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DECOMPOSITIONS OF REFLEXIVE BIMODULES OVER MAXIMAL ABELIAN SELFADJOINT ALGEBRAS

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ABSTRACT. We generalize the notion of "diagonal" from the class of CSL algebras to masa bimodules. We prove that a reflexive masa bimodule decomposes as a sum of two bimodules, the diagonal and a module generalizing the w^*-closure of the Jacobson radical of a CSL algebra. The latter module turns out to be reflexive, a result which is new even for CSL algebras. We show that the projection onto the direct summand contained in the diagonal is contractive and preserves compactness and reduces rank of operators. Stronger results are obtained when the module is the reflexive hull of its rank-one subspace.

KEYWORDS: *Operator algebras, reflexivity, TRO, masa-bimodules.*

MSC (2000): Primary 47L05; Secondary 47L35, 46L10.

1. INTRODUCTION

In this paper we attempt a generalisation of the concept of the diagonal of a CSL algebra to reflexive spaces of operators which are modules over maximal abelian selfadjoint algebras (masas).

Recall [2] that a CSL algebra is an algebra $\mathcal{A}$ of operators on a Hilbert space H which can be written in the form

$$\mathcal{A} = \{A \in B(H) : AP = PAP \text{ for all } P \in \mathcal{S}\}$$

where $\mathcal{S}$ is a commuting family of projections. Note that $\mathcal{A}$ contains any masa containing $\mathcal{S}''$. More generally, a reflexive masa bimodule $\mathcal{U}$ of operators from H to another Hilbert space K can be written in the form

$$\mathcal{U} = \{T \in B(H,K) : TP = \phi(P)TP \text{ for all } P \in \mathcal{S}\}$$

where $\mathcal{S}$ is a commuting family of projections on H and ϕ maps them to commuting projections on K (see below for details). The diagonal $\mathcal{A} \cap \mathcal{A}^*$ of a CSL algebra $\mathcal{A}$ is a von Neumann algebra, which equals the commutant

$$\mathcal{S}' = \{A \in B(H) : AP = PA \text{ for all } P \in \mathcal{S}\}$$

of the corresponding invariant projection family. The natural corresponding object for a reflexive masa bimodule $\mathcal{U}$ is a ternary ring of operators (TRO)

$$\Delta(\mathcal{U}) = \{T \in B(H,K) : TP = \phi(P)T \text{ for all } P \in \mathcal{S}\}$$

which is also a reflexive masa bimodule. This "diagonal" $\Delta(\mathcal{U})$ is the primary object of study of the present paper.

We decompose $\mathcal{U}$ as a sum $\mathcal{U}_0 + \Delta(\mathcal{U})$, where $\mathcal{U}_0$ also turns out to be reflexive (Theorem 5.2). This is new even for the case of CSL algebras; note, however, that for nest algebras reflexivity of w^*-closed bimodules is automatic [7]. An analogous decomposition for the case of nest subalgebras of von Neumann algebras is in [11]. We also prove (Corollary 5.3) that the bimodule $\mathcal{U}_0$ has in our context the role corresponding to the w^*-closure of the Jacobson radical of a CSL algebra.

The diagonal $\Delta(\mathcal{U})$ is proved to be generated by a partial isometry and natural von Neumann algebras associated to $\mathcal{U}$ (Theorem 4.1).

The above decomposition may be further refined to a direct sum: $\mathcal{U} = \mathcal{U}_0 \oplus \mathcal{M}$ where $\mathcal{M}$ is a TRO ideal of the diagonal $\Delta(\mathcal{U})$ (Theorem 3.4), containing the compact operators of the diagonal (Proposition 6.3). In case $\mathcal{U}$ is strongly reflexive (that is, coincides with the reflexive hull of the rank one operators it contains) we show (Theorem 7.4) that $\mathcal{M}$ coincides with the w^*-closed linear span of the finite rank operators of the diagonal, an equality which fails in general. As in the case of von Neumann algebras, we show that every TRO decomposes in an "atomic" and a "nonatomic" part. The "atomic" part of the diagonal $\Delta(\mathcal{U})$ is contained (properly in general) in $\mathcal{M}$ (Proposition 6.3).

We also study the projection $\theta : \mathcal{U} \longrightarrow \mathcal{M}$ defined by the above direct sum decomposition. We prove that it is contractive and maps compact operators to compact operators and finite rank operators to operators of at most the same rank. In case $\mathcal{U}$ is strongly reflexive, we show that $\theta = D|_{\mathcal{U}}$, where D is the natural projection onto the "atomic" part of the diagonal $\Delta(\mathcal{U})$.

A main tool used to obtain these results is an appropriate sequence of projections (U_n) on $B(H,K)$ which depend on $\mathcal{U}$. This sequence behaves analogously to the net of "diagonal sums" used in nest algebras (see for example [2]). In nest algebra theory, the net of diagonal sums of a compact operator converges in norm to a compact operator in the "atomic" part of the diagonal. This has been generalised to CSL algebras by Katsoulis [10]. Here we show (Proposition 6.10) that for every compact operator K, the sequence $(U_n(K))$ converges in norm to $D(K)$.

We present some definitions and concepts used in this work. All Hilbert spaces will be assumed separable.

If S is a set of operators $R_1(S)$ denotes the subset of S consisting of rank 1 operators and the zero operator. If H is a Hilbert space and $S \subset B(H)$, the set of orthogonal projections of S is denoted by $\mathcal{P}(S)$.

If H_1, H_2 are Hilbert spaces, $C_1(H_1, H_2)$ are the trace class operators and $\mathcal{R}$ a subset of $C_1(H_1, H_2)$, we denote by $\mathcal{R}^0$ the set of operators which are annihilated by $\mathcal{R}$:

$$\mathcal{R}^0 = \{T \in B(H_2, H_1) : \mathrm{tr}(TS) = 0 \text{ for all } S \in \mathcal{R}\}.$$

Let H_1, H_2 be Hilbert spaces and $\mathcal{U}$ a subset of $B(H_1, H_2)$. The *reflexive hull* of $\mathcal{U}$ is defined [12] to be the space

$$\mathrm{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) : Tx \in \overline{[\mathcal{U}x]} \text{ for each } x \in H_1\}.$$

Simple arguments show that

$$\mathrm{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) : \text{for all projections } E, F \text{ such that } E\mathcal{U}F = 0 \Rightarrow ETF = 0\}.$$

A subspace $\mathcal{U}$ is called *reflexive* if $\mathcal{U} = \mathrm{Ref}(\mathcal{U})$. It is called *strongly reflexive* if there exists a set $L \subset B(H_1, H_2)$ of rank 1 operators such that $\mathcal{U} = \mathrm{Ref}(L)$.

Now we present some concepts introduced by Erdos [5].

Let $\mathcal{P}_i = \mathcal{P}(B(H_i)), i = 1, 2$. Define $\phi = \mathrm{Map}(\mathcal{U})$ to be the map $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ which associates to every $P \in \mathcal{P}_1$ the projection onto the subspace $[TPy : T \in \mathcal{U}, y \in H_1]^-$. The map ϕ is $\vee$-continuous (that is, it preserves arbitrary suprema) and 0 preserving.

Let $\phi^* = \mathrm{Map}(\mathcal{U}^*), \mathcal{S}_{1,\phi} = \{\phi^*(P)^\perp : P \in \mathcal{P}_2\}, \mathcal{S}_{2,\phi} = \{\phi(P) : P \in \mathcal{P}_1\}$. Erdos has proved that $\mathcal{S}_{1,\phi}$ is meet complete and contains the identity projection, $\mathcal{S}_{2,\phi}$ is join complete and contains the zero projection, while $\phi|_{\mathcal{S}_{1,\phi}} : \mathcal{S}_{1,\phi} \to \mathcal{S}_{2,\phi}$ is a bijection. In fact

$$(\phi|_{\mathcal{S}_{1,\phi}})^{-1}(Q) = \phi^*(Q^\perp)^\perp \tag{1.1}$$

for all $Q \in \mathcal{S}_{2,\phi}$ and

$$\mathrm{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) : \phi(P)^\perp TP = 0 \text{ for each } P \in \mathcal{S}_{1,\phi}\}.$$

We call the families $\mathcal{S}_{1,\phi}, \mathcal{S}_{2,\phi}$ the *semilattices* of $\mathcal{U}$.

A *CSL* is a complete lattice of commuting projections which contains the identity and the zero projection.

If $\mathcal{A}_1 \subset B(H_1)$, and $\mathcal{A}_2 \subset B(H_2)$ are algebras, a subspace $\mathcal{U} \subset B(H_1, H_2)$ is called an $\mathcal{A}_1$,$\mathcal{A}_2$-*bimodule* if $\mathcal{A}_2\mathcal{U}\mathcal{A}_1 \subset \mathcal{U}$.

A subspace $\mathcal{M}$ of $B(H_1, H_2)$ is called a *ternary ring of operators* (TRO) if $\mathcal{M}\mathcal{M}^*\mathcal{M} \subset \mathcal{M}$. Katavolos and Todorov [9] have proved that a TRO $\mathcal{M}$ is w^*-closed if and only if it is *wot*-closed if and only if it is reflexive. In this case, if $\chi = \mathrm{Map}(\mathcal{M})$, then

$$\mathcal{M} = \{T \in B(H_1, H_2) : TP = \chi(P)T \text{ for all } P \in \mathcal{S}_{1,\chi}\}.$$

They also proved that for every strongly reflexive TRO $\mathcal{M}$ there exist families of mutually orthogonal projections $(F_n), (E_n)$ such that

$$\mathcal{M} = \sum_{n=1}^{\infty} \bigoplus E_n B(H_1, H_2) F_n.$$

We present a new proof of this result in Corollary 6.9.

