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A FINITENESS RESULT FOR COMMUTING SQUARES OF MATRIX ALGEBRAS

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ABSTRACT. We consider a condition for non-degenerate commuting squares of matrix algebras (finite dimensional von Neumann algebras) called the *span condition*, which in the case of the n-dimensional standard spin models is shown to be satisfied if and only if n is prime. We prove that the commuting squares satisfying the span condition are isolated among all commuting squares (modulo isomorphisms). In particular, they are finitely many for any fixed dimension. Also, we give a conceptual proof of previous constructions of certain one-parameter families of complex Hadamard matrices.

KEYWORDS: *Commuting squares, complex Hadamard matrices, subfactors.*

MSC (2000): 46L10, 46L37, 15A57, 05B20.

INTRODUCTION

In this paper we prove some finiteness results for commuting squares of matrix algebras, i.e. finite dimensional von Neumann algebras. Commuting squares were introduced in [9], as invariants and construction data in Jones' theory of subfactors. They encode the generalized symmetries of the subfactor, in a lot of situations being complete invariants [9],[10]. In particular, all finite groups and finite dimensional C^*-Hopf algebras can be encoded in commuting squares.

One of the simplest examples of commuting squares is

$$\mathfrak{C} = \begin{pmatrix} D & \subset & M_n(\mathbb{C}) \\ \cup & & \cup & ,\tau \\ \mathbb{C} & \subset & U^*DU \end{pmatrix}$$

where D is the algebra of diagonal matrices, $U = \big(\frac{1}{\sqrt{n}}\varepsilon^{(i-1)(j-1)}\big)_{i,j}$ with $\varepsilon = \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}$, so U^*DU is the algebra of circulant permutation matrices [11].

We call U the *standard biunitary matrix* of order n. More generally, one can ask for what unitaries U is $\mathfrak{C}$ a commuting square. The commuting square condition asks that D, U^*DU be orthogonal modulo $\mathbb{C}$, which is equivalent to U having all entries of the same absolute value $\frac{1}{\sqrt{n}}$. Such a matrix is called a *biunitary matrix* or *complex Hadamard matrix*.

In [8] Petrescu showed that, for n positive integer, the standard biunitary of order n is isolated among all normalized biunitary matrices of order n if and only if n is prime.

We introduce a condition for arbitrary non-degenerate commuting squares, which we call *the span condition*, and prove that it is sufficient to ensure isolation. We show that when the commuting square is given by the standard biunitary of order n the span condition is satisfied if and only if n is prime. Thus our result generalizes Petrescu's finiteness theorem and the span condition can be regarded as a *primeness condition*.

We also show how one can use our theorem to check if a given biunitary is isolated. As an application we show that all circulant biunitaries of order 7 (computed in [3]) are isolated among all biunitary matrices.

Conversely, we find sufficient conditions for the span condition to fail and prove that if these conditions are satisfied then there exists a continuum of non-isomorphic commuting squares.

It is not known if for every $n > 5$ prime there exists a one-parameter family of (different) normalized biunitary matrices. For $n = 7, 13, 19, 31$ Petrescu found such examples, using a computer; we find a conceptual explanation for these examples. A main point of interest of Petrescu's result is that it might produce examples of one-parameter families of non-isomorphic subfactors of same index n and same graph, conjectured to be A_∞.

1. PRELIMINARIES AND A TECHNICAL RESULT

We recall the following definition from [10] (see also [11], [9]):

DEFINITION 1.1. A *commuting square* of matrix algebras is a square of inclusions:

$$\begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup \\ Q_{-1} & \subset & Q_0 \end{pmatrix}, \tau$$

with P_0, P_{-1}, Q_0, Q_{-1} finite dimensional von Neumann algebras (i.e. algebras of the form $\bigoplus_i \mathbb{M}_{n_i}(\mathbb{C})$, or equivalently $*$-subalgebras of $\mathbb{M}_n(\mathbb{C})$ for some $n \geqslant 1$) and τ a faithful positive trace on P_0, $\tau(1) = 1$, satisfying the condition:

$$E_{P_{-1}} E_{Q_0} = E_{Q_{-1}} \tag{1.1}$$

where $E_A = E_A^{P_0}$ denotes the τ-invariant conditional expectation of P_0 onto the subalgebra $A \subset P_0$. We say that the commuting square is *non-degenerate* if $P_0 = \mathrm{span} P_{-1} Q_0$.

The following definition is from [2]:

DEFINITION 1.2. Let A be a finite dimensional von Neumann algebra with identity I and normalized trace τ. Denote $\mathcal{S}(A)$ the set of all $*$-subalgebras of A containing I. For $B_1, B_2 \in \mathcal{S}(A)$ and $\delta > 0$ we say that B_1 is *δ-contained* in B_2 if for every element $x \in B_1$ of $\|x\| = 1$ there exists $y \in B_2$ such that $\|x - y\|_2 < \delta$. If B_1 is δ-contained in B_2 and B_2 is δ-contained in B_1 we write $\|B_1 - B_2\|_{2,A} < \delta$.

REMARK 1.3. Arguments from [2] show that there exists a continuous increasing function $f : [0,\infty) \to [0,\infty), f(0) = 0$, such that if δ is small and $\|B_1 - B_2\|_{2,A} < \delta$, then $B_2 = \mathrm{Ad}(U)(B_1)$ for some unitary element $U \in A$, $\|U - I\|_2 < f(\delta)$ (where $\|x\|_2 = \tau(x^*x)^{1/2}$).

DEFINITION 1.4. We say that the commuting square

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ Q_{-1} & \subset & Q_0 & \end{pmatrix}$$

is *isomorphic* to the commuting square

$$\widetilde{\mathfrak{C}} = \begin{pmatrix} \widetilde{P}_{-1} & \subset & \widetilde{P}_0 & \\ \cup & & \cup & ,\widetilde{\tau} \\ \widetilde{Q}_{-1} & \subset & \widetilde{Q}_0 & \end{pmatrix}$$

with trace $\widetilde{\tau}$, if there exists a trace-invariant $*$-isomorphism $\phi : P_0 \to \widetilde{P}_0$ such that $\phi(P_{-1}) = \widetilde{P}_{-1}, \phi(Q_{-1}) = \widetilde{Q}_{-1}, \phi(Q_0) = \widetilde{Q}_0$.

