $n \geqslant 1$ and all $y_j \in X$. Indeed, we have

$$\sum_{j=1}^{n} S^{-j} \pi(y_j) = \sum_{j=1}^{n} S^{-j} [y_j, 0] = \sum_{j=1}^{n} [y_j, j].$$

Therefore

$$\Big| \sum_{j=1}^{n} S^{-j} \pi(y_j) \Big|^p \leqslant \sum_{j=1}^{n} c_j^p \|y_j\|^p.$$

In fact, the same arguments provide the stronger (if $p > 1$) inequality

$$\Big|\sum_{j=0}^{n} S^{-j}\pi(y_j)\Big|^p \leqslant \sum_{j=0}^{n} c_j^p \|y_j\|^p,$$

for all $y_j \in X$, $j \geqslant 0$.

The space Y. We take the Banach space Y to be the completion of X_1 with the norm $|\cdot|$ and extend S continuously to an operator (also denoted by) S on Y.

We show now that Y is an $SQ_p(X)$-space. Let $[a_{ij}]$ be an $n \times n$ matrix with complex entries such that $\|a\|_{p,X} \leqslant 1$ (the definition of $\|a\|_{p,X}$ is recalled in the Introduction). Let $[x_j, t_j]$ be elements of X_1 with decompositions

$$[x_j, t_j] = \sum_{r=0}^{n^{(j)}} [w_r^{(j)}, r].$$

We have

$$\begin{aligned}\sum_i \Big|\sum_j [a_{i,j}x_j, t_j]\Big|^p &= \sum_i \Big|\sum_j \sum_r [a_{i,j}w_r^{(j)}, r]\Big|^p \\ &\leqslant \sum_i \sum_r c_r^p \Big\|\sum_j a_{i,j}w_r^{(j)}\Big\|^p = \sum_r c_r^p \sum_i \Big\|\sum_j a_{i,j}w_r^{(j)}\Big\|^p \\ &\leqslant \sum_r c_r^p \sum_j \|w_r^{(j)}\|^p = \sum_j \sum_r c_r^p \|w_r^{(j)}\|^p.\end{aligned}$$

By taking the infimum over all possible decompositions, we get

$$\sum_i \Big|\sum_j [a_{i,j}x_j, t_j]\Big|^p \leqslant \sum_j |[x_j, t_j]|^p.$$

Thus $\|a\|_{p,Y} \leqslant 1$, and so [13] X_1 and Y are $SQ_p(X)$-spaces.

Spectrum behaviour. Suppose that $T - \lambda$ is invertible in $B(X)$. Define L on X_1 by $L[x, t] = [(T - \lambda)^{-1}x, t]$. It is easy to see that the definition of L is correct. We have

$$(S - \lambda)[x, t] = [x, t - 1] - [\lambda x, t] = [(T - \lambda)x, t]$$

and $L(S - \lambda)[x, t] = (S - \lambda)L[x, t] = [x, t]$. Hence $S - \lambda$ is invertible in $B(Y)$. ∎

REMARKS 3.2. (i) The embedding π becomes isometric if $M = p = 1$ (for instance). The case $M = p = 1$ and $c_j = 1$ for $j \geqslant 1$ was considered in [5].

(ii) An alternative definition of the norm in X_1 is

$$|[x, t]|^p = \inf\Big\{\sum_{i=1}^{n} \|x_i\|^p c_i^p : n \in \mathbb{N}, \sum_{i=1}^{n} [x_i, i] = [x, t]\Big\}.$$

The difference is that decompositions of $[x, t]$ start now at $i = 1$. The construction of Y, S and $\pi : X \to Y$ remains unchanged. The embedding π satisfies in this case

$$\frac{\|x\|}{M} \leqslant |\pi(x)| \leqslant c_1\|T\| \cdot \|x\| \quad (x \in X).$$

The remaining properties are without any change.

(iii) Note that $\sigma_{\text{ap}}(T) \subset \sigma_{\text{ap}}(S)$.

(iv) We also note that Theorem 3.1 has a generalization to representations of semigroups (like in [5]).

DEFINITION 3.3. Let X be a Banach space, $T \in B(X)$, and let $p \geqslant 1$ be a fixed real number. Let $(c_j)_{j=1}^{\infty}$ be a sequence of positive numbers which is submultiplicative. We say that (c_j) *satisfies condition* $(*)_p$ *for* $T \in B(X)$ if there exists an increasing sequence of integers (k_n) such that $0 = k_0 < k_1 < k_2 < \cdots$ and

$$(*)_p \qquad c_j \geqslant \frac{2^{(n+1)(p-1)/p}(k_{n+1}-k_n)^{(p-1)/p}}{m(T^{k_1})m(T^{k_2-k_1})\cdots m(T^{k_{n+1}-k_n})}\|T^{k_{n+1}-j}\|$$

for all $n \geqslant 0$ and j satisfying $k_n < j \leqslant k_{n+1}$.

We say that (c_j) *satisfies condition* $(*)_\infty$ *for* $T \in B(X)$ if there exists an increasing sequence of integers (k_n) such that $0 = k_0 < k_1 < k_2 < \cdots$ and

$$(*)_\infty \qquad c_j \geqslant \frac{2^{n+1}(k_{n+1}-k_n)}{m(T^{k_1})m(T^{k_2-k_1})\cdots m(T^{k_{n+1}-k_n})}\|T^{k_{n+1}-j}\|$$

for all $n \geqslant 0$ and j satisfying $k_n < j \leqslant k_{n+1}$.