The following proposition is easily proved.

PROPOSITION 1.1. *Let H_1, H_2 be Hilbert spaces, $\mathcal{A}_1 \subset B(H_1), \mathcal{A}_2 \subset B(H_2)$ masas and $\mathcal{U}$ a $\mathcal{A}_1$,$\mathcal{A}_2$-bimodule. Then*

$$\mathrm{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) : E \in \mathcal{P}(\mathcal{A}_2),\ F \in \mathcal{P}(\mathcal{A}_1),\ E\mathcal{U}F = 0 \Rightarrow ETF = 0\}.$$

The next section contains some preliminary results.

2. DECOMPOSITION OF A REFLEXIVE TRO

In this section we show that a reflexive TRO decomposes into a "nonatomic" and a "totally atomic" part.

Let H_1, H_2 be Hilbert spaces, $\mathcal{M} \subset B(H_1, H_2)$ be a w^*-closed TRO and $\mathcal{B}_1 = (\mathcal{M}^*\mathcal{M})'', \mathcal{B}_2 = (\mathcal{M}\mathcal{M}^*)''$.

REMARK 2.1. We suppose that $\mathcal{M}_0$ is a w^*-closed TRO ideal of $\mathcal{M}$; namely, $\mathcal{M}_0$ is a linear subspace of $\mathcal{M}$ and

$$\mathcal{M}_0\mathcal{M}^*\mathcal{M} \subset \mathcal{M}_0, \quad \mathcal{M}\mathcal{M}^*\mathcal{M}_0 \subset \mathcal{M}_0.$$

It follows that $\mathcal{M}\mathcal{M}_0^*\mathcal{M} \subset \mathcal{M}_0$ [4]. Now, we observe that there exist projections Q_i in the centre of $\mathcal{B}_i$, $i = 1, 2$ such that $\mathcal{M}_0 = \mathcal{M}Q_1 = Q_2\mathcal{M}$. Hence $\mathcal{M}_0$ is a $\mathcal{B}_1, \mathcal{B}_2$-bimodule.

Indeed, let $\mathcal{J}_1 = [\mathcal{M}_0^*\mathcal{M}_0]^{-w^*}$ and $\mathcal{J}_2 = [\mathcal{M}_0\mathcal{M}_0^*]^{-w^*}$. We can easily verify that $\mathcal{J}_i$ is an ideal of $\mathcal{B}_i, i = 1, 2$. Hence there is a projection Q_i in the centre of $\mathcal{B}_i$ so that $\mathcal{J}_i = \mathcal{B}_iQ_i, i = 1, 2$.

One easily checks that

$$\mathcal{M}\mathcal{B}_1 \subset \mathcal{M}, \quad \mathcal{B}_2\mathcal{M} \subset \mathcal{M},$$
$$\mathcal{M}\mathcal{J}_1 \subset \mathcal{M}_0, \quad \mathcal{J}_2\mathcal{M} \subset \mathcal{M}_0.$$

We observe that $\mathcal{M}Q_1 \subset \mathcal{M}\mathcal{J}_1 \subset \mathcal{M}_0$. For every $T \in \mathcal{M}_0$, $T^*T \in \mathcal{J}_1$, so $T^*T = T^*TQ_1$ and thus $T = TQ_1$. Hence $T \in \mathcal{M}Q_1$. We conclude that $\mathcal{M}_0 \subset \mathcal{M}Q_1$ and hence equality holds. Similarly one shows that $\mathcal{M}_0 = Q_2\mathcal{M}$.

Since $[R_1(\mathcal{M})]^{-w^*}$ is a strongly reflexive TRO, by Proposition 3.5 in [9] there exist mutually orthogonal projections (F_n) in the centre of $\mathcal{B}_1$ and (E_n) in the centre of $\mathcal{B}_2$ such that $[R_1(\mathcal{M})]^{-w^*} = \sum_{n=1}^{\infty} \bigoplus E_nB(H_1, H_2)F_n$. We write $E = \bigvee_n E_n$, $F = \bigvee_n F_n$.

THEOREM 2.2. *The space $\mathcal{M}$ decomposes in the following direct sum*

$$\mathcal{M} = (\mathcal{M} \cap (R_1(\mathcal{M})^*)^0) \oplus [R_1(\mathcal{M})]^{-w^*}.$$

The spaces $\mathcal{M} \cap (R_1(\mathcal{M})^*)^0$ *and* $[R_1(\mathcal{M})]^{-w^*}$ *are* TRO *ideals of* $\mathcal{M}$. *Moreover*

$$[R_1(\mathcal{M})]^{-w^*} = \mathcal{M}F = E\mathcal{M} = E\mathcal{M}F,$$
$$\mathcal{M} \cap (R_1(\mathcal{M})^*)^0 = \mathcal{M}F^\perp = E^\perp\mathcal{M} = E^\perp\mathcal{M}F^\perp.$$

Proof. We observe that $[R_1(\mathcal{M})]^{-w^*}$ is a TRO ideal of $\mathcal{M}$. By Remark 2.1 there exists a projection Q in the centre of $\mathcal{B}_1$ such that $[R_1(\mathcal{M})]^{-w^*} = \mathcal{M}Q$.

For every $m \in \mathbb{N}$, we have $E_mB(H_1,H_2)F_m \subset \mathcal{M}Q$. It follows that $E_mB(H_1,H_2)F_m = E_mB(H_1,H_2)F_mQ$, so $F_m = F_mQ$. We conclude that $\bigvee\limits_m F_m = F \leqslant Q$. Since $F \in \mathcal{B}_1$ we get $\mathcal{M}F \subset \mathcal{M}$, therefore $\mathcal{M}F = \mathcal{M}FQ \subset \mathcal{M}Q$. It follows that

$$[R_1(\mathcal{M})]^{-w^*} = \mathcal{M}Q \supset \mathcal{M}F \supset [R_1(\mathcal{M})]^{-w^*}F = [R_1(\mathcal{M})]^{-w^*}.$$

We proved that $[R_1(\mathcal{M})]^{-w^*} = \mathcal{M}F$.

If $M \in \mathcal{M}$ and $R \in R_1(\mathcal{M})$, then $R = RF$ so $\operatorname{tr}(MF^\perp R^*) = \operatorname{tr}(M(RF^\perp)^*) = \operatorname{tr}(M0) = 0$. We conclude that

$$\mathcal{M}F^\perp \subset \mathcal{M} \cap (R_1(\mathcal{M})^*)^0.$$

Hence $\mathcal{M} = \mathcal{M}F^\perp + \mathcal{M}F \subset \mathcal{M} \cap (R_1(\mathcal{M})^*)^0 + [R_1(\mathcal{M})]^{-w^*} \subset \mathcal{M}$. It follows that

$$\mathcal{M} = (\mathcal{M} \cap (R_1(\mathcal{M})^*)^0) + [R_1(\mathcal{M})]^{-w^*}.$$

We shall prove that this sum is direct. If $T \in [R_1(\mathcal{M})]^{-w^*} \cap (R_1(\mathcal{M})^*)^0$ then $T = \sum\limits_{n=1}^{\infty} E_nTF_n$. If R is a rank 1 operator then $\operatorname{tr}(TR) = \sum\limits_{n=1}^{\infty} \operatorname{tr}(E_nTF_nR) = \sum\limits_{n=1}^{\infty} \operatorname{tr}(TF_nRE_n)$.

But for all $n \in \mathbb{N}$, $\operatorname{tr}(TF_nRE_n) = \operatorname{tr}(T(E_nR^*F_n)^*) = 0$ since $E_nR^*F_n \in R_1(\mathcal{M})$ and $T \in (R_1(\mathcal{M})^*)^0$. Thus $\operatorname{tr}(TR) = 0$ for every rank 1 operator R, hence $T = 0$. This shows that $[R_1(\mathcal{M})]^{-w^*} \cap (R_1(\mathcal{M})^\perp)^* = 0$. We have shown that $\mathcal{M} = (\mathcal{M} \cap (R_1(\mathcal{M})^*)^0) \oplus [R_1(\mathcal{M})]^{-w^*}$.

Since $\mathcal{M} = \mathcal{M}F^\perp \oplus \mathcal{M}F$, $[R_1(\mathcal{M})]^{-w^*} = \mathcal{M}F$ and $\mathcal{M}F^\perp \subset \mathcal{M} \cap (R_1(\mathcal{M})^*)^0$ we conclude that $\mathcal{M}F^\perp = \mathcal{M} \cap (R_1(\mathcal{M})^*)^0$.

The equalities $E^\perp\mathcal{M} = \mathcal{M} \cap (R_1(\mathcal{M})^*)^0, E\mathcal{M} = [R_1(\mathcal{M})]^{-w^*}$ are proved similarly. ∎

PROPOSITION 2.3. *Suppose that* $\theta : \mathcal{M} \to \mathcal{M}$ *is the projection onto* $[R_1(\mathcal{M})]^{-w^*}$ *defined by the decomposition in Theorem* 2.2. *Then* $\theta(T) = \sum\limits_{n=1}^{\infty} E_nTF_n$ *for every* $T \in \mathcal{M}$.