DEFINITION 1.5. We say that the commuting square of matrix algebras

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ Q_{-1} & \subset & Q_0 & \end{pmatrix}$$

is *isolated* if there exists $\delta > 0$ such that if

$$\widetilde{\mathfrak{C}} = \begin{pmatrix} \widetilde{P}_{-1} & \subset & \widetilde{P}_0 & \\ \cup & & \cup & ,\widetilde{\tau} \\ \widetilde{Q}_{-1} & \subset & \widetilde{Q}_0 & \end{pmatrix}$$

is a commuting square and $\phi : P_0 \to \widetilde{P}_0$ a trace-invariant $*$-isomorphism satisfying

$$\|\phi(P_{-1}) - \widetilde{P}_{-1}\|_{2,\widetilde{P}_0} < \delta, \quad \|\phi(Q_{-1}) - \widetilde{Q}_{-1}\|_{2,\widetilde{P}_0} < \delta, \quad \|\phi(Q_0) - \widetilde{Q}_0\|_{2,\widetilde{P}_0} < \delta$$

then $\widetilde{\mathfrak{C}}$ is isomorphic to $\mathfrak{C}$.

For algebras $B \subset A$ we will use the notation: $B' \cap A = \{a \in A \text{ such that } ab = ba, \forall b \in B\}$.

LEMMA 1.6. *Let P_0, P_{-1}, Q_0, Q_{-1} be finite dimensional von Neumann algebras, and U a unitary element of P_0 such that*

$$\mathfrak{C}(U) = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ Q_{-1} & \subset & U^*Q_0U & \end{pmatrix}$$

is a commuting square. Let $q \in Q_0, q' \in Q_0^{'} \cap P_{-1}, p \in Q'_{-1} \cap P_{-1}, p' \in P_{-1}^{'} \cap P_0$ be unitary elements. Then $\mathfrak{C}(qq^{'}Upp^{'})$ is a commuting square isomorphic to $\mathfrak{C}(U)$.

Proof. Modifying U to the left by q, q' does not change the algebra U^*Q_0U and thus does not change the commuting square: $\mathfrak{C}(qq^{'}Upp^{'}) = \mathfrak{C}(Upp^{'})$. By applying $\mathrm{Ad}(pp')$ to $\mathfrak{C}(Upp^{'})$ (which leaves P_0, P_{-1}, Q_{-1} invariant) we see that $\mathfrak{C}(Upp^{'})$ is isomorphic to $\mathfrak{C}$. ∎

To check in practical situations if a certain commuting square is isolated, we need the following lemma:

LEMMA 1.7. *Let*

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ Q_{-1} & \subset & Q_0 & \end{pmatrix}$$

be a commuting square of finite dimensional von Neumann algebras, with trace τ. Then $\mathfrak{C}$ is isolated if and only if there exists $\varepsilon > 0$ such that if $U \in Q'_{-1} \cap P_0$ is a unitary, $\|U - I\|_2 < \varepsilon$, and

$$\mathfrak{C}(U) = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ Q_{-1} & \subset & U^*Q_0U & \end{pmatrix}$$

is a commuting square; then $\mathfrak{C}(U)$ is isomorphic to $\mathfrak{C}$.

Proof. We only have to show the implication from right to left. Assume $\mathfrak{C}$ is isolated among commuting squares of the form $\mathfrak{C}(U), U \in Q'_{-1} \cap P_0, \|U - I\|_2 < \varepsilon$ and let $\delta > 0$ be such that $f(2f(\delta)) + f(\delta) + f(f(\delta) + f(2f(\delta))) < \varepsilon$, where f is as in Remark 1.3. We show that $\mathfrak{C}, \delta$ satisfy the definition of isolation in Definition 1.5. Assume $\widetilde{\mathfrak{C}}$ is δ-close to $\mathfrak{C}$ as in Definition 1.5. For δ small the inclusions $\widetilde{Q}_{-1} \subset \widetilde{P}_{-1} \subset \widetilde{P}_0$ and $\phi(Q_{-1}) \subset \phi(P_{-1}) \subset \widetilde{P}_0$ are unitary conjugate. Because our definition of isolation is invariant to isomorphisms of commuting squares, it follows that to check if $\mathfrak{C}$ is isolated it is enough to check isolation among commuting squares of the form:

$$\begin{matrix} P_{-1} & \subset & P_0 \\ \cup & & \cup \\ Q_{-1} & \subset & \widetilde{Q}_0 \end{matrix}.$$

If $\|\widetilde{Q}_0 - Q_0\|_{2,P_0} < \delta$ then by Remark 1.3 we have $\widetilde{Q}_0 = U^*Q_0U$, for some unitary $U \in P_0$, $\|U - I\|_2 < f(\delta)$.

Since $UQ_{-1}U^* \subset Q_0$ and $\|UQ_{-1}U^* - Q_{-1}\|_{2,Q_0} < 2f(\delta)$, Remark 1.3 implies the existence of a unitary $r_1 \in Q_0$, $\|r_1 - I\|_2 < f(2f(\delta))$, such that $UQ_{-1}U^* = r_1 Q_{-1} r_1^*$. So $\mathrm{Ad}(r_1^* U)$ is an isomorphism of Q_{-1} $f(\delta) + f(2f(\delta))$-close to identity, therefore: $\mathrm{Ad}(r_1^* U)_{|Q_{-1}} = \mathrm{Ad}(r_2)$, for some $r_2 \in Q_{-1}$, $\|r_2 - I\|_2 < f(f(\delta) + f(2f(\delta)))$. Thus, by changing U to $r_1^* U r_2^*$ (which does not change the isomorphism class of the commuting square), we may assume that $U \in Q'_{-1} \cap P_0$ and, since $\varepsilon > f(2f(\delta)) + f(\delta) + f(f(\delta) + f(2f(\delta)))$, we obtain $\widetilde{\mathfrak{C}}$ isomorphic to $\mathfrak{C}$. ∎

According to Lemma 1.7, if a commuting square $\mathfrak{C}$ is not isolated then there exists a sequence of unitaries $U_n \to I$ such that $\mathfrak{C}(U_n)$ are non-isomorphic to $\mathfrak{C}, \forall n \geqslant 1$. In our main theorem we prove that commuting squares satisfying a certain *span condition* are isolated. To do this, we contradict isolation by assuming the existence of such U_n, then we write the commuting square relations for each n and take the "derivative" of this relations along some "direction of convergence" of U_n. We start by giving a clear meaning to the notion of "direction of convergence".