The condition $(*)_1$ is the same as condition $(*)$ considered above for Banach algebra elements. Clearly $(*)_p$ implies $(*)_q$ whenever $\infty \geqslant p \geqslant q \geqslant 1$; in particular, $(*)_\infty$ implies all other conditions $(*)_p$, $p \geqslant 1$.

LEMMA 3.4. *Let $p \geqslant 1$ be a fixed real number. Suppose that (c_j) is a sequence of positive numbers satisfying condition $(*)_p$ for $T \in B(X)$. Then*

$$\|x\|^p \leqslant c_m^p\|x_0\|^p + c_{m-1}^p\|x_1\|^p + \cdots + c_1^p\|x_{m-1}\|^p,$$

whenever $T^m x = x_0 + Tx_1 + \cdots + T^{m-1}x_{m-1}$.

Proof. Suppose that $k_n < m \leqslant k_{n+1}$ and that the conclusion of the lemma was proved for decompositions of form $T^{k_{n+1}}x = x_0 + Tx_1 + \cdots + T^{k_{n+1}-1}x_{k_{n+1}-1}$. If $T^m y = y_0 + Ty_1 + \cdots + T^{m-1}y_{m-1}$, then

$$T^{k_{n+1}}y = 0 + \cdots + T^{k_{n+1}-m}y_0 + T^{k_{n+1}-m+1}y_1 + \cdots + T^{k_{n+1}-1}y_{m-1}$$

and the lemma will be also proved for decompositions starting with $T^m y$.

So suppose that

$$T^{k_{n+1}}x = \sum_{j=1}^{k_{n+1}} T^{k_{n+1}-j}x_{k_{n+1}-j}.$$

Then, using the inequality $\|a-b\|^p \geqslant \frac{1}{2^{p-1}}\|a\|^p - \|b\|^p$, we have

$$\begin{aligned}
\|x_0\|^p &= \Big\| T^{k_{n+1}}x - \sum_{j=1}^{k_{n+1}-1} T^{k_{n+1}-j}x_{k_{n+1}-j} \Big\|^p \\
&= \Big\| T^{k_{n+1}-k_n}\Big(T^{k_n}x - \sum_{j=1}^{k_n} T^{k_n-j}x_{k_{n+1}-j}\Big) - \sum_{j=k_n+1}^{k_{n+1}-1} T^{k_{n+1}-j}x_{k_{n+1}-j} \Big\|^p \\
&\geqslant \frac{1}{2^{p-1}} m(T^{k_{n+1}-k_n})^p \Big\| T^{k_n}x - \sum_{j=1}^{k_n} T^{k_n-j}x_{k_{n+1}-j} \Big\|^p - \Big\| \sum_{j=k_n+1}^{k_{n+1}-1} T^{k_{n+1}-j}x_{k_{n+1}-j} \Big\|^p .
\end{aligned}$$

Using now the inequality

$$\Big\| \sum_{i=1}^{N} a_i \Big\|^p \leqslant N^{p-1}\Big(\sum_{i=1}^{N} \|a_i\|^p \Big),$$

we obtain

$$\begin{aligned}
&\|x_0\|^p + (k_{n+1}-k_n-1)^{p-1} \sum_{j=k_n+1}^{k_{n+1}-1} \|T^{k_{n+1}-j}\|^p \|x_{k_{n+1}-j}\|^p \\
&\qquad \geqslant \frac{1}{2^{p-1}} m(T^{k_{n+1}-k_n})^p \Big\| T^{k_n}x - \sum_{j=1}^{k_n} T^{k_n-j}x_{k_{n+1}-j} \Big\|^p .
\end{aligned}$$

Writing again

$$T^{k_n}x - \sum_{j=1}^{k_n} T^{k_n-j}x_{k_{n+1}-j}$$

as

$$T^{k_n-k_{n-1}}\Big(T^{k_{n-1}}x - \sum_{j=1}^{k_{n-1}} T^{k_{n-1}-j}x_{k_{n+1}-j}\Big) - \sum_{j=k_{n-1}+1}^{k_n} T^{k_n-j}x_{k_{n+1}-j}$$

and applying the same inequalities, we arrive after several steps at

$$\begin{aligned}
&\|x_0\|^p + (k_{n+1}-k_n-1)^{p-1} \sum_{j=k_n+1}^{k_{n+1}-1} \|T^{k_{n+1}-j}\|^p \|x_{k_{n+1}-j}\|^p \\
&\quad + \sum_{r=1}^{n} \Big(\frac{1}{2^{p-1}}\Big)^r m(T^{k_{n+1}-k_n})^p \cdots m(T^{k_{n-r+2}-k_{n-r+1}})^p (k_{n-r+1}-k_{n-r})^{p-1} \\
&\quad \times \sum_{j=k_{n-r}+1}^{k_{n-r+1}} \|T^{k_{n-r+1}-j}\|^p \|x_{k_{n+1}-j}\|^p \\
&\qquad \geqslant \Big(\frac{1}{2^{p-1}}\Big)^{n+1} m(T^{k_{n+1}-k_n})^p \cdots m(T^{k_2-k_1})^p m(T^{k_1})^p \|x\|^p .
\end{aligned}$$

This yields

$$\|x\|^p \leqslant \sum_{r=0}^{n} \frac{(2^{p-1})^{n+1-r}(k_{n-r+1}-k_{n-r})^{p-1}}{m(T^{k_{n-r+1}-k_{n-r}})^p \cdots m(T^{k_1})^p} \sum_{j=k_{n-r}+1}^{k_{n-r+1}} \|T^{k_{n-r+1}-j}\|^p \|x_{k_{n+1}-j}\|^p$$

$$\leqslant \sum_{r=0}^{n} \sum_{j=k_{n-r}+1}^{k_{n-r+1}} c_j^p \|x_{k_{n+1}-j}\|^p = \sum_{j=1}^{k_{n+1}} c_j^p \|x_{k_{n+1}-j}\|^p. \quad ▮$$

The above results imply the following generalization of Theorem 2.4.