Proof. Since $\mathcal{M}$ decomposes as the direct sum of the $\mathcal{B}_1,\mathcal{B}_2$-bimodules $\mathcal{M} \cap (R_1(\mathcal{M})^*)^0$ and $[R_1(\mathcal{M})]^{-w^*}$, θ is a $\mathcal{B}_1,\mathcal{B}_2$-bimodule map:

$$\theta(B_2TB_1) = B_2\theta(T)B_1$$

for every $T \in \mathcal{M}, B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2$. Since $(F_n) \subset \mathcal{B}_1, (E_n) \subset \mathcal{B}_2$ we have that:

$$\theta(T) = \sum_{n=1}^{\infty} E_n \theta(T) F_n = \sum_{n=1}^{\infty} \theta(E_n T F_n) = \sum_{n=1}^{\infty} E_n T F_n. \quad \blacksquare$$

3. DECOMPOSITION OF A REFLEXIVE MASA BIMODULE

Let H_1, H_2 be Hilbert spaces, $\mathcal{P}_i = \mathcal{P}(B(H_i)), i = 1,2$, $\mathcal{D}_i \subset B(H_i), i = 1,2$ be masas, $\mathcal{U} \subset B(H_1, H_2)$ be a reflexive $\mathcal{D}_1$,$\mathcal{D}_2$-bimodule. Write

$$\begin{aligned} \phi &= \mathrm{Map}(\mathcal{U}), & \phi^* &= \mathrm{Map}(\mathcal{U}^*), \\ \mathcal{S}_{2,\phi} &= \phi(\mathcal{P}_1), & \mathcal{S}_{1,\phi} &= \{P^{\perp} : P \in \phi^*(\mathcal{P}_2)\}, \\ \mathcal{A}_2 &= (\mathcal{S}_{2,\phi})', & \mathcal{A}_1 &= (\mathcal{S}_{1,\phi})'. \end{aligned}$$

Observe that $\mathcal{S}_{i,\phi} \subset \mathcal{D}_i$ hence $\mathcal{D}_i \subset \mathcal{A}_i, i = 1,2$. We define

$$\mathcal{U}_0 = [\phi(P) T P^{\perp} : T \in \mathcal{U}, P \in \mathcal{S}_{1,\phi}]^{-w^*},$$
$$\Delta(\mathcal{U}) = \{T : TP = \phi(P) T \text{ for all } P \in \mathcal{S}_{1,\phi}\}.$$

We remark that $\mathcal{U}_0$ and $\Delta(\mathcal{U})$ are $\mathcal{D}_1$,$\mathcal{D}_2$-bimodules contained in $\mathcal{U}$ and $\Delta(\mathcal{U})$ is a reflexive TRO. We call $\Delta(\mathcal{U})$ the *diagonal* of $\mathcal{U}$.

This object, in case the masa bimodule is a CSL algebra coincides with the usual diagonal of the algebra.

EXAMPLE 3.1. Every bimodule over nest algebras is unitarily equivalent to a bimodule $\mathcal{U} \subset B(L^2(X,\mu), L^2(Y,\nu))$ for Borel spaces $(X,\mu), (Y,\nu)$. This bimodule has support, in the sense of Erdos, Katavolos, Shulman [6], a set of the form $\{(x,y) \in X \times Y : f(x) \leqslant g(y)\}$ for approriate real valued Borel functions f, g (see [13]). It can be shown that the diagonal $\Delta(\mathcal{U})$ is the reflexive masa bimodule whose support is the set $\{(x,y) \in X \times Y : f(x) = g(y)\}$.

THEOREM 3.2. $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})$.

Proof. As noted in the introduction

$$\mathcal{U} = \{T \in B(H_1, H_2) : \phi(P)^{\perp} T P = 0 \text{ for all } P \in \mathcal{S}_{1,\phi}\}.$$

Since the Hilbert spaces H_1, H_2 are separable we can choose a sequence $(P_n) \subset \mathcal{S}_{1,\phi}$ such that

$$\mathcal{U} = \{T \in B(H_1, H_2) : \phi(P_n)^{\perp} T P_n = 0 \text{ for all } n \in \mathbb{N}\}.$$

We define

$$V_n : B(H_1, H_2) \to B(H_1, H_2) : V_n(T) = \phi(P_n) T P_n + \phi(P_n)^{\perp} T P_n^{\perp}, \quad n \in \mathbb{N}.$$

One easily checks that V_n is idempotent and a norm contraction.

We also define $U_n = V_n \circ V_{n-1} \circ \cdots \circ V_1, n \in \mathbb{N}$. If $T \in \mathcal{U}$, then

$$T = U_1(T) + \phi(P_1)TP_1^{\perp},$$
$$U_1(T) = U_2(T) + \phi(P_2)U_1(T)P_2^{\perp},$$

by induction

$$U_{n-1}(T) = U_n(T) + \phi(P_n)U_{n-1}(T)P_n^{\perp}$$

for all $n \in \mathbb{N}$. Adding the previous equalities we obtain

$$T = U_n(T) + M_n$$

where

$$M_n = \phi(P_1)TP_1^{\perp} + \phi(P_2)U_1(T)P_2^{\perp} + \cdots + \phi(P_n)U_{n-1}(T)P_n^{\perp} \in \mathcal{U}_0$$

for all $n \in \mathbb{N}$.

The sequence $(U_n(T))$ is bounded since $\|U_n(T)\| \leqslant \|U_{n-1}(T)\| \leqslant \cdots \leqslant \|T\|$ for all $n \in \mathbb{N}$. So there exists a subsequence $(U_{n_m}(T))$ that converges in the weak*-topology to an operator L. Then $M_{n_m} = T - U_{n_m}(T) \xrightarrow{w^*} T - L = M \in \mathcal{U}_0$.

We observe that $\phi(P_i)^{\perp}U_n(T)P_i = \phi(P_i)U_n(T)P_i^{\perp} = 0$ for $i = 1, 2, \ldots, n$. It follows that $\phi(P_i)^{\perp}LP_i = \phi(P_i)LP_i^{\perp} = 0$ for all $i \in \mathbb{N}$. The conclusion is that $L \in \Delta(\mathcal{U})$ and $T = M + L \in \mathcal{U}_0 + \Delta(\mathcal{U})$. ∎

REMARK 3.3. The following are equivalent:

(i) $\mathcal{U}$ is a TRO.

(ii) $\mathcal{U} = \Delta(\mathcal{U})$.

(iii) $\mathcal{U}_0 = 0$.

THEOREM 3.4. *There exist projections $Q_i \in \mathcal{D}_i, i = 1, 2$ such that:*

$$\mathcal{U} = \mathcal{U}_0 \oplus (I - Q_2)\Delta(\mathcal{U})(I - Q_1) = \mathcal{U}_0 \oplus (I - Q_2)\Delta(\mathcal{U}) = \mathcal{U}_0 \oplus \Delta(\mathcal{U})(I - Q_1).$$

Proof. We make the following observations:

Step 1. $\mathcal{U}\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{U}, \quad \Delta(\mathcal{U})\Delta(\mathcal{U})^*\mathcal{U} \subset \mathcal{U}$.

Let $T \in \mathcal{U}, M, N \in \Delta(\mathcal{U})$. For every $P \in \mathcal{S}_{1,\phi}$ we have

$$\phi(P)^{\perp}TM^*NP = \phi(P)^{\perp}TM^*\phi(P)N = \phi(P)^{\perp}TPM^*N = 0M^*N = 0.$$

Thus $TM^*N \in \mathcal{U}$. Similarly we have that $MN^*T \in \mathcal{U}$.

Step 2. $\mathcal{U}_0\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{U}_0, \quad \Delta(\mathcal{U})\Delta(\mathcal{U})^*\mathcal{U}_0 \subset \mathcal{U}_0$.

Let $T \in \mathcal{U}, M, N \in \Delta(\mathcal{U})$. For every $P \in \mathcal{S}_{1,\phi}$ we have

$$\phi(P)TP^{\perp}M^*N = \phi(P)TM^*\phi(P)^{\perp}N = \phi(P)TM^*NP^{\perp}.$$

It follows by Step 1 that $TM^*N \in \mathcal{U}$ so $\phi(P)TP^{\perp}M^*N \in \mathcal{U}_0$. Taking the w^*-closed linear span we get $SM^*N \in \mathcal{U}_0$ for all $S \in \mathcal{U}_0, M, N \in \Delta(\mathcal{U})$. Similarly we have that $\Delta(\mathcal{U})\Delta(\mathcal{U})^*\mathcal{U}_0 \subset \mathcal{U}_0$.

Step 3. The space $\mathcal{U}_0 \cap \Delta(\mathcal{U})$ is a TRO ideal of $\Delta(\mathcal{U})$.

Since $\Delta(\mathcal{U})$ is a TRO $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \Delta(\mathcal{U})$. By Step 2 we have that $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{U}_0$. It follows that $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{U}_0 \cap \Delta(\mathcal{U})$.

Analogously we get $\Delta(\mathcal{U})\Delta(\mathcal{U})^*$ $(\mathcal{U}_0 \cap \Delta(\mathcal{U})) \subset \mathcal{U}_0 \cap \Delta(\mathcal{U})$. We conclude that the space $\mathcal{U}_0 \cap \Delta(\mathcal{U})$ is a TRO ideal of $\Delta(\mathcal{U})$. Thus there exist projections $Q_i \in \mathcal{D}_i, i = 1,2$, such that $\mathcal{U}_0 \cap \Delta(\mathcal{U}) = \Delta(\mathcal{U})Q_1 = Q_2\Delta(\mathcal{U})$ (Remark 2.1).