Let P_0 be a finite dimensional von Neumann algebra and let $U_n = \exp(\mathrm{i}h_n)$, $n \geqslant 1$, for some $h_n \in P_0$ hermitian non-zero elements converging to 0. Because of the compactness of the unit ball in the finite dimensional algebra P_0 we may assume, after eventually passing to a subsequence, that $\frac{h_n}{\|h_n\|} \to h \in P_0$, $\|h\| = 1$. We will refer to h as a *direction of convergence* of $(U_n)_n$.

Since $\frac{U_n - I}{\mathrm{i}\|h_n\|} \to h$ as $n \to \infty$, it follows $\frac{\|U_n - I\|}{\|h_n\|} \to \|h\| = 1$ so:

$$h = \lim_{n\to\infty} \frac{U_n - I}{\mathrm{i}\|U_n - I\|}. \tag{1.2}$$

The following technical lemma is essential for the proof of the main theorem. It gives a normalization of h, obtained by modifying U_n as in Lemma 1.6.

LEMMA 1.8. *With the notations of Lemma* 1.6, *assume that* $U_n \in Q'_{-1} \cap P_0, n \geqslant 1$ *are unitary elements converging to* I *such that* $\mathfrak{C}(U_n)$ *are commuting squares non-isomorphic to* $\mathfrak{C}$.

Then, after replacing $(U_n)_n$ *with one of its subsequences, there exist unitaries* $q_n \in Q'_{-1} \cap Q_0, q'_n \in Q'_0 \cap P_{-1}, p_n \in Q'_{-1} \cap P_{-1}, p'_n \in P'_{-1} \cap P_0$ *such that:*

$$\widetilde{U}_n = q_n {q_n}' U_n p'_n p_n \to I, \quad \lim_{n\to\infty} \frac{\widetilde{U}_n - I}{\mathrm{i}\|\widetilde{U}_n - I\|} = \widetilde{h} \in P_0,$$

and

$$E_{P'_{-1}\cap P_0}(\widetilde{h}) = E_{Q'_0\cap P_0}(\widetilde{h}) = E_{Q'_{-1}\cap P_{-1}}(\widetilde{h}) = E_{Q'_{-1}\cap Q_0}(\widetilde{h}) = 0, \quad [\widetilde{h}, Q_{-1}] = 0.$$

Proof. Let $\mathfrak{X} = \mathfrak{U}(Q'_{-1} \cap Q_0) \times \mathfrak{U}(Q'_0 \cap P_0) \times \mathfrak{U}(P'_{-1} \cap P_0) \times \mathfrak{U}(Q'_{-1} \cap P_{-1})$ be the set of quadruples of unitaries in the four algebras. $\mathfrak{X}$ being compact in $\|\cdot\|_2$, for every n there exist elements $q_n \in Q'_{-1} \cap Q_0, p_n \in Q'_0 \cap P_{-1}, q'_n \in Q'_{-1} \cap$

P_{-1}, $p'_n \in P'_{-1} \cap P_0$ that realize the minimum:

$$\|q_n q_n' U_n p'_n p_n - I\|_2 = \inf_{(q,q',p,p') \in \mathfrak{X}} \|qq' U_n p' p - I\|_2.$$

Define $\widetilde{U}_n = q_n q_n' U_n p'_n p_n$. Then $\widetilde{U}_n \to I$, since for $p = p' = q = q' = I$ we have: $\|\widetilde{U}_n - I\|_2 \leqslant \|U_n - I\|_2$. Note that $U_n \neq I$ because the commuting squares were assumed non-isomorphic. Since for every unitary U we have $\|U - I\|_2^2 = 2 - 2\Re\tau(U)$ (where $\Re\tau$ is the real part of τ), it follows:

$$\Re\tau(\widetilde{U}_n) \geqslant \Re\tau(qq' U_n p' p), \quad \forall (q, q', p, p') \in \mathfrak{X}.$$

Let λ be a real number, let $q_0 \in Q'_{-1} \cap Q_0$ be a hermitian element, and let $q = \exp(\mathrm{i}\lambda q_0) q_n$, $q' = q'_n$, $p = p_n$, $p' = p'_n$. Then:

$$\Re\tau(\widetilde{U}_n) \geqslant \Re\tau(\exp(\mathrm{i}\lambda q_0)\widetilde{U}_n) \Longrightarrow \Re\tau((\exp(\mathrm{i}\lambda q_0) - I)\widetilde{U}_n) \leqslant 0.$$

By dividing with $\lambda > 0$ and taking limit as λ approaches 0, we obtain $\Re\tau(\mathrm{i}q_0\widetilde{U}_n) \leqslant 0$; doing the same for $\lambda < 0$, we have $\Re\tau(\mathrm{i}q_0\widetilde{U}_n) \geqslant 0$, and thus

$$\Re\tau(\mathrm{i}q_0\widetilde{U}_n) = 0.$$

Since for hermitians q_0 we have $\Re\tau(\mathrm{i}q_0 I) = 0$, we can rewrite the previous equality as

$$\Re\tau(\mathrm{i}q_0(\widetilde{U}_n - I)) = 0.$$

Let now $\widetilde{h} = \lim\limits_{n\to\infty} \frac{\widetilde{U}_n - I}{\mathrm{i}\|\widetilde{U}_n - I\|}$ (after passing to a subsequence if needed) . Dividing the previous equality by the real number $\|\widetilde{U}_n - I\|$ and taking the limit we have:

$$\Re\tau(\mathrm{i}q_0(\mathrm{i}\widetilde{h})) = 0 \Longrightarrow \tau(q_0\widetilde{h}) = 0.$$

Here we used that $\tau(q_0\widetilde{h})$ is a real number, since $q_0, \widetilde{h}$ are hermitians. Since $Q'_{-1} \cap Q_0$ is the span of its self-adjoint elements, it follows that $E_{Q'_{-1}\cap Q_0}(h) = 0$.

