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THE KALMAN-YAKUBOVICH-POPOV INEQUALITY FOR DISCRETE TIME SYSTEMS OF INFINITE DIMENSION

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ABSTRACT. Infinite dimensional discrete time dissipative scattering systems are introduced in terms of generalized (possibly unbounded) solutions of the Kalman-Yakubovich-Popov inequality (KYP-inequality). It is shown that for a minimal system the KYP-inequality has a generalized solution if and only if the transfer function of the system coincides with a Schur class function θ in a neighborhood of zero. The set of solutions of the KYP-inequality, its order structure, and the corresponding contractive systems are studied in terms of θ. Also using the KYP-inequality a number of stability theorems are derived.

KEYWORDS: *Dissipative linear systems, optimal control, stability.*

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1. INTRODUCTION

Consider the *linear time-invariant system* with discrete time n:

$$\Sigma \left\{ \begin{array}{rcl} x_{n+1} & = & Ax_n + Bu_n, \\ y_n & = & Cx_n + Du_n, \end{array} \right. \quad (n \geqslant 0). \tag{1.1}$$

Here $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$ and $D : \mathcal{U} \to \mathcal{Y}$ are bounded linear operators acting between separable Hilbert spaces. We refer to A as the *state operator* of Σ, and to D as the *external operator*. Starting from the initial state x_0, one computes the output $y_0, y_1, y_2, \ldots$ of the system Σ from the input sequence $u_0, u_1, u_2, \ldots$ via the system equations (1.1). In fact, for $k = 0, 1, 2, \ldots$ we have

$$y_k = CA^k x_0 + \sum_{j=0}^{k-1} CA^j B u_{k-j-1} + Du_k.$$

To simplify notation we will write $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. The spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are called the *state space*, the *input space*, and the *output space*, respectively.

Since the fundamental work of Kalman [24], Yakubovich [34], and Popov [29] in optimal control theory and the stability theory for non-linear systems, the notion of a dissipative system has become a classical object. To introduce dissipativity in our setting we follow [33] and use the concepts of a supply rate and storage function. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a system. A *supply rate function* for Σ is a function

$$w(u, y) = \langle \Phi(u, y), (u, y) \rangle, \tag{1.2}$$

defined on the Hilbert space direct sum $\mathcal{U} \oplus \mathcal{Y}$, where Φ is a bounded selfadjoint operator acting on $\mathcal{U} \oplus \mathcal{Y}$. A first example, which originates from network theory, concerns the case of impedance systems when the input space $\mathcal{U}$ coincides with the output space $\mathcal{Y}$ and the function w is given by $w(u_1, u_2) = \mathrm{Re}\, \langle u_1, u_2 \rangle$. In this case the selfadjoint operator Φ in (1.2) equals $\frac{1}{2}J$, where J is the signature operator

$$J = \begin{bmatrix} 0 & I_{\mathcal{U}} \\ I_{\mathcal{U}} & 0 \end{bmatrix},$$

acting on $\mathcal{U} \oplus \mathcal{U}$.

In this paper we will deal with the scattering supply rate function

$$w(u, y) = \|u\|^2 - \|y\|^2, \tag{1.3}$$

which plays an important role in scattering theory and also in the analysis of H^∞- optimal control problems. In (1.3) the norms $\|\cdot\|$ are the usual Hilbert space norms on the input space $\mathcal{U}$ and output space $\mathcal{Y}$, respectively. The corresponding selfadjoint operator Φ is given by

$$\Phi = \begin{bmatrix} I_{\mathcal{U}} & 0 \\ 0 & -I_{\mathcal{Y}} \end{bmatrix}.$$

A system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is called *dissipative with respect to the supply rate function* w if there exists a (possibly unbounded) positive operator H in $\mathcal{X}$ such that

$$A\mathcal{D}(H^{1/2}) \subset \mathcal{D}(H^{1/2}), \quad B\mathcal{U} \subset \mathcal{D}(H^{1/2}), \tag{1.4}$$

and for each initial state $x_0 \in \mathcal{D}(H^{1/2})$ and each sequence of inputs $u_0, u_1, u_2, \ldots$ from $\mathcal{U}$ we have

$$w(u_n, y_n) \geqslant \|H^{1/2}x_{n+1}\|^2 - \|H^{1/2}x_n\|^2, \quad n = 0, 1, 2, \ldots. \tag{1.5}$$

Throughout this paper a (possibly unbounded) operator H acting in a Hilbert space $\mathcal{X}$ is said to be *positive* if H is selfadjoint and $\langle Hx, x \rangle > 0$ for each $x \neq 0$ in the domain $\mathcal{D}(H)$ of H.

Here $H^{1/2}$ is the square root of H, i.e., the unique non-negative selfadjoint operator Y in $\mathcal{X}$ such that $Y^2 = H$. For $n = 0, 1, 2, \ldots$ the vectors x_{n+1} and y_n in (1.5) are derived from the initial vector x_0 and the input sequence $u_0, u_1, u_2, \ldots$ via the system equations. It follows from (1.4) that the state vectors x_n, $n \geqslant 0$, all belong to the domain $\mathcal{D}(H^{1/2})$ of $H^{1/2}$. For a positive operator H in $\mathcal{X}$ satisfying (1.4) we refer to the function

$$Q_H(x) = \|H^{1/2}x\|^2, \quad x \in \mathcal{D}(H^{1/2}),$$

as the *storage function* for Σ defined by H.

By rewriting the system equations (1.1) in the following form

$$\begin{bmatrix} x_{n+1} \\ y_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_n \\ u_n \end{bmatrix} \quad (n = 0, 1, 2, \ldots) \tag{1.6}$$

we see that for the scattering supply rate function (1.3) the dissipativity condition (1.5) is just equivalent to the requirement that there exists a positive operator H in $\mathcal{X}$ satisfying (1.4) and

$$K_\Sigma(H)(x, u) \geqslant 0, \ \ x \in \mathcal{D}(H^{1/2}), \quad u \in \mathcal{U}, \tag{1.7}$$

where

$$\begin{aligned} K_\Sigma(H)(x, u) & \\ &= \left\| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2. \end{aligned} \tag{1.8}$$

If the state space $\mathcal{X}$ is finite dimensional, then the operator H is automatically defined on the whole space and is a bounded (and boundedly invertible) positive selfadjoint operator. In the latter case the inequality (1.7) reduces to the usual Kalman-Yakubovich-Popov inequality

$$\begin{bmatrix} H - A^*HA - C^*C & -C^*D - A^*HB \\ -D^*C - B^*HA & I - D^*D - B^*HB \end{bmatrix} \geqslant 0. \tag{1.9}$$

However for systems with an infinite dimensional state space, it may happen (an example is given in Section 4.5) that no bounded positive operator H satisfies (1.9) while there exist unbounded positive operators H satisfying (1.4) and (1.7). Moreover, it may happen that H^{-1} is unbounded too.

This connection between (1.7) and the Kalman-Yakubovich-Popov inequality justifies the following terminology. We say that a (possibly unbounded) positive operator H in $\mathcal{X}$ is a *generalized solution of the Kalman-Yakubovich-Popov inequality* (for short, KYP-*inequality*) for Σ if (1.4) and (1.7) are satisfied. Summarizing: *a system Σ is dissipative with respect to the supply rate* (1.3) *if and only if the* KYP-*inequality for Σ has a generalized solution*.

The main purpose of this paper is to present a generalization to the infinite dimensional case of the classical *Kalman-Yakubovich-Popov lemma* which can be found in textbooks (see, e.g., [35]). Here we state this lemma for the case when the supply rate function is given by (1.3) and the state space is finite dimensional. The terminology from the theory of systems will be explained in the next section.

LEMMA 1.1. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system with finite dimensional state space $\mathcal{X}$. Then the set*

$$\mathcal{K}_\Sigma = \{H : H > 0 \text{ and } H \text{ satisfies } (1.9)\} \tag{1.10}$$

is non-empty if and only if the transfer function θ_Σ of the system Σ belongs to the Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$. In that case $\mathcal{K}_\Sigma$ contains an element H_0 and an element $H_\bullet$ such that

$$H_0 \leqslant H \leqslant H_\bullet, \quad H \in \mathcal{K}_\Sigma.$$

The Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$ is the set of functions θ, which are analytic in the open unit disk $\mathbb{D} = \{\lambda : |\lambda| < 1\}$, and of which the values are contractive linear operators acting between the separable Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, i.e.,

$$\mathcal{S}(\mathcal{U},\mathcal{Y}) = \{\theta : \theta \in H_\infty(\mathcal{U},\mathcal{Y}), \|\theta\|_\infty \leqslant 1\}, \tag{1.11}$$

where $\|\theta\|_\infty = \sup\{\|\theta(\lambda)\| : \lambda \in \mathbb{D}\}$.

We remark that in Lemma 1.1 for the case when the spectrum of A is contained in the closed unit disk the set $\mathcal{K}_\Sigma$ in (1.10) does not change if the condition $H > 0$ is replaced by the requirement that H is selfadjoint and invertible (see, for instance, page 550 of [28]).

There exist various generalizations of this lemma for the case that the state space of Σ is infinite dimensional (see [20]). In each of these generalizations the positive solution H to the inequality (1.7) is required to be a bounded operator. Nevertheless, the unbounded solutions to (1.7) are interesting and important in their own right. In this paper we obtain a generalization of Lemma 1.1 in which the solutions H may be unbounded selfadjoint operators. Moreover, the transfer function of the system Σ can be an arbitrary operator valued function, which is analytic in a neighborhood of 0, and which coincides with a Schur class function in this neighborhood. The next theorem is our first main result.

THEOREM 1.2. *Let $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y})$ be a minimal system. Then the* KYP*-inequality for Σ has a generalized solution if and only if its transfer function θ_Σ coincides with a Schur class function in a neighborhood of zero.*

In our second main result (Theorem 5.1 in Section 5) we identify solutions of the KYP-inequality that play the same role (relative to an appropriate ordering of positive operators that may be unbounded) as the minimal and maximal elements $H_\circ$ and $H_\bullet$ in Lemma 1.1.

An important aspect of the KYP-inequality is its connection to stability. For systems with an infinite dimensional state space this connection is subtle and very different from what is known for systems with a finite dimensional state space. For instance, if H is a generalized solution to the KYP-inequality of the minimal system $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y})$, then from the finite dimensional case one would expect that

$$\lim_{n\to\infty} \|H^{1/2}A^n x\| = 0, \quad x \in \mathcal{D}(H^{1/2}). \tag{1.12}$$

In Section 6 we shall see that in general for systems with an infinite dimensional state space this is not true. In fact, it may happen that Σ is a dissipative minimal system, and that (1.12) does not hold for any generalized solution H to the KYP-inequality of Σ. Since in the finite dimensional case all solutions H of the KYP-inequality are bounded and strictly positive, it follows that for this case (1.12) holds for all solutions H of the KYP-inequality whenever it holds for one. The latter property also does not carry over to the infinite dimensional case. Furthermore, in general in the infinite dimensional case, formula (1.12) does not imply stability in the usual sense. In Section 6 we shall also present a number of positive stability results based on [4].

This paper consists of seven sections, this introduction being the first. In the second section we review the general theory of infinite dimensional discrete time systems, and define notions as transfer function, dilation, restriction and minimality. In the third section we introduce the notion of pseudo-similarity, and prove that minimal systems with the same transfer function in a neighborhood of zero are pseudo-similar. In the fourth section we show that a system is dissipative with respect to the supply rate (1.3) if and only if it is pseudo-similar to a contractive system. The fourth section also contains the proof of Theorem 1.2. Our second main theorem (Theorem 5.1) is stated and proved in the fifth section. The sixth section concerns the connection between the solvability of the KYP-inequality and stability of the corresponding systems. In the final section we present some additional information on the set of solutions of the KYP-inequality and the corresponding contractive systems, using results from [8] and [9]. A preliminary version of this paper is the report [6].

In conclusion we mention that the results derived in this paper also hold with appropriate modifications for scattering dissipative continuous time systems and for dissipative systems with other supply rate functions (impedance and transmission systems), both in discrete time and in continuous time. In fact (see, e.g., [3]) there are standard ways to translate results about discrete time dissipative scattering systems into results about other dissipative systems of the above mentioned type (by using the Cayley transform, the Potapov-Ginzburg transform). The connection between solutions of the KYP-inequality and the solutions of the algebraic Ricatti inequality and equality will be developed in a further paper.

2. PRELIMINARIES ABOUT INFINITE DIMENSIONAL DISCRETE TIME SYSTEMS

In this section we review a number of fundamental concepts of the theory of infinite dimensional discrete time Hilbert space systems that are used throughout this paper. The main source for this section are the papers [22] and [3]. Some of the material can also be found in books; see, e.g., [11], and pages 79ff of [23].

2.1. TRANSFER FUNCTION AND REALIZATION. The *transfer function* of the system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is the operator valued function θ_Σ given by

$$\theta_\Sigma(\lambda) = D + \lambda C(I - \lambda A)^{-1} B, \tag{2.1}$$

which is defined on the set consisting of all $\lambda \in \mathbb{C}$ such that $I - \lambda A$ is boundedly invertible. Its values are bounded linear operators acting between the Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. Obviously, θ_Σ is analytic at 0. Given a sequence of inputs $u_0, u_1, u_2, \ldots$ and initial state $x_0 = 0$, one can obtain the sequence of outputs $y_0, y_1, y_2, \ldots$ from the transfer function by multiplication of the following two formal power series

$$\theta_\Sigma(\lambda) = D + \sum_{j \geqslant 1} CA^{j-1} B \lambda^j, \quad u(\lambda) = \sum_{j \geqslant 0} u_j \lambda^j.$$

Indeed, $\theta_\Sigma(\lambda) u(\lambda) = y(\lambda)$, where $y(\lambda)$ is the formal power series $\sum_{j \geqslant 0} y_j \lambda^j$. If the series $\sum_{j \geqslant 0} u_j$ is convergent, i.e., if $u(\lambda)$ is analytic at 0, then $y(\lambda)$ is analytic at 0 too.

Let $\theta(\lambda) : \mathcal{U} \to \mathcal{Y}$ be an operator valued function which is analytic in a neighborhood of 0. Then there exists a system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with transfer function θ (see [2], [11], [16], and [22]). In that case the system Σ is called a *realization* of θ.

In this connection, we introduce the following notation. Let θ and θ_1 be two operator valued functions which are analytic in a neighborhood of 0. We write $\theta \sim \theta_1$ if $\theta(\lambda) = \theta_1(\lambda)$ in a neighborhood of 0. In this case we say that θ and θ_1 *coincide in a neighborhood of* 0.

The equivalence relation $\sim$ of transfer functions might seem to be weak, however, if θ_Σ and $\theta_{\Sigma'}$ coincide in a neighborhood of zero and their state operators are contractions, then $\theta_\Sigma(\lambda) = \theta_{\Sigma'}(\lambda)$ for each λ in the open unit disc. More refined statements of this type involve the complement of the spectra of the state operators.

2.2. DILATION AND RESTRICTION. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}})$ be two given systems. Then $\widetilde{\Sigma}$ is called a *dilation* of the system Σ if $\widetilde{\mathcal{U}} = \mathcal{U}$, $\widetilde{\mathcal{Y}} = \mathcal{Y}$, $\widetilde{D} = D$, and the state space $\widetilde{\mathcal{X}}$ admits an orthogonal sum decomposition $\widetilde{\mathcal{X}} = \mathcal{E} \oplus H \oplus \mathcal{E}_*$ such that relative to this decomposition the system $\widetilde{\Sigma}$ can be written as

$$\widetilde{\Sigma} = \left(\begin{bmatrix} A_1 & A_3 & A_4 \\ 0 & A & A_5 \\ 0 & 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C & C_1 \end{bmatrix}, D; \mathcal{E} \oplus \mathcal{X} \oplus \mathcal{E}_*, \mathcal{U}, \mathcal{Y} \right). \tag{2.2}$$

Explicitly,

$$A = P_{\mathcal{X}} \widetilde{A} | \mathcal{X}, \quad B = P_{\mathcal{X}} \widetilde{B}, \quad C = \widetilde{C} | \mathcal{X}, \tag{2.3}$$

$$A\mathcal{E} \subset \mathcal{E}, \quad A^* \mathcal{E}_* \subset \mathcal{E}_*, \quad C\mathcal{E} = \{0\}, \quad B^* \mathcal{E}_* = \{0\}. \tag{2.4}$$

If $\widetilde{\Sigma}$ is a dilation of Σ, then the system Σ is called a *restriction* of $\widetilde{\Sigma}$.

Notice that dilating or restricting a system does not change the Taylor coefficients of its transfer function at zero. Since these Taylor coefficients determine the transfer function in a neighborhood of zero, it follows that dilating or restricting a system does not change its transfer function in a neighborhood of zero. In other words, if $\widetilde{\Sigma}$ is a dilation of Σ, then $\theta_{\widetilde{\Sigma}} \sim \theta_{\Sigma}$.

2.3. MINIMALITY. A system is called *minimal* if it is not a dilation of any other (different) system. In other words a system is minimal if and only if it does not have a proper restriction. Minimality can be characterized in term of controllability and observability. For this purpose we need the following notation and terminology.

Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a system. The linear manifold

$$\mathrm{Im}\,(A|B) = \mathrm{span}\,\{A^n Bu : u \in \mathcal{U}, n \in \mathbb{N}_0\} \tag{2.5}$$

consists of all vectors in the state space which can be reached in finite time. We call the set $\mathrm{Im}\,(A|B)$ the *reachable manifold* of Σ. The *controllable subspace* is by definition the closure of this set. The system Σ is said to be *(approximately) controllable* if the controllable subspace is equal to $\mathcal{X}$ or, equivalently, the reachable manifold is dense in $\mathcal{X}$.

The *unobservable subspace* of $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is by definition the subspace

$$\mathrm{Ker}\,(C|A) = \bigcap_{n \geqslant 0} \mathrm{Ker}\, CA^n. \tag{2.6}$$

The system Σ is called *observable* if $\mathrm{Ker}\,(C|A) = \{0\}$.

The next theorem is classical for finite dimensional systems (see, e.g., [23] and the references therein) and can be found in [3], [4] for infinite dimensional time invariant systems. The result also has a time variant analog (see [18]).

THEOREM 2.1. *A system is minimal if and only if it is controllable and observable.*

2.4. THE FIRST AND SECOND MINIMAL RESTRICTION. Each system appears in two fundamental ways as a dilation of a minimal system (see also [5]). In the proof of the next theorem one such construction is carried out.

THEOREM 2.2. *Each system is a dilation of a minimal system.*

Proof. Introduce the subspaces:

$$\mathcal{X}_1 = \mathrm{Ker}\,(C|A), \quad \mathcal{X}_0 = \left(\overline{\mathrm{Ker}\,(C|A) + \mathrm{Im}\,(A|B)}\right) \ominus \mathrm{Ker}\,(C|A),$$

$$\mathcal{X}_2 = \left(\overline{\mathrm{Ker}\,(C|A) + \mathrm{Im}\,(A|B)}\right)^{\perp}.$$

Then $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_0 \oplus \mathcal{X}_2$ and relative to this decomposition A, B, and C partition as:

$$A = \begin{bmatrix} * & * & * \\ 0 & A_0 & * \\ 0 & 0 & * \end{bmatrix}, \quad B = \begin{bmatrix} * \\ B_0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & C_0 & * \end{bmatrix}.$$

The system $\Sigma_{\mathrm{res},1} = (A_0, B_0, C_0, D; \mathcal{X}_0, \mathcal{U}, \mathcal{Y})$ is a restriction of Σ, and is minimal. ∎

The system $\Sigma_{\mathrm{res},1}$ defined in the above proof will be referred to as the *first minimal restriction* of Σ. There is also a second minimal restriction, which is defined as follows.