By Theorem 3.2 we have

$$\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U}) = \mathcal{U}_0 + \Delta(\mathcal{U})Q_1 + \Delta(\mathcal{U})(I - Q_1) = \mathcal{U}_0 + \Delta(\mathcal{U})(I - Q_1).$$

Clearly $\mathcal{U}_0 \cap \Delta(\mathcal{U})(I - Q_1) = 0$. Similarly one shows that $\mathcal{U} = \mathcal{U}_0 \oplus (I - Q_2)\Delta(\mathcal{U})$ and it therefore follows that $\mathcal{U} = \mathcal{U}_0 \oplus (I - Q_2)\Delta(\mathcal{U})(I - Q_1)$. ∎

REMARK 3.5. The projection $\theta : \mathcal{U} \to \mathcal{U}$ onto $(I - Q_2)\Delta(\mathcal{U})(I - Q_1)$ defined by the decomposition in Theorem 3.4 is a contraction.

Indeed, if $T \in \mathcal{U}$, as in Theorem 3.2 we have $T = M + S$ where $M \in \Delta(\mathcal{U}), S \in \mathcal{U}_0$ and $\|M\| \leqslant \|T\|$ (see the proof). Since $\theta(T) = (I - Q_2)M(I - Q_1)$, we obtain $\|\theta(T)\| \leqslant \|T\|$.

Let $\mathcal{N}_i = \mathrm{Alg}(\mathcal{S}_{i,\phi}) = \{T : P^\perp TP = 0 \text{ for all } P \in \mathcal{S}_{i,\phi}\}, i = 1,2$, and $\mathcal{L}_i = [PTP^\perp : T \in \mathcal{N}_i, P \in \mathcal{S}_{i,\phi}]^{-w^*}$, $i = 1,2$.

LEMMA 3.6. (i) $\mathcal{A}_2\Delta(\mathcal{U})\mathcal{A}_1 \subset \Delta(\mathcal{U})$.
(ii) $\Delta(\mathcal{U})^*\mathcal{A}_2\Delta(\mathcal{U}) \subset \mathcal{A}_1, \Delta(\mathcal{U})\mathcal{A}_1\Delta(\mathcal{U})^* \subset \mathcal{A}_2$.
(iii) $\mathcal{U} = \mathcal{N}_2\mathcal{U}\mathcal{N}_1$.
(iv) $\mathcal{U}_0 = \mathcal{N}_2\mathcal{U}_0\mathcal{N}_1$.
(v) $\mathcal{U}\mathcal{L}_1 \subset \mathcal{U}_0,\ \mathcal{L}_2\mathcal{U} \subset \mathcal{U}_0$.
(vi) $\Delta(\mathcal{U})^*\mathcal{U} \subset \mathcal{N}_1,\ \mathcal{U}\Delta(\mathcal{U})^* \subset \mathcal{N}_2$.
(vii) $\Delta(\mathcal{U})^*\mathcal{U}_0 \subset \mathcal{L}_1,\ \mathcal{U}_0\Delta(\mathcal{U})^* \subset \mathcal{L}_2$.

Proof. Claims (i),(ii) are obvious and (iii) is Lemma 1.1 in [9].

(iv) If $N_1 \in \mathcal{N}_1, N_2 \in \mathcal{N}_2, T \in \mathcal{U}$ and $P \in \mathcal{S}_{1,\phi}$ then

$$N_2\phi(P)TP^\perp N_1 = \phi(P)N_2\phi(P)TP^\perp N_1P^\perp \in \mathcal{U}_0$$

since $N_2\phi(P)TP^\perp N_1 \in \mathcal{U}$ by (iii). Taking the w^*-closed linear span we get $\mathcal{N}_2\mathcal{U}_0\mathcal{N}_1 \subset \mathcal{U}_0$.

(v) If $N_1 \in \mathcal{N}_1, T \in \mathcal{U}$ and $P \in \mathcal{S}_{1,\phi}$ then

$$TPN_1P^\perp = \phi(P)TPN_1P^\perp \in \mathcal{U}_0$$

since $TPN_1 \in \mathcal{U}\mathcal{N}_1 \subset \mathcal{U}$. Taking the w^*-closed linear span we get $TK \in \mathcal{U}_0$ for every $K \in \mathcal{L}_1$. The second inclusion follows by symmetry.

(vi) If $M \in \Delta(\mathcal{U}), T \in \mathcal{U}, P \in \mathcal{S}_{1,\phi}$ we have $PM^*TP = M^*\phi(P)TP = M^*TP$ so $M^*T \in \mathcal{N}_1$. Similarly one shows that $TM^* \in \mathcal{N}_2$.

(vii) If $M \in \Delta(\mathcal{U}), T \in \mathcal{U}, P \in \mathcal{S}_{1,\phi}$ we have $M^*\phi(P)TP^\perp = PM^*TP^\perp \in \mathcal{L}_1$ since $M^*T \in \Delta(\mathcal{U})^*\mathcal{U} \subset \mathcal{N}_1$. Taking the w^*-closed linear span we get $M^*S \in \mathcal{L}_1$ for every $S \in \mathcal{U}_0$. Similarly one shows that $\mathcal{U}_0\Delta(\mathcal{U})^* \subset \mathcal{L}_2$. ∎

PROPOSITION 3.7. *The following are equivalent:*
(i) $\mathcal{U} = \mathcal{U}_0$.

(ii) $\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{L}_1 \cap \mathcal{A}_1$.
(iii) $\Delta(\mathcal{U})\Delta(\mathcal{U})^* \subset \mathcal{L}_2 \cap \mathcal{A}_2$.

Proof. If $\mathcal{U} = \mathcal{U}_0$ then $\Delta(\mathcal{U}) \subset \mathcal{U}_0$, hence $\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \Delta(\mathcal{U})^*\mathcal{U}_0 \subset \mathcal{L}_1$ by the previous lemma. Since $\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{A}_1$ we get $\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{L}_1 \cap \mathcal{A}_1$.

If conversely $\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{L}_1 \cap \mathcal{A}_1$, then $\Delta(\mathcal{U})^*\Delta(\mathcal{U})(I - Q_1) \subset \mathcal{L}_1 \cap \mathcal{A}_1$, so by the previous lemma $\Delta(\mathcal{U})\Delta(\mathcal{U})^*\Delta(\mathcal{U})(I - Q_1) \subset \mathcal{U}\mathcal{L}_1 \subset \mathcal{U}_0$ (Q_1 is the projection in Theorem 3.4).

Since $\mathcal{U}_0 \cap \Delta(\mathcal{U})$ is a TRO ideal of $\Delta(\mathcal{U})$ (Theorem 3.4) we have that

$$\Delta(\mathcal{U})\Delta(\mathcal{U})^*\Delta(\mathcal{U})Q_1 = \Delta(\mathcal{U})\Delta(\mathcal{U})^*(\Delta(\mathcal{U}) \cap \mathcal{U}_0) \subset \Delta(\mathcal{U}) \cap \mathcal{U}_0 \subset \mathcal{U}_0.$$

We conclude that $\Delta(\mathcal{U})\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{U}_0$. Since $\Delta(\mathcal{U})$ is a TRO its subspace $\Delta(\mathcal{U})$ $\Delta(\mathcal{U})^*\Delta(\mathcal{U})$ is norm-dense [4]. Therefore $\Delta(\mathcal{U}) \subset \mathcal{U}_0$ and so $\mathcal{U} = \mathcal{U}_0$.

The equivalence (i)$\Leftrightarrow$(iii) is proved similarly. ∎

PROPOSITION 3.8. *The following are equivalent:*
(i) $\mathcal{U} = \mathcal{U}_0 \oplus \Delta(\mathcal{U})$.
(ii) $\Delta(\mathcal{U})\ (\mathcal{L}_1 \cap \mathcal{A}_1) = 0$.
(iii) $(\mathcal{L}_2 \cap \mathcal{A}_2)\ \Delta(\mathcal{U}) = 0$.

Proof. Note by Lemma 3.6 that $\Delta(\mathcal{U})(\mathcal{L}_1 \cap \mathcal{A}_1) \subset \Delta(\mathcal{U})\mathcal{A}_1 \subset \Delta(\mathcal{U})$ and $\Delta(\mathcal{U})(\mathcal{L}_1 \cap \mathcal{A}_1) \subset \mathcal{U}\mathcal{L}_1 \subset \mathcal{U}_0$. Thus if the sum $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})$ is direct then $\Delta(\mathcal{U})\ (\mathcal{L}_1 \cap \mathcal{A}_1) = 0$.

Suppose conversely that $\Delta(\mathcal{U})(\mathcal{L}_1 \cap \mathcal{A}_1) = 0$. Using again Lemma 3.6 we have that $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^*(\mathcal{U}_0 \cap \Delta(\mathcal{U})) \subset \Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{A}_1$ and $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^*(\mathcal{U}_0 \cap \Delta(\mathcal{U})) \subset \Delta(\mathcal{U})^*\mathcal{U}_0 \subset \mathcal{L}_1$ so $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^*(\mathcal{U}_0 \cap \Delta(\mathcal{U})) \subset \mathcal{L}_1 \cap \mathcal{A}_1$ hence $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^*(\mathcal{U}_0 \cap \Delta(\mathcal{U})) \subset \Delta(\mathcal{U})(\mathcal{L}_1 \cap \mathcal{A}_1) = 0$. But since $\mathcal{U}_0 \cap \Delta(\mathcal{U})$ is a TRO (Theorem 3.3), its subspace $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^*(\mathcal{U}_0 \cap \Delta(\mathcal{U}))$ is norm-dense [4]. Therefore $\mathcal{U}_0 \cap \Delta(\mathcal{U}) = 0$. This shows that (i) and (ii) are equivalent.