THEOREM 3.5. *Let $p \geqslant 1$. Let T be an operator acting on a Banach space X. Let $(c_j)_{j=1}^{\infty}$ be a sequence of positive numbers satisfying condition $(*)_p$ for $T \in B(X)$. Then there exists a Banach space Y which is a $SQ_p(X)$-space, an isomorphic embedding $\pi : X \mapsto Y$ satisfying $\frac{\|x\|}{2^{(p-1)/p}} \leqslant \|\pi(x)\| \leqslant \|x\|$ $(x \in X)$ and an invertible operator $S \in B(Y)$ such that $S\pi = \pi T$, $\|S^{-j}\| \leqslant c_j$ $(j \geqslant 1)$ and $\|S^j\| \leqslant \|T^j\|$ $(j \geqslant 1)$. Moreover, S^{-1} is (c, p)-near the null operator modulo $\pi(X)$ and $\sigma(S) \subset \sigma(T)$.*

4. APPLICATIONS

The previous extension results give a general way of constructing invertible extensions of an operator with prescribed growth conditions. For an operator $T \in B(X)$ we write for short

$$v_n(T) = \max\{\|T^n\|, m(T^n)^{-1}\} \quad (n \geqslant 0).$$

We consider the following growth conditions for T:

(P(s)) (Polynomial growth condition) there are $C > 0$ and $s \geqslant 0$ such that $v_n(T) \leqslant Cn^s$ $(n \geqslant 1)$;

(B) (Beurling-type condition) $\sum\limits_{n=1}^{\infty} \frac{\log v_n(T)}{n^2} < \infty$;

(E(s)) (Exponential growth) there are $C > 0$ and $0 < s < 1$ such that $v_n(T) \leqslant Ce^{n^s}$ $(n \geqslant 1)$.

Note that condition (P(s)) implies (E(s')) (for any $s' > 0$), which implies (B). Also [20] if T satisfies (B) and T is invertible, then $\sigma(T) = \sigma_{\mathrm{ap}}(T) \subset \mathbb{T}$. If T satisfies (B) and $0 \in \sigma(T)$, then $\sigma_{\mathrm{ap}}(T) = \mathbb{T}$ and $\sigma(T) = \{z : |z| \leqslant 1\}$.

Other growth conditions can be also considered.

4.1. $\mathcal{E}(\mathbb{T})$-SUBSCALAR OPERATORS. We denote as usually by $\mathcal{E}(\mathbb{C})$ the Fréchet algebra of all C^{∞}-functions on $\mathbb{C}$ with the topology of uniform convergence of derivatives of all orders on compact subsets of $\mathbb{C}$. An operator $S \in B(X)$ is said [7] to be *generalized scalar* (or $\mathcal{E}(\mathbb{C})$-scalar) if there is a continuous algebra homomorphism $\Phi : \mathcal{E}(\mathbb{C}) \to B(X)$ for which $\Phi(1) = I$ and $\Phi(z) = S$. A bounded linear operator is $\mathcal{E}(\mathbb{C})$-*subscalar* if it is similar to the restriction of a $\mathcal{E}(\mathbb{C})$-scalar operator to one of its closed invariant subspaces. According to a result of J. Eschmeier and M. Putinar (see Section 6.4 of [10]), a Banach space operator T is $\mathcal{E}(\mathbb{C})$-subscalar if and only if T has property $(\beta)_{\mathcal{E}}$, i.e., for every open set $U \subset \mathbb{C}$, the operator T_U on $\mathcal{E}(U, X)$ (the space of C^∞-functions from U into X), defined by $T_U(f)(z) = (T - z)f(z)$, is injective and has closed range.

The following statements are equivalent (see [7]) :

(1) T is $\mathcal{E}(\mathbb{T})$-scalar (by definition, this means that T has a continuous functional calculus on the Fréchet algebra $\mathcal{E}(\mathbb{T}) = C^\infty(\mathbb{T})$ of smooth functions on the unit circle $\mathbb{T}$);

(2) T is generalized scalar with $\sigma(T) \subset \mathbb{T}$;

(3) T is invertible, and there exist constants $C > 0$ and $s \geqslant 0$ such that

$$\|T^n\| \leqslant C(1 + |n|)^s \quad (n \in \mathbb{Z}).$$

K.B. Laursen and M.M. Neumann ([17], Problem 6.1.15) and M. Didas [9] asked if $\mathcal{E}(\mathbb{T})$-subscalar operators are characterized by the polynomial growth condition $(\mathrm{P}(s))$ above. We refer to [9], [17], [20], [23], [22],[21] for several partial results. By [8] the hard implication holds for $s = 0$ and $C = 1$.

Since condition $(\mathrm{P}(s))$ implies that $\sigma_{\mathrm{ap}}(T) \subset \mathbb{T}$, it follows [24], [27] that T has an invertible extension S such that $\sigma(S) = \sigma_{\mathrm{ap}}(T) \subset \mathbb{T}$. By [28], if T acts on a Hilbert space, then S acts also on a Hilbert space. However, no control on the norms of inverses is guaranteed by this method.