Given $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ introduce the subspaces:

$$\widetilde{\mathcal{X}}_1 = \mathrm{Ker}\,(C|A) \cap \overline{\mathrm{Im}\,(A|B)}, \quad \widetilde{\mathcal{X}}_0 = \overline{\mathrm{Im}\,(A|B)} \ominus (\mathrm{Ker}\,(C|A) \cap \overline{\mathrm{Im}\,(A|B)}),$$
$$\widetilde{\mathcal{X}}_2 = \left(\overline{\mathrm{Im}\,(A|B)}\right)^{\perp}.$$

Then $\mathcal{X} = \widetilde{\mathcal{X}}_1 \oplus \widetilde{\mathcal{X}}_0 \oplus \widetilde{\mathcal{X}}_2$, and relative to this decomposition A, B, and C partition as

$$A = \begin{bmatrix} * & * & * \\ 0 & \widetilde{A}_0 & * \\ 0 & 0 & * \end{bmatrix}, \quad B = \begin{bmatrix} * \\ \widetilde{B}_0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \widetilde{C}_0 & * \end{bmatrix}.$$

The system $\Sigma_{\mathrm{res},2} := (\widetilde{A}_0, \widetilde{B}_0, \widetilde{C}_0, D; \widetilde{\mathcal{X}}_0, \mathcal{U}, \mathcal{Y})$ is a restriction of Σ, and is minimal. We call $\Sigma_{\mathrm{res},2}$ the *second minimal restriction* of Σ.

2.5. ADJOINT SYSTEMS. Given $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ we define its *adjoint* Σ^* to be the system

$$\Sigma^* = (A^*, C^*, B^*, D^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}).$$

Notice that $\widetilde{\Sigma}$ is a dilation of Σ if and only if $(\widetilde{\Sigma})^*$ is a dilation of Σ^*. Hence the system Σ is minimal if and only if the same is true for Σ^*. Also, Σ is observable (controllable) if and only if Σ^* is controllable (observable).

The construction of the second minimal restriction given in the previous subsection is the dual of that of the first minimal restriction, in the sense that

$$\Sigma_{\mathrm{res},2} = ((\Sigma^*)_{\mathrm{res},1})^*. \tag{2.7}$$

2.6. SIMILARITY AND UNITARY EQUIVALENCE. Two systems $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}})$ are called *similar* if $\widetilde{\mathcal{U}} = \mathcal{U}$, $\widetilde{\mathcal{Y}} = \mathcal{Y}$, $\widetilde{D} = D$, and

$$\widetilde{A} = SAS^{-1}, \quad \widetilde{B} = SB, \quad \widetilde{C} = CS^{-1}, \tag{2.8}$$

for some bounded and boundedly invertible operator S from $\mathcal{X}$ onto $\widetilde{\mathcal{X}}$. The systems Σ and $\widetilde{\Sigma}$ are said to be *unitarily equivalent* if $\widetilde{D} = D$ and there exists a unitary operator $S : \mathcal{X} \to \widetilde{\mathcal{X}}$ such that the identities in (2.8) hold true.

If two systems Σ and $\widetilde{\Sigma}$ are similar, then their transfer functions coincide in a neighborhood of 0, that is, $\theta_{\widetilde{\Sigma}} \sim \theta_\Sigma$. The converse is also true for minimal systems with a finite dimensional state space. More precisely, if the transfer functions of two minimal systems $\Sigma_i = (A_i, B_i, C_i, D_i; \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i)$, $i = 1, 2$, with finite dimensional state spaces coincide in a neighborhood of 0, then these systems are similar. It is known (see, e.g., page 267 of [16]) that this result does not carry over to the infinite dimensional case; in the next subsection we present an example (related to but somewhat different from the one in [16]) that also will be used in Subsection 4.4 for other purposes.

2.7. AN EXAMPLE OF NON-SIMILAR MINIMAL SYSTEMS OF WHICH THE TRANSFER FUNCTIONS COINCIDE IN A NEIGHBORHOOD OF ZERO. Let θ be the entire function $\theta(z) = \mathrm{e}^{z-1}$. Notice that for t real we have

$$|\theta(\mathrm{e}^{\mathrm{i}t})| = \mathrm{e}^{\cos t - 1} = \mathrm{e}^{-2\sin^2 \frac{1}{2}t} \leqslant 1.$$

This together with the analyticity of θ shows that θ is a scalar Schur class function. We shall show that θ has minimal realizations that are not similar.

Let T be the backward shift on the Hardy space $H^2(\mathbb{D})$, that is,

$$(Th)(z) = z^{-1}(h(z) - h(0)), \quad z \in \mathbb{D}.$$

Recall that $H^2(\mathbb{D})$ consists of all analytic functions h on $\mathbb{D}$ with square summable Taylor coefficients. For each $\rho > 0$ consider the system $\Sigma_\rho = (A_\rho, B_\rho, C, D; H^2(\mathbb{D}), \mathbb{C}, \mathbb{C})$, where

$$A_\rho = \rho T, \quad (B_\rho c)(z) = \frac{\theta(\rho^{-1}z) - \theta(0)}{\rho^{-1}z} c \quad (c \in \mathbb{C}) \tag{2.9}$$

$$Ch = h(0) \quad (h \in H^2(\mathbb{D})), \qquad Dc = \theta(0)c \quad (c \in \mathbb{C}). \tag{2.10}$$

The operators A_ρ, B_ρ, C, and D are bounded linear operators, and the spectrum of A_ρ is equal to the closed disk with center zero and radius ρ. A straightforward computation shows that

$$C(I - \lambda A_\rho)^{-1} h = h(\rho\lambda), \quad |\lambda| < \rho^{-1}. \tag{2.11}$$

It follows that

$$D + \lambda C(I - \lambda A_\rho)^{-1} B_\rho = D + \lambda \frac{\theta(\lambda) - \theta(0)}{\lambda} = \theta(\lambda), \quad |\lambda| < \rho^{-1}.$$

Hence for each ρ the system Σ_ρ is a realization of θ.

All these realizations are non-similar. Indeed, if Σ_{ρ_1} and Σ_{ρ_2} are similar, then the operators A_{ρ_1} and A_{ρ_2} are similar, and hence in that case A_{ρ_1} and A_{ρ_2} must have the same spectra. Since the spectrum of A_ρ is equal to the closed disk with center zero and radius ρ, it follows that Σ_{ρ_1} and Σ_{ρ_2} are similar if and only if $\rho_1 = \rho_2$.

Next we show (using Theorem 2.1) that the systems Σ_ρ are all minimal. It is straightforward to check (use (2.11)) that Σ_ρ is observable. To prove controllability, let $\phi_\rho = B_\rho 1$. Then

$$\begin{aligned}\operatorname{Im} & \begin{bmatrix} B_\rho & A_\rho B_\rho & A_\rho^2 B_\rho & \dots & A_\rho^{k-1} B_\rho \end{bmatrix} \\ &= \operatorname{span}\{\phi_\rho, \rho T\phi_\rho, \rho^2 T^2 \phi_\rho, \dots, \rho^{k-1} T^{k-1}\phi_\rho\} \\ &= \operatorname{span}\{\phi_\rho, T\phi_\rho, T^2\phi_\rho, \dots, T^{k-1}\phi_\rho\}.\end{aligned}$$

It follows that Σ_ρ is controllable if and only if function ϕ_ρ is cyclic with respect to backward shift T on $H^2(\mathbb{D})$. According to a well-known theorem of Douglas, Shields and Shapiro ([14], Theorem 2.2.1) the latter happens if and only if ϕ_ρ does not allow for a pseudo-continuation across the circle $\mathbb{T}$. Recall that a meromorphic function η on $\mathbb{D}_e$, where $\mathbb{D}_e = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, is called a *pseudo-continuation* of $\psi \in H^2(\mathbb{D})$ if η is of bounded Nevanlinna type, i.e., η is the quotient of two functions in $H^\infty(\mathbb{D}_e)$, and the non-tangential boundary values of ψ and η coincide on the unit circle almost everywhere (see pages 267ff. of [15], pages 285ff. of [26], pages 81ff. of [27], and [13]). Since

$$\phi_\rho(z) = \frac{\theta(\rho^{-1}z) - \theta(0)}{\rho^{-1}z} = \frac{\mathrm{e}^{\rho^{-1}z-1} - \mathrm{e}^{-1}}{\rho^{-1}z}$$

has an essential singularity at infinity, the function ϕ_ρ does not have a pseudo-continuation across the circle $\mathbb{T}$, and therefore Σ_ρ is controllable. (One can prove the cyclicity of ϕ_ρ also by using the condition appearing in Problem and Solution 160 of [19].)

Summarizing we have that for each $\rho > 0$ the system Σ_ρ is a minimal realization of the Schur class function θ, and that all these realizations are mutually non-similar.

In conclusion let us mention that in this subsection the special form of θ is not important; one only has to require that θ is a non-rational entire function which is bounded by one on the unit disk. More generally, if we restrict the values of ρ to $\rho > 1$, then it suffices to require that the function $\widetilde{\theta}$, given by $\widetilde{\theta}(z) = \theta(\rho^{-1}z)$, does not have a pseudo-continuation across the circle.

3. PSEUDO-SIMILARITY

Consider two systems $\Sigma_\nu = (A_\nu, B_\nu, C_\nu, D_\nu; \mathcal{X}_\nu, \mathcal{U}, \mathcal{Y}), \nu = 1, 2$. We say that Σ_1 and Σ_2 are *pseudo-similar*, if $D_1 = D_2$, and there exists an injective closed linear operator $S(\mathcal{X}_1 \to \mathcal{X}_2)$ such that

$$\overline{\mathcal{D}(S)} = \mathcal{X}_1, \qquad \overline{\operatorname{Im}(S)} = \mathcal{X}_2, \tag{3.1}$$

$$A_1\mathcal{D}(S) \subset \mathcal{D}(S), \qquad SA_1|\mathcal{D}(S) = A_2S, \tag{3.2}$$

$$B_1\mathcal{U} \subset \mathcal{D}(S), \qquad B_2 = SB_1, \tag{3.3}$$

$$C_1|\mathcal{D}(S) = C_2S. \tag{3.4}$$

In this case we call S a *pseudo-similarity* from Σ_1 to Σ_2. (Some authors use the term weak similarity, see e.g., [30]; the term quasi-similarity is usually used for the case when $\mathcal{D}(S)$ is the full space and hence S is bounded). The vertical bar $|$ in conditions (3.2) and (3.4) means *restriction to*; for instance, $C_1|\mathcal{D}(S)$ stands for the restriction of the operator C_1 to the linear manifold $\mathcal{D}(S)$.

Conditions (3.2) and (3.3) imply that $A_1^jB_1\mathcal{U} \subset \mathcal{D}(S)$ and $SA_1^jB_1 = A_2^jB_2$ for each $j \geqslant 0$, and thus

$$\operatorname{Im}(A_1|B_1) \subset \mathcal{D}(S), \quad S[\operatorname{Im}(A_1|B_1)] = \operatorname{Im}(A_2|B_2). \tag{3.5}$$

From (3.2) – (3.4) we get that $C_1A_1^jB_1 = C_2SA_1^jB_1 = C_2A_2^jB_2$ for each $j \geqslant 0$. Hence if two systems Σ and $\widetilde{\Sigma}$ are pseudo-similar, then $\theta_{\widetilde{\Sigma}} \sim \theta_\Sigma$.

3.1. BASIC PROPERTIES. The following proposition establishes some basic properties of pseudo-similarity of systems.

PROPOSITION 3.1. *Consider two systems $\Sigma_\nu = (A_\nu, B_\nu, C_\nu, D; \mathcal{X}_\nu, \mathcal{U}, \mathcal{Y})$, $\nu = 1,2$. Suppose $S(\mathcal{X}_1 \to \mathcal{X}_2)$ is a densely defined closed injective operator with dense range. Then S is a pseudo-similarity from Σ_1 to Σ_2 if and only if the graph of S*

$$G(S) = \left\{ \begin{bmatrix} x \\ Sx \end{bmatrix} : x \in \mathcal{D}(S) \right\}$$

satisfies the following inclusions:

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} G(S) \subset G(S), \quad \operatorname{Im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset G(S) \subset \operatorname{Ker} \begin{bmatrix} C_1 & -C_2 \end{bmatrix}. \tag{3.6}$$

Moreover, if $S(\mathcal{X}_1 \to \mathcal{X}_2)$ is a pseudo-similarity from Σ_1 to Σ_2, then $S^{-1}(\mathcal{X}_2 \to \mathcal{X}_1)$ is a pseudo-similarity from Σ_2 to Σ_1, and $S^*(\mathcal{X}_2 \to \mathcal{X}_1)$ is a pseudo-similarity from Σ_2^* to Σ_1^*.

Proof. It is straightforward to check the first part of the proposition. Indeed, it suffices to note that the first inclusion in (3.6) is equivalent to condition (3.2), and that the two other inclusions in (3.6) are equivalent to conditions (3.3) and (3.4).

It remains to prove the statements appearing after formula (3.6). Therefore in what follows we assume that $S(\mathcal{X}_1 \to \mathcal{X}_2)$ is a pseudo-similarity from Σ_1 to Σ_2.

Let us prove that $S^{-1}(\mathcal{X}_2 \to \mathcal{X}_1)$ is pseudo-similarity from Σ_2 to Σ_1. Obviously, S^{-1} is a densely defined closed injective operator with dense range. Take

$\begin{bmatrix} y & S^{-1}y \end{bmatrix}^{t}$ in $G(S^{-1})$. Thus $y \in \operatorname{Im} S$ and $\begin{bmatrix} y & S^{-1}y \end{bmatrix}^{t} = \begin{bmatrix} Sx & x \end{bmatrix}^{t}$ for some $x \in \mathcal{D}(S)$. Then

$$\begin{bmatrix} A_2 & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} y \\ S^{-1}y \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ Sx \end{bmatrix} \subset \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} G(S) = G(S^{-1}).$$

Take $u \in \mathcal{U}$. Then

$$\begin{bmatrix} B_2 \\ B_1 \end{bmatrix} u = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \subset \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} G(S) = G(S^{-1}).$$

Finally,

$$G(S^{-1}) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} G(S) \subset \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \operatorname{Ker} \begin{bmatrix} C_1 & -C_2 \end{bmatrix} = \operatorname{Ker} \begin{bmatrix} C_2 & -C_1 \end{bmatrix}.$$

From these inclusions and the first part of the above proposition it follows that $S^{-1}(\mathcal{X}_2 \to \mathcal{X}_1)$ is a pseudo-similarity from Σ_2 to Σ_1.

To prove the final statement we first note that $S^*(\mathcal{X}_2 \to \mathcal{X}_1)$ is a densely defined closed injective operator with dense range (see, for instance, Chapter 3, Section 5.5 of [25]). Next, observe that $G(S)^{\perp} = G'(-S^*)$, where

$$G'(-S^*) = \left\{ \begin{bmatrix} -S^*y \\ y \end{bmatrix} : y \in \mathcal{D}(S^*) \right\}.$$

Since

$$\left(\operatorname{Im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right)^{\perp} = \operatorname{Ker} \begin{bmatrix} B_1^* & B_2^* \end{bmatrix}, \quad \left(\operatorname{Ker} \begin{bmatrix} C_1 & -C_2 \end{bmatrix}\right)^{\perp} = \overline{\operatorname{Im} \begin{bmatrix} C_1^* \\ -C_2^* \end{bmatrix}},$$

it is now simple to see by taking orthogonal complements in (3.6) that S^* is a pseudo-similarity from Σ_2^* to Σ_1^*. ∎

3.2. THE STATE SPACE PSEUDO-SIMILARITY THEOREM. The next theorem, which is an analog of the classical state space similarity theorem, has appeared as Theorem 3b.1 in [22], and Theorem 3.2 in [10] (see Theorem 9.2.3 in [30] for a continuous time version). The closedness of the constructed similarity has been proved in Proposition 6 of [2].

THEOREM 3.2. *Let Σ_1 and Σ_2 be minimal systems, and suppose that their transfer functions coincide in a neighborhood of zero. Then the two systems are pseudo-similar.*

From the above theorem it follows that for minimal systems pseudo-similarity is transitive. Indeed, if Σ_1, Σ_2, and Σ_3 are minimal systems such that Σ_1 and Σ_2 are pseudo-similar, and Σ_2 and Σ_3 are pseudo-similar, then Σ_1 and Σ_3 are pseudo-similar. To see this, notice that we have $\theta_{\Sigma_1} \sim \theta_{\Sigma_2}$, and $\theta_{\Sigma_2} \sim \theta_{\Sigma_3}$, so $\theta_{\Sigma_1} \sim \theta_{\Sigma_3}$. Since Σ_1 and Σ_3 are minimal, they are pseudo-similar by Theorem 3.2.

REMARK. In general, in contrast to systems with a finite dimensional state space, minimality of a system is not preserved under pseudo-similarity, and a pseudo-similarity between two systems does not have to be unique.

We shall see that the above statements can already be proved for systems for which the state operator and external operator are both zero, that is for systems with a transfer function of the form $\Theta(\lambda) = \lambda K$. In fact, for minimal systems with transfer functions of this simple form the statements in the above remark reduce to statements about minimal representations of K as a product of two bounded linear operators which we derived in [7]. For this purpose we need the following lemma.

LEMMA 3.3. *Let $\Theta(\lambda) = \lambda K$, where K is a bounded linear operator form $\mathcal{U}$ into $\mathcal{Y}$. Then Σ is a minimal realization of $\Theta(\lambda) = \lambda K$ if and only if*

$$\Sigma = (0, B, C, 0; \mathcal{X}, \mathcal{U}, \mathcal{Y}), \quad \overline{\operatorname{Im} B} = \mathcal{X}, \quad \operatorname{Ker} C = \{0\}, \quad \textit{and} \quad K = CB. \tag{3.7}$$

Proof. First we construct a special minimal realization of $\Theta(\lambda) = \lambda K$. Put $\mathcal{X}_\circ = \overline{K\mathcal{U}} \subset \mathcal{Y}$, and let $\tau_\circ$ be the canonical embedding of $\mathcal{X}_\circ$ into $\mathcal{Y}$. Consider the system

$$\Sigma_{\circ K} = (0, B_\circ, C_\circ, 0; \mathcal{X}_\circ, \mathcal{U}, \mathcal{Y}), \quad B_\circ = \tau_\circ^* K, \quad C_\circ = \tau_\circ. \tag{3.8}$$

Since $\tau_\circ \tau_\circ^*$ acts as the identity operator on $\operatorname{Im} K$, we have $C_\circ B_\circ = K$, and hence $\Sigma_{\circ K}$ is a realization of Θ. This realization is minimal. Indeed, $\operatorname{Im} B_\circ = \operatorname{Im} K$ and hence $\overline{\operatorname{Im} B_\circ} = \mathcal{X}_\circ$. Obviously, $\operatorname{Ker} C_\circ = \{0\}$. Thus $\Sigma_{\circ K}$ is controllable and observable, and hence minimal. Next, let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an arbitrary minimal realization of $\Theta(\lambda) = \lambda K$. Then the transfer functions of Σ and $\Sigma_{\circ K}$ coincide in the neighborhood of zero. By Theorem 3.2 there exists a pseudo-similarity S from Σ to $\Sigma_{\circ K}$. It follows that $D = 0$ (because the external operator of $\Sigma_{\circ K}$ is zero), and $SAx = 0$ for each $x \in \mathcal{D}(S)$ (because the state operator of $\Sigma_{\circ K}$ is zero). Since S is one to one, we see that $Ax = 0$ for each $x \in \mathcal{D}(S)$. But $\mathcal{D}(S)$ is dense in $\mathcal{X}$ and A is bounded. So $A = 0$. Thus the state operator and the external operator of Σ are zero as desired. The fact that $A = 0$ implies that $\operatorname{Im}(A|B) = \operatorname{Im} B$ and $\operatorname{Ker}(C|A) = \ker C$. Thus minimality of Σ implies $\overline{\operatorname{Im} B} = \mathcal{X}$ and $\operatorname{Ker} C = \{0\}$. Formula (3.7) is proved. The reverse implication is trivial. ∎

From the previous lemma it follows that $\Sigma = (0, B, C, 0; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a minimal realization of $\Theta(\lambda) = \lambda K$ if and only if $(B, C; \mathcal{X})$ is a minimal multiplicative representation of the operator K as defined in [7]. But then we can use the examples in Sections 2.3.1 and 2.3.2 of [7] to derive the statements in the above remark.