The proof of the equivalence of (i) and (iii) is analogous. ∎

4. THE DIAGONAL

Let $\mathcal{U}, \mathcal{U}_0, \Delta(\mathcal{U}), \phi$ be as in Section 3 and $\chi = \mathrm{Map}(\Delta(\mathcal{U}))$.

THEOREM 4.1. *There exists a partial isometry $V \in \Delta(\mathcal{U})$ such that*

$$\Delta(\mathcal{U}) = [\mathcal{A}_2 V \mathcal{A}_1]^{-w^*}$$

(recall that $\mathcal{A}_i = (\mathcal{S}_{i,\phi})'$).

Proof. If $T \in \Delta(\mathcal{U})$ and $T = U|T|$ is the polar decomposition of T, then $U \in \Delta(\mathcal{U})$ and $|T| \in \mathcal{A}_1$ (Proposition 2.6 in [9]).

By Zorn's lemma there exists a maximal family of partial isometries $(V_n) \subset \Delta(\mathcal{U})$ such that: $V_n^* V_n \perp V_m^* V_m, V_n V_n^* \perp V_m V_m^*$ for $n \neq m$. The partial isometry $V = \sum_{n=1}^{\infty} V_n$ belongs to $\Delta(\mathcal{U})$.

First we show that

$$\Delta(\mathcal{U}) = \{T \in B(H_1, H_2) : E \in \mathcal{P}(\mathcal{A}_1'), F \in \mathcal{P}(\mathcal{A}_2'), FVE = 0 \Rightarrow FTE = 0\}. \tag{4.1}$$

Let T be such that, if $FVE = 0$ for $E \in \mathcal{P}(\mathcal{A}_1')$ and $F \in \mathcal{P}(\mathcal{A}_2')$, then $FTE = 0$. Since $\phi(P)^\perp VP = \phi(P)VP^\perp = 0$ for every $P \in \mathcal{S}_{1,\phi}$ and $\mathcal{S}_{i,\phi} \subset \mathcal{A}_i', i = 1, 2$ we have $\phi(P)^\perp TP = \phi(P)TP^\perp = 0$ for every $P \in \mathcal{S}_{1,\phi}$ so $T \in \Delta(\mathcal{U})$.

For the converse let $T \in \Delta(\mathcal{U})$ and $T = U|T|$ be the polar decomposition of T. If $E \in \mathcal{P}(\mathcal{A}_1'), F \in \mathcal{P}(\mathcal{A}_2')$ are such that $FVE = 0$, since $|T| \in \mathcal{A}_1$, we have $FTE = FU|T|E = FUE|T|$. Hence it suffices to show that $FUE = 0$.

We observe that:

$$\begin{aligned} V^*V(FUE)^*FUE &= (V^*V)EU^*FUE = E(V^*V)U^*FUE \quad (V^*V \in \mathcal{A}_1) \\ &= EV^*(VU^*)FUE = EV^*F(VU^*)UE \quad (VU^* \in \mathcal{A}_2) \\ &= 0VU^*UE = 0 \end{aligned}$$

hence

$$(FUE)^*FUE \leqslant I - V^*V. \tag{4.2}$$

Similarly, one shows that

$$FUE(FUE)^* \leqslant I - VV^*. \tag{4.3}$$

Since FUE is a partial isometry in $\Delta(\mathcal{U})$, the maximality of V and (4.2),(4.3) imply that $FUE = 0$. Thus claim (4.1) holds.

Let $\mathcal{M} = [\mathcal{A}_2 V \mathcal{A}_1]^{-w^*}$. We observe that $\mathcal{M}$ is a TRO which is contained in $\Delta(\mathcal{U})$. Since $\mathcal{M}$ is w^*-closed, it is reflexive. If $\zeta = \mathrm{Map}(\mathcal{M})$, for every projection P,

$$\zeta(P) = [A_2 V A_1 P y : A_i \in \mathcal{A}_i, i = 1, 2, y \in H_1]^-.$$

We observe that $\zeta(P) \in \mathcal{A}_2'$ for every projection P so $\mathcal{S}_{2,\zeta} \subset \mathcal{A}_2'$. Similarly if $\zeta^* = \mathrm{Map}(\mathcal{M}^*)$ then $\mathcal{S}_{2,\zeta^*} \subset \mathcal{A}_1'$ but $\mathcal{S}_{1,\zeta} = \{P^\perp : P \in \mathcal{S}_{2,\zeta^*}\}$ so we have that $\mathcal{S}_{1,\zeta} \subset \mathcal{A}_1'$. Now since $V \in \mathcal{M}$ we conclude that $\zeta(P)^\perp VP = 0$ for every $P \in \mathcal{S}_{1,\zeta}$. From claim (4.1) we obtain $\zeta(P)^\perp \Delta(\mathcal{U})P = 0$ for every $P \in \mathcal{S}_{1,\zeta}$, so since $\mathcal{M}$ is reflexive $\Delta(\mathcal{U}) \subset \mathcal{M}$. ∎

By the previous theorem it follows that if $\mathcal{M}$ is a w^*-closed TRO masa bimodule and $\zeta = \mathrm{Map}(\mathcal{M})$ then there exists a partial isometry $V \in \mathcal{M}$ so that $\mathcal{M} = [(\mathcal{S}_{2,\zeta})' V (\mathcal{S}_{1,\zeta})']^{-w^*}$. But we shall prove a stronger result:

THEOREM 4.2. *Let $\mathcal{M}$ be a w^*-closed* TRO *masa bimodule and the algebras $\mathcal{B}_1 = [\mathcal{M}^*\mathcal{M}]^{-w^*}, \mathcal{B}_2 = [\mathcal{M}\mathcal{M}^*]^{-w^*}$. Then there exists a partial isometry V such that $\mathcal{M} = [\mathcal{B}_2 V \mathcal{B}_1]^{-w^*}$.*

Proof. Let $\mathcal{D}_i \subset B(H_i), i = 1, 2$ be masas such that $\mathcal{D}_2\mathcal{M}\mathcal{D}_1 \subset \mathcal{M}$ and put $\zeta = \mathrm{Map}(\mathcal{M})$. We shall prove that $\mathcal{B}_2'\mathcal{M}\mathcal{B}_1' \subset \mathcal{M}$. In Theorem 2.10 of [9], it is shown that

$$\mathcal{B}_2' = (\mathcal{M}\mathcal{M}^*)' \subset \mathcal{D}_2|_{\zeta(I)} \oplus B(\zeta(I)^\perp(H_2))$$

and

$$\mathcal{B}_1' = (\mathcal{M}^*\mathcal{M})' \subset \mathcal{D}_1|_{\zeta^*(I)} \oplus B(\zeta^*(I)^\perp(H_1)).$$

So it suffices to show that

$$(\mathcal{D}_2|_{\zeta(I)} \oplus B(\zeta(I)^\perp(H_2)))\ \mathcal{M}\ (\mathcal{D}_1|_{\zeta^*(I)} \oplus B(\zeta^*(I)^\perp(H_1))) \subset \mathcal{M}.$$

But this is true because

$$\mathcal{D}_2\mathcal{M}\mathcal{D}_1 \subset \mathcal{M}, \quad \zeta(I) \in \mathcal{D}_2, \quad \zeta^*(I) \in \mathcal{D}_1 \quad \text{and} \quad \mathcal{M} = \zeta(I)\mathcal{M}\zeta^*(I).$$

Now, we shall follow the proof of the previous theorem: By Zorn's lemma there exists a maximal family of partial isometries $(V_n) \subset \mathcal{M}$ such that $V_n^*V_n \perp V_m^*V_m$, $V_nV_n^* \perp V_mV_m^*$ for $n \neq m$. The partial isometry $V = \sum\limits_{n=1}^{\infty} V_n$, belongs to $\mathcal{M}$. We shall show that

$$\mathcal{M} \subset \{T \in B(H_1, H_2) : E \in \mathcal{P}(\mathcal{B}_1'),\ F \in \mathcal{P}(\mathcal{B}_2'),\ FVE = 0 \ \Rightarrow\ FTE = 0\}. \tag{4.4}$$

Let $T \in \mathcal{M}$ and let $T = U|T|$ be the polar decomposition of T. By Proposition 2.6 in [9], $|T| \in (\mathcal{M}^*\mathcal{M})''$ and $U \in \mathcal{M}$. If $E \in \mathcal{P}(\mathcal{B}_1')$, $F \in \mathcal{P}(\mathcal{B}_2')$ are such that $FVE = 0$, since $|T| \in (\mathcal{M}^*\mathcal{M})''$ and $E \in \mathcal{B}_1' = (\mathcal{M}^*\mathcal{M})'$, we have $FTE = FU|T|E = FUE|T|$. Hence it suffices to show that $FUE = 0$.