The following result gives a complete positive answer.

THEOREM 4.1. (i) *An operator $T \in B(X)$ is $\mathcal{E}(\mathbb{T})$-subscalar if and only if there exist constants $C > 0$ and $s \geqslant 0$ such that*

$$\frac{1}{Cn^s}\|x\| \leqslant \|T^n x\| \leqslant Cn^s\|x\| \quad (x \in X,\ n \in \mathbb{N}). \tag{$\mathrm{P}(s)$}$$

Moreover, given $p \geqslant 1$, there exist a $SQ_p(X)$-space Y, an invertible $\mathcal{E}(\mathbb{T})$-scalar operator S on Y and a closed subspace $M \subset Y$ invariant with respect to S such that T is similar to the restriction $S_{|M}$. We also have $\sigma(S) = \sigma_{\mathrm{ap}}(T)$.

For $p = 1$ the operator S is an extension of T.

(ii) *If the Hilbert space operator $T \in B(H)$ verifies*

$$\frac{1}{Cn^s}\|h\| \leqslant \|T^n h\| \leqslant Cn^s\|h\| \quad (h \in H,\ n \in \mathbb{N}), \tag{$\mathrm{P}(s)$}$$

then there exists a Hilbert space K and a $\mathcal{E}(\mathbb{T})$-scalar extension $S \in B(K)$ with $\sigma(S) = \sigma_{\mathrm{ap}}(T)$.

Proof. (i) Suppose that T is similar to an operator having a $\mathcal{E}(\mathbb{T})$-scalar extension S. According to the above mentioned result, S is $\mathcal{E}(\mathbb{T})$-scalar if and only if S is invertible and $\|S^n\|$ is bounded by a constant times $(1 + |n|)^s$, for each $n \in \mathbb{Z}$. Therefore, restrictions of $\mathcal{E}(\mathbb{T})$-scalar operators satisfy the growth condition $(\mathrm{P}(s))$ from the theorem. Consequently, T satisfies $(\mathrm{P}(s))$.

Suppose now that T satisfies the growth condition $(\mathrm{P}(s))$. Let $C > 0$ and

$s \geqslant 0$ satisfy $v_n := v_n(T) \leqslant Cn^s$ $(n \geqslant 1)$. Let $\varepsilon > 0$. Then $\lim_{n\to\infty} \frac{v_n}{n^{s+\varepsilon/6}} = 0$. Choose $k_1 \geqslant e^4$ such that $v_n \leqslant n^{s+\varepsilon/6}$ for all $n \geqslant k_1$.

Let

$$K = \max\{2k_1\|T^j\| \cdot m(T^{k_1})^{-1} : 0 \leqslant j \leqslant k_1\}$$

and set $c_j = K(j+1)^{6s+3+\varepsilon}$. Clearly (c_j) is a submultiplicative sequence.

We show that (c_j) satisfies condition $(*)_\infty$ for T. Set $k_n = k_1^{2^{n-1}}$ $(n \geqslant 1)$.

For $j \leqslant k_1$ we have

$$2k_1 m(T^{k_1})^{-1} \cdot \|T^{k_1-j}\| \leqslant K \leqslant c_j.$$

Let $n \geqslant 1$ and $k_n < j \leqslant k_{n+1}$. Then $2^{n-1}\log k_1 \leqslant \log j$ and

$$\begin{aligned} 2^{n+1}(k_{n+1}-k_n)m(T^{k_1})^{-1}\cdots m(T^{k_{n+1}-k_n})^{-1}\|T^{k_{n+1}-j}\| \\ &\leqslant 2^{n+1}k_{n+1}(k_1k_2\cdots k_{n+1}k_{n+1})^{s+\varepsilon/6} \\ &\leqslant \Big(\frac{2^2}{\log k_1}\log j\Big)k_1^{2^n}(k_1k_1^2\cdots k_1^{2^n}k_1^{2^n})^{s+\varepsilon/6} \\ &\leqslant (\log j)(k_1^{2^{n-1}})^2(k_1^{3\cdot 2^n})^{s+\varepsilon/6} \\ &\leqslant j(k_1^{2^{n-1}})^{2+6s+\varepsilon} \leqslant j^{6s+\varepsilon+3} \leqslant c_j. \end{aligned}$$

Thus (c_j) satisfies condition $(*)_\infty$. If $p \geqslant 1$ is fixed, then (c_j) also satisfies condition $(*)_p$. By Theorem 3.5, there exists an invertible operator S on a $SQ_p(X)$-space Y extending T up to a similarity and satisfying $\|S^j\| = \|T^j\|$ and $\|S^{-j}\| \leqslant c_j$ for all $j \geqslant 1$. Clearly S has property $(\mathrm{P}(6s+\varepsilon+3))$. Moreover, S^{-1} is (c,p)-near the null operator modulo X.

For $p = 1$, the space X is isometrically embedded into Y, and so S is an extension of T.

Since $\sigma(S) \subset \mathbb{T}$, we have $\sigma_{\mathrm{ap}}(S) = \sigma(S)$. By the spectral radius formula we have $\sigma(T) \subset \{z : |z| \leqslant 1\}$. By [19],

$$\min\{|z| : z \in \sigma_{\mathrm{ap}}(T)\} = \lim_{n\to\infty} m(T^n)^{1/n} \geqslant 1.$$

Thus $\sigma_{\mathrm{ap}}(T) = \sigma(T) \cap \mathbb{T}$. By Theorem 3.1, $\sigma_{\mathrm{ap}}(T) \subset \sigma(S) \subset \sigma(T)$. Hence $\sigma(S) = \sigma_{\mathrm{ap}}(T)$.