3.3. NON-UNIQUENESS IN THE STATE SPACE PSEUDO-SIMILARITY THEOREM. As we have seen in the previous section a pseudo-similarity between two systems does not have to be unique. The next two propositions present a full description of the freedom one has in the choice of the pseudo-similarity in the state space pseudo-similarity theorem. Recall (see page 166 of [25]) that a linear submanifold $\mathcal{M}$ in the domain $\mathcal{D}(T)$ of a closed linear operator $T(\mathcal{X} \to \mathcal{Y})$ is said to be a *core* of T if the closure of $T|\mathcal{M}$ is equal to T. In particular, in that case $\mathcal{M}$ is dense in $\mathcal{D}(T)$.

PROPOSITION 3.4. *Let Σ_1 and Σ_2 be minimal systems, and suppose their transfer functions coincide in a neighborhood of zero. Then there exist unique pseudo-similarities S_0 and S_1 from Σ_1 to Σ_2 such that*

$$G(S_0) \subset G(S) \subset G(S_1) \tag{3.9}$$

for each pseudo-similarity S from Σ_1 to Σ_2. In fact, S_0 is the unique pseudo-similarity from Σ_1 to Σ_2 such that $\operatorname{Im}(A_1|B_1)$ is a core for S_0, and S_1 is the unique pseudo-similarity determined by

$$G(S_1) = \bigcap_{j=0}^{\infty} \operatorname{Ker} \begin{bmatrix} C_1 A_1^j & -C_2 A_2^j \end{bmatrix}.$$

Proof. Let S be an arbitrary pseudo-similarity from Σ_1 to Σ_2, and define S_0 to be the closure of $S|\operatorname{Im}(A_1|B_1)$. Obviously, S_0 is a closed operator and $\operatorname{Im}(A_1|B_1)$ is a core for S_0. Since $G(S)$ is closed and $G(S_0)$ is the closure of $G(S|\operatorname{Im}(A_1|B_1))$, we have $G(S_0) \subset G(S)$, which proves the first inclusion in (3.9). It remains to prove that S_0 is a pseudo-similarity. Since S is a pseudo-similarity, we have

$$SA_1^j B_1 u = A_2^j B_2 u, \quad u \in \mathcal{U}, j = 0, 1, 2, \ldots,$$

and hence $S_0 \operatorname{Im}(A_1|B_1) = \operatorname{Im}(A_2|B_2)$. We proceed by showing that (3.1)–(3.4) are fulfilled. By definition, $\operatorname{Im}(A_1|B_1) \subset \mathcal{D}(S_0)$, and thus the minimality of Σ_1 yields $\overline{\mathcal{D}(S_0)} = \mathcal{X}_1$. Similarly, $\operatorname{Im} S_0 \supset \operatorname{Im}(A_2|B_2)$, and thus $\overline{\operatorname{Im} S_0} = \mathcal{X}_2$ because of the minimality of Σ_2. Thus (3.1) holds. Next, take $x \in \mathcal{D}(S_0)$. So there exist $x_1, x_2, \ldots$ in $\operatorname{Im}(A_1|B_1)$ such that $x_n \to x$ and $S_0 x_n \to S_0 x$ for $n \to \infty$. Now

$$\begin{gathered} A_1 x_n \in \operatorname{Im}(A_1|B_1) \subset \mathcal{D}(S_0), \quad A_1 x_n \to A_1 x \quad (n \to \infty); \\ S_0 A_1 x_n = S A_1 x_n = A_2 S x_n = A_2 S_0 x_n \to A_2 S_0 x \quad (n \to \infty). \end{gathered}$$

Since S_0 is closed, this shows that $A_1 x \in \mathcal{D}(S_0)$ and $S_0 A_1 x = A_2 S_0 x$. Thus (3.2) holds. Since $B_1 \mathcal{U} \subset \operatorname{Im}(A_1|B_1)$, we have $B_1 \mathcal{U} \subset \mathcal{D}(S_0)$ and $S_0 B_1 = S B_1 = B_2$, because S is a pseudo-similarity. Finally, to prove (3.4), take $x \in \mathcal{D}(S_0)$. Again there exist $x_1, x_2, \ldots$ in $\operatorname{Im}(A_1|B_1)$ such that $x_n \to x$ and $S_0 x_n \to S_0 x$ for $n \to \infty$. For the vectors x_n formula (3.4) is valid and $S_0 x_n = S x_n$. It follows that

$$C_1 x = \lim_{n\to\infty} C_1 x_n = \lim_{n\to\infty} C_2 S x_n = \lim_{n\to\infty} C_2 S_0 x_n = C_2 S_0 x,$$

which proves (3.4).

To define S_1, put

$$G_1 = \bigcap_{j=0}^{\infty} \operatorname{Ker} \begin{bmatrix} C_1 A_1^j & -C_2 A_2^j \end{bmatrix}.$$

From the definition of a pseudo-similarity it follows that $C_1 A_1^j x = C_2 A_2^j S x$ for each $x \in \mathcal{D}(S)$. Thus $G(S) \subset G_1$. Obviously, G_1 is closed. We claim that G_1 is a graph space. Indeed, we have

$$\begin{bmatrix} 0 \\ x \end{bmatrix} \in G_1 \Leftrightarrow C_2 A_2^j x = 0 \quad (j \geqslant 0) \Leftrightarrow x = 0,$$

because Σ_2 is minimal (and hence observable). Thus there exists an operator $S_1(\mathcal{X}_1 \to \mathcal{X}_2)$ such that $G_1 = G(S_1)$. With this choice of S_1 formula (3.9) is proved.

Let us prove that S_1 is a pseudo-similarity. From $G(S) \subset G(S_1)$ we see that $\mathcal{D}(S) \subset \mathcal{D}(S_1)$ and $\operatorname{Im} S \subset \operatorname{Im} S_1$, and thus the domain and range of S_1 are dense in $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. Notice that

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \in G(S_1) = G_1 \;\Leftrightarrow\; C_1 A_1^j x = 0 \quad (j \geqslant 0) \;\Leftrightarrow\; x = 0,$$

because Σ_1 is minimal. Thus S_1 is injective. From the definition of $G_1 = G(S_1)$ and (3.9) we immediately see that (3.6) holds for S_1 in place of S. Thus S_1 is a pseudo-similarity from Σ_1 to Σ_2. Finally, notice that (3.9) determines S_0 and S_1 uniquely. ∎

Let Σ_1 and Σ_2 be minimal systems, and suppose their transfer functions coincide in a neighborhood of zero. Let us write $S_{\min}$ for the pseudo-similarity S_0 and $S_{\max}$ for the pseudo-similarity S_1 appearing in (3.9). We shall refer to $S_{\min}$ and $S_{\max}$ as the *minimal* and *maximal* pseudo-similarities from Σ_1 to Σ_2 with respect to graph space inclusion. We write $S_{*,\min}$ and $S_{*,\max}$ for the minimal and maximal pseudo-similarities from $(\Sigma_2)^*$ to $(\Sigma_1)^*$. We claim that

$$(S_{\min})^* = S_{*,\max}\,, \quad (S_{\max})^* = S_{*,\min}\,. \tag{3.10}$$

Indeed, an arbitrary pseudo-similarity E from $(\Sigma_2)^*$ to $(\Sigma_1)^*$ is of the form $E = S^*$, where S is a pseudo-similarity from Σ_1 to Σ_2. Thus, by taking orthogonal complements in (3.9), we see that

$$G((S_{\max})^*) \subset G(E) \subset G((S_{\min})^*).$$

Since $(S_{\max})^*$ and $(S_{\min})^*$ are pseudo-similarities from $(\Sigma_2)^*$ to $(\Sigma_1)^*$ and E is an arbitrary one, the above inclusions yield (3.10) because of Proposition 3.4.

PROPOSITION 3.5. *Let Σ_1 and Σ_2 be minimal systems, and suppose their transfer functions coincide in a neighborhood of zero. Let G be a closed subspace of $\mathcal{X}_1 \oplus \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are the state spaces of Σ_1 and Σ_2, respectively. Then $G = G(S)$ for some pseudo-similarity S from Σ_1 to Σ_2 if and only if*

$$G(S_{\min}) \subset G \subset G(S_{\max}), \quad \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} G \subset G. \tag{3.11}$$

Here $S_{\min}$ and $S_{\max}$ are the minimal and maximal pseudo-similarities from Σ_1 to Σ_2 with respect to graph space inclusion, and A_1 and A_2 are the state operators of Σ_1 and Σ_2, respectively.

Proof. Assume $G = G(S)$ for some pseudo-similarity from Σ_1 to Σ_2. Then the first part of (3.11) is covered by (3.9). The first inclusion in (3.6) yields the second part of (3.11).

To prove the converse, assume (3.11) holds. Since $G \subset G(S_{\max})$ and G is a linear space, it follows that G is a graph space, that is, there exists an operator S with domain $\mathcal{D}(S)$ in $\mathcal{X}_1$ and range in $\mathcal{X}_2$ such that $G = G(S)$. The fact that G is closed implies that S is a closed operator. From $G(S_{\min}) \subset G(S)$ it follows that $\mathcal{D}(S_{\min}) \subset \mathcal{D}(S)$ and $\operatorname{Im} S_{\min} \subset \operatorname{Im} S$. Thus, as $S_{\min}$, the operator S is densely defined and has a dense range. On the other hand the inclusion $G(S) \subset G(S_{\max})$ shows that S is injective. Thus in order to show that S is a pseudo-similarity it suffices to show that S satisfies (3.6). The first inclusion in (3.6) is fulfilled because we assume (3.11) holds. By applying the second part of (3.6) to $S_{\min}$ and $S_{\max}$ we see that

$$\operatorname{Im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset G(S_{\min}) \subset G(S), \quad G(S) \subset G(S_{\max}) \subset \operatorname{Ker} \begin{bmatrix} C_1 & -C_2 \end{bmatrix}.$$

From these inclusions it follows that S satisfies the second part of (3.6) too. Thus S is a pseudo-similarity. ∎

COROLLARY 3.6. *Let Σ_1 and Σ_2 be minimal systems, and let S be a pseudo-similarity from Σ_1 to Σ_2. If $\mathcal{D}(S) = \mathcal{X}_1$ (and hence $S \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$) or $\operatorname{Im} S = \mathcal{X}_2$ (and hence $S^{-1} \in \mathcal{L}(\mathcal{X}_2, \mathcal{X}_1)$), then S is the only pseudo-similarity from Σ_1 to Σ_2.*

Proof. Since S^{-1} is a pseudo-similarity from Σ_2 to Σ_1 and $\mathcal{D}(S^{-1}) = \operatorname{Im} S$, it suffices to prove the corollary for $\mathcal{D}(S) = \mathcal{X}_1$.

So assume $\mathcal{D}(S) = \mathcal{X}_1$. Let $S_{\min}$ and $S_{\max}$ be the minimal and maximal pseudo-similarities from Σ_1 to Σ_2 with respect to graph space inclusion. Since S is closed, the assumption $\mathcal{D}(S) = \mathcal{X}_1$ implies that S is bounded. According to (3.9) we have $G(S_{\min}) \subset G(S)$, and thus

$$\|S_{\min}x\| = \|Sx\| \leqslant \|S\|\|x\|, \quad x \in \mathcal{D}(S_{\min}).$$

Thus $S_{\min}$ is bounded too. This can only happen when $\mathcal{D}(S_{\min}) = \mathcal{X}_1$, because $S_{\min}$ is closed and densely defined. Thus $S_{\min} = S$. On the other hand, from $\mathcal{D}(S) = \mathcal{X}_1$ and $G(S) \subset G(S_{\max})$ it also follows that $\mathcal{D}(S_{\max}) = \mathcal{X}_1$. Therefore $S = S_{\max}$, and hence S is the only pseudo-similarity from Σ_1 to Σ_2. ∎

One can construct an example (use Lemma 3.3 and the operators S and $\widehat{S}$ in Section 2.3.2 of [7]) such that $G(S_{\max})/G(S_{\min})$ has dimension one, and hence in that case $S_{\min}$ and $S_{\max}$ are the only two pseudo-similarities.

To conclude this chapter let us return to the systems

$$\Sigma_\rho = (A_\rho, B_\rho, C, D; H^2(\mathbb{D}), \mathbb{C}, \mathbb{C}), \quad \rho > 0,$$

considered in Subsection 2.7. Thus $A_\rho, B_\rho, C,$ and D are the operators defined in (2.9) and (2.10). Recall that for each $\rho > 0$ the system Σ_ρ is minimal and in a neighborhood of zero its transfer function coincides with the function $\theta(z) =$

e^{z-1}. Nevertheless, as we have seen in Subsection 2.7, the systems Σ_ρ, $\rho > 0$, are not mutually similar. On the other hand, according to Theorem 3.2, they must be mutually pseudo-similar. In fact, in this case the pseudo-similarity from Σ_{ρ_1} to Σ_{ρ_2} is unique and easy to describe. Indeed, assume $\rho_1 \neq \rho_2$ and put $\eta = \rho_1/\rho_2$. Let S be the operator in $H^2(\mathbb{D})$ defined by

$$\mathcal{D}(S) = \{h \in H^2(\mathbb{D}) : \lambda \mapsto h(\eta\lambda) \text{ belongs to } H^2(\mathbb{D})\},$$
$$(Sh)(\lambda) = h(\eta\lambda), \quad \lambda \in \mathbb{D}.$$

For $0 < \eta < 1$ we have $\mathcal{D}(S) = H^2(\mathbb{D})$ and for $\eta > 1$ we have $\operatorname{Im} S = H^2(\mathbb{D})$. It is straightforward to check that S is a pseudo-similarity from Σ_{ρ_1} to Σ_{ρ_2}. Since either $\mathcal{D}(S)$ or $\operatorname{Im} S$ is equal to the state space $H^2(\mathbb{D})$, there are no other pseudo-similarities from Σ_{ρ_1} to Σ_{ρ_2} by Corollary 3.6.

4. THE KALMAN-YAKUBOVICH-POPOV INEQUALITY FOR THE SCATTERING CASE

In this section we will prove the first main theorem of this article (Theorem 1.2). First we will introduce contractive systems, and give some elementary properties. A system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is called *contractive* if for each initial state $x_0 \in H$ and each input sequence $(u_k)_{k \geqslant 0}$ we have

$$\|u_n\|^2 - \|y_n\|^2 \geqslant \|x_{n+1}\|^2 - \|x_n\|^2 \quad (n \geqslant 0). \tag{4.1}$$

Here for $n \geqslant 0$ the vectors x_{n+1} and y_n are determined from u_n and x_n via the equations (1.1) from the introduction. In this case the adjoint system Σ^* is also contractive. To see this, notice that the system Σ is contractive if and only if its system matrix M_Σ,

$$M_\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

is contractive. Since $M_{(\Sigma^*)} = (M_\Sigma)^*$, it follows that Σ is contractive if and only if Σ^* is contractive. We will show the following theorem.

THEOREM 4.1. *A system is dissipative with respect to the supply rate function* (1.3) *if and only if it is pseudo-similar to a contractive system.*

In the above theorem one cannot replace the word pseudo-similar by just similar. Indeed, it is possible that a system which is dissipative with respect to the supply rate $w(u, y) = \|u\|^2 - \|y\|^2$ is not similar to any contractive system. An example will be given in Subsection 4.4.

In the next subsection we show that with each generalized solution of the Kalman-Yakubovich-Popov inequality we can associate in a canonical way a contractive system. The proof of the above theorem is given in the second subsection. In the third subsection we use Theorem 4.1 to prove Theorem 1.2.

4.1. THE SYSTEM ASSOCIATED WITH THE KYP-INEQUALITY. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a dissipative system with respect to the supply rate function (1.3). In other words, there exists a generalized solution H to the Kalman-Yakubovich-Popov inequality for Σ. With Σ and H, we shall associate a system Σ_H in a canonical way. Since $H(\mathcal{X} \to \mathcal{X})$ is a positive operator, the same is true for $H^{1/2}(\mathcal{X} \to \mathcal{X})$. Moreover, since H is injective, $H^{1/2}$ is injective, and $\operatorname{Im} H^{1/2}$ is dense in $\mathcal{X}$. By specifying (1.7) for the vectors $(x, 0)$ and $(0, u)$ we see that

$$\|H^{1/2}x\|^2 - \|H^{1/2}Ax\|^2 - \|Cx\|^2 \geqslant 0, \quad \|u\|^2 - \|Du\|^2 - \|H^{1/2}Bu\|^2 \geqslant 0, \tag{4.2}$$

for each $x \in \mathcal{D}(H^{1/2})$ and each $u \in \mathcal{U}$. Introduce the operator

$$A_H : \operatorname{Im} H^{1/2} \to \mathcal{X}; \quad A_H(H^{1/2}x) = H^{1/2}Ax \quad (x \in \mathcal{D}(H^{1/2})). \tag{4.3}$$

Then A_H is well-defined, because $H^{1/2}$ is injective. Since

$$\|A_H(H^{1/2}x)\| = \|H^{1/2}Ax\| \leqslant \|H^{1/2}x\|, \quad x \in \mathcal{D}(H^{1/2}),$$

the operator A_H is contractive on $\operatorname{Im} H^{1/2}$. We extend A_H by continuity to a contraction, also denoted by A_H, on $\mathcal{X} = \overline{\operatorname{Im} H^{1/2}}$. Define $B_H : \mathcal{U} \to \mathcal{X}$ by $B_H u = H^{1/2}Bu$. Then

$$\|B_H u\| = \|H^{1/2}Bu\| \leqslant \|u\|, \quad u \in \mathcal{U},$$

hence B_H is a contractive operator. Define $C_H : \operatorname{Im} H^{1/2} \to \mathcal{Y}$ by $C_H H^{1/2}x = Cx$, for $x \in \mathcal{D}(H^{1/2})$. Then

$$\|C_H H^{1/2}x\| = \|Cx\| \leqslant \|H^{1/2}x\|,$$

hence C_H is a contractive operator. The operator C_H extends by continuity to a contraction from $\mathcal{X} = \overline{\operatorname{Im} H^{1/2}}$ into $\mathcal{Y}$. The system $\Sigma_H = (A_H, B_H, C_H, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is well-defined, and will be called the *system associated to the generalized solution H of the* KYP-*inequality for Σ*. Sometimes we also refer to Σ_H as the *system associated to H and Σ*.