Similarly it follows that all four expectations are zero. For instance, choose q' to be $\exp(\mathrm{i}\lambda q'_0) q'_n$ and do the same trick, using the fact that $\exp(\mathrm{i}\lambda q'_0)$ commutes with q_n, so it can be moved to the left of the formula for $\widetilde{U}_n$.

Since we only modified U_n by elements commuting with Q_{-1}, we also have $[\widetilde{h}, Q_{-1}] = 0$. ∎

2. THE SPAN CONDITION

We introduce the *span condition* and show that a commuting square satisfying it is isolated among all commuting square (modulo isomorphisms).

In the next lemmas we will often use the following relation that holds true for every $a, b, c \in P_0$:

$$\tau([a,b]c) = \tau(a[b,c]) = \tau([c,a]b) \tag{2.1}$$

as it can be easily checked: $\tau([a,b]c) = \tau(abc - bac) = \tau(abc) - \tau(bac) = \tau(abc) - \tau(acb) = \tau(a[b,c]) = \tau(cab) - \tau(acb)$.

We present a lemma that justifies the definition of the span condition. For V, W vector subspaces of the algebra P_0, denote

$$V + W = \{v + w : v \in V, w \in W\},$$
$$[V, W] = \text{span}\{vw - wv : v \in V, w \in W\}.$$

LEMMA 2.1. *Let*

$$\begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup \ , \tau \\ Q_{-1} & \subset & Q_0 \end{pmatrix}$$

be a commuting square with normalized trace τ. Then the vector space $Q'_{-1} \cap P_{-1} + Q'_{-1} \cap Q_0 + P'_{-1} \cap P_0 + Q'_0 \cap P_0$ is orthogonal on $[P_{-1}, Q_0]$, with respect to the inner product defined by τ on P_0.

Proof. Let $p \in P_{-1}$ and $q \in Q_0$. The commuting square condition $E_{P_{-1}} E_{Q_0} = E_{Q_{-1}}$ implies $E_{P_{-1}}(q) = E_{Q_{-1}}(q)$ so $E_{P_{-1}}(q - E_{Q_{-1}}(q)) = 0$, which implies $\tau((q - E_{Q_{-1}}(q))p) = 0$, wich in turn implies

$$\tau(qp) = \tau(E_{Q_{-1}}(q)p) = \tau(E_{Q_{-1}}(q)E_{Q_{-1}}(p)) = \tau(qE_{Q_{-1}}(p)).$$

Let $[p_0, q_0] \in [P_{-1}, Q_0]$, and $p_1 \in Q'_{-1} \cap P_{-1}, q_1 \in Q'_{-1} \cap P_{-1}, p'_1 \in P'_{-1} \cap P_0, q'_1 \in Q'_0 \cap P_0$. Using Lemma (2.1) and $[p_1, p_0] \in P_{-1}$ we obtain:

$$\tau([[p_0, q_0]p_1]) = \tau([p_1, p_0]q_0) = \tau([p_1, p_0]E_{Q_{-1}}(q_0)) = \tau([E_{Q_{-1}}(q_0), p_1]p_0) = 0,$$

since $[E_{Q_{-1}}(q_0), p_1] = 0$. Similarly $\tau([p_0, q_0]q_1) = 0$. We also have:

$$\tau([p_0, q_0]p'_1) = \tau([p'_1, p_0]q_0) = 0, \quad \tau([p_0, q_0]q'_1) = \tau(p_0[q_0, q'_1]) = 0,$$

which ends the proof of the lemma. ∎

DEFINITION 2.2. We say that the commuting square from Lemma 2.1 satisfies *the span condition* if:

$$[P_{-1}, Q_0] + (Q'_{-1} \cap P_{-1}) + (Q'_{-1} \cap Q_0) + (P'_{-1} \cap P_0) + (Q'_0 \cap P_0) = P_0.$$

REMARK 2.3. Lemma 2.1 implies that

$$\dim[P_{-1}, Q_0] \leqslant \dim(P_0) - \dim(Q'_{-1} \cap P_{-1} + Q'_{-1} \cap Q_0 + P'_{-1} \cap P_0 + Q'_0 \cap P_0),$$

so in some sense the span condition asks for the dimension of the commutator $[P_{-1}, Q_0]$ to be maximal.

The span condition is a reasonable restriction as long as we assume that the commuting square satisfies some non-degeneracy properties, like $\dim(P'_{-1} \cap Q_0) = \dim(Q_{-1})$, $P_0 = \text{span} P_{-1} Q_0$. Indeed, the dimension of $[P_{-1}, Q_0]$ is typically big, P_{-1}, Q_0 are mutually orthogonal (modulo their intersection Q_{-1}) and in most of the examples (like the commuting squares associated to groups, Hopf algebras [12], or those corresponding to complex Hadamard matrices [8]) their commutants are also orthogonal, because of the existence of some modular involutions.

We can now prove our main result, which shows that the span condition is sufficient for isolation.