(ii) Since (c_j) satisfies condition $(*)_2$ for T, it follows from Theorem 3.5 that there exists a Hilbert space K, an isomorphic embedding $\pi : H \mapsto K$ and an $\mathcal{E}(\mathbb{T})$-scalar operator $S \in B(K)$ satisfying $S\pi = \pi T$. We can introduce a new equivalent Hilbert space norm on K such that π becomes an isometry. Indeed, let P be the orthogonal projection onto $\pi(H)$. Define the new norm on K by

$$|||u||| = (\|\pi^{-1}Pu\|_H^2 + \|(I-P)u\|_K^2)^{1/2} \quad (u \in K).$$

We have $|||\pi(x)||| = \|x\|_H$ for all $x \in H$. Then S, acting on the Hilbert space $(K, |||\cdot|||)$, is the required $\mathcal{E}(\mathbb{T})$-scalar extension of T. ∎

REMARK 4.2. Let H be the Hilbert space with an orthonormal basis (e_n) $(n = 0, 1, \dots)$. It is easy to see that the Bergman shift on H, given by

$$Be_n = \sqrt{\frac{n+1}{n+2}}e_{n+1},$$

satisfies the polynomial growth condition $(\mathrm{P}(1/2))$. Therefore, the Bergman shift has a generalized scalar extension with spectrum the unit circle. This has to be compared to the known fact that B is subnormal, with minimal normal extension (the multiplication by the variable z on $L^2(\mathbb{D}, \mu)$, where μ is the Lebesgue measure in $\mathbb{D}$) having as spectrum the closed unit disk $\overline{\mathbb{D}}$.

PROBLEM 4.3. *Let $s \geqslant 0$. What is the optimal value of $s' = f(s)$ such that every $T \in B(X)$ satisfying $(\mathrm{P}(s))$ has an invertible extension satisfying $(\mathrm{P}(s'))$? What is the optimal value of $s' = g(s)$ such that every $T \in B(H)$ satisfying $(\mathrm{P}(s))$ has an invertible Hilbert space extension satisfying $(\mathrm{P}(s'))$?*

The proof of Theorem 4.1 can be modified to give, for fixed $\varepsilon > 0$ and $T \in B(X)$, a Banach space Y and an extension (with an isometric embedding) $S \in B(Y)$ satisfying condition $(\mathrm{P}(6s + \varepsilon))$. Indeed, with k_1 as in the proof of Theorem 4.1, let

$$K = \max\{\|T^j\| \cdot m(T^{k_1})^{-1} : 0 \leqslant j \leqslant k_1\}$$

and set $c_j = K(j+1)^{6s+\varepsilon}$. Then a similar proof shows that the sequence (c_j) satisfies condition $(*)_1$ for $k_n = k_1^{2^{n-1}}$ $(n \geqslant 1)$.

We also notice that $g(0) = 0$. Indeed, if a Hilbert space operator $T \in B(H)$ satisfies $(\mathrm{P}(0))$, then by [30] there exists an invertible operator $L \in B(H)$ such that $V = L^{-1}TL$ is an isometry. Let U be a unitary extension of V on a larger Hilbert space $K = H \oplus H^{\perp}$. Then $(L \oplus I)U(L \oplus I)^{-1}$ is an extension of T satisfying $(\mathrm{P}(0))$.

We can consider representations of $\mathbb{N}^n$ to deal with $\mathcal{E}(\mathbb{T}^n)$-subscalar operators. The proof of the following result follows a different approach.

THEOREM 4.4. *An n-tuple of commuting Banach space operators is $\mathcal{E}(\mathbb{T}^n)$-subscalar if and only if each of the n operators is $\mathcal{E}(\mathbb{T})$-subscalar.*

Proof. The previous characterization of $\mathcal{E}(\mathbb{T})$-subscalar operators implies that if $T_1, \dots, T_n$ are commuting $\mathcal{E}(\mathbb{T})$-subscalar operators, then the product operator $T_1 \cdots T_n$ is also $\mathcal{E}(\mathbb{T})$-subscalar. The result follows from Theorem 2.2.7 in [9]. ∎

4.2. OPERATORS WITH BISHOP'S PROPERTY (β). Recall that an equivalent definition of decomposable operators is the following : $T \in B(X)$ is *decomposable* if, for every open cover $\mathbb{C} = U \cup V$, there are closed invariant (for T) subspaces Y and Z of X such that $X = Y + Z$ and $\sigma(T \mid Y) \subset U$, $\sigma(T \mid Z) \subset V$. We refer for instance to [7] and [17]. An operator $T \in B(X)$ has *Bishop's property* (β) if, for every open set $U \subset \mathbb{C}$, the operator T_U defined by $T_U(f)(z) = (T - z)f(z)$ on the set $\mathcal{O}(U, X)$ of holomorphic functions from U into X is injective and has closed range. According to a result by E. Albrecht and J. Eschmeier (see [17], [10]), $T \in B(X)$ is *subdecomposable* (i.e., T is similar to the restriction of a decomposable operator) if and only if T has Bishop's property (β).

It was proved in Theorem 5.3.2 of [7] that an invertible operator $S \in B(X)$ is decomposable provided that

$$\sum_{n=-\infty}^{\infty} \frac{\log \|S^n\|}{1+n^2} < \infty.$$

The following result answers in the affirmative a question from [20].