As in the proof of the previous theorem we have that $V^*V \perp (FUE)^*(FUE)$ and $VV^* \perp (FUE)(FUE)^*$. But $FUE \in \mathcal{B}_2'\mathcal{M}\mathcal{B}_1' \subset \mathcal{M}$, so by the maximality of V we have that $FUE = 0$.

Let $\mathcal{W} = [\mathcal{B}_2 V \mathcal{B}_1]^{-w^*}$. We observe that $\mathcal{W} \subset \mathcal{M}$. For the converse, we follow the proof of the previous theorem and we use relation (4.4). ∎

An alternative proof of the previous theorem was communicated to us by I. Todorov, based on his paper [14].

THEOREM 4.3. *The semilattices of $\Delta(\mathcal{U})$ are the following:*

$$\mathcal{S}_{1,\chi} = \chi^*(I)^\perp \oplus \chi^*(I)\mathcal{P}((\mathcal{S}_{1,\phi})''),$$
$$\mathcal{S}_{2,\chi} = \chi(I)\mathcal{P}((\mathcal{S}_{2,\phi})'').$$

The map $\chi : \mathcal{S}_{1,\chi} \longrightarrow \mathcal{S}_{2,\chi}$ is such that

$$\chi(\chi^*(I)^\perp \oplus \chi^*(I)Q) = \chi(I)\phi(Q) \quad \textit{for every } Q \in \mathcal{S}_{1,\phi}. \tag{4.5}$$

Proof. In Theorem 4.1 we showed that there exists a partial isometry V in $\Delta(\mathcal{U})$ such that $\Delta(\mathcal{U}) = [(\mathcal{S}_{2,\phi})'V(\mathcal{S}_{1,\phi})']^{-w^*}$. So if $P \in \mathcal{S}_{1,\chi}$ then $\chi(P)$ is the projection onto $[(\mathcal{S}_{2,\phi})'V(\mathcal{S}_{1,\phi})'P(H_1)]^-$. We conclude that $\chi(P) \in (\mathcal{S}_{2,\phi})''$. Hence $\mathcal{S}_{2,\chi} \subset (\mathcal{S}_{2,\phi})''$.

If H is a Hilbert space, $\mathcal{B}$ is a subset of $B(H)$ and Q a projection in $\mathcal{B}'$ the set $\{T|_{Q(H)} : T \in \mathcal{B}\}$ is denoted by $\mathcal{B}|_Q$.

We have shown that $(\mathcal{S}_{2,\chi})''|_{\chi(I)} \subset (\mathcal{S}_{2,\phi})''|_{\chi(I)}$.

Let $P \in \mathcal{S}_{1,\phi}$. Since $\Delta(\mathcal{U})P = \phi(P)\Delta(\mathcal{U})$ it follows that $\chi(P) = \phi(P)\chi(I)$. So $\chi(I)\mathcal{S}_{2,\phi} \subset \mathcal{S}_{2,\chi}$ hence, $(\mathcal{S}_{2,\phi})''|_{\chi(I)} \subset (\mathcal{S}_{2,\chi})''|_{\chi(I)}$. We proved that

$$(\mathcal{S}_{2,\phi})''|_{\chi(I)} = (\mathcal{S}_{2,\chi})''|_{\chi(I)}.$$

Since $\Delta(\mathcal{U})$ is a TRO, using Theorem 2.10 in [9] (see the proof) we have that

$$\mathcal{S}_{2,\chi}|_{\chi(I)} = \mathcal{P}((\mathcal{S}_{2,\chi})''|_{\chi(I)}).$$

It follows that

$$\mathcal{S}_{2,\chi} = \chi(I)\mathcal{P}((\mathcal{S}_{2,\phi})'').$$

Applying this to $\Delta(\mathcal{U})^* = \Delta(\mathcal{U}^*)$,

$$\mathcal{S}_{2,\chi^*} = \chi^*(I)\mathcal{P}((\mathcal{S}_{2,\phi^*})'').$$

Since $\mathcal{S}_{1,\phi} = \{Q^\perp : Q \in \mathcal{S}_{2,\phi^*}\}$ (see the introduction) we have that

$$\mathcal{S}_{2,\chi^*} = \chi^*(I)\mathcal{P}((\mathcal{S}_{1,\phi})'').$$

But

$$\begin{aligned}\mathcal{S}_{1,\chi} = \{Q^\perp : Q \in \mathcal{S}_{2,\chi^*}\} &= \{(\chi^*(I)Q)^\perp : Q \in \mathcal{P}((\mathcal{S}_{1,\phi})'')\} \\ &= \{\chi^*(I)^\perp \oplus \chi^*(I)Q : Q \in \mathcal{P}((\mathcal{S}_{1,\phi})'')\}.\end{aligned}$$

If $Q \in \mathcal{S}_{1,\phi}$ then

$$\begin{aligned}\chi(\chi^*(I)^\perp \oplus \chi^*(I)Q) &= \chi(\chi^*(I)Q) && (\chi(\chi^*(I)^\perp) = 0) \\ &= \chi(Q) && (\Delta(\mathcal{U})\chi^*(I) = \Delta(\mathcal{U})) \\ &= \phi(Q)\chi(I) && (\Delta(\mathcal{U})Q = \phi(Q)\Delta(\mathcal{U})). \quad \blacksquare\end{aligned}$$

REMARK 4.4. The smallest ortholattice containing the commutative family $\chi(I)\mathcal{S}_{2,\phi}$ is easily seen to be $\chi(I)\mathcal{P}((\mathcal{S}_{2,\phi})'')$, which equals $\mathcal{S}_{2,\chi}$; similarly the family $\chi^*(I)^\perp \oplus \chi^*(I)\mathcal{S}_{1,\phi}$ generates the complete ortho-lattice $\mathcal{S}_{1,\chi}$. Therefore, since $\chi|_{\mathcal{S}_{1,\chi}}$ is a complete ortho-lattice isomorphism (Theorem 2.10 in [9]) equality (4.5) determines the map χ.

PROPOSITION 4.5. *The families $\chi^*(I)\mathcal{S}_{1,\phi}$ and $\chi(I)\mathcal{S}_{2,\phi}$ are complete lattices and the map*

$$\vartheta : \chi^*(I)\mathcal{S}_{1,\phi} \to \chi(I)\mathcal{S}_{2,\phi} \text{ with } \vartheta(\chi^*(I)P) = \chi(I)\phi(P)$$

is a complete lattice isomorphism.

Proof. We use Theorem 4.3 and the fact [9] that the map $\chi|_{\mathcal{S}_{1,\chi}}$ is a complete ortholattice isomorphism .

Let $(P_i)_{i\in I} \subset \mathcal{S}_{1,\phi}$. We claim that

$$\bigwedge_{i\in I} \chi(I)\phi(P_i) = \chi(I)\phi\Big(\bigwedge_{i\in I} P_i\Big). \tag{4.6}$$

Indeed, by (4.5),

$$\begin{aligned}\bigwedge_{i\in I}\chi(I)\phi(P_i) =& \bigwedge_{i\in I}\chi(\chi^*(I)^\perp\oplus\chi^*(I)P_i)\\ =&\chi\Big(\bigwedge_{i\in I}(\chi^*(I)^\perp\oplus\chi^*(I)P_i)\Big)=\chi\Big(\chi^*(I)^\perp\oplus\chi^*(I)\Big(\bigwedge_{i\in I}P_i\Big)\Big).\end{aligned}$$

Since $\bigwedge_{i\in I} P_i \in \mathcal{S}_{1,\phi}$ we get that $\chi\Big(\chi^*(I)^\perp\oplus\chi^*(I)\Big(\bigwedge_{i\in I}P_i\Big)\Big) = \chi(I)\phi\Big(\bigwedge_{i\in I}P_i\Big)$ again using (4.5).

By (1.1), there exist $(Q_i)_{i\in I}\subset\mathcal{S}_{1,\phi^*}$ such that $\phi^*(Q_i)^\perp = P_i$ for every $i\in I$. We shall prove that

$$\bigvee_{i\in I}\chi^*(I)P_i = \chi^*(I)\Big(\phi^*\Big(\bigwedge_{i\in I}Q_i\Big)\Big)^\perp. \tag{4.7}$$

Since $\Delta(\mathcal{U}^*) = \Delta(\mathcal{U})^*$ we have that $\chi^* = \mathrm{Map}(\Delta(\mathcal{U}^*))$ and so applying equation (4.6) to χ^* we have that

$$\begin{aligned}\bigwedge_{i\in I}\chi^*(I)\phi^*(Q_i) =&\chi^*(I)\phi^*\Big(\bigwedge_{i\in I}Q_i\Big) && \Rightarrow\\ \bigvee_{i\in I}(\chi^*(I)\phi^*(Q_i))^\perp =&\chi^*(I)\phi^*\Big(\bigwedge_{i\in I}Q_i\Big)^\perp && \Rightarrow\\ \bigvee_{i\in I}(\chi^*(I)^\perp\oplus\chi^*(I)(\phi^*(Q_i))^\perp) =&\chi^*(I)^\perp\oplus\chi^*(I)\Big(\phi^*\Big(\bigwedge_{i\in I}Q_i\Big)\Big)^\perp && \Rightarrow\\ \bigvee_{i\in I}(\chi^*(I)^\perp\oplus\chi^*(I)P_i) =&\chi^*(I)^\perp\oplus\chi^*(I)\Big(\phi^*\Big(\bigwedge_{i\in I}Q_i\Big)\Big)^\perp && \Rightarrow\\ \bigvee_{i\in I}\chi^*(I)P_i =&\chi^*(I)\Big(\phi^*\Big(\bigwedge_{i\in I}Q_i\Big)\Big)^\perp.\end{aligned}$$

From equalities (4.6) and (4.7) we conclude that the families $\chi^*(I)\mathcal{S}_{1,\phi}$, and $\chi(I)\mathcal{S}_{2,\phi}$ are complete lattices.