THEOREM 2.4. *If the commuting square of finite dimensional von Neumann algebras*

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ Q_{-1} & \subset & Q_0 & \end{pmatrix}$$

satisfies the span condition of Definition 2.2*, then* $\mathfrak{C}$ *is isolated.*

Proof. Assume, by contradiction, that $\mathfrak{C}$ satisfies the span condition but it is not isolated. According to Lemma 1.7, this implies the existence of unitaries $U_n \in P_0, n \geqslant 1$ converging to I such that:

$$\mathfrak{C}_n = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ Q_{-1} & \subset & U_n^* Q_0 U_n & \end{pmatrix}$$

are commuting squares non-isomorphic to $\mathfrak{C}$. Using Lemma 1.8 we may assume:

$$\lim_{n\to\infty} \frac{U_n - I}{\mathrm{i}\|U_n - I\|} = h \in Q'_{-1} \cap P_0\,,$$
$$E_{P'_{-1}\cap P_0}(h) = E_{Q'_0\cap P_0}(h) = E_{Q'_{-1}\cap P_{-1}}(h) = E_{Q'_{-1}\cap Q_0}(h) = 0.$$

Also

$$\lim_{n\to\infty} \frac{U_n^* - I}{\mathrm{i}\|U_n - I\|} = -h.$$

Let $p \in P_{-1}$ such that $E_{Q_{-1}}(p) = 0$ and let $q \in Q_0$. The commuting square condition implies $E_{U_n^* Q_0 U_n}(p) = E_{Q_{-1}}(p) = 0$, thus

$$\tau(pU_n^* q U_n) = 0 = \tau(pq) \Longrightarrow \tau(p(U_n - I)^* q U_n) + \tau(pq(U_n - I)) = 0.$$

Dividing by $\mathrm{i}\|U_n - I\|$ and taking the limit as $n \to \infty$ it follows

$$\tau(p(-h)q) + \tau(pqh) = 0 \Longrightarrow \tau([p,q]h) = 0.$$

Thus h is orthogonal on all vectors $[p,q]$ with $E_{Q_{-1}}(p) = 0$. We show that h is in fact orthogonal on all vectors in $[P_{-1}, Q_0]$. Indeed, if p_1 is an arbitrary element of P_{-1}, using $E_{Q_{-1}}(p_1 - E_{Q_{-1}}(p_1)) = 0$ we obtain:

$$\begin{aligned} \tau([p_1,q]h) &= \tau([p_1 - E_{Q_{-1}}(p_1), q]h + [E_{Q_{-1}}(p_1), q]h) \\ &= 0 + \tau([E_{Q_{-1}}(p_1), q]h) = \tau([h, E_{Q_{-1}}(p_1)]q) = 0. \end{aligned}$$

We used formula (2.1) and $[h, Q_{-1}] = 0$. This shows that h is orthogonal on $[P_{-1}, Q_0]$. Since h is also orthogonal on the algebras $Q'_{-1} \cap P_{-1}, Q'_{-1} \cap Q_0, P'_{-1} \cap P_0, Q'_0 \cap P_0$, it follows that if the span condition holds we must have $E_{P_0}(h) = 0$ so $h = 0$, which contradicts $\|h\| = 1$. ∎

COROLLARY 2.5. *For every $N \geqslant 2$ there are only finitely many isomorphism classes of commuting squares $\mathfrak{C}$ with $\dim(P_0) = N$, satisfying the span condition.*

3. ONE-PARAMETER FAMILIES OF NON-ISOMORPHIC COMMUTING SQUARES

In the previous section we have showed that the span condition is sufficient for isolation, but we did not discuss wether it is also necessary. We give partial converses to Theorem 2.4, which consider some of the simplest cases in which the span condition fails. The next theorem shows that one can construct a continuum of commuting squares if there exist two non-trivial elements $p_0 \in P_{-1}, q_0 \in Q_0$ that commute.

THEOREM 3.1. *Let*

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ Q_{-1} & \subset & Q_0 & \end{pmatrix}$$

be a commuting square of finite dimensional von Neumann algebras, and assume there exist hermitian elements $p_0 \in Q'_{-1} \cap P_{-1}, q_0 \in Q'_{-1} \cap Q_0$, that are not in Q_{-1}, such that $p_0q_0 - q_0p_0 = 0$. If $U_t = \exp(\mathrm{i}tp_0q_0), t \in \mathbb{R}$, then

$$\mathfrak{C}_t = \begin{pmatrix} P_{-1} & \subset & P_0 & \\ \cup & & \cup & ,\tau \\ Q_{-1} & \subset & U_t^* Q_0 U_t & \end{pmatrix}$$

is a one-parameter family of commuting squares.

Proof. We show that the commuting square condition holds for each t. Let $p \in P_{-1}$ such that $E_{Q_{-1}}(p) = 0$, and $q \in Q_0$. We need to show that $\tau(pU_t^* qU_t) = 0$. Writing $U_t = \exp(\mathrm{i}pqt) = \sum\limits_k \frac{\mathrm{i}^k t^k}{k!} p^k q^k$ we have:

$$\begin{aligned} \tau(pU_t^* qU_t) &= \sum_{k,l} \frac{(-1)^l \mathrm{i}^{k+l} t^{k+l}}{k!l!} \tau(pp_0^l q_0^l q q_0^k p_0^k) \\ &= \sum_{k,l} \frac{(-1)^l \mathrm{i}^{k+l} t^{k+l}}{(k+l)!} C_{k+l}^l \tau(p_0^k p p_0^l q_0^l q q_0^k) \\ &= \sum_n \sum_{k+l=n} \frac{(-1)^l \mathrm{i}^n t^n}{n!} C_n^l \tau(p_0^k p p_0^l q_0^l q q_0^k) \\ &= \sum_n \Big(\frac{\mathrm{i}^n t^n}{n!} \tau(p_0^n p q_0^n q) \Big(\sum_{k+l=n} (-1)^l C_n^l \Big) \Big) = \tau(pq) = \tau(p)\tau(q) = 0. \end{aligned}$$

We used:

$$\begin{aligned}\tau(p_0^k p p_0^l q_0^l q q_0^k) &= \tau(E_{Q_{-1}}(p_0^k p p_0^l) E_{Q_{-1}}(q_0^l q q_0^k)) \\ &= \tau(E_{Q_{-1}}(p_0^n p) E_{Q_{-1}}(q_0^n q)) = \tau(p_0^n p q_0^n q),\end{aligned}$$

since $p_0 \in Q'_{-1} \cap P_{-1}, q_0 \in Q'_{-1} \cap Q_0$. We also used $\sum\limits_l (-1)^l C_n^l = 0$ for $n \geqslant 1$. ∎

If p, q are projections then the unitaries in Theorem 3.1 can be written as $U(\lambda) = I + (\lambda - 1)pq$, $\lambda = \mathrm{e}^{\mathrm{i}t} \in \mathbb{T}$. This justifies the class of unitaries we construct in the next theorem, that applies to situations when there exists a linear dependence relation between 2 commutators in the span.