THEOREM 4.5. *Let $T \in B(X)$ be a Banach space operator such that*

$$\sum_{n=1}^{\infty} \frac{\log \max(\|T^n\|, m(T^n)^{-1})}{n^2} < \infty.$$

Then there exists a Banach space $Y \supset X$ and an invertible operator $S \in B(Y)$ such that $T = S_{|X}$ and S satisfies

$$\sum_{n=-\infty}^{\infty} \frac{\log \|S^n\|}{1+n^2} < \infty.$$

In particular, T has Bishop's property (β). Moreover, $\sigma(S) = \sigma_{\mathrm{ap}}(T) = \sigma(T) \cap \mathbb{T}$.

If $X = H$ is a Hilbert space, then $Y = K$ can be chosen to be a Hilbert space too.

Proof. Let $T \in B(X)$ satisfy (B). By Theorem 3.5, it is sufficient to show the existence of a submultiplicative sequence (d_n) satisfying $\sum\limits_{n=1}^{\infty} \frac{\log d_n}{n^2} < \infty$ and the condition $(*)_\infty$ for T.

Write $r_n = v_{2^n}$ $(n \geqslant 0)$. Clearly $r_{n+1} \leqslant r_n^2$ for all n.

Claim 1. $\sum\limits_{n=0}^{\infty} \frac{\log r_n}{2^n} < \infty$.

Proof. Fix $n \geqslant 2$. For $1 \leqslant j \leqslant 2^{n-3}$ we have

$$v_{2^n} \leqslant v_{2^{n-1}+j} \cdot v_{2^{n-1}-j}.$$

Thus $\log r_n \leqslant \log v_{2^{n-1}+j} + \log v_{2^{n-1}-j}$ and

$$\frac{\log r_n}{2^{2n}} \leqslant \frac{\log v_{2^{n-1}+j}}{2^{2n}} + \frac{\log v_{2^{n-1}-j}}{2^{2n}} \leqslant \frac{\log v_{2^{n-1}+j}}{(2^{n-1}+j)^2} + \frac{\log v_{2^{n-1}-j}}{(2^{n-1}-j)^2}.$$

Hence

$$2^{n-3} \cdot \frac{\log r_n}{2^{2n}} \leqslant \sum_{j=1}^{2^{n-3}} \Big(\frac{\log v_{2^{n-1}+j}}{(2^{n-1}+j)^2} + \frac{\log v_{2^{n-1}-j}}{(2^{n-1}-j)^2} \Big)$$

and

$$\frac{1}{8} \sum_{n=2}^{\infty} \frac{\log r_n}{2^n} \leqslant \sum_{j=1}^{\infty} \frac{\log v_j}{j^2} < \infty. \quad ∎$$

Let n be a non-negative integer and let $n = \sum_{j=0}^{\infty} \alpha_j 2^j$, where $\alpha_j \in \{0,1\}$, be its binary representation. Define

$$b_n = \prod_{j=0}^{\infty} r_j^{\alpha_j}, \quad c_n = \max\{b_j^2 : n \leqslant j \leqslant 2n\} \quad \text{and} \quad d_n = 4n^2 c_n.$$

Claim 2. (b_n) is submultiplicative, i.e., $b_{n+m} \leqslant b_n b_m$ for all $m, n \geqslant 0$.

Proof. Let $n = \sum_{j=0}^{\infty} \alpha_j 2^j$ and $m = \sum_{j=0}^{\infty} \beta_j 2^j$ be the binary representations of n and m, respectively.

By induction on j_0, we prove the following statement:

> There are numbers γ_j $(0 \leqslant j)$ such that $n + m = \sum_{j=0}^{\infty} \gamma_j 2^j$, $\gamma_j \in \{0,1\}$ $(j < j_0)$, $\gamma_{j_0} \in \{0,1,2,3\}$, $\gamma_j \in \{0,1,2\}$ $(j > j_0)$ and $b_n b_m \geqslant \prod_{j=0}^{\infty} r_j^{\gamma_j}$.

For $j_0 = 0$ the statement is clear for the numbers $\gamma_j = \alpha_j + \beta_j$.

Suppose that the statement is true for some j_0. We show it for $j_0 + 1$. If $\gamma_{j_0} \leqslant 1$ then the statement is clear. Let $\gamma_{j_0} \in \{2,3\}$. Then

$$n + m = \sum_{j=0}^{\infty} \gamma_j' 2^j,$$

where $\gamma_j' = \gamma_j$ $(j \neq j_0, j_0 + 1)$, $\gamma_{j_0}' = \gamma_{j_0} - 2$ and $\gamma_{j_0+1}' = \gamma_{j_0+1} + 1$. Then

$$b_n b_m \geqslant \prod_{j=0}^{\infty} r_j^{\gamma_j} \geqslant \prod_{j=0}^{\infty} r_j^{\gamma_j'}.$$

The statement for $j_0 > \log_2(n+m)$ gives the inequality $b_n b_m \geqslant b_{n+m}$. ∎

Claim 3. (d_n) is submultiplicative.

Proof. Notice that $16m^2n^2 \geqslant 4(m+n)^2$ for all positive integers m and n. We have

$$\begin{aligned} d_n d_m &\geqslant 4(m+n)^2 \max\{b_i^2 b_j^2 : n \leqslant i \leqslant 2n, m \leqslant j \leqslant 2m\} \\ &\geqslant 4(m+n)^2 \max\{b_l^2 : n + m \leqslant l \leqslant 2(n+m)\} = d_{n+m}. \end{aligned}$$ ∎

Claim 4. $\sum_{n=1}^{\infty} \frac{\log d_n}{n^2} < \infty$.