PROPOSITION 4.2. *Assume H is a generalized solution to the Kalman-Yakubovich-Popov inequality for the system Σ, and let $\Sigma_H = (A_H, B_H, C_H, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be the associated system. Then Σ_H is contractive, the systems Σ and Σ_H are pseudo-similar, and $H^{1/2}$ is a pseudo-similarity from Σ to Σ_H.*

Proof. The system Σ_H is contractive, because for each $x \in \mathcal{D}(H^{1/2})$ and $u \in \mathcal{U}$ we have

$$\begin{aligned} 0 &\leqslant K_\Sigma(H) \begin{bmatrix} x \\ u \end{bmatrix} \\ &= \left\| \begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} H^{1/2}A & H^{1/2}B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} A_H H^{1/2} & B_H \\ C_H H^{1/2} & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} A_H & B_H \\ C_H & D \end{bmatrix} \begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \right\|^2. \end{aligned}$$

By continuity it follows that Σ_H is a contractive system.

The operator $H^{1/2}(\mathcal{X} \to \mathcal{X})$ is closed, injective, and densely defined. Since $H^{1/2}$ is selfadjoint, $\operatorname{Im} H^{1/2}$ is dense in $\mathcal{X}$. Take $x \in \mathcal{D}(H^{1/2})$. Then

$$\begin{bmatrix} A & 0 \\ 0 & A_H \end{bmatrix} \begin{bmatrix} x \\ H^{1/2}x \end{bmatrix} = \begin{bmatrix} Ax \\ A_H H^{1/2}x \end{bmatrix} = \begin{bmatrix} Ax \\ H^{1/2}Ax \end{bmatrix} \in G(H^{1/2}),$$

by the first inclusion in (1.4). The second inclusion of (1.4) yields

$$\begin{bmatrix} B \\ B_H \end{bmatrix} u = \begin{bmatrix} Bu \\ H^{1/2}Bu \end{bmatrix} \in G(H^{1/2}).$$

Take $x \in \mathcal{D}(H^{1/2})$. From

$$\begin{bmatrix} C & -C_H \end{bmatrix} \begin{bmatrix} x \\ H^{1/2}x \end{bmatrix} = Cx - C_H H^{1/2}x = Cx - Cx = 0,$$

it follows that $G(H^{1/2}) \subset \operatorname{Ker} \begin{bmatrix} C & -C_H \end{bmatrix}$. Thus, the operator $H^{1/2}$ establishes a pseudo-similarity from Σ to Σ_H. ∎

PROPOSITION 4.3. *Let H be a generalized solution to the* KYP-*inequality for the system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, and let Σ_H be the associated system. Then Σ_H is minimal if and only if*

$$\overline{H^{1/2}\operatorname{Im}(A|B)} = \mathcal{X}, \quad \overline{(H^{1/2})^{-1}\operatorname{Im}(A^*|C^*)} = \mathcal{X}. \tag{4.4}$$

Proof. From the identity

$$\operatorname{Im}(A_H|B_H) = \operatorname*{span}_{n\geqslant 0} \operatorname{Im} A_H^n B_H = H^{1/2}\operatorname{Im}(A|B), \tag{4.5}$$

we see that Σ_H is controllable if and only if $H^{1/2}\operatorname{Im}(A|B)$ is dense in $\mathcal{X}$.

Since, by Proposition 4.2, the operator $H^{1/2}$ is a pseudo-similarity from Σ to Σ_H, we know that $(H^{1/2})^* = H^{1/2}$ is a pseudo-similarity from $(\Sigma_H)^*$ to Σ^*. Consequently,

$$A_H^{*n} C_H^* \mathcal{Y} \subset \mathcal{D}(H^{1/2}) \quad \text{and} \quad H^{1/2} A_H^{*n} C_H^* = A^{*n} C^*$$

for each $n \geqslant 0$. We conclude that $H^{1/2} \mathrm{Im}\,(A_H^* | C_H^*) = \mathrm{Im}\,(A^* | C^*)$. It follows that the system Σ_H is observable if and only if $(H^{1/2})^{-1} \mathrm{Im}\,(A^* | C^*)$ is dense in $\mathcal{X}$. ∎

PROPOSITION 4.4. *Let H be a generalized solution to the* KYP*-inequality for the system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, and assume that* $\mathrm{Im}\,(A|B)$ *is a core for $H^{1/2}$. Then* $\overline{H^{1/2} \mathrm{Im}\,(A|B)} = \mathcal{X}$.

Proof. Since $\mathrm{Im}\, H^{1/2}$ is dense in $\mathcal{X}$, it suffices to show that

$$\mathrm{Im}\, H^{1/2} \subset \overline{H^{1/2} \mathrm{Im}\,(A|B)}. \tag{4.6}$$

Take $y \in \mathrm{Im}\, H^{1/2}$. Thus $y = H^{1/2} x$ for some $x \in \mathcal{D}(H^{1/2})$. Since $\mathrm{Im}\,(A|B)$ is a core for $H^{1/2}$, there exists a sequence $x_1, x_2, \ldots$ in $\mathrm{Im}\,(A|B)$ such that $x_n \to x$ and $H^{1/2} x_n \to y$. Obviously, $H^{1/2} x_n \in H^{1/2} \mathrm{Im}\,(A|B)$. Thus $y \in \overline{H^{1/2} \mathrm{Im}\,(A|B)}$, and (4.6) is proved. ∎

4.2. PROOF OF THEOREM 4.1.

Proof. Assume the system Σ is dissipative with respect to (1.3). Thus there exists a generalized solution H to the KYP-inequality for Σ. Let Σ_H be the system associated to H and Σ. By Proposition 4.2, the system Σ_H is contractive, and Σ and Σ_H are pseudo-similar. Thus Σ is pseudo-similar to contractive system.

To prove the converse implication, let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be pseudo-similar to the contractive system $\Upsilon = (\widetilde{A}, \widetilde{B}, \widetilde{C}, D; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$, and let the pseudo-similarity be given by $S(\mathcal{X} \to \widetilde{\mathcal{X}})$. We shall show that $H = S^* S$ is a generalized solution to the KYP-inequality with respect to Σ. Since S is closed and densely defined, the operator $H(\mathcal{X} \to \mathcal{X})$ is selfadjoint (by Chapter 5, Theorem 3.24 in [25]). The operator S is injective, hence

$$\langle Hx, x \rangle = \|Sx\|^2 > 0, \quad (x \in \mathcal{D}(H), x \neq 0)$$

and the operator H is positive. Since $\mathcal{D}(H^{1/2}) = \mathcal{D}(S)$ (see Chapter 6, Theorem 2.23 of [25], and also formula (2.22) in the same chapter), the similarity conditions (3.2) and (3.3) yield

$$A \mathcal{D}(H^{1/2}) \subset \mathcal{D}(H^{1/2}), \quad B \mathcal{U} \subset \mathcal{D}(H^{1/2}).$$

By the polar decomposition (see page 334 of [25]), we have $U H^{1/2} = S$, where $U : \mathcal{X} \to \widetilde{\mathcal{X}}$ is a partial isometry with initial space $\overline{\mathrm{Im}\, H^{1/2}}$ and final space $\overline{\mathrm{Im}\, S}$. Since S is a pseudo-similarity, $\overline{\mathrm{Im}\, S} = \widetilde{\mathcal{X}}$, and since $H^{1/2}$ is injective and selfadjoint, $\overline{\mathrm{Im}\, H^{1/2}} = \mathcal{X}$. It follows that U is unitary.

Take $x \in \mathcal{D}(H^{1/2})$ and $u \in \mathcal{U}$. Then

$$
\begin{aligned}
K_\Sigma(H)\begin{bmatrix} x \\ u \end{bmatrix} &= \left\| \begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2 \\
&= \left\| \begin{bmatrix} Sx \\ u \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} SA & SB \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2 \\
&= \left\| \begin{bmatrix} Sx \\ u \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & D \end{bmatrix} \begin{bmatrix} Sx \\ u \end{bmatrix} \right\|^2 \geqslant 0,
\end{aligned}
$$

because Υ is a contractive system. Thus H is a generalized solution to the KYP-inequality for Σ. ∎

The proof of Theorem 4.1 also yields the first part of the following proposition.

PROPOSITION 4.5. *Let S be a pseudo-similarity from Σ to Σ_1, and assume that Σ_1 is contractive. Then $H = S^*S$ is a generalized solution to the* KYP*-inequality for Σ. Moreover, the polar decomposition of S is given by $S = UH^{1/2}$, with $U : \mathcal{X} \to \mathcal{X}_1$ being a unitary operator, and the system Σ_H associated to H and Σ is unitarily equivalent to Σ_1 with U providing the unitary equivalence. In particular, if $S = H^{1/2}$, then $\Sigma_H = \Sigma_1$.*

Proof. The proof of the first statement is contained in (the second and third paragraph of) the proof of Theorem 4.1. In the proof of this theorem it was also shown that $S = UH^{1/2}$, with $U : \mathcal{X} \to \mathcal{X}_1$ a unitary operator. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. We show that the system $\Sigma_H = (A_H, B_H, C_H, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ associated to H and Σ, and the system $\Sigma_1 = (A_1, B_1, C_1, D; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ are unitarily equivalent to the unitary equivalence being provided by U. For $x \in \mathcal{D}(H^{1/2}) = \mathcal{D}(S)$ the identities

$$A_H(H^{1/2}x) = H^{1/2}Ax = U^*SAx = U^*A_1Sx = U^*A_1U(H^{1/2}x) \tag{4.7}$$

$$C_H H^{1/2}x = Cx = C_1Sx = C_1U(H^{1/2}x) \tag{4.8}$$

hold, and since $H^{1/2}$ is densely defined, it follows by continuity that $A_H = U^*A_1U$, and $C_H = C_1U$. Finally, for $u \in \mathcal{U}$ we have $UB_Hu = UH^{1/2}Bu = SBu = B_1u$. The proposition follows. ∎

PROPOSITION 4.6. *If H is a generalized solution to the* KYP*-inequality for the system Σ, then H^{-1} is a generalized solution to the* KYP*-inequality for the system Σ^*,*

$$(\Sigma_H)^* = (\Sigma^*)_{H^{-1}}. \tag{4.9}$$

Proof. By Proposition 4.2 the selfadjoint operator $H^{1/2}$ establishes a pseudo-similarity from Σ to Σ_H. Hence $(H^{1/2})^* = H^{1/2}$ is a pseudo-similarity from $(\Sigma_H)^*$ to Σ^*, and thus $(H^{1/2})^{-1}$ is a pseudo-similarity from Σ^* to $(\Sigma_H)^*$. The operator $H^{1/2}$ is defined as the unique non-negative selfadjoint operator such that $(H^{1/2})^2 = H$ (see Chapter 5, Theorem 3.35 of [25]). Hence $\mathcal{D}(H) = \{x \in \mathcal{D}(H^{1/2}) : H^{1/2}x \in \mathcal{D}(H^{1/2})\}$. It follows that

$$(H^{1/2})^{-1}(H^{1/2})^{-1}Hx = (H^{1/2})^{-1}H^{1/2}x = x, \quad x \in \mathcal{D}(H).$$

Put $K = (H^{1/2})^{-1}(H^{1/2})^{-1}$. The previous identity shows that K is an extension of H^{-1}. Since $K = S^*S$, where S is the selfadjoint operator $(H^{1/2})^{-1}$, we know that K is selfadjoint. Thus K is a selfadjoint extension of the selfadjoint operator H^{-1}, which implies that $K = H^{-1}$, that is, $H^{-1} = (H^{1/2})^{-1}(H^{1/2})^{-1}$. Since $(\Sigma_H)^*$ is a contractive system, we can use Proposition 4.5 to show that H^{-1} is a generalized solution to the KYP-inequality for Σ^*.

It remains to prove (4.9). From $H^{-1} = (H^{1/2})^{-1}(H^{1/2})^{-1}$ and $(H^{1/2})^{-1}$ nonnegative it follows that $(H^{1/2})^{-1} = (H^{-1})^{1/2}$. As we have shown in the previous paragraph, the operator $(H^{-1})^{1/2}$ is a pseudo-similarity from Σ^* to $(\Sigma_H)^*$. Now apply Proposition 4.5 with $S = (H^{-1})^{1/2}$. It follows that (4.9) holds, and the proof is complete. ∎

4.3. PROOF OF THEOREM 1.2.

Proof. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system. Assume first that the KYP-inequality for Σ has a generalized solution. In other words, assume Σ is dissipative with respect to (1.3). By Theorem 4.1 this implies that Σ is pseudo-similar to a contractive system $\widetilde{\Sigma}$. Because of the pseudo-similarity, the transfer function θ_Σ coincides with a Schur class function $\theta_{\widetilde{\Sigma}}$ in a neighborhood of zero.

Assume now that the transfer function θ_Σ coincides with a Schur class function θ in a neighborhood of 0. Let $\widetilde{\Sigma}$ be a minimal contractive realization of θ. Since Σ and $\widetilde{\Sigma}$ are both minimal, the fact that $\theta_\Sigma \sim \theta_{\widetilde{\Sigma}}(= \theta)$ in a neighborhood of zero implies (see Theorem 3.2) that there exists a pseudo-similarity S from Σ to $\widetilde{\Sigma}$. Proposition 4.5 shows that $H = S^*S$ is a generalized solution to the KYP-inequality for Σ. ∎

4.4. PSEUDO-SIMILARITY VERSUS ORDINARY SIMILARITY. In Theorem 4.1 it is shown that a system is dissipative with respect to the supply rate (1.3) if and only if it is pseudo-similar to a contractive system. In this statement the condition of pseudo-similarity cannot be replaced by ordinary similarity (i.e., with a bounded and boundedly invertible similarity operator). In fact, it may happen that a system Σ which is dissipative with respect to the supply rate function (1.3) is not similar (with a bounded and bounded invertible similarity) to any contractive system. To present an example, take $\rho > 1$, and consider the system

$$\Sigma_\rho = (A_\rho, B_\rho, C, D; H^2(\mathbb{D}), \mathbb{C}, \mathbb{C}), \tag{4.10}$$

where A_ρ, B_ρ, C, and D are the operators defined in (2.9) and (2.10). Notice that the spectrum $\sigma(A_\rho) = \rho\overline{\mathbb{D}}$ contains points outside the closed unit disk (because $\rho > 1$). Thus Σ_ρ is not similar to any contractive system. Next we show that Σ_ρ is dissipative with respect to the supply rate (1.3). To do this, notice that the transfer function of Σ_ρ coincides with the Schur class function $\theta(z) = \mathrm{e}^{z-1}$ in a neighborhood of 0 (see Subsection 2.7). From Subsection 2.7 we also know that Σ_ρ is minimal. By Theorem 1.2 the KYP-inequality for the system Σ_ρ has a generalized solution. By Proposition 4.2 the system Σ_ρ is pseudo-similar to a contractive system. By Theorem 4.1 the system Σ_ρ is dissipative with respect to the supply rate (1.3).

4.5. AN EXAMPLE OF A KYP-INEQUALITY WITH ALL GENERALIZED SOLUTIONS UNBOUNDED. Let Σ be the system Σ_ρ in (4.10), with $\rho > 1$ being fixed. We conclude this section by showing that all generalized solutions to the KYP-inequality for this Σ are unbounded. Indeed, let H be a generalized solution to the KYP-inequality for Σ, and assume $H \in \mathcal{L}(\mathcal{X})$, where $\mathcal{X}$ is the state space of $\Sigma = \Sigma_\rho$. Then $H^{1/2}$ is a pseudo-similarity from Σ to Σ_H. In particular, using $\mathcal{D}(H) = \mathcal{X}$, we have

$$A_H H^{1/2}\phi = H^{1/2}A_\rho\phi = H^{1/2}\rho T\phi, \quad \phi \in \mathcal{X}. \tag{4.11}$$

Recall that the state space $\mathcal{X}$ of $\Sigma = \Sigma_\rho$ is the Hardy space $H^2(\mathbb{D})$, and T is the backward shift on this space. It follows that every point z in $\mathbb{C}$ with $|z| < \rho$ is an eigenvalue of ρT with $\phi_z(\lambda) = (1 - \rho^{-1}z\lambda)^{-1}$ as corresponding eigenvector. So, for $1 < |z| < \rho$ the function $H^{1/2}\phi_z$ is an eigenvector of A_H with eigenvalue z, because of (4.11). This is impossible. Indeed, A_H is a contraction and hence the eigenvalues of A are in the closed unit disk. Thus H cannot be bounded. One can construct more elaborate examples showing that both H and H^{-1} are unbounded operators.

5. ORDER PROPERTIES OF THE GENERALIZED SOLUTIONS OF THE KYP-INEQUALITY

To state our second main theorem we need the following partial ordering on the set of non-negative selfadjoint operators, which is taken from page 330, formula (2.17), and the remark below, in [25]. Let H_1, H_2 be non-negative selfadjoint operators acting in $\mathcal{X}$. We define

$$H_1 \prec H_2$$

if $\mathcal{D}(H_2^{1/2}) \subset \mathcal{D}(H_1^{1/2})$ and $\|H_1^{1/2}x\| \leqslant \|H_2^{1/2}x\|$ for each $x \in \mathcal{D}(H_2^{1/2})$. Notice that if H_1 and H_2 are bounded, then $H_1 \prec H_2$ means $H_1 \leqslant H_2$. The next theorem is the main theorem of this section.

THEOREM 5.1. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system, which is dissipative with respect to the supply rate* (1.3). *Then the set of all generalized solutions H to the* KYP-*inequality for Σ which have the following two additional properties*

(i) $H^{1/2}\mathrm{Im}\,(A|B)$ *and* $(H^{1/2})^{-1}\mathrm{Im}\,(A^*|C^*)$ *are dense in* $\mathcal{X}$,
(ii) $\mathrm{Im}\,(A|B)$ *is a core for the operator* $H^{1/2}$,

is not empty and this set contains a minimal element $H_\circ$ and a maximal element $H_\bullet$ with respect to the ordering $\prec$.