THEOREM 3.2. *Let*

$$\mathfrak{C} = \begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup \ , \tau \\ Q_{-1} & \subset & Q_0 \end{pmatrix}$$

be a commuting square of finite dimensional von Neumann algebras, and assume there exist orthogonal projections $p_1, p_2 \in Q'_{-1} \cap P_{-1}$ and orthogonal projections $q_1, q_2 \in Q'_{-1} \cap Q_0$, that are not in Q_{-1}, satisfying $[p_1, q_1] - [p_2, q_2] = 0$. Let

$$U(\lambda) = I + (\lambda - 1)p_1 q_1 + (\overline{\lambda} - 1)p_2 q_2,$$

for $\lambda \in \mathbb{T}$. Then

$$\mathfrak{C}_\lambda = \begin{pmatrix} P_{-1} & \subset & P_0 \\ \cup & & \cup & , \tau \\ Q_{-1} & \subset & U(\lambda)^* Q_0 U(\lambda) \end{pmatrix}$$

is a one-parameter family of commuting squares.

Proof. Since

$$[p_1, q_1] - [p_2, q_2] = 0 \Longrightarrow p_1 q_1 + q_2 p_2 = p_2 q_2 + q_1 p_1\,,$$

multiplying by p_1 to the right, and then by p_2 to the left, we have:

$$p_1 q_1 p_1 = p_2 q_2 p_1 + q_1 p_1, \quad p_2 q_2 p_1 + p_2 q_1 p_1 = 0.$$

Similary, multiplying by p_2 to the left we have:

$$p_2 q_2 p_2 = p_2 q_2 + p_2 q_1 p_1$$

and summing up the last relations

$$\begin{aligned}p_1 q_1 p_1 + p_2 q_2 p_2 &= p_2 q_2 p_1 + q_1 p_1 + p_2 q_2 + p_2 q_1 p_1 \\ &= q_1 p_1 + p_2 q_2 + (p_2 q_2 p_1 + p_2 q_1 p_1) = q_1 p_1 + p_2 q_2.\end{aligned} \tag{3.1}$$

We now show that $U(\lambda)$ is a unitary:

$$\begin{aligned}U(\lambda)U(\lambda)^* &= (I + (\lambda - 1)p_1 q_1 + (\overline{\lambda} - 1)p_2 q_2)(I + (\overline{\lambda} - 1)q_1 p_1 + (\lambda - 1)q_2 p_2) \\ &= I + (\lambda - 1)(p_1 q_1 + q_2 p_2) + (\overline{\lambda} - 1)(p_2 q_2 + q_1 p_1) \\ &\quad + (\lambda - 1)(\overline{\lambda} - 1)(p_1 q_1 p_1 + p_2 q_2 p_2) = I.\end{aligned}$$

We used: $q_1p_1 + p_2q_2 = p_2q_2 + q_1p_1, (\lambda - 1)(\overline{\lambda} - 1) = -(\lambda - 1) - (\overline{\lambda} - 1)$ and equation (3.1):

$$p_1q_1p_1 + p_2q_2p_2 = q_1p_1 + p_2q_2.$$

Let's now check that $\mathfrak{C}(U)$ is a commuting square: for $p \in P_{-1}$ with $E_{Q_{-1}}(p) = 0$ and $q \in Q_0$ we have:

$$\begin{aligned}
&\tau(pU(\lambda)qU(\lambda)^*) \\
&\qquad = \tau(pq) + (\lambda - 1)\tau(pp_1q_1q + pqq_2p_2) + (\overline{\lambda} - 1)\tau(pp_2q_2q + pqq_1p_1) \\
&\qquad\qquad + (\lambda - 1)(\overline{\lambda} - 1)\tau(pp_1q_1qq_1p_1 + pp_2q_2qq_2p_2) \\
&\qquad\qquad + (\lambda - 1)^2\tau(pp_1q_1qq_2p_2) + (\overline{\lambda} - 1)^2\tau(pp_2q_2qq_1p_1).
\end{aligned}$$

But $\tau(pp_1q_1qq_2p_2) = \tau(p_2pp_1q_1qq_2) = \tau(E_{Q_{-1}}(p_2pp_1)E_{Q_{-1}}(q_1qq_2)) = 0$, because $E_{Q_{-1}}(q_1qq_2) = E_{Q_{-1}}(qq_2q_1) = 0$, since $q_2q_1 = 0$ and $[q_1, Q_{-1}] = 0$. Similarly $\tau(pp_2q_2qq_1p_1) = 0$. Also:

$$\begin{aligned}
\tau(pp_1q_1q + pqq_2p_2) &= \tau(pp_2q_2q + pqq_1p_1) = \tau(pp_1q_1qq_1p_1 + pp_2q_2qq_2p_2) \\
&= \tau(pq(q_1p_1 + q_2p_2)).
\end{aligned}$$

Indeed:

$$\begin{aligned}
\tau(pp_1q_1q + pqq_2p_2) &= \tau(pp_1q_1q) + \tau(qq_2p_2p) \\
&= \tau(E_{Q_{-1}}(pp_1)E_{Q_{-1}}(q_1q)) + \tau(E_{Q_{-1}}(qq_2)E_{Q_{-1}}(p_2p)) \\
&= \tau(E_{Q_{-1}}(p_1p)E_{Q_{-1}}(qq_1)) + \tau(E_{Q_{-1}}(p_2p)E_{Q_{-1}}(qq_2)) \\
&= \tau(p_1pqq_1) + \tau(p_2pqq_2) = \tau(pq(q_1p_1 + q_2p_2))
\end{aligned}$$

and the other equalities follow similarly. Thus, using $(\lambda - 1)(\overline{\lambda} - 1) + (\lambda - 1) + (\overline{\lambda} - 1) = 0$ we have:

$$\tau(pU(\lambda)qU(\lambda)^*) = 0$$

which ends the proof. ∎

4. COMMENTS ON PETRESCU'S RESULTS

We discuss consequences of the theorems from the previous sections for commuting squares of the form:

$$\begin{pmatrix} D & \subset & \mathbb{M}_n(\mathbb{C}) & \\ \cup & & \cup & , \tau \\ \mathbb{C} & \subset & U^*DU & \end{pmatrix}$$

with D the diagonal matrices, U unitary in $\mathbb{M}_n(\mathbb{C})$ and $\tau = \frac{1}{n}\mathrm{Tr}$ the normalized trace.