Proof. It is sufficient to show the analogue claim for the sequence (c_n).

For $2^j \leqslant n < 2^{j+1}$ we have $c_n = b_i^2$ for some i, $i \leqslant 2n < 2^{j+2}$. So $c_n \leqslant b^2_{2^{j+2}-1} = \prod_{i=0}^{j+1} r_i^2$. Thus

$$\sum_{n=2}^{\infty} \frac{\log c_n}{n^2} \leqslant \sum_{j=1}^{\infty} \frac{2^j \sum_{i=0}^{j+1} 2\log r_i}{2^{2j}} \leqslant 2\sum_{i=0}^{\infty} \log r_i \cdot \sum_{j=i-1}^{\infty} 2^{-j} \leqslant 8 \sum_{i=0}^{\infty} \frac{\log r_i}{2^i} < \infty. \quad ∎$$

Claim 5. (d_n) satisfies condition $(*)_\infty$ for T.

Proof. Set $k_n = 2^n - 1$. For $k_n < j \leqslant k_{n+1}$ we have $2^n \leqslant j < 2^{n+1}$, and so $c_j \geqslant \prod_{i=0}^{n} r_i^2$. Hence

$$\begin{aligned} 2^{n+1}(k_{n+1}-k_n)(m(T^{k_1})m(T^{k_2-k_1})\cdots m(T^{k_{n+1}-k_n}))^{-1}\|T^{k_{n+1}-j}\| & \\ \leqslant 2(2^n)^2 r_0 r_1 \cdots r_n \cdot b_{k_{n+1}-j} & \\ \leqslant 2j^2 \prod_{i=0}^{n} r_i^2 & \\ \leqslant d_j. & \end{aligned}$$

The inequality for $j = 1 = k_1$ is clear. ∎

Thus (d_n) also satisfies condition $(*)_1$, and so there is an invertible extension S of T such that $\|S^{-n}\| \leqslant d_n$ $(n > 0)$. Hence S is decomposable.

The equalities $\sigma(S) = \sigma_{\mathrm{ap}}(T) = \sigma(T) \cap \mathbb{T}$ can be shown as in Theorem 4.1.

If $X = H$ is a Hilbert space, then the sequence (d_n) satisfies condition $(*)_2$. By Theorem 3.5, there is a Hilbert space K, an invertible operator $S \in B(K)$ and an isomorphic embedding $\pi : H \to K$ with $\pi T = S\pi$ and $\|S^{-n}\| \leqslant d_n$ $(n > 0)$. As in the proof of Theorem 4.1, K can be given a new equivalent Hilbertian norm such that π becomes an isometry. ∎

4.3. CONDITION $(\mathrm{E}(s))$. The following consequence of Theorem 4.5 implies that condition (b) from Theorem 3.2 in [22] is superfluous.

COROLLARY 4.6. *Let $T \in B(X)$ satisfying the exponential condition $(\mathrm{E}(s))$, that is, there are $C > 0$ and $0 < s < 1$ such that $v_n(T) \leqslant C\mathrm{e}^{n^s}$ $(n \geqslant 0)$. Then T has property (β).*

The following result answers an open question from [20].

THEOREM 4.7. *Let $T \in B(X)$ satisfy (E). Then there exist a Banach space $Y \supset X$ and an invertible operator S on a larger space such that T is a restriction of S and S satisfies $(\mathrm{E}(s'))$ for suitable $s' < 1$. The construction is Hilbertian.*

Proof. Let ε be an arbitrary positive number. Set $k_n = 2^n$ $(n \geqslant 1)$. It is now a matter of routine to verify that the sequence $c_j = K \cdot \mathrm{e}^{j^{s+\varepsilon}}$ satisfies condition $(*)_\infty$ for T, where K is a suitable constant. Thus T can be extended to an invertible operator satisfying condition $(\mathrm{E}(s+\varepsilon))$. The construction is Hilbertian in the sense that if $X = H$ is Hilbert, then $Y = K$ can be chosen a Hilbert space too. We omit the details. ∎

4.4. A HILBERTIAN COUNTERPART OF ARENS' RESULT. We obtain the following Hilbertian counterpart of Arens' result.

COROLLARY 4.8. *Let $T \in B(H)$ be an operator on Hilbert space with $m(T) > 0$. Then there exist a Hilbert space K, an isometric embedding $\pi : H \mapsto K$ and an invertible operator $S \in B(K)$ such that $S\pi = \pi T$, $\|S^j\| \leqslant \|T^j\|$ $(j \geqslant 1)$, $\|S^{-1}\| \leqslant \frac{2}{m(T)}$ and*

$$\Big\| \sum_{j=0}^{N} S^{-j}\pi(x_j) \Big\|^2 \leqslant 2 \sum_{j=0}^{N} \Big(\frac{\sqrt{2}}{m(T)} \Big)^{2j} \|x_j\|^2$$

for every $N \in \mathbb{N}$ and all $x_j \in H$.