The conditions (i) and (ii) in the above theorem are not independent. In fact, if (ii) holds, then $H^{1/2}\mathrm{Im}\,(A|B)$ is dense in $\mathcal{X}$ by Proposition 4.4. On the other hand, condition (ii) does not imply (i). To see the latter, we note that from the examples in Section 2.3 in [7] we can obtain (using Lemma 3.3) two pseudo-similar contractive systems,

$$\Sigma_1 = (0, B_1, C_1, 0; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \quad \text{and} \quad \Sigma_2 = (0, B_2, C_2, 0; \mathcal{X}_2, \mathcal{U}, \mathcal{Y})$$

such that Σ_1 is minimal while Σ_2 is not. Furthermore, we can choose (see formulas (2.8) and (2.9) in [7]) a pseudo-similarity $\widehat{S}$ from Σ_1 to Σ_2 such that $\widehat{S}^* = C_1^*$, and hence $\mathrm{Im}\,\widehat{S}^* = \mathrm{Im}\,C_1^*$. Now put $\Sigma = \Sigma_1^*$. Notice that $(\widehat{S}^{-1})^*$ is a pseudo-similarity from Σ to Σ_2^*. Put $H = (\widehat{S}^{-1})(\widehat{S}^{-1})^*$. Since Σ_2 is contractive, the same holds true for Σ_2^*, and hence we can apply Proposition 4.5 to show that H is a generalized solution to the KYP-inequality for Σ, and that Σ_H is unitarily equivalent to Σ_2^*. Thus Σ_H is not minimal, because Σ_2^* is not minimal. According to Proposition 4.3, this implies that for this choice of Σ and H condition (i) in the above theorem is not satisfied. Next, notice that $\Sigma = \Sigma_1^* = (0, C_1^*, B_1^*, 0; \mathcal{X}_1, \mathcal{Y}, \mathcal{U})$. Using $(\widehat{S}^{-1})^* = (\widehat{S}^*)^{-1}$, we have

$$\mathrm{Im}\,C_1^* = \mathrm{Im}\,\widehat{S}^* = \mathcal{D}((\widehat{S}^*)^{-1}) = \mathcal{D}((\widehat{S}^{-1})^*).$$

In particular (see Proposition 4.5) the domain of $H^{1/2}$ is equal to $\mathrm{Im}\,C_1^*$, and hence condition (ii) in Theorem 5.1 is trivially satisfied. Thus (ii) does not imply (i).

We proceed with some notation. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system. The set of all generalized solutions H of the KYP-inequality for Σ will be denoted by $\mathcal{GK}_\Sigma$, and we write $\mathcal{CK}_\Sigma$ for all classical solutions H of the KYP-inequality for Σ, i.e., all generalized solutions H that are bounded and boundedly invertible. When the state space $\mathcal{X}$ is finite dimensional, then the sets $\mathcal{GK}_\Sigma$ and $\mathcal{CK}_\Sigma$ coincide, and are equal to the set $\mathcal{K}_\Sigma$ defined by (1.10). The following two subsets of $\mathcal{GK}_\Sigma$ will be important in the sequel:

$$\mathcal{GK}_\Sigma^{\min} = \{H \in \mathcal{GK}_\Sigma : \Sigma_H \text{ is minimal}\}, \tag{5.1}$$

$$\mathcal{GK}_{\Sigma,\text{core}}^{\min} = \{H \in \mathcal{GK}_\Sigma^{\min} : \mathrm{Im}\,(A|B) \text{ is a core for the operator } H^{1/2}\}. \tag{5.2}$$

Recall (see Proposition 4.3) that Σ_H is minimal if and only if condition (i) in Theorem 5.1 is satisfied. Thus, using the above notation, Theorem 5.1 can be reformulated as follows. *If Σ is minimal and dissipative with respect to the supply rate* (1.3), *then the set $\mathcal{GK}_{\Sigma,\text{core}}^{\min}$ is non-empty and with respect to the ordering $\prec$ this set has a minimal and a maximal element.*

Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system, and let H be a generalized solution of the KYP-inequality for Σ which is bounded and boundedly invertible, i.e., H is a classical solution. Then, trivially, $\mathrm{Im}\,(A|B)$ is a core for $H^{1/2}$. Furthermore, $H^{1/2}$ is a usual (i.e., bounded and boundedly invertible) similarity from Σ to Σ_H. Since Σ is assumed to be minimal, the same holds true for Σ_H. We conclude that $H \in \mathcal{GK}_{\Sigma,\text{core}}^{\min}$. Hence we have the following inclusions:

$$\mathcal{CK}_\Sigma \subset \mathcal{GK}_{\Sigma,\text{core}}^{\min} \subset \mathcal{GK}_{\Sigma}^{\min} \subset \mathcal{GK}_\Sigma. \tag{5.3}$$

However, notice that for a minimal dissipative system it may happen (as we know from Subsection 4.5) that $\mathcal{CK}_\Sigma$ is empty while for such a system $\mathcal{GK}_{\Sigma,\text{core}}^{\min}$ is always non-empty. In particular, the first inclusion in (5.3) can be strict. The second inclusion in (5.3) can also be strict (see Subsection 5.5).

As a first step towards the proof of Theorem 5.1 we shall establish the following result.

THEOREM 5.2. *Let Σ be a minimal system which is dissipative with respect to the supply rate* (1.3), *and let θ be the Schur class function coinciding with the transfer function of Σ in a neighborhood of* 0. *Then each minimal and contractive realization of θ is unitarily equivalent to a system Σ_H for some unique generalized solution $H \in \mathcal{GK}_{\Sigma,\text{core}}^{\min}$.*

For the case when the state operator and external operator of Σ are both zero, Theorems 5.1 and 5.2 reduce to Theorems 1.4 and 1.3 in [7], respectively.

In the proof of the second main theorem optimal and star optimal systems play an essential role. We review the theory of these systems in the next subsection. Some auxiliary results on the ordering $\prec$ will be presented in Subsection 5.2.

5.1. OPTIMAL AND STAR-OPTIMAL SYSTEMS. In this subsection we consider two classes of contractive systems that have extremal properties. A contractive system $\Sigma_\circ = (A_\circ, B_\circ, C_\circ, D, \mathcal{X}_\circ, \mathcal{U}, \mathcal{Y})$ with transfer function θ is called *optimal* if for each contractive realization $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ of θ the estimate

$$\Big\| \sum_{j=0}^{n} A_\circ^{n-j} B_\circ u_j \Big\| \leqslant \Big\| \sum_{j=0}^{n} A^{n-j} B u_j \Big\| \tag{5.4}$$

holds for each $u_0, u_1, \ldots, u_n \in \mathcal{U}$ and each $n \geqslant 0$. To prove that $\Sigma_\circ$ is optimal it suffices to check (5.4) for minimal contractive realizations of θ. Each Schur class function θ appears as the transfer function of a minimal and optimal system, which is determined by θ up to unitary equivalence (see [4]). Moreover, given a Schur class function θ, a minimal and optimal realization can be constructed as follows. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a unitary realization of θ. Define the subspace

$$\mathcal{X}_\circ = \overline{P_{\operatorname{Ker}(C|A)^\perp} \operatorname{Im}(A|B)},$$

let $\tau_{\mathcal{X}_\circ}$ be the canonical embedding of $\mathcal{X}_\circ$ into $\mathcal{X}$, and consider the operators

$$A_\circ = \tau_{\mathcal{X}_\circ}^* A \tau_{\mathcal{X}_\circ} : \mathcal{X}_\circ \to \mathcal{X}_\circ, \quad B_\circ = \tau_{\mathcal{X}_\circ}^* B : \mathcal{U} \to \mathcal{X}_\circ, \quad B_\circ = C\tau_{\mathcal{X}_\circ} : \mathcal{X}_\circ \to \mathcal{Y}.$$

Then (see [5]) the system $\Sigma_\circ = (A_\circ, B_\circ, C_\circ, D; \mathcal{X}_\circ, \mathcal{U}, \mathcal{Y})$ is a minimal and optimal realization of θ. Notice that we obtained the minimal and optimal system as the first minimal restriction of a unitary system.

The other class of contractive systems is defined as follows. Let $\Sigma_\bullet = (A_\bullet, B_\bullet, C_\bullet, D; \mathcal{X}_\bullet, \mathcal{U}, \mathcal{Y})$ be an observable contractive system with transfer function θ. The system $\Sigma_\bullet$ is called *star-optimal* if for each observable contractive realization $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ of θ and for each input sequence $u_0, u_1, u_2, \ldots, u_n$ in $\mathcal{U}$, we have

$$\Big\| \sum_{j=0}^{n} A_\bullet^{n-j} B_\bullet u_j \Big\| \geqslant \Big\| \sum_{j=0}^{n} A^{n-j} B u_j \Big\| \quad (n \geqslant 0). \tag{5.5}$$

Each Schur class function θ admits a minimal and star-optimal realization, which is determined by θ up to unitary equivalence (see [4]). Given a Schur class function θ, a minimal and star-optimal realization can be constructed as follows (see [5]): let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a unitary realization of θ. Define the subspace

$$\mathcal{X}_\bullet = \overline{P_{\overline{\mathrm{Im}\,(A|B)}} \mathrm{Ker}\,(C|A)^\perp},$$

and the operators

$$A_\bullet = \tau_{\mathcal{X}_\bullet}^* A \tau_{\mathcal{X}_\bullet} : \mathcal{X}_\bullet \to \mathcal{X}_\bullet, \quad B_\bullet = \tau_{\mathcal{X}_\bullet}^* B : \mathcal{U} \to \mathcal{X}_\bullet, \quad B_\bullet = C \tau_{\mathcal{X}_\bullet} : \mathcal{X}_\bullet \to \mathcal{Y}.$$

Then the system $\Sigma_\bullet = (A_\bullet, B_\bullet, C_\bullet, D; \mathcal{X}_\bullet, \mathcal{U}, \mathcal{Y})$ is a minimal and star-optimal realization of θ. Notice again, that we obtained the minimal and star-optimal system as the second minimal restriction of a unitary system. Using (2.7) we see that Σ is minimal and star-optimal if and only if the adjoint system Σ^* is minimal and optimal. For further information on optimal and star-optimal systems, see [4] and [5].

5.2. AUXILIARY RESULTS ON THE ORDERING $\prec$. In this subsection we present a few auxiliary results on the ordering $\prec$ that will play a role in the proofs of Theorems 5.1 and 5.2 or that will be useful in later sections. It is straightforward to check that the relation $\prec$ is transitive. The first of the next two propositions shows that the ordering $\prec$ is also antisymmetric.

PROPOSITION 5.3. *Let H_1 and H_2 be non-negative selfadjoint operators acting in $\mathcal{X}$ such that $H_1 \prec H_2$ and $H_2 \prec H_1$. Then $H_1 = H_2$.*

PROPOSITION 5.4. *Let H_1 and H_2 be positive selfadjoint operators acting in $\mathcal{X}$. Then $H_1 \prec H_2$ is equivalent to $H_2^{-1} \prec H_1^{-1}$.*

The above propositions and their proofs can be found in Section 3.2 of [7]. In the sequel we shall need the following lemma.

LEMMA 5.5. *For $\nu = 1, 2$, let $H_\nu(\mathcal{X} \to \mathcal{X})$ be a non-negative selfadjoint operator, and let $\mathcal{D}$ be a linear sub-manifold of both $\mathcal{D}(H_1)$ and $\mathcal{D}(H_2)$. If $\|H_1^{1/2} x\| \leqslant \|H_2^{1/2} x\|$ for each $x \in \mathcal{D}$, and $\mathcal{D}$ is a core for $H_2^{1/2}$, then $H_1 \prec H_2$.*

Proof. Take $x \in \mathcal{D}(H_2^{1/2})$. Since $\mathcal{D}$ is a core for $H_2^{1/2}$, there exists a sequence $x_1, x_2, \ldots$ in $\mathcal{D}$ such that $x_n \to x$ and $H_2^{1/2} x_n \to H_2^{1/2} x$ if $n \to \infty$. The second limit and the assumption that $\|H_1^{1/2} x\| \leqslant \|H_2^{1/2} x\|$ for each $x \in \mathcal{D}$ imply that

$(H_1^{1/2}x_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{X}$. Thus $y = \lim_{n\to\infty} H_1^{1/2}x_n$ exists. But $H_1^{1/2}$ is closed. Therefore, $x \in \mathcal{D}(H_1^{1/2})$ and $H_1^{1/2}x = y$. We have now proved that $\mathcal{D}(H_2^{1/2}) \subset \mathcal{D}(H_1^{1/2})$. Furthermore, again using that $\|H_1^{1/2}x\| \leqslant \|H_2^{1/2}x\|$ for each $x \in \mathcal{D}$, we see that

$$\|H_1^{1/2}x\| = \lim_{n\to\infty} \|H_1^{1/2}x_n\| \leqslant \lim_{n\to\infty} \|H_2^{1/2}x_n\| = \|H_2^{1/2}x\|.$$

Thus $H_1 \prec H_2$. ∎

PROPOSITION 5.6. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system which is dissipative with respect to the supply rate* (1.3)*, and let H_1 and H_2 belong to $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$. Then $H_1 \prec H_2$ if and only if there exists a contraction R on $\mathcal{X}$ such that*

$$RA_{H_2} = A_{H_1}R, \quad RB_{H_2} = B_{H_1}, \quad C_{H_2} = C_{H_1}R. \tag{5.6}$$

Moreover, (5.6) *determines R uniquely.*

Proof. Notice that both Σ_{H_1} and Σ_{H_2} are minimal. For each $H \in \mathcal{GK}_\Sigma$ and each set of vectors $u_0, u_1, \ldots, u_N$ in $\mathcal{U}$ we have

$$\sum_{j=0}^N A^j B u_j \in \mathcal{D}(H^{1/2}), \quad \sum_{j=0}^N A_H^j B_H u_j = H^{1/2}\Big(\sum_{j=0}^N A^j B u_j\Big). \tag{5.7}$$

Now assume $H_1 \prec H_2$. Then using the definition of $\prec$, we obtain

$$\Big\|\sum_{j=0}^N A_{H_1}^j B_{H_1} u_j\Big\| \leqslant \Big\|\sum_{j=0}^N A_{H_2}^j B_{H_2} u_j\Big\|, \quad u_0, u_1, \ldots, u_N \text{ in } \mathcal{U}. \tag{5.8}$$

From (5.8) and the fact that $\mathrm{Im}\,(A_{H_2}|B_{H_2})$ is dense in $\mathcal{X}$ (because Σ_{H_2} is minimal) it follows that there exists a unique contraction R on $\mathcal{X}$ such that

$$R\Big(\sum_{j=0}^N A_{H_2}^j B_{H_2} u_j\Big) = \sum_{j=0}^N A_{H_1}^j B_{H_1} u_j, \quad u_0, u_1, \ldots, u_N \text{ in } \mathcal{U}. \tag{5.9}$$

Again using the density of $\mathrm{Im}\,(A_{H_2}|B_{H_2})$ in $\mathcal{X}$, we see that (5.9) yields the first two identities in (5.6). Next, recall that the transfer functions of Σ_{H_1} and Σ_{H_2} coincide in a neighborhood of zero. Thus $C_{H_2}A_{H_2}^j B_{H_2} = C_{H_1}A_{H_1}^j B_{H_1}$ for each $j = 0, 1, 2, \ldots$. By using (5.9) it follows that

$$\begin{aligned} C_{H_1}R\Big(\sum_{j=0}^N A_{H_2}^j B_{H_2} u_j\Big) &= \sum_{j=0}^N C_{H_1}A_{H_1}^j B_{H_1} u_j = \sum_{j=0}^N C_{H_2}A_{H_2}^j B_{H_2} u_j \\ &= C_{H_2}\Big(\sum_{j=0}^N A_{H_2}^j B_{H_2} u_j\Big). \end{aligned}$$

Using $\mathrm{Im}\,(A_{H_2}|B_{H_2})$ is dense in $\mathcal{X}$, we get $C_{H_1}R = C_{H_2}$, and (5.6) is proved.

To prove the reverse implication, assume that R is a contraction such that (5.6) holds. Then we also have

$$R\Big(\sum_{j=0}^{N} A_{H_2}^{j} B_{H_2} u_j\Big) = \sum_{j=0}^{N} A_{H_1}^{j} B_{H_1} u_j, \quad u_0, u_1, \ldots, u_N \text{ in } \mathcal{U}.$$

Since $\mathrm{Im}\,(A_{H_2}|B_{H_2})$ is dense in $\mathcal{X}$ this determines R uniquely. The fact that R is a contraction implies

$$\Big\| \sum_{j=0}^{N} A_{H_1}^{j} B_{H_1} u_j \Big\| \leqslant \Big\| \sum_{j=0}^{N} A_{H_2}^{j} B_{H_2} u_j \Big\|, \quad u_0, u_1, \ldots, u_N \text{ in } \mathcal{U}.$$

Now put $\mathcal{D} = \mathrm{Im}\,(A|B)$. Using (5.7) and the preceding norm inequality, we see that $\|H_1^{1/2}x\| \leqslant \|H_2^{1/2}x\|$ for each $x \in \mathcal{D}$. From the definition of $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ we know that $\mathcal{D}$ is a core for $H_2^{1/2}$. Thus Lemma 5.5 shows that $H_1 \prec H_2$. ∎

REMARK. Notice that in the first paragraph of the above proof we did not use that $\mathrm{Im}\,(A|B)$ is a core for H_1 and H_2. Thus if H_1 and H_2 are generalized solutions to the KYP-inequality for the minimal system Σ, such that $H_1 \prec H_2$, and the associated systems Σ_{H_1} and Σ_{H_2} are minimal, then there exists a unique contraction R on $\mathcal{X}$ such that (5.6) holds.

COROLLARY 5.7. *Let Σ be a minimal system which is dissipative with respect to the supply rate* (1.3), *and let H_1 and H_2 belong to $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$. Then Σ_{H_1} and Σ_{H_2} are unitarily equivalent if and only if $H_1 = H_2$.*

Proof. Assume Σ_{H_1} and Σ_{H_2} are unitarily equivalent. Then there exists a unitary operator R on $\mathcal{X}$ such that (5.6) holds. Since R is contractive, it follows that $H_1 \prec H_2$. Interchanging the roles of Σ_{H_1} and Σ_{H_2} we also get $H_2 \prec H_1$. Hence $H_1 = H_2$ by Proposition 5.3. The reverse implication is trivial. ∎

5.3. PROOFS OF THEOREMS 5.1 AND 5.2.

Proof of Theorem 5.2. Let Σ be a minimal system which is dissipative with respect to the supply rate (1.3), and let θ be the Schur class function coinciding with the transfer function of Σ in a neighborhood of 0. Let Υ be a minimal and contractive realization of θ. Let S be the unique pseudo-similarity from Σ to Υ such that $\mathrm{Im}\,(A|B)$ is a core for S. Put $H = S^*S$. By Proposition 4.5, the operator H is a generalized solution to the KYP-inequality for Σ. Let Σ_H denote the system associated to the generalized solution H of the KYP-inequality for Σ. By Proposition 4.5, the systems Σ_H and Υ are unitarily equivalent, and the unitary operator U, that establishes the unitary equivalence, satisfies $S = UH^{1/2}$. It follows by unitary equivalence that Σ_H is minimal. This is equivalent to the requirement that

$$\overline{H^{1/2}\mathrm{Im}\,(A|B)} = \mathcal{X}, \quad \overline{(H^{1/2})^{-1}\mathrm{Im}\,(A^*|C^*)} = \mathcal{X},$$

by Proposition 4.3. Since $\operatorname{Im}(A|B)$ is a core for S, the linear manifold $\operatorname{Im}(A|B)$ is also a core for $H^{1/2}$, because $H^{1/2} = U^{-1}S$. Thus $H \in \mathcal{GK}^{\min}_{\Sigma,\text{core}}$, and Σ_H and Υ are unitarily equivalent.

It remains to prove the uniqueness of H. Let H' be a second operator in $\mathcal{GK}^{\min}_{\Sigma,\text{core}}$ such that $\Sigma_{H'}$ and Υ are unitarily equivalent. Then $\Sigma_{H'}$ and Σ_H are unitarily equivalent, and we can apply Corollary 5.7 to show that $H' = H$, which completes the proof. ∎

The proof of Theorem 5.1 follows from the first two statements of the next proposition.