Denote by $(A_{i,j})_{i,j}$ the matrix units of $M_n(\mathbb{C})$, $A_{i,j}$ the matrix having 1 at the intersection of the i^{th} row and $j^t h$ column, and only zeros on the other positions. Also, let $D_k = A_{k,k}, k = 1, \ldots, n$ be an orthogonal basis of D.

The commuting square condition $\tau(D_i U^* D_j U) = \tau(D_i)\tau(D_j)$ can be rewritten as $\overline{u}_{ji} u_{ij} = \frac{1}{n}$ if $U = (u_{ij})_{1 \leqslant i,j \leqslant n}$. Thus it amounts to all entries of U having the same absolute value $\frac{1}{\sqrt{n}}$. Such a U is called a *biunitary matrix* or *complex Hadamard matrix*. We say that two biunitaries are *equivalent* if the corresponding commuting squares are isomorphic. For every n there exists at least one biunitary of order n such that $U = \frac{1}{\sqrt{n}}(\varepsilon^{(i-1)(j-1)})_{i,j}, \varepsilon = \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}$, called the *standard biunitary matrix* of order n.

We can apply Theorem 2.4 to commuting squares given by biunitary matrices. Since the algebras D and U^*DU are abelian and orthogonal modulo their intersection $\mathbb{C}I$ the span condition becomes:

$$\dim([D, U^*DU]) = n^2 - 2n + 1.$$

Thus, we have the following:

PROPOSITION 4.1. *If $U \in M_n(\mathbb{C})$ is a biunitary matrix such that the dimension of the vector space $[D, U^*DU]$ is $n^2 - 2n + 1$, then U is isolated among all biunitaries (up to equivalence).*

COROLLARY 4.2 (Petrescu's Theorem). *The standard biunitary of order n is isolated if and only if n is prime.*

Proof. Assume n is prime and let $U = (\varepsilon^{(i-1)(j-1)})_{i,j}$ be the standard biunitary matrix of order n, and $U^*DU = S$ is the algebra of circulant permutation matrices. Then $S_k = \sum\limits_i A_{i,i+k}, k = 1, \dots, n$ give a basis for S (all the indices are considered modulo n).

Thus $X_{k,l} = [D_k, S_l] = A_{k,k+l} - A_{k-l,k}$ is a set of generators for $[D, U^*DU]$. Assume that for some complex numbers $c_{k,l}$ we have:

$$\sum_{k,l} c_{k,l} X_{k,l} = 0.$$

It follows

$$\sum_{k,l}(c_{k,l}A_{k,k+l} - c_{k,l}A_{k-l,k}) = 0 \Longrightarrow \sum_{i,j}(c_{i,j-i} - c_{j,j-i})A_{i,j} = 0,$$

so $c_{i,j-i} = c_{j,j-i}$ and if we denote by $s = j - i$ we have $c_{i,s} = c_{i+s,s}$ so $c_{i,s} = c_{i+ms,s}, \forall m = 0, 1, \dots, n-1$. Since n is prime, for every s different from zero the elements $0, s, 2s, \dots, (n-1)s$ cover all possible residues mod n, so $c_{i,s} = c_{0,s}$.

Thus the dimension of the kernel of the linear transformation

$$(c_{k,l})_{k,l} \to \sum_{k,l} c_{k,l} X_{k,l}$$

is $(2n-1)$, so its range has dimension $n^2 - 2n + 1$, which shows that the span condition holds.

Conversely, if n is not prime, $n = n_1 n_2$ with $n_1, n_2 > 1$, then the matrices $p = \sum\limits_{j \leqslant n_2} A_{jn_1, jn_1} \in D$ and $q = \sum\limits_{i,j} A_{j, in_1 + j} \in U^* D U$ commute, so by Theorem 3.1 we can construct a one-parameter family of biunitaries $U(t) = \exp(\mathrm{i}pqt)$. ∎

For $n = 5$ the standard biunitary matrix is the only complex Hadamard matrix up to equivalence, as proven by U. Haagerup [3]. For all $n > 5$ prime there exists at least another biunitary which is a circulant matrix [1], [4], [7], and for every n non prime one can easily construct infinitely many biunitaries.

S. Popa conjectured that for every $n > 5$ prime there exist only finitely many normalized biunitaries [11]. Surprisingly, this turned out to be false: one-parameter families of normalized biunitaries where constructed by M. Petrescu for $n = 7, 13, 19, 31, 79$ [8]. A main point of interest in this result is that it might produce one-parameter families of non-isomorphic subfactors with the same graph, conjectured to be A_∞. While Petrescu's examples have been constructed using the computer, we give a conceptual proof of their existence as a consequence of Theorem 3.2 . We will work the details for one of the two examples for $n = 7$, the other examples having similar proofs.

COROLLARY 4.3 (Petrescu's biunitaries). *Let $\lambda \in \mathbb{T}$, $w = \cos\frac{2\pi}{6} + \mathrm{i}\sin\frac{2\pi}{6}$ and*

$$U(\lambda) = \frac{1}{\sqrt{7}} \begin{pmatrix} \lambda w & \lambda w^4 & w^5 & w^3 & w^3 & w & 1 \\ \lambda w^4 & \lambda w & w^3 & w^5 & w^3 & w & 1 \\ w^5 & w^3 & \overline{\lambda} w & \overline{\lambda} w^4 & w & w^3 & 1 \\ w^3 & w^5 & \overline{\lambda} w^4 & \overline{\lambda} w & w & w^3 & 1 \\ w^3 & w^3 & w & w & w^4 & w^5 & 1 \\ w & w & w^3 & w^3 & w^5 & w^4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{4.1}$$

Then $U(\lambda)$ is a 1-parameter family of (non-equivalent) biunitaries.