Proof. Let $c_j = \Big(\frac{\sqrt{2}}{m(T)} \Big)^j$, $j \geqslant 1$. Then the sequence (c_j) satisfies the condition $(*)_2$ for T (take $k_n = n$). It follows from the proof of Theorem 3.5 that there exist a Hilbert space K, an isomorphic embedding $\pi : H \mapsto K$ satisfying $\frac{1}{\sqrt{2}}\|x\| \leqslant \|\pi(x)\| \leqslant \|x\|$ for any $x \in H$, and an invertible operator $S \in B(K)$ such that $S\pi = \pi T$, $\|S^j\| \leqslant \|T^j\|$ $(j \geqslant 1)$, $\|S^{-1}\| \leqslant \frac{\sqrt{2}}{m(T)}$ and

$$\Big\| \sum_{j=0}^{N} S^{-j}\pi(x_j) \Big\|^2 \leqslant \sum_{j=0}^{N} \Big(\frac{\sqrt{2}}{m(T)} \Big)^{2j} \|x_j\|^2$$

for every $N \in \mathbb{N}$ and all $x_j \in H$. We now introduce a new equivalent Hilbert space norm on K such that π becomes an isometry as in the proof of Theorem 4.1. So let P be the orthogonal projection onto πH and define the new norm on K by

$$|||x||| = (\|\pi^{-1}Px\|_H^2 + \|(I-P)x\|_K^2)^{1/2}.$$

Then $|||x|||^2 \leqslant 2\|Px\|_K^2 + \|(I-P)x\|_K^2 \leqslant 2\|x\|_K^2$. In the same way a lower bound can be obtained; we get $\|x\| \leqslant |||x||| \leqslant \sqrt{2}\|x\|$ for every $x \in K$. Then S, acting on the Hilbert space $(K, |||\cdot|||)$, verifies the required inequalities. ∎

4.5. OPERATORS WITH COUNTABLE SPECTRUM. In the following two results we assume that the spectrum of T is countable. We refer to [5], [4], [15] and their references for related results.

THEOREM 4.9. *Let $T \in B(X)$ be a Banach space operator. Suppose that there are positive constants $M > 0$, $C > 0$ and $0 < s < \frac{1}{2}$ such that*

$$\frac{1}{Ce^{n^s}}\|x\| \leqslant \|T^n x\| \leqslant M\|x\|$$

for every $x \in X$ and $n \in \mathbb{N}$. Suppose also that the spectrum $\sigma(T)$ of T is countable. Then

$$\text{(P(0))} \qquad \frac{1}{M}\|x\| \leqslant \|T^n x\| \leqslant M\|x\|$$

for every $x \in X$. In particular, T is $\mathcal{E}(\mathbb{T})$-subscalar.

Proof. We have $\|T^n\| \leqslant M$ and $m(T^n))^{-1} \leqslant Ce^{n^s}$. Let $\varepsilon > 0$ be a positive number such that $s + \varepsilon < \frac{1}{2}$. Using (the proofs of) Theorems 4.7 and 2.4 (or 3.5), there exists a constant $K > 0$ such that T has an invertible extension S on a Banach space Y verifying $\|S^n\| \leqslant M$ and $\|S^{-n}\| \leqslant K \exp(n^{s+\varepsilon})$ for all $n \in \mathbb{N}$. Moreover, it is possible to have an extension satisfying $\sigma(S) \subset \sigma(T)$. We obtain in particular that

$$\lim_{n\to\infty} \frac{\log \|S^{-n}\|}{\sqrt{n}} = 0$$

and that the spectrum $\sigma(S)$ of S is countable. From Remarque 2, p. 259 of [32] we obtain $\|S^p\| \leqslant M$ for all $p \in \mathbb{Z}$. This yields $m(T^n)^{-1} \leqslant M$ for $n \geqslant 1$ and the stated inequality $(\mathrm{P}(0))$. ∎

We obtain the following consequence in the case of Hilbert space operators.

COROLLARY 4.10. *Let $T \in B(H)$ be a power bounded operator on a Hilbert space H. Suppose that there are positive constants C and $s < \frac{1}{2}$ such that*

$$m(T^n)^{-1} \leqslant Ce^{n^s} \quad (n \geqslant 1)$$

and that $\sigma(T)$ is countable. Then T is similar to a unitary operator.

Proof. By the previous theorem, the operator T satisfies $(\mathrm{P}(0))$ on H, a condition which characterizes Hilbert space operators similar to isometries [30]. As the spectrum is a similarity invariant, T is similar to an isometry with a countable spectrum. Since the spectrum of a non-invertible isometry is the entire closed unit disk, we obtain that T is similar to a unitary operator. ∎

4.6. CONTRACTIONS WITH SPECTRUM A CARLESON SET. Recall that a closed set E of $\mathbb{T}$ is said to be a *Carleson set* if

$$\int_0^{2\pi} \log\Big(\frac{2}{\mathrm{dist}\,(\mathrm{e}^{\mathrm{i}t}, E)}\Big)\,\mathrm{d}t < +\infty.$$

THEOREM 4.11. *Let $T \in B(H)$ be a Hilbert space contraction such that $\sigma_{\mathrm{ap}}(T) \subset \mathbb{T}$ is a Carleson set. Suppose that there exist $C > 0$ and $s \geqslant 0$ such that $m(T^n)^{-1} \leqslant Cn^s$. Then T is an isometry.*

Proof. Using Theorem 4.1, (ii), there exist $K > 0$, $s' \geqslant 0$, a Hilbert space K and an invertible operator $S \in B(K)$ which is an extension of T such that $\|S\| \leqslant 1$, $\|S^{-n}\| \leqslant Kn^{s'}$ and $\sigma(S) = \sigma_{\mathrm{ap}}(T)$. We obtain in particular that $\sigma(S) = \sigma_{\mathrm{ap}}(T)$ is a Carleson set. By a theorem of Esterle [11] (see also [14]), S is unitary. Therefore its restriction T is an isometry. ∎

Several results for unitaries (or operators similar to unitaries) can be transferred to results for isometries (or operators similar to isometries) in an analogous manner.

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