PROPOSITION 5.8. *Let $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y})$ be a minimal system, which is dissipative with respect to the supply rate* (1.3), *and let θ be a Schur class function coinciding with the transfer function of Σ in a neighborhood of zero.*

(i) *If $\Sigma_\circ$ is a minimal and optimal realization of θ, and $S_\circ$ is the unique pseudo-similarity from Σ to $\Sigma_\circ$ such that* $\operatorname{Im}(A|B)$ *is a core for $S_\circ$, then $S_\circ^* S_\circ$ is the minimal element of $\mathcal{GK}^{\min}_{\Sigma,\text{core}}$.*

(ii) *If $\Sigma_\bullet$ is a minimal and star-optimal realization of θ, and $S_\bullet$ is the unique pseudo-similarity from Σ to $\Sigma_\bullet$ such that* $\operatorname{Im}(A|B)$ *is a core for $S_\bullet$, then $S_\bullet^* S_\bullet$ is the maximal element of $\mathcal{GK}^{\min}_{\Sigma,\text{core}}$. Conversely, if $H_\circ$ is the minimal and $H_\bullet$ is the maximal element in $\mathcal{GK}^{\min}_{\Sigma,\text{core}}$, then $\Sigma_{H_\circ}$ is a minimal and optimal system, and $\Sigma_{H_\bullet}$ is a minimal and star-optimal system.*

Proof. We split the proof in three parts. In each one of these parts, we show the corresponding statement of the theorem.

Part (a). Let $\Sigma_\circ = (A_\circ, B_\circ, C_\circ, D; \mathcal{X}_\circ, \mathcal{U}, \mathcal{Y})$ be a minimal and optimal realization of θ. Let $S_\circ$ be the unique pseudo-similarity from Σ to $\Sigma_\circ$ such that $\operatorname{Im}(A|B)$ is a core for $S_\circ$, which exists by Proposition 3.4. Put $H_\circ = S_\circ^* S_\circ$. The proof of Theorem 5.2 shows that $H_\circ \in \mathcal{GK}^{\min}_{\Sigma,\text{core}}$.

We will show that $H_\circ$ is minimal with respect to the ordering $\prec$. Take $H \in \mathcal{GK}^{\min}_{\Sigma,\text{core}}$, and construct $\Sigma_H = (A_H, B_H, C_H, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. The system Σ_H is minimal. Notice that $\operatorname{Im}(A|B)$ is in the domain of both $H^{1/2}$ and $H_\circ^{1/2}$. Thus

$$\Big\| H_\circ^{1/2}\Big(\sum_{j=0}^n A^j B u_j\Big)\Big\| = \Big\| \sum_{j=0}^n A_\circ^j B_\circ u_j \Big\| \leqslant \Big\| \sum_{j=0}^n A_H^j B_H u_j \Big\| = \Big\| H^{1/2}\Big(\sum_{j=0}^n A^j B u_j\Big)\Big\|.$$

The inequality follows from the optimality of $\Sigma_\circ$, and the last equality follows from Proposition 4.2. Since the linear manifold $\operatorname{Im}(A|B)$ is a core for $H^{1/2}$, and since the inequality $\|H_\circ^{1/2}x\| \leqslant \|H^{1/2}x\|$ holds for each for each $x \in \operatorname{Im}(A|B)$, by Lemma 5.5 we conclude that $H_\circ \prec H$.

Part (b). Let $\Sigma_\bullet = (A_\bullet, B_\bullet, C_\bullet, D; \mathcal{X}_\bullet, \mathcal{U}, \mathcal{Y})$ be a minimal and star-optimal realization of θ. Let $S_\bullet$ be the unique pseudo-similarity from Σ to $\Sigma_\bullet$ such that $\operatorname{Im}(A|B)$ is a core for $S_\bullet$. Put $H_\bullet = S_\bullet^* S_\bullet$. The proof of Theorem 5.2 shows that $H_\bullet \in \mathcal{GK}^{\min}_{\Sigma,\text{core}}$.

We will show that $H_\bullet$ is maximal in $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ with respect to the ordering $\prec$. Take $H \in \mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$, and construct $\Sigma_H = (A_H, B_H, C_H, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. The system Σ_H is minimal. Notice that $\mathrm{Im}\,(A|B)$ is in the domain of both the operators $H^{1/2}$ and $H_\bullet^{1/2}$. By star-optimality of $\Sigma_\bullet$ we obtain the inequality

$$\Big\|H_\bullet^{1/2}\Big(\sum_{j=0}^{n} A^j B u_j\Big)\Big\| = \Big\|\sum_{j=0}^{n} A_\bullet^j B_\bullet u_j\Big\| \geqslant \Big\|\sum_{j=0}^{n} A_H^j B_H u_j\Big\| = \Big\|H^{1/2}\Big(\sum_{j=0}^{n} A^j B u_j\Big)\Big\|.$$

The last equality follows from Proposition 4.2. Since the linear manifold $\mathrm{Im}\,(A|B)$ is a core for $H_\bullet^{1/2}$ and the inequality $\|H^{1/2}x\| \leqslant \|H_\bullet^{1/2}x\|$ holds for each for each $x \in \mathrm{Im}\,(A|B)$, we conclude again by Lemma 5.5 that $H \prec H_\bullet$.

Part (c). It remains to show the last statement of the proposition. Let $H_\circ$ be the minimal element in $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$. We have to show, that the system $\Sigma_{H_\circ}$ is minimal and optimal. It is a minimal system, because $H_\circ$ is an element in $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$. Let $\Upsilon = (\alpha, \beta, \gamma, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an arbitrary minimal contractive realization of the Schur class function θ. By Theorem 5.2 the system Υ is unitarily equivalent to Σ_H with $H \in \mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$. The optimality of $\Sigma_{H_\circ}$ follows from the inequality

$$\Big\|\sum_{j=0}^{n} A_{H_\circ}^j B_\circ u_j\Big\| = \Big\|H_\circ^{1/2}\Big(\sum_{j=0}^{n} A^j B u_j\Big)\Big\| \leqslant \Big\|H^{1/2}\Big(\sum_{j=0}^{n} A^j B u_j\Big)\Big\| = \Big\|\sum_{j=0}^{n} \alpha^j a\beta u_j\Big\|.$$

The proof that $H_\bullet$ is the maximal element in $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ is obtained in an analogous way. ∎

Proof of Theorem 5.1. The theorem follows from the first two statements in Proposition 5.8. ∎

The last statement of Proposition 5.8 is summarized in the following proposition.

PROPOSITION 5.9. *Let Σ be a minimal system which is dissipative with respect to the supply rate* (1.3). *Let $H_\circ$ be the minimal element and $H_\bullet$ be the maximal element in $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ with respect to the ordering $\prec$. Then $\Sigma_{H_\circ}$ is a minimal and optimal system, and $\Sigma_{H_\bullet}$ is a minimal and star-optimal system.*

5.4. FURTHER PROPERTIES OF THE SET $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$. Let Σ and $\widetilde{\Sigma}$ be pseudo-similar minimal systems which are dissipative with respect to the supply rate (1.3). The first result of this subsection shows that the sets $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ and $\mathcal{GK}^{\min}_{\widetilde{\Sigma},\mathrm{core}}$ are order isomorphic with respect to the ordering $\prec$.

To define the order isomorphism referred to in the previous paragraph, take H in $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$. Then Σ_H is a minimal contractive realization of the Schur class function θ coinciding with the transfer function of Σ in a neighborhood of zero. Since Σ and $\widetilde{\Sigma}$ are pseudo-similar, the transfer function of $\widetilde{\Sigma}$ also coincides with θ in a neighborhood of zero. Now, apply Theorem 5.2 to $\widetilde{\Sigma}$. The fact that Σ_H is a minimal contractive realization of θ implies that there exists a unique $\widetilde{H} \in$

$\mathcal{GK}^{\min}_{\widetilde{\Sigma},\mathrm{core}}$ such that Σ_H and $\Sigma_{\widetilde{H}}$ are unitarily equivalent. Let J be the map given by

$$J : \mathcal{GK}^{\min}_{\Sigma,\mathrm{core}} \to \mathcal{GK}^{\min}_{\widetilde{\Sigma},\mathrm{core}}, \quad J(H) = \widetilde{H}. \tag{5.10}$$

The next proposition shows that J is an order isomorphism with respect to $\prec$.

PROPOSITION 5.10. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, D; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ be pseudo-similar minimal systems, which are dissipative with respect to the supply rate* (1.3), *and let J be the map defined by* (5.10). *Then J is a bijective map preserving the order relation $\prec$, that is,*

$$H_1 \prec H_2 \iff J(H_1) \prec J(H_2). \tag{5.11}$$

Proof. Let us write $J_{\Sigma,\widetilde{\Sigma}}$ for the map J defined by (5.10). By interchanging the roles of Σ and $\widetilde{\Sigma}$ we can also consider the map $J_{\widetilde{\Sigma},\Sigma}$ which transforms $\mathcal{GK}^{\min}_{\widetilde{\Sigma},\mathrm{core}}$ into $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$. Using the uniqueness statements in Theorem 5.2 and Corollary 5.7 it is straightforward to show the products $J_{\Sigma,\widetilde{\Sigma}} J_{\widetilde{\Sigma},\Sigma}$ and $J_{\widetilde{\Sigma},\Sigma} J_{\Sigma,\widetilde{\Sigma}}$ are the identity maps on $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ and $\mathcal{GK}^{\min}_{\widetilde{\Sigma},\mathrm{core}}$, respectively. In particular, the map $J = J_{\Sigma,\widetilde{\Sigma}}$ is a bijection.

Next, we prove (5.11). Since $J_{\Sigma,\widetilde{\Sigma}} = (J_{\widetilde{\Sigma},\Sigma})^{-1}$, it suffices to show that $H_1 \prec H_2$ implies $J(H_1) \prec J(H_2)$. For $i = 1, 2$ put $\widetilde{H}_i = J(H_i)$, and consider the systems

$$\Sigma_{H_i} = (A_{H_i}, B_{H_i}, C_{H_i}, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}), \quad \Sigma_{\widetilde{H}_i} = (\widetilde{A}_{\widetilde{H}_i}, \widetilde{B}_{\widetilde{H}_i}, \widetilde{C}_{\widetilde{H}_i}, D; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}).$$

Here $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are the state spaces of the systems Σ and $\widetilde{\Sigma}$, respectively. Let $U_i : \mathcal{X} \to \widetilde{\mathcal{X}}$, $i = 1, 2$, be the unitary operator providing the unitary equivalence from Σ_{H_i} to $\Sigma_{\widetilde{H}_i}$. Thus

$$U_i A_{H_i} = \widetilde{A}_{\widetilde{H}_i} U_i, \quad U_i B_{H_i} = \widetilde{B}_{\widetilde{H}_i}, \quad C_{H_i} = C_{\widetilde{H}_i} U_i, \quad i = 1, 2. \tag{5.12}$$

Recall that we assume that $H_1 \prec H_2$. Thus, by Proposition 5.6, there exists a contraction R on $\mathcal{X}$ such that (5.6) holds. Now, let $\widetilde{R}$ be the contraction on $\widetilde{\mathcal{X}}$ defined by $\widetilde{R} = U_1 R U_2^{-1}$. Then using the identities in (5.6) and (5.12) it is straightforward to check that

$$\widetilde{R}\widetilde{A}_{\widetilde{H}_2} = \widetilde{A}_{\widetilde{H}_1}\widetilde{R}, \quad \widetilde{R}\widetilde{B}_{\widetilde{H}_2} = \widetilde{B}_{\widetilde{H}_1}, \quad \widetilde{C}_{\widetilde{H}_2} = \widetilde{C}_{\widetilde{H}_1}\widetilde{R}.$$

According to Proposition 5.6 this implies that $\widetilde{H}_1 \prec \widetilde{H}_2$, which completes the proof. ∎

The following similarity result will be used in the next section.

PROPOSITION 5.11. *Let Σ be a minimal system which is dissipative with respect to the supply rate* (1.3), *and let $H_\circ$ be the minimal and $H_\bullet$ the maximal element in $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ with respect to the ordering $\prec$. Then all Σ_H with $H \in \mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ are mutually similar if and only if $H_\bullet \prec \gamma H_\circ$ for some $\gamma > 0$.*

Proof. Since $H_\circ \prec H_\bullet$, by Proposition 5.6 there exists a unique contraction R on $\mathcal{X}$ such that

$$RA_{H_\bullet} = A_{H_\circ}R, \quad RB_{H_\bullet} = B_{H_\circ}, \quad C_{H_\bullet} = C_{H_\circ}R. \tag{5.13}$$

Now assume that all Σ_H with $H \in \mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ are mutually similar. In particular, $\Sigma_{H_\circ}$ and $\Sigma_{H_\bullet}$ are similar. It follows that the unique R in (5.13) is boundedly invertible. Put $\gamma = \|R^{-1}\|^2$. For $u_0, \ldots, u_N$ in $\mathcal{U}$ and using (5.13) we have $\sum_{j=0}^{N} A^j_{H_\bullet} B_{H_\bullet} u_j = R^{-1} \sum_{j=0}^{N} A^j_{H_\circ} B_{H_\circ} u_j$, and hence

$$\Big\| \sum_{j=0}^{N} A^j_{H_\bullet} B_{H_\bullet} u_j \Big\| \leqslant g^{1/2} \Big\| \sum_{j=0}^{N} A^j_{H_\circ} B_{H_\circ} u_j \Big\|.$$

Put $\mathcal{D} = \mathrm{Im}\,(A|B)$. Using (5.7) and the previous norm inequality we see that

$$\|H_\bullet^{1/2} x\| \leqslant \|\gamma^{1/2} H_\circ^{1/2} x\|, \quad x \in \mathcal{D}.$$

Since $\mathcal{D}$ is a core for $\gamma^{1/2} H_\circ^{1/2}$, we can apply Lemma 5.5 to show that $H_\bullet \prec \gamma H_\circ$.

Conversely, assume that $H_\bullet \prec \gamma H_\circ$ for some $\gamma > 0$. Let H be an arbitrary operator from $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$. It suffices to show that Σ_H is similar to $\Sigma_{H_\circ}$. Since $H_\circ \prec H$ we know from Proposition 5.6 that the exists a unique contraction $R_\circ$ on $\mathcal{X}$ such that

$$R_\circ A_H = A_{H_\circ} R_\circ, \quad R_\circ B_H = B_{H_\circ}, \quad C_H = C_{H_\circ} R_\circ.$$

Notice that $H \prec H_\bullet \prec \gamma H_\circ$. Thus we also have $H \prec \gamma H_\circ$. It follows that

$$\Big\| \sum_{j=0}^{N} A^j_H B_H u_j \Big\| \leqslant \gamma^{1/2} \Big\| \sum_{j=0}^{N} A^j_{H_\circ} B_{H_\circ} u_j \Big\| = \gamma^{1/2} \Big\| R_\circ \Big(\sum_{j=0}^{N} A^j_H B_H u_j \Big) \Big\|$$

for $u_0, \ldots, u_N$ in $\mathcal{U}$. Thus $\|R_\circ w\| \geqslant \gamma^{1/2} \|w\|$ for each $w \in \mathrm{Im}\,(A_H|B_H)$. Since $\mathrm{Im}\,(A_H|B_H)$ is dense in $\mathcal{X}$, we conclude that $R_\circ$ is one to one and has closed range. But the range of $R_\circ$ contains the set $\mathrm{Im}\,(A_{H_\circ}|B_{H_\circ})$ which is also dense in $\mathcal{X}$. Thus $R_\circ$ is boundedly invertible, and hence $\Sigma_{H_\circ}$ and Σ_H are similar. ∎

Proposition 5.10 above shows that the order properties of the set $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ are determined by the transfer function of the system Σ, and do not depend on the particular choice of the Σ. This fact will be developed further in Section 7.

5.5. THE SET $\mathcal{GK}^{\min}_{\Sigma}$ AND ITS EXTREMAL ELEMENTS. Throughout this subsection $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a minimal system which is dissipative with respect to the supply rate function (1.3). Recall (compare with (5.1)) that

$$\mathcal{GK}^{\min}_{\Sigma} = \{H \in \mathcal{GK}_\Sigma : \Sigma_H \text{ minimal}\}. \tag{5.14}$$

Thus $\mathcal{GK}^{\min}_{\Sigma}$ denotes the set of all generalized solutions H of the KYP-inequality for Σ such that Σ_H is minimal while $\mathrm{Im}\,(A|B)$ is not required to be a core for $H^{1/2}$.

Obviously, $\mathcal{GK}_{\Sigma}^{\min} \supset \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$. The two sets can be different. To see this, recall (see the paragraph directly after Theorem 5.1) that there exist two pseudo-similar contractive systems

$$\Sigma_1 = (0, B_1, C_1, 0; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \quad \text{and} \quad \Sigma_2 = (0, B_2, C_2, 0; \mathcal{X}_2, \mathcal{U}, \mathcal{Y})$$

such that Σ_1 is minimal while Σ_2 is not. Furthermore, we can choose (see formulas (2.8) and (2.9) in [7]) a pseudo-similarity $\widehat{S}$ from Σ_1 to Σ_2 such that $\operatorname{Im} B_1$ is not a core for $\widehat{S}$. In particular, Σ_1 is a minimal system which is dissipative with respect to the supply rate function (1.3). Now take $\Sigma = \Sigma_1$, and let $H = \widehat{S}^*\widehat{S}$. Then H is a generalized solution to the KYP-inequality for Σ, and Σ_H is minimal (because Σ_H is unitarily equivalent to Σ_2, by Proposition 4.5). We know that $\operatorname{Im}(A|B) = \operatorname{Im}(A_1|B_1) = \operatorname{Im} B_1$ is not a core for $\widehat{S}$. Thus H belongs to $\mathcal{GK}_{\Sigma}^{\min}$ but not to $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$. Thus for this choice of Σ the sets $\mathcal{GK}_{\Sigma}^{\min}$ and $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ are different. We shall prove the following theorem.

THEOREM 5.12. *Given H' in $\mathcal{GK}_{\Sigma}^{\min}$, there exists a unique H in $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ such that $\Sigma_{H'}$ and Σ_H are unitarily equivalent. Moreover, $H' \prec H$. Finally, with respect to the ordering $\prec$ the maximal element of $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ is also maximal in $\mathcal{GK}_{\Sigma}^{\min}$.*

In Proposition 5.16 below we shall identify the minimal element in $\mathcal{GK}_{\Sigma}^{\min}$. Before we prove Theorem 5.12 we make some preparations. Let $\widetilde{\Sigma}$ be any minimal contractive system with the property that in a neighborhood of zero the transfer function of $\widetilde{\Sigma}$ coincides with the transfer function of Σ. From Theorem 3.2 we know that Σ and $\widetilde{\Sigma}$ are pseudo-similar. By $\mathcal{P}_{\Sigma,\widetilde{\Sigma}}$ we denote the set of all pseudo-similarities from Σ to $\widetilde{\Sigma}$. Since a pseudo-similarity between two minimal systems does not have to be unique, it can happen that the set $\mathcal{P}_{\Sigma,\widetilde{\Sigma}}$ consists of more than one element.