Proof. Let $U = U(1)$, $P_0 = \mathbb{M}_n(\mathbb{C})$, $P_{-1} = D$, $Q_0 = U^* D U$, $Q_{-1} = \mathbb{C}$ and

$$p_1 = A_{1,1} + A_{2,2}, \qquad p_2 = A_{3,3} + A_{4,4} \in P_{-1}$$
$$q_1 = U^*(A_{1,1} + A_{2,2})U, \qquad q_2 = U^*(A_{3,3} + A_{4,4})U \in Q_0.$$

It is easy to check that

$$[p_1, q_1] - [p_2, q_2] = 0, \quad p_1 p_2 = q_1 q_2 = 0.$$

Thus we are in the conditions of Theorem 3.2, so

$$U(\lambda) = (I + (\lambda - 1) p_1 q_1 + (\overline{\lambda} - 1) p_2 q_2) U$$

are biunitaries for all λ complex numbers of absolute value 1. One can easily verify that $U(\lambda)$ are the biunitaries from (4.1). ∎

REMARK 4.4. One can try to find more examples of complex Hadamard matrices using the following algorithm: fix $p_1, p_2, p_3, p_4 \in D$, with $p_1 p_2 = p_3 p_4 = 0$, and find, with the help of a computer and local minimum algorithms, matrices U satisfying the biunitarity condition and the condition $\|[p_1, U^* p_3 U] + [p_2, U^* p_4 U]\| = 0$. One can construct from U a one-parameter family of biunitaries as before.

5. APPLICATION TO CIRCULANT MATRICES

A matrix S is called *circulant* if all rows are obtained from consecutive circular permutations of the first row, i.e. $S = (s_{j-i})_{i,j \in \mathbb{Z}/n\mathbb{Z}}$. The problem of classifying circulant biunitaries is equivalent to Bjorck's problem of classifying *cyclic n-roots* [1]. For every n prime there exists at least one circulant complex Hadamard matrix. If n is a prime of the form $n = 4k + 3, k \in \mathbb{Z}$, this matrix can be defined as $s_i = \frac{1}{\sqrt{n}}$ for i quadratic residue modulo n and $s_i = a$ in rest, where a is the root of a certain quadratic equation over $\mathbb{Q}$. A similar but slightly more complicated formula defines a circulant complex Hadamard matrix of order $n = 4k + 1$ [1], [4], [7].

U. Haagerup proved that for every n prime there exist finitely many circulant biunitaries (up to equivalence). We conjecture that in fact every circulant biunitary matrix satisfies the span condition, and thus is isolated among all biunitary matrices. We show this is true for $n = 7$, by using Haagerup's classification of biunitaries of order 7 [3]. We give an algorithm that can be used, more generally, to check if a given U satisfies the span condition, and thus is isolated among all normalized biunitary matrices.

PROPOSITION 5.1. *If U is a circulant biunitary matrix of order 7 then U is isolated among all biunitary matrices.*

Proof. Let $U = (u_{ij})_{i,j \in \mathbb{Z}_7}$ be a circulant biunitary of order 7. According to Proposition 4.1 it is enough to show that $\dim(\mathrm{span}[D, U^* D U]) = 7^2 - 27 + 1 = 36$. Let $D_i, i = 0, \ldots, 6$ be a basis for D, where D_i is the diagonal having 1 on the $i + 1$ position on the diagonal and only 0's on the other positions. Let: $a^{ij}_{kl} = [D_i, U^* D_j U]_{kl} = (D_i U^* D_j U - U^* D_j U D_i)_{kl} = \delta^k_i \overline{u}_{ji} u_{jl} - \delta^l_i \overline{u}_{jk} u_{ji} = (\delta^k_i - \delta^l_i) \overline{u}_{jk} u_{jl}$.

Let A be the 49×49 matrix given by $A_{(i,j),(k,l)} = a^{ij}_{kl}$. We need to check that $\mathrm{rank}(A) = 36$. Because $\sum_i a^{ij}_{kl} = \sum_i [D_i, U^* D_j U]_{kl} = [I, U^* D_j U]_{kl} = 0$ and similarly $\sum_j a^{ij}_{kl} = 0$, to find the rank of A we may remove the 13 rows of A indexed after $(i, 0), (0, j), 0 \leqslant i, j \leqslant 6$. Similar arguments show that we can remove the 13 columns of A indexed after $(i, i), (0, j), 0 \leqslant i, j \leqslant 6$. If we denote by M the 36×36 matrix left, we need to check that $\det(M) \neq 0$.

We include this computation for one of the circulant matrices of order 7 (of the type described at the beginning of this section, for $a = -\frac{3}{4} + \mathrm{i}\frac{\sqrt{7}}{4}$). Similarly, we checked that all circulant biunitaries of order 7 (computed in [3]) are isolated among all biunitaries. The pairs $(i,j), 0 \leqslant i,j \leqslant 6$ are encoded in the vector w, and the selection of rows and columns of A is encoded in v.

```
with(LinearAlgebra):
    ID:=Matrix(7,7,shape=identity);a:=-3/4+I*sqrt(7)/4;
    U:=1/sqrt(7)Matrix([[1,1,1,a,1,a,a],[a,1,1,1,a,1,a],[a,a,1,1,1,a,1],
    [1,a,a,1,1,1,a],[a,1,a,a,1,1,1],[1,a,1,a,a,1,1],[1,1,a,1,a,a,1]]);
    A:=(i,j,k,l)->(ID[k,i]-ID[l,i])*conj(U[j,k])*U[j,l];
    w:=m->(1+((m-(1+((m-1) mod 6)))/6),1+((m-1) mod 6));
    v := array(1..72,[1,2,1,3,1,4,1,5,1,6,1,7,2,1,2,3,2,4,2,5,2,6,2,7,3,1,3,
    2,3,4,3,5,3,6,3,7,4,1,4,2,4,3,4,5,4,6,4,7,5,1,5,2,5,3,5,4,5,6,5,7,6,1,
    6,2,6,3,6,4,6,5,6,7]);
    g:=(n,m)->A(w(m),v[2*n-1],v[2*n]);
    M:=Matrix(36,g); Determinant(M);
```
∎

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