Since $\chi(\chi^*(I)^\perp\oplus\chi^*(I)Q) = \chi(I)\phi(Q)$ for every $Q\in\mathcal{S}_{1,\phi}$ and $\chi|_{\mathcal{S}_{1,\chi}}$ is 1-1, the map ϑ is a bijection. It remains to show that ϑ is sup and inf continuous.

Let $(P_i)_{i\in I}\subset\mathcal{S}_{1,\phi}$ and $(Q_i)_{i\in I}\subset\mathcal{S}_{1,\phi^*}$ be such that $\phi^*(Q_i)^\perp = P_i$, equivalently by equation (1.1) $\phi(P_i)^\perp = Q_i$ for every $i\in I$. Then, since $\bigwedge_{i\in I}P_i\in\mathcal{S}_{1,\phi}$, by the definition of ϑ we have

$$\begin{aligned}\vartheta\Big(\bigwedge_{i\in I}\chi^*(I)P_i\Big) &= \vartheta\Big(\chi^*(I)\Big(\bigwedge_{i\in I}P_i\Big)\Big) = \chi(I)\phi\Big(\bigwedge_{i\in I}P_i\Big)\\ &= \bigwedge_{i\in I}\chi(I)\phi(P_i) = \bigwedge_{i\in I}\vartheta(\chi^*(I)P_i).\end{aligned}$$

Using equations (4.7) and (1.1) we have that

$$\begin{aligned}\vartheta\Big(\bigvee_{i\in I}\chi^*(I)P_i\Big) &= \vartheta\Big(\chi^*(I)\Big(\phi^*\Big(\bigwedge_{i\in I}Q_i\Big)\Big)^{\perp}\Big) = \chi(I)\phi\Big(\Big(\phi^*\Big(\bigwedge_{i\in I}Q_i\Big)\Big)^{\perp}\Big)\\ &= \chi(I)\Big(\bigwedge_{i\in I}Q_i\Big)^{\perp} = \bigvee_{i\in I}\chi(I)Q_i^{\perp}\\ &= \bigvee_{i\in I}\chi(I)\phi(P_i) = \bigvee_{i\in I}\vartheta(\chi^*(I)P_i). \qquad \blacksquare\end{aligned}$$

We call the diagonal $\Delta(\mathcal{U})$ *essential* if $\chi(I) = I$ and $\chi^*(I) = I$. By the previous proposition if the diagonal is essential the semilattices of the bimodule are CSL's and its map is a complete lattice isomorphism. The implications of this result are studied in a paper in preparation.

5. THE SPACE $\mathcal{U}_0$ IS REFLEXIVE

Let $\mathcal{U}$, $\mathcal{U}_0$, $\Delta(\mathcal{U})$ and ϕ be as in Section 3 and $\chi = \mathrm{Map}(\Delta(\mathcal{U}))$, $\psi = \mathrm{Map}(\mathcal{U}_0)$.

LEMMA 5.1. *If $\Delta(\mathcal{U})$ is essential, i.e. $\chi(I) = I, \chi^*(I) = I$, then $\mathcal{S}_{1,\psi} \subset \mathcal{S}_{1,\phi}$ and $\mathcal{S}_{2,\psi} \subset \mathcal{S}_{2,\phi}$.*

Proof. Since $\Delta(\mathcal{U})$ is essential by Theorem 4.5 the semilattices $\mathcal{S}_{1,\phi}, \mathcal{S}_{2,\phi}$, are CSL's.

If E is a projection, then $\mathrm{Alg}(\mathcal{S}_{2,\phi})\mathcal{U}_0 E \subset \mathcal{U}_0 E$ (Lemma 3.6). It follows that $\psi(E)^{\perp}\mathrm{Alg}(\mathcal{S}_{2,\phi})\psi(E) = 0$. Hence $\psi(E) \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{S}_{2,\phi}))$. Since commutative subspace lattices are reflexive [1], it follows that $\psi(E) \in \mathcal{S}_{2,\phi}$. Thus $\mathcal{S}_{2,\psi} \subset \mathcal{S}_{2,\phi}$. Analogously $\mathcal{U}_0\mathrm{Alg}(\mathcal{S}_{1,\phi}) \subset \mathrm{Alg}(\mathcal{S}_{1,\phi})$ so $\mathrm{Alg}(\mathcal{S}_{1,\phi}^{\perp})\mathcal{U}_0^* \subset \mathcal{U}_0^*$. As above we obtain $\mathcal{S}_{2,\psi^*} \subset \mathcal{S}_{1,\phi}^{\perp}$ hence $\mathcal{S}_{1,\psi} \subset \mathcal{S}_{1,\phi}$. $\blacksquare$

THEOREM 5.2. *The space $\mathcal{U}_0$ is reflexive.*

Proof. Firstly, we suppose that $\Delta(\mathcal{U})$ is essential ($\chi(I) = I, \chi^*(I) = I$). Now, by Theorem 4.3 we have that $\mathcal{S}_{1,\chi} = \mathcal{P}((\mathcal{S}_{1,\phi})'')$, $\mathcal{S}_{2,\chi} = \mathcal{P}((\mathcal{S}_{2,\phi})'')$ and $\chi|_{\mathcal{S}_{1,\phi}} = \phi$.

If $E \in \mathcal{S}_{1,\phi}$, then $\phi(E), \psi(E) \in \mathcal{P}((\mathcal{S}_{2,\phi})'')$ so there exists a unique $F \in \mathcal{P}((\mathcal{S}_{1,\phi})'')$ such that $\chi(F) = \phi(E) - \psi(E)$. We observe that $\chi(F) \leqslant \phi(E) = \chi(E)$. Since χ is a lattice isomorphism, $F \leqslant E$ and so $\psi(F) \leqslant \psi(E)$; therefore $\chi(F) \perp \psi(F)$. It follows that $\Delta(\mathcal{U})F(H_1) \perp \mathrm{Ref}(\mathcal{U}_0)F(H_1)$ and $\Delta(\mathcal{U})F \cap \mathrm{Ref}(\mathcal{U}_0)F = 0$.

By Theorem 3.1, $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})$, hence $\mathcal{U}F = \mathrm{Ref}(\mathcal{U}_0)F \oplus \Delta(\mathcal{U})F$ and $\mathcal{U}F = \mathcal{U}_0 F \oplus \Delta(\mathcal{U})F$. It follows that $\mathcal{U}_0 F = \mathrm{Ref}(\mathcal{U}_0)F$ and therefore $\mathcal{U}_0 F$ is reflexive.

Let

$$P = \bigvee\{F \in \mathcal{P}((\mathcal{S}_{1,\phi})'') : \chi(F) = \phi(E) - \psi(E), E \in \mathcal{S}_{1,\phi}\}.$$

By the previous arguments the space $\mathcal{U}_0P$ is reflexive. Since χ is $\vee$-continuous we have that

$$\chi(P) = \bigvee\{\phi(E) - \psi(E), E \in \mathcal{S}_{1,\phi}\}.$$

Let $Q = \chi(P)^{\perp}$. Since $Q\phi(E) = Q\psi(E)$ for all $E \in \mathcal{S}_{1,\phi}$, it follows that

$$\begin{aligned} Q\mathcal{U} &= \{T : Q\phi(E)^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}_{1,\phi}\} \\ &= \{T : Q\psi(E)^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}_{1,\phi}\}. \end{aligned}$$

Using the previous lemma ($\mathcal{S}_{1,\psi} \subset \mathcal{S}_{1,\phi}$) we obtain that $Q\mathcal{U}$ is contained in the space:

$$\{T : Q\psi(E)^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}_{1,\psi}\} = Q\mathrm{Ref}(\mathcal{U}_0) = \mathrm{Ref}(Q\mathcal{U}_0) \subset Q\mathcal{U}.$$

This shows that $Q\mathcal{U} = \mathrm{Ref}(Q\mathcal{U}_0)$.

Katavolos and Todorov [9] have proved that $\Delta(\mathcal{U}) \subset (\mathcal{U})_{\min}$ where $(\mathcal{U})_{\min}$ is the smallest w^*-closed masa bimodule such that $\mathrm{Ref}((\mathcal{U})_{\min}) = \mathcal{U}$.

So $Q\Delta(\mathcal{U}) \subset Q(\mathcal{U})_{\min} = (Q\mathcal{U})_{\min}$. But since $Q\mathcal{U}_0$ is a w^*-closed masa bimodule such that $\mathrm{Ref}(Q\mathcal{U}_0) = Q\mathcal{U}$ it follows that $Q\Delta(\mathcal{U}) \subset Q\mathcal{U}_0$. Now $Q\Delta(\mathcal{U}) = \chi(P)^{\perp}\Delta(\mathcal{U}) = \Delta(\mathcal{U})P^{\perp}$, hence $\Delta(\mathcal{U})P^{\perp} \subset \mathcal{U}_0$. So $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})P^{\perp} + \Delta(\mathcal{U})P = \mathcal{U}_0 + \Delta(\mathcal{U})P$ and therefore $\mathcal{U}P^{\perp} = \mathcal{U}_0P^{\perp}$. We conclude that $\mathcal{U}_0P^{\perp}$ is reflexive. Since $\mathcal{U}_0P$ is reflexive too, $\mathcal{U}_0$ is reflexive.