Given $S \in \mathcal{P}_{\Sigma,\widetilde{\Sigma}}$, put $H_S = S^*S$. According to Proposition 4.5 the operator H_S is a generalized solution to the KYP-inequality for Σ. Let Σ_{H_S} be the system associated to H_S and Σ. From Proposition 4.5 we also know that Σ_{H_S} is unitarily equivalent to $\widetilde{\Sigma}$. Hence $H_S \in \mathcal{GK}_{\Sigma}^{\min}$.

PROPOSITION 5.13. *All systems Σ_{H_S} with $S \in \mathcal{P}_{\Sigma,\widetilde{\Sigma}}$ are mutually unitarily equivalent.*

Proof. The statement follows from the fact that Σ_{H_S} is unitarily equivalent to $\widetilde{\Sigma}$, by Proposition 4.5, and from the fact that unitary equivalence is transitive. ∎

As we have seen in Proposition 3.4, with respect to graph space inclusion, the set $\mathcal{P}_{\Sigma,\widetilde{\Sigma}}$ contains a minimal and a maximal element, which we shall denote by $S_{\min}$ and $S_{\max}$, respectively.

PROPOSITION 5.14. *For $S \in \mathcal{P}_{\Sigma,\widetilde{\Sigma}}$ we have $H_S \in \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ if and only if $S = S_{\min}$.*

Proof. Since Σ_{H_S} and $\widetilde{\Sigma}$ are unitarily equivalent (by Proposition 4.5) and $\widetilde{\Sigma}$ is minimal, we have $H_S \in \mathcal{GK}_{\Sigma,\,\mathrm{core}}^{\min}$ if and only if $\mathrm{Im}\,(A|B)$ is a core for $H_S^{1/2}$. But $S = UH_S^{1/2}$ for some unitary operator U from $\mathcal{X}$ onto $\widetilde{\mathcal{X}}$, where $\widetilde{\mathcal{X}}$ is the state space of $\widetilde{\Sigma}$; see Proposition 4.5. It follows that $H_S \in \mathcal{GK}_{\Sigma,\,\mathrm{core}}^{\min}$ if and only if $\mathrm{Im}\,(A|B)$ is a core for S, or, equivalently, $S = S_{\min}$. ∎

Next we show that

$$H_{S_{\max}} \prec H_S \prec H_{S_{\min}}, \quad S \in \mathcal{P}_{\Sigma,\,\widetilde{\Sigma}}. \tag{5.15}$$

In fact, this order relation is a corollary of the following proposition.

PROPOSITION 5.15. *For S_1 and S_2 in $\mathcal{P}_{\Sigma,\,\widetilde{\Sigma}}$ we have*

$$G(S_2) \subset G(S_1) \Longrightarrow H_{S_1} \prec H_{S_2}. \tag{5.16}$$

Proof. From Proposition 4.5 we know that $S_1 = U_1 H_{S_1}^{1/2}$ and $S_2 = U_2 H_{S_2}^{1/2}$ for some unitary operators U_1 and U_2. In particular, $\mathcal{D}(H_{S_1}^{1/2}) = \mathcal{D}(S_1)$ and $\mathcal{D}(H_{S_2}^{1/2}) = \mathcal{D}(S_2)$. Now, assume that $G(S_2) \subset G(S_1)$. Then $\mathcal{D}(S_2) \subset \mathcal{D}(S_1)$, and the operators S_1 and S_2 coincide on $\mathcal{D}(S_2)$. It follows that $\mathcal{D}(H_{S_2}^{1/2}) \subset \mathcal{D}(H_{S_1}^{1/2})$, and

$$\|H_{S_1}^{1/2}x\| = \|S_1 x\| = \|S_2 x\| = \|H_{S_2}^{1/2}x\|, \quad x \in \mathcal{D}(H_{S_2}^{1/2}).$$

This shows that $H_{S_1} \prec H_{S_2}$. ∎

Proof of Theorem 5.12. Let $H' \in \mathcal{GK}_{\Sigma}^{\min}$, and consider $\Sigma_{H'}$. From Proposition 4.2 we know that $S = (H')^{1/2}$ is a pseudo-similarity from Σ to $\Sigma_{H'}$. Notice that $H_S = H'$. Now put $\widetilde{\Sigma} = \Sigma_{H'}$, which is minimal and contractive, and consider the corresponding set $\mathcal{P}_{\Sigma,\,\widetilde{\Sigma}}$. Let $S_{\min}$ be the minimal element of $\mathcal{P}_{\Sigma,\,\widetilde{\Sigma}}$ with respect to graph space inclusion. Put $H = H_{S_{\min}}$. Then $H \in \mathcal{GK}_{\Sigma,\,\mathrm{core}}^{\min}$, by Proposition 5.14, and the systems $\Sigma_{H'}$ and Σ_H are unitarily equivalent, by Proposition 5.13.

Next, let $\widehat{H}$ be an arbitrary element in $\mathcal{GK}_{\Sigma,\,\mathrm{core}}^{\min}$ such that the systems $\Sigma_{\widehat{H}}$ and $\Sigma_{H'}$ are unitarily equivalent. Then $\Sigma_{\widehat{H}}$ and Σ_H are unitarily equivalent, and we can apply Corollary 5.7 to show that $\widehat{H} = H$. According to formula (5.15) we have $H' = H_S \prec H_{S_{\min}} = H$. Thus $H' \prec H$.

Finally, let $H_\bullet$ be the maximal element in $\mathcal{GK}_{\Sigma,\,\mathrm{core}}^{\min}$ relative to the ordering $\prec$. Thus $H = H_{S_{\min}} \prec H_\bullet$. Therefore, since $\prec$ is transitive, $H' \prec H_\bullet$. ∎

PROPOSITION 5.16. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system, which is dissipative with respect to the supply rate* (1.3)*, and let $\Sigma_\circ = (A_\circ, B_\circ, C_\circ, D; \mathcal{X}_\circ, \mathcal{U}, \mathcal{Y})$ be an optimal minimal realization of θ_Σ. Let $\widehat{S}_\circ$ be the unique pseudo-similarity from Σ to $\Sigma_\circ$ such that*

$$G(\widehat{S}_\circ) = \bigcap_{j \geqslant 0} \mathrm{Ker} \begin{bmatrix} CA^j & -C_\circ A_\circ^j \end{bmatrix}. \tag{5.17}$$

Then $\widehat{H}_\circ = \widehat{S}_\circ^* \widehat{S}_\circ$ *is the minimal element of* $\mathcal{GK}_\Sigma^{\min}$.

Proof. Notice that $(\widehat{S}_\circ^{-1})^*$ is a pseudo-similarity from Σ^* to $(\Sigma_\circ)^* = (\Sigma^*)_\bullet$. Since $\Sigma_\circ$ is minimal and optimal, the system $(\Sigma^*)_\bullet$ is minimal and star optimal. Moreover, from (5.17) we obtain that

$$\begin{aligned} G((\widehat{S}_\circ^{-1})^*) &= G((\widehat{S}_\circ^*)^{-1}) = G((\widehat{S}_\circ^{-1})^*) = G'(\widehat{S}_\circ^*) \\ &= G(-\widehat{S}_\circ)^\perp = \overline{\operatorname{Im}\left(\begin{bmatrix} A^* & 0 \\ 0 & A_\circ^* \end{bmatrix} \middle| \begin{bmatrix} C^* \\ C_\circ^* \end{bmatrix} \right)}. \end{aligned}$$

In particular, $\operatorname{Im}(A^*|C^*)$ is a core for $(\widehat{S}_\circ^{-1})^*$. Thus (cf., Part (c) of the proof of Theorem 5.1) the map $K = (\widehat{S}_\circ^{-1})(\widehat{S}_\circ^{-1})^*$ is the maximal element of $\mathcal{GK}_{\Sigma^*,\mathrm{core}}^{\min}$. According to Theorem 5.12 this implies that K is the maximal element of $\mathcal{GK}_{\Sigma^*}^{\min}$. Notice that $K = \widehat{S}_\circ^{-1}(\widehat{S}_\circ^*)^{-1} = \widehat{H}_\circ^{-1}$.

Thus $\widehat{H}_\circ^{-1}$ is the maximal element of $\mathcal{GK}_{\Sigma^*}^{\min}$. Now, let $H \in \mathcal{GK}_\Sigma^{\min}$ be arbitrary. Then, by Proposition 4.6, $H^{-1} \in \mathcal{GK}_{\Sigma^*}^{\min}$. Thus $H^{-1} \prec \widehat{H}_\circ^{-1}$. But then $\widehat{H}_\circ \prec H$ by Proposition 5.4. Thus $\widehat{H}_\circ$ is the minimal element of $\mathcal{GK}_\Sigma^{\min}$. ∎

We don't know whether or not the minimal elements of $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ and $\mathcal{GK}_\Sigma^{\min}$ coincide. The next proposition shows that under certain additional conditions the two minimal elements are the same. We conjecture that in general they will be different.

PROPOSITION 5.17. *Let* Σ *be a minimal system which is dissipative with respect to the supply rate* (1.3). *If* $H \in \mathcal{GK}_\Sigma^{\min}$ *is bounded, then* $H \in \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$. *Furthermore, if the minimal element* $H_\circ$ *of* $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ *is bounded, then* $H_\circ$ *is also the minimal element of* $\mathcal{GK}_\Sigma^{\min}$.

Proof. Since Σ is minimal, $\operatorname{Im}(A|B)$ is dense in the state space $\mathcal{X}$. Thus, if H is bounded on $\mathcal{X}$, then trivially $\operatorname{Im}(A|B)$ is a core for the bounded operator $H^{1/2}$. Thus $H \in \mathcal{GK}_\Sigma^{\min}$ and H bounded imply that $H \in \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$.

Now assume that the minimal element of $H_\circ$ of $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ is bounded. We want to show that $H_\circ$ is minimal in $\mathcal{GK}_\Sigma^{\min}$. Take $H \in \mathcal{GK}_\Sigma^{\min}$, and assume $H \prec H_\circ$. This implies that $\mathcal{D}(H_\circ^{1/2}) \subset \mathcal{D}(H^{1/2})$. From $H_\circ$ is bounded, it follows that $H_\circ^{1/2}$ is also bounded. In particular, $\mathcal{D}(H_\circ^{1/2}) = \mathcal{X}$. But then $\mathcal{D}(H^{1/2}) = \mathcal{X}$ too. Thus $H^{1/2}$ is bounded. It follows that H is bounded, and by the result of the first paragraph $H \in \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$. But then $H_\circ \prec H$. Since the relation $\prec$ is antisymmetric (Proposition 5.3) we conclude that $H = H_\circ$. Thus $H_\circ$ is the minimal element of $\mathcal{GK}_\Sigma^{\min}$. ∎

6. STABILITY AND THE KALMAN-YAKUBOVICH-POPOV INEQUALITY

An important aspect of the KYP-inequality is the connection with stability. In this section we describe these connections and some of their corollaries. We begin by defining the notions of stability involved.

6.1. VARIOUS NOTIONS OF STABILITY. An operator A on a Hilbert space $\mathcal{X}$ is called *exponentially stable* if there exists constants $M \geqslant 0$, $0 < q < 1$, such that

$$\|A^n x\| \leqslant Mq^n \|x\|, \quad n = 0, 1, 2, \ldots, x \in \mathcal{X}, \tag{6.1}$$

and A on $\mathcal{X}$ is called (*pointwise*) *stable* if

$$\lim_{n\to\infty} \|A^n x\| = 0, \quad x \in \mathcal{X}. \tag{6.2}$$

In the sequel we shall omit the word pointwise, and simply speak about stable operators. Finally, the operator A is called *star-stable* if A^* is stable, i.e.,

$$\lim_{n\to\infty} \|(A^*)^n x\| = 0, \quad x \in \mathcal{X}. \tag{6.3}$$

In the finite dimensional case these three conditions of stability are the same, but in the infinite dimensional case all three are different. Of course, (6.1) implies (6.2) and (6.3), but the converse is not true. Also, (6.2) and (6.3) are not equivalent, not even when A is a contraction. For instance, the forward shift on the Hardy space $H^2(\mathbb{D})$ is star-stable but not stable, and the backward shift is stable but not star-stable. The following lemma will play useful role later.

LEMMA 6.1. *Let A on $\mathcal{X}$ be power bounded, that is, $\|A^n\| \leqslant M < \infty$ for $n \geqslant 0$. Then A is stable whenever A is stable on a dense subset $\mathcal{L}$ of $\mathcal{X}$, that is*

$$\lim_{n\to\infty} \|A^n y\| = 0, \quad y \in \mathcal{L}. \tag{6.4}$$

Proof. Take $x \in \mathcal{X}$, and let $\varepsilon > 0$. First we choose $y \in \mathcal{L}$ such that $\|x - y\| < (M+1)^{-1}\varepsilon$. From (6.4) we know that there exists a positive integer N such that $\|A^n y\| < \varepsilon$ for each $n \geqslant N$. Now

$$\|A^n x\| \leqslant \|A^n x - A^n y\| + \|A^n y\| \leqslant M\|x - y\| + \|A^n y\| < 2\varepsilon \quad (n \geqslant N).$$

Hence $A^n x \to 0$ for $n \to \infty$. Thus A is stable. ∎

In the sequel we shall say that a system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is *exponentially stable*, *stable* or *star-stable* if its state operator A has the corresponding property.

For a minimal system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with finite dimensional state space $\mathcal{X}$ the fact that it is dissipative with respect to the supply rate (1.3) implies that the system is exponentially stable. This statement is also known as the bounded real lemma (see, for instance, page 549 of [35]). In the infinite dimensional case the connection between stability and the KYP-inequality is much more subtle. For instance, in the infinite dimensional case it may happen (see below for further details) that a minimal system which is dissipative with respect to the supply rate (1.3) is neither stable nor star-stable.

Another difficulty is that in the infinite dimensional case the stability depends on the solution of the KYP-inequality one is dealing with, that is, the state operator A may be stable (or star-stable) in the inner product defined by one solution of the KYP-inequality but not with respect to the inner product defined by another solution. More precisely, given a minimal system which is dissipative with respect to the supply rate (1.3) it can happen that for two solutions H_1 and H_2 of the KYP-inequality for Σ the associated system Σ_{H_1} is stable while Σ_{H_2} is not stable. In the finite dimensional case this phenomenon does not appear because in that case all solutions of the KYP-inequality are bounded and boundedly invertible.

The following proposition explains what stability and star-stability means for a system Σ_H.

PROPOSITION 6.2. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a system which is dissipative with respect to the supply rate* (1.3)*, and let H be a generalized solution to the* KYP*-inequality for Σ. Then Σ_H is stable if and only if*

$$\lim_{n\to\infty} \|H^{1/2}A^n x\| = 0, \quad x \in \mathcal{D}(H^{1/2}), \tag{6.5}$$

and Σ_H is star-stable if and only if

$$\lim_{n\to\infty} \|H^{-1/2}(A^*)^n x\| = 0, \quad x \in \mathcal{D}(H^{-1/2}). \tag{6.6}$$

Proof. From Proposition 4.2 we know that Σ_H is contractive. In particular, A_H is a contraction, and hence A_H is power bounded. Let $\mathcal{L} = \operatorname{Im} H^{1/2}$. Then $\mathcal{L}$ is dense in $\mathcal{X}$, and by Lemma 6.1 the operator A_H is stable if and only if A_H is stable on $\mathcal{L}$, that is,

$$\lim_{n\to\infty} \|(A_H)^n H^{1/2} x\| = 0, \quad x \in \mathcal{D}(H^{1/2}). \tag{6.7}$$

From the definition of A_H in (4.3) we conclude that $(A_H)^n H^{1/2} x = H^{1/2} A^n x$ for each $n \geqslant 0$ and each $x \in \mathcal{D}(H^{1/2})$. Thus (6.7) is equivalent to (6.5) which completes the proof of the first statement.

By definition Σ_H is star-stable if and only if $(\Sigma_H)^*$ is stable. From Proposition 4.6 we know that H^{-1} is a generalized solution of the KYP-inequality for Σ^* (which mean that Σ^* is dissipative with respect to the scattering supply rate function), and that $(\Sigma^*)_{H^{-1}} = (\Sigma_H)^*$. Thus we have to consider the stability of $(\Sigma^*)_{H^{-1}}$. Since $(H^{-1})^{1/2} = (H^{1/2})^{-1} = H^{-1/2}$ (see the proof of Proposition 4.6), the result of the first paragraph yields that $(\Sigma^*)_{H^{-1}}$ is stable if and only if (6.6) holds. ∎

6.2. MAIN STABILITY THEOREMS. To describe in more detail the connection between the KYP-inequality and stability we shall combine results from [4] with those of the preceding sections. To do this we need the following notions.

Let θ be an element of the Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$. Then the factorization problem

$$\phi(\zeta)^*\phi(\zeta) = I - \theta(\zeta)^*\theta(\zeta) \quad \text{(a.e. for } |\zeta| = 1) \tag{6.8}$$

is said to have *a solution* ϕ if there exists an auxiliary Hilbert space $\mathcal{Y}_\phi$, and a Schur class function $\phi \in \mathcal{S}(\mathcal{U},\mathcal{Y}_\phi)$ such that (6.8) holds almost everywhere on the unit circle. In that case, by inner-outer factorization, the factorization problem has also an outer solution, which after an appropriate normalization is unique. This unique outer solution will be denoted by ϕ_θ. The normalization of ϕ_θ means that ϕ_θ is required to satisfy the following additional conditions:

$$\mathcal{Y}_{\phi_\theta} \subset \mathcal{U}, \quad \phi_\theta(0)|\mathcal{Y}_{\phi_\theta} \text{ is a positive operator on } \mathcal{Y}_{\phi_\theta}. \tag{6.9}$$

Analogously, the factorization problem

$$\psi(\zeta)\psi(\zeta)^* = I - \theta(\zeta)\theta(\zeta)^* \tag{6.10}$$

is said to have *a solution* ψ if there exists an auxiliary Hilbert space $\mathcal{U}_\psi$, and a Schur class function $\psi \in \mathcal{S}(\mathcal{U}_\psi,\mathcal{Y})$ such that (6.10) holds almost everywhere on the unit circle. By outer-inner factorization the factorization problem (6.10) has also a star-outer solution which is unique after an appropriate normalization. This unique star-outer solution will be denoted by ψ_θ. In this case the normalization of ψ_θ means that ψ_θ is required to satisfy the following additional conditions:

$$\mathcal{U}_{\psi_\theta} \subset \mathcal{Y}, \quad \psi_\theta(0)^*|\mathcal{U}_{\psi_\theta} \text{ is a positive operator on } \mathcal{U}_{\psi_\theta}. \tag{6.11}$$

For the definitions of outer, star-outer and inner functions, and for the existence of inner-outer and outer-inner factorizations we refer the reader to the book [32].

Now suppose equations (6.8) and (6.10) have solutions, and let ϕ_θ and ψ_θ be the unique normalized outer and star-outer solutions introduced in the previous paragraph. Then there exists $h_\theta \in L^\infty(\mathbb{T},\mathcal{L}(\mathcal{U}_{\psi_\theta},\mathcal{Y}_{\phi_\theta}))$ a unique operator valued function defined on the unit circle, such that

$$h_\theta(\zeta)^*\phi_\theta(\zeta) = -\psi_\theta(\zeta)^*\theta(\zeta) \tag{6.12}$$

almost everywhere on the unit circle (see formula 11 of [4]).

In the following three theorems Σ is a minimal system which is dissipative with respect to the supply rate (1.3). Thus by Theorem 1.2 the transfer function of Σ coincides with a Schur class function θ in a neighborhood of 0. Furthermore, by Theorem 5.1, the set $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ is non-empty, and with respect to the ordering $\prec$ it contains a minimal and a maximal element which are denoted by $H_\circ$ and $H_\bullet$, respectively. Recall that $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min} \subset \mathcal{GK}_{\Sigma}^{\min}$, where $\mathcal{GK}_{\Sigma}^{\min}$ is defined by (5.14).

THEOREM 6.3. *Let Σ be a minimal system which is dissipative with respect to the supply rate* (1.3)*, and let θ be the Schur class function coinciding in a neighborhood of* 0 *with the transfer function of Σ. Then there exists an element $H \in \mathcal{GK}_{\Sigma}^{\min}$ such that Σ_H is stable if and only if for θ the factorization problem* (6.8) *has a solution, and in that case, the system $\Sigma_{H_\circ}$, where $H_\circ$ is the minimal element of $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$, is also stable. Moreover, the system $\Sigma_{H_\circ}$ is stable and star-stable if and only if the following two conditions are* *satisfied:* (I) *the factorization problems* (6.8) *and* (6.10) *both have solutions, and* (II) *the unique function* h_θ *defined in* (6.12) *can be represented as*

$$h_\theta(\zeta) = s_\circ(\zeta) b_\circ(\zeta)^*, \tag{6.13}$$

where $b_\circ$ *is a bi-inner function and* $s_\circ$ *is a Schur class function.*

THEOREM 6.4. *Let* Σ *and* θ *be as in Theorem* 6.3. *Then there exists an element* $H \in \mathcal{GK}_\Sigma^{\min}$ *such that* Σ_H *is star-stable if and only if the factorization problem* (6.10) *has a solution, and in this case, the system* $\Sigma_{H_\bullet}$, *where* $H_\bullet$ *is the maximal element of* $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$, *is also star-stable. Moreover, the system* $\Sigma_{H_\bullet}$ *is both stable and star-stable if and only if the following two conditions are satisfied:* (I) *the two factorization problems* (6.8) *and* (6.10) *have solutions, and* (II) *the unique function* h_θ *defined in* (6.12) *has a representation*

$$h_\theta(\zeta) = b_\bullet(\zeta)^* s_\bullet(\zeta), \tag{6.14}$$

where $b_\bullet$ *is a bi-inner function and* $s_\bullet$ *is a Schur class function.*

THEOREM 6.5. *Let* Σ *and* θ *be as in Theorem* 6.3. *Then for each* $H \in \mathcal{GK}_\Sigma^{\min}$ *the system* Σ_H *is both stable and star-stable if and only if two factorization problems* (6.8) *and* (6.10) *have solutions and the unique function* h_θ *defined in* (6.12) *has representations* (6.13) *and* (6.14).

Formulas (6.12), (6.13), and (6.14) are closely related to the notion of Darlington synthesis. Indeed, let the factorization problems (6.8) and (6.10) be solvable, let ϕ_θ and ψ_θ be the unique normalized outer and star-outer solutions, and let h be defined by (6.12). Then the operator-valued function

$$\begin{bmatrix} \psi_\theta(\zeta) & \theta(\zeta) \\ h_\theta(\zeta) & \phi_\theta(\zeta) \end{bmatrix}$$

is well-defined and its values are unitary almost everywhere on the unit circle. Now assume that condition (6.13) is fulfilled. Then

$$\begin{bmatrix} \psi_\theta(\zeta) b_\circ(\zeta) & \theta(\zeta) \\ s_\circ(\zeta) & \phi_\theta(\zeta) \end{bmatrix}$$

is bi-inner. Furthermore, the function $\widetilde{\psi}_\theta = \psi_\theta b_\circ$ is a Schur class function, and is a solution to the factorization problem (6.10). In this case, one says that the triple ϕ_θ, $\widetilde{\psi}_\theta$, $s_\circ$ is a solution for the *Darlington synthesis problem for* θ. A similar remark applies to condition (6.14). See [4] for further details.

6.3. PROOFS OF THE MAIN STABILITY THEOREMS. To prove Theorems 6.3, 6.4, and 6.5 the following results will be useful.

LEMMA 6.6. *Let* $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *be a contractive controllable system. Then* Σ *is stable whenever*

$$\lim_{n\to\infty} \|A^n B u\| = 0 \quad (u \in \mathcal{U}). \tag{6.15}$$

Proof. We apply Lemma 6.1. Since Σ is contractive, the state operator A is a contraction, and hence it is power bounded. The controllability of Σ means that the set $\mathcal{L} = \operatorname{Im}(A|B)$ is dense in $\mathcal{X}$. Now, take $y \in \mathcal{L}$. Then we can find $u_0, u_1, \ldots, u_N$ such that $y = \sum_{j=0}^{N} A^j B u_j$. Thus (6.15) implies that (6.4) holds for $\mathcal{L} = \operatorname{Im}(A|B)$. But then Lemma 6.1 shows that Σ is stable. ∎

COROLLARY 6.7. *If the operator valued function θ has a stable contractive realization, then any optimal minimal realization of θ is stable too.*

Proof. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a stable contractive realization of θ, and let $\Sigma_\circ = (A_\circ, B_\circ, C_\circ, D; \mathcal{X}_\circ, \mathcal{U}, \mathcal{Y})$ be an optimal minimal realization of θ. By the previous lemma, since $\Sigma_\circ$ is contractive and controllable, it suffices to show that for each $u \in \mathcal{U}$ we have $A_\circ^n B_\circ u \to 0$ if $n \to \infty$. According to (5.4) we have $\|A_\circ^n B_\circ u\| \leqslant \|A^n B u\|$. Since Σ is stable, $A^n B u \to 0$ if $n \to \infty$, and thus $A_\circ^n B_\circ u \to 0$ for $n \to \infty$ too. Hence $\Sigma_\circ$ is stable. ∎

PROPOSITION 6.8. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system which is dissipative with respect to the supply rate* (1.3)*, and let H_1 and H_2 belong to $\mathcal{GK}_\Sigma^{\min}$. Assume that $H_1 \prec H_2$. Then the following holds:*

(i) *Σ_{H_2} is stable implies that Σ_{H_1} is stable;*
(ii) *Σ_{H_1} is star-stable implies that Σ_{H_2} is star-stable.*

Proof. (i). Assume $H_1 \prec H_2$ and Σ_{H_2} is stable. Since Σ_{H_1} is minimal (because $H \in \mathcal{GK}_\Sigma^{\min}$) and contractive (by Proposition 4.2), we see from Lemma 6.6 that it suffices to show that $\lim_{n\to\infty} A_{H_1}^n B_{H_1} u = 0$ for each $u \in \mathcal{U}$. Fix $u \in \mathcal{U}$. Since $A^n B\mathcal{U} \subset \mathcal{D}(H_2^{1/2})$ for each n and $H_1 \prec H_2^{1/2}$, we have

$$\|A_{H_1}^n B_{H_1} u\| = \|H_1^{1/2} A^n B u\| \leqslant \|H_2^{1/2} A^n B u\| = \|A_{H_2}^n B_{H_2} u\|.$$

Now use that Σ_{H_2} is stable. Thus $A_{H_2}^n B_{H_2} u \to 0$ when $n \to \infty$. It follows that $A_{H_1}^n B_{H_1} u$ goes to 0 if $n \to \infty$, and hence Σ_{H_1}.

(ii). Assume $H_1 \prec H_2$ and Σ_{H_1} is star-stable. It suffices to show that $(\Sigma_{H_2})^*$ is stable. Since $H_1 \prec H_2$, we know from the first paragraph of the proof of Proposition 5.6 (see the remark preceding Corollary 5.7) that there exists a contraction R on $\mathcal{X}$ such that

$$RA_{H_2} = A_{H_1} R, \quad RB_{H_2} = B_{H_1}, \quad C_{H_2} = C_{H_1} R.$$

By taking the adjoint of the first and third identity in the preceding formula we obtain

$$R^*(A_{H_1})^* = (A_{H_2})^* R^*, \quad R^*(C_{H_1})^* = (C_{H_2})^*.$$

Thus

$$R^*(A_{H_1})^{*n}(C_{H_1})^* y = (A_{H_2})^{*n}(C_{H_2})^* y \quad (y \in \mathcal{Y}).$$

Since R^* is a contraction, it follows that

$$\|(A_{H_2})^{*n}(C_{H_2})^*y\| \leqslant \|(A_{H_1})^{*n}(C_{H_1})^*y\| \quad (y \in \mathcal{Y}). \tag{6.16}$$

Since Σ_{H_2} is minimal and contractive, the same holds true for $(\Sigma_{H_2})^*$, and hence by Lemma 6.6 it suffices to show that for each $y \in \mathcal{Y}$ we have

$$\lim_{n\to\infty}(A_{H_2})^{*n}(C_{H_2})^*y = 0.$$

Since $(\Sigma_{H_1})^*$ is stable, the latter limit holds with H_1 in place of H_2. But then we can use the inequality (6.16) to show that it holds for H_2 too. Hence $(\Sigma_{H_2})^*$ is stable. ∎

Proof of Theorem 6.3. Assume there exists an element $H \in \mathcal{GK}_\Sigma^{\min}$ such that Σ_H is stable. Since Σ_H is a stable and contractive system, we can use Proposition 4 of [4], to show that the factorization problem (6.8) has a solution. To show the reverse implication, assume the problem (6.8) has a solution ϕ. Then θ has a contractive stable realization Σ by Proposition 4 of [4]. According to Corollary 6.7 any optimal realization of θ is stable. In particular, by Corollary 5.9, the system $\Sigma_{H_\circ}$ is stable.

The final statement of the theorem is a reformulation of Theorem 8 from [4]. ∎

Proof of Theorem 6.4. The proof follows by employing the duality between optimal and star-optimal systems, and using Theorem 6.3 together with Theorem 8 from [4]. ∎

Proof of Theorem 6.5. We claim that for each $H \in \mathcal{GK}_\Sigma^{\min}$ the system Σ_H is stable and star-stable if and only if the two systems $\Sigma_{H_\circ}$ and $\Sigma_{H_\bullet}$ are both stable and star-stable. Since $H_\circ$ and $H_\bullet$ belong to $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$, and $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ is contained in $\mathcal{GK}_\Sigma^{\min}$, the 'only if' part is trivial. The 'if' part follows from Proposition 6.8. Indeed, by Proposition 6.8 (i), stability of $\Sigma_{H_\bullet}$ implies stability of Σ_H for each $H \in \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ because $H \prec H_\bullet$. Similarly, by Proposition 6.8 (ii), star-stability of $\Sigma_{H_\circ}$ implies star-stability of Σ_H for each $H \in \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ because $H_\circ \prec H$. By applying Theorem 6.3 and Theorem 6.4 we see that stability and star-stability of both $\Sigma_{H_\circ}$ and $\Sigma_{H_\bullet}$ implies that Σ_H is stable and star-stable for each $H \in \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$. Now take an arbitrary $H' \in \mathcal{GK}_\Sigma^{\min}$. Then $\Sigma_{H'}$ is unitarily equivalent to Σ_H for some $H \in \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ by Theorem 5.12. Thus $\Sigma_{H'}$ is stable and star-stable whenever Σ_H has these properties. This completes the proof. ∎

6.4. COROLLARIES OF THE MAIN STABILITY THEOREMS. For the next two corollaries we need the notion of pseudo-continuation across the unit circle for an operator valued Schur class function. To define this notion let $\mathbb{D}_e = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$. Recall that a meromorphic $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function ϕ is of *bounded Nevanlinna type* on $\mathbb{D}_e$ if $\phi = \phi_1^{-1}\phi_2$, where ϕ_1 and ϕ_2 are bounded analytic functions on $\mathbb{D}_e$, the function ϕ_1 is scalar-valued and ϕ_2 is $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued (cf., Subsection 2.7, where the scalar case is considered). Such a function ϕ has non-tangential boundary values almost everywhere on the unit circle. A Schur class function θ is said to admit a *pseudo-continuation across the unit circle* if there exists a $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function ϕ of bounded Nevanlinna type on $\mathbb{D}_e$ such that θ and ϕ have the same non-tangential boundary values almost everywhere on the unit circle, that is, $\theta(\zeta) = \phi(\zeta)$ for almost every $\zeta \in \mathbb{T}$.

COROLLARY 6.9. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system which is dissipative with respect to the supply rate* (1.3)*, and let θ be the (unique) Schur class function coinciding with the transfer function θ_Σ in a neighborhood of zero. If θ admits a pseudo-continuation across the unit circle, then for each $H \in \mathcal{GK}_\Sigma^{\min}$ the system Σ_H is stable and star-stable.*

Proof. Let θ be a Schur class function that has a pseudo-continuation across the unit circle. In Section 3 of [4] it is shown that both factorization problems (6.8) and (6.10) have a solution and the unique function h_θ defined in (6.12) has representations (6.13) and (6.14). Thus the result follows from Theorem 6.5. ∎

Let Σ, θ_Σ, and θ be as in the previous corollary, and assume that θ admits a pseudo-continuation across the unit circle. Then from [4] we also know that the spectrum $\sigma(A_H)$, with H from $\mathcal{GK}_\Sigma^{\min}$, does not depend on the particular choice of H. Moreover, taking into account the property of pseudo-continuation, we have

$$\theta(\lambda) = D + \lambda C_H(I - \lambda A_H)^{-1}B_H \ \text{ for all } \ \lambda \ \text{ such that } \ I - \lambda A_H \ \text{ is invertible.}$$

COROLLARY 6.10. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathbb{C}^p, \mathbb{C}^m)$ be a minimal system (with finite dimensional input and output spaces) which is dissipative with respect to the supply rate* (1.3)*. Then the following statements are equivalent:*

(i) *Σ_H is stable and star-stable for each $H \in \mathcal{GK}_\Sigma^{\min}$;*

(ii) *$\Sigma_{H_\circ}$ is stable and star-stable;*

(iii) *$\Sigma_{H_\bullet}$ is stable and star-stable;*

(iv) *the transfer function θ_Σ coincides with a Schur class function θ in a neighborhood of zero that has a pseudo-continuation across the unit circle.*

Proof. The equivalence of (i) and (iv) follows from the previous corollary and the fact that a matrix-valued Schur class function θ admits a pseudo-continuation across the unit circle if and only if the two factorization problems (6.8) and (6.10) have solutions and the unique function h_θ defined in (6.12) has representations (6.13) and (6.14) (see [4]).

Next we use again a result from [4] which shows that the representation (6.13) exists if and only if (6.14) exists whenever the input and output spaces are finite dimensional. Using this result and our main stability theorems it is then straightforward to prove the remaining equivalences. ∎

7. ADDITIONAL INFORMATION ON $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$

Let Σ be a minimal system which is dissipative with respect to the supply rate (1.3). In this section we combine results from the present paper with results from [8] and [9] to derive criteria in order that $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ consists of one element only (i.e., $H_\circ = H_\bullet$) or that all systems Σ_H with $H \in \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ are mutually similar (i.e., $H_\bullet \prec \gamma H_\circ$ for some $\gamma > 0$). The criteria will be stated in terms of the Schur class function θ coinciding with the transfer function of Σ in a neighborhood of 0.

To formulate these criteria we need the *inner scattering sub-operator function* s_θ associated with θ. For the definition of this notion we refer to [12]. (See also Section 3.1 in [9].) Here we only mention that s_θ is an $L(\mathcal{U}_\theta, \mathcal{Y}_\theta)$-valued L^∞-function on the unit circle (where $\mathcal{U}_\theta$ and $\mathcal{Y}_\theta$ are auxiliary Hilbert spaces) which coincides with the function h_θ defined in (6.12) provided the two factorization problems (6.8) and (6.10) have solutions.

THEOREM 7.1. *Let Σ be a minimal system which is dissipative with respect to the supply rate* (1.3), *and let θ be the Schur class function coinciding with the transfer function of Σ in a neighborhood of* 0. *Let s_θ be the inner scattering sub-operator function associated with θ. Then $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ consists of one element only if and only if s_θ is the boundary value function of a Schur class function.*

Proof. Notice that the statement $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ consists of one element only is equivalent to the statement that $H_\circ = H_\bullet$. From Theorem 5.2 and Corollary 5.7 we know that $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$ consists of one element only is equivalent to the statement that all minimal contractive systems with transfer function θ are unitarily equivalent. From [8] (see, also Theorem 2 in [9]) we know that the latter happens if and only if s_θ is the boundary value function of a Schur class function. ∎

Let Σ and θ be as in the previous theorem, and assume that the two factorization problems (6.8) and (6.10) have solutions, and hence $s_\theta = h_\theta$. If $H_\circ = H_\bullet$, then $\Sigma_{H_\circ}$ is stable and star-stable. This follows from Theorems 6.3 and 6.4 and the fact that h_θ is the boundary value function of a Schur class function (according to the previous theorem).

For the next theorem we need the Hankel operator with symbol s_θ, that is, the operator

$$\Gamma_{s_\theta} : H^2(\mathcal{U}_\theta) \to K^2(\mathcal{Y}_\theta), \quad \Gamma_{s_\theta} = P_{K^2(\mathcal{Y}_\theta)} M_{s_\theta} | H^2(\mathcal{U}_\theta). \tag{7.1}$$

Here M_{s_θ} is the operator of multiplication by s_θ from $L^2(\mathbb{T}, \mathcal{U}_\theta)$ into $L^2(\mathbb{T}, \mathcal{Y}_\theta)$, the space $H^2(\mathcal{U}_\theta)$ is the Hardy space consisting of all functions in $L^2(\mathbb{T}, \mathcal{U}_\theta)$ of which the Fourier coefficients with negative index are zero, the space $K^2(\mathcal{Y}_\theta)$ is the orthogonal complement of the Hardy space $H^2(\mathcal{Y}_\theta)$ in $L^2(\mathbb{T}, \mathcal{Y}_\theta)$, and $P_{K^2(\mathcal{Y}_\theta)}$ is the orthogonal projection of $L^2(\mathbb{T}, \mathcal{Y}_{\phi_\theta})$ onto $K^2(\mathcal{Y}_\theta)$. If one of the (or both) spaces $\mathcal{U}_\theta$ or $\mathcal{Y}_\theta$ are zero, then we define $\Gamma_{s_\theta} = 0$.

THEOREM 7.2. *Let Σ be a minimal system which is dissipative with respect to the supply rate* (1.3), *and let θ be the Schur class function coinciding with the transfer function of Σ in a neighborhood of* 0. *Let s_θ be the inner scattering sub-operator function associated with θ. Then all systems Σ_H with $H \in \mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ are mutually similar if and only if the Hankel operator Γ_{s_θ} in* (7.1) *associated with s_θ has closed range.*

Proof. The condition that all systems Σ_H with $H \in \mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ are mutually similar is equivalent to the condition that all minimal contractive realizations of θ are mutually similar. But then we can use Theorem 3 in [9] to finish the proof. ∎

When one applies Theorems 7.1 and 7.2 to $\theta(\lambda) = \lambda K$, where $K : \mathcal{U} \to \mathcal{Y}$ is a contraction, one obtains Corollary 3.7 and Proposition 3.8 in [7]; see Section 8 in [6] for further details.

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