Now, relax the assumption that $\Delta(\mathcal{U})$ is essential. Let $\mathcal{W} = \chi(I)\mathcal{U}|_{\chi^*(I)}$. This is a masa bimodule in $B(\chi^*(I)(H_1), \chi(I)(H_2))$.

We have that

$$\mathcal{W} = \{T : \chi(I)\phi(L)^{\perp}TL|_{\chi^*(I)} = 0 \text{ for all } L \in \mathcal{S}_{1,\phi}\}.$$

By Proposition 4.5 the families $\mathcal{S}_{1,\phi}|_{\chi^*(I)}, \mathcal{S}_{2,\phi}|_{\chi(I)}$ are complete lattices and the map $\mathcal{S}_{1,\phi}|_{\chi^*(I)} \to \mathcal{S}_{2,\phi}|_{\chi(I)} : P|_{\chi^*(I)} \to \phi(P)|_{\chi(I)}$ is a complete lattice isomorphism. By the Lifting theorem of J. Erdos [5] it follows that the (semi)lattices of $\mathcal{W}$ are the families $\mathcal{S}_{1,\phi}|_{\chi^*(I)}, \mathcal{S}_{2,\phi}|_{\chi(I)}$.

Therefore, $\mathcal{W}_0 = [\chi(I)\phi(L)TL^{\perp}|_{\chi^*(I)} : T \in \mathcal{W}, L \in \mathcal{S}_{1,\phi}]^{-w^*} = \chi(I)\mathcal{U}_0|_{\chi^*(I)}$. Since

$$\chi(I)\Delta(\mathcal{U})|_{\chi^*(I)} = \{T : TP|_{\chi^*(I)} = \phi(P)|_{\chi(I)}T \text{ for all } P \in \mathcal{S}_{1,\phi}\} = \Delta(\mathcal{W})$$

the diagonal of $\mathcal{W}$ is essential. It follows by the first part of the proof that the space $\chi(I)\mathcal{U}_0\chi^*(I)$ is reflexive.

But $\chi(I)^{\perp}\mathcal{U} = \chi(I)^{\perp}\mathcal{U}_0$ and $\mathcal{U}\chi^*(I)^{\perp} = \mathcal{U}_0\chi^*(I)^{\perp}$ so the spaces $\chi(I)^{\perp}\mathcal{U}_0$ and $\mathcal{U}_0\chi^*(I)^{\perp}$ are reflexive. Finally the space $\mathcal{U}_0$ is reflexive. ∎

For the rest of this section let $\mathcal{S}$ be a CSL and $\mathcal{U} = \mathrm{Alg}(\mathcal{S})$. Let $\mathcal{J}$ be the ideal $[PTP^{\perp} : T \in \mathcal{U}, P \in \mathcal{S}]^{-\|\cdot\|}$, let $\mathcal{U}_0 = \mathcal{J}^{-w^*}$ and $\psi = \mathrm{Map}(\mathcal{U}_0)$. It is known that $\mathcal{J} \subset \mathrm{Rad}(\mathcal{U})$, where $\mathrm{Rad}(\mathcal{U})$ is the radical of $\mathcal{U}$. The equality $\mathcal{J} = \mathrm{Rad}(\mathcal{U})$ is an open problem (Hopenwasser's conjecture), [8], [3]. I. Todorov [13] has proved that $\mathcal{J}$ and $\mathrm{Rad}(\mathcal{U})$ have the same reflexive hull. We improve this by showing the next corollary.

COROLLARY 5.3. *The spaces $\mathcal{J}$ and* $\mathrm{Rad}(\mathcal{U})$ *have the same w^*-closure.*

Proof.

$$\mathcal{U}_0 = \mathcal{J}^{-w^*} \subset \mathrm{Rad}(\mathcal{U})^{-w^*} \subset \mathrm{Ref}(\mathrm{Rad}(\mathcal{U})) = \mathrm{Ref}(\mathcal{J}) = \mathcal{U}_0. \quad ∎$$

COROLLARY 5.4. $\mathrm{Rad}(\mathcal{U})^{-w^*} = \{T : \psi(E)^\perp TE = 0 \text{ for all } E \in \mathcal{S}\}$.

Proof. $\mathrm{Rad}(\mathcal{U})^{-w^*} = \mathcal{U}_0 = \{T : \psi(E)^\perp TE = 0$ for every projection $E\} \subset \{T : \psi(E)^\perp TE = 0$ for all $E \in \mathcal{S}\}$. Using Lemma 5.1 the last space is contained in the space: $\{T : \psi(E)^\perp TE = 0$ for all $E \in \mathcal{S}_{1,\psi}\} = \mathcal{U}_0 = \mathrm{Rad}(\mathcal{U})^{-w^*}$. ∎

Now we are ready to give the form of the decomposition of $\mathcal{U}$ in the case that $\mathcal{U}$ is a CSL algebra:

PROPOSITION 5.5. *Let $Q = \bigvee\{E - \psi(E) : E \in \mathcal{S}\}$ then*

$$\mathcal{U} = \mathrm{Rad}(\mathcal{U})^{-w^*} \oplus Q\mathcal{S}'.$$

Proof. We observe that $Q^\perp E = Q^\perp \psi(E)$ for all $E \in \mathcal{S}$, so we have:

$$Q^\perp \mathcal{U} = \{T : Q^\perp E^\perp TE = 0 \text{ for all } E \in \mathcal{S}\} = \{T : Q^\perp \psi(E)^\perp TE = 0 \text{ for all } E \in \mathcal{S}\}.$$

By the previous corollary the last space is the space $Q^\perp \mathrm{Rad}(\mathcal{U})^{-w^*}$. So we have that $Q^\perp \mathcal{S}' \subset Q^\perp \mathrm{Rad}(\mathcal{U})^{-w^*} \subset \mathrm{Rad}(\mathcal{U})^{-w^*}$. Since $\mathcal{U} = \mathrm{Rad}(\mathcal{U})^{-w^*} + \mathcal{S}'$ we obtain $\mathcal{U} = \mathrm{Rad}(\mathcal{U})^{-w^*} + Q\mathcal{S}'$.

It suffices to show that $\mathrm{Rad}(\mathcal{U})^{-w^*} \cap Q\mathcal{S}' = 0$. Taking $E \in \mathcal{S}$ and $T \in \mathcal{U}_0 \cap (E - \psi(E))\mathcal{S}'$ we have $T = (E - \psi(E))T = \psi(E)^\perp ET = \psi(E)^\perp TE = 0$, because $T \in \mathcal{U}_0$. If $T \in \mathcal{U}_0 \cap Q\mathcal{S}'$ then $(E - \psi(E))T \in \mathcal{U}_0 \cap (E - \psi(E))\mathcal{S}' = 0$. So $(E - \psi(E))T = 0$ for all $E \in \mathcal{S}$. But $T = (\bigvee\{E - \psi(E) : E \in \mathcal{S}\})T$. It follows that $T = 0$. ∎

6. DECOMPOSITIONS OF COMPACT OPERATORS IN REFLEXIVE MASA BIMODULES

Let $\mathcal{U}$, $\mathcal{U}_0$, $\Delta(\mathcal{U})$, ϕ, $\mathcal{D}_1$, $\mathcal{D}_2$, Q_1 be as in Section 3 and $\chi = \mathrm{Map}(\Delta(\mathcal{U}))$.

We denote by $\mathcal{K}$ the set of compact operators and by C_p the set of p-Schatten class operators in $B(H_1, H_2)$.

PROPOSITION 6.1. *If $T \in R_1(\mathcal{U})$, there exist $L \in R_1(\Delta(\mathcal{U}))$ and $S \in [R_1(\mathcal{U}_0)]^{-\|\cdot\|_1}$ such that $T = L + S$.*

Proof. Write $\mathcal{U} = \{X : \phi(P_n)^\perp X P_n = 0$ for all $n \in \mathbb{N}\}$ for an appropriate sequence $(P_n) \subset \mathcal{S}_{1,\phi}$ and let $T \in R_1(\mathcal{U})$. As in the proof of Theorem 3.2

$$T = L_1 + \phi(P_1)TP_1^\perp, \quad \text{where } L_1 = \phi(P_1)TP_1 + \phi(P_1)^\perp TP_1^\perp.$$

Since $\phi(P_1)^\perp TP_1 = 0$ and T has rank 1, either $\phi(P_1)^\perp T = 0$ or $TP_1 = 0$, hence either $L_1 = \phi(P_1)TP_1$ or $L_1 = \phi(P_1)^\perp TP_1^\perp$. Now

$$L_1 = L_2 + \phi(P_2)L_1P_2^\perp, \quad \text{where } L_2 = \phi(P_2)L_1P_2 + \phi(P_2)^\perp L_1P_2^\perp.$$

Since $\phi(P_2)^\perp L_1 P_2 = 0$, either $L_2 = \phi(P_2)L_1P_2$ or $L_2 = \phi(P_2)^\perp L_1 P_2^\perp$. Similarly

$$L_{n-1} = L_n + \phi(P_n)L_{n-1}P_n^\perp, \quad \text{where } L_n = \phi(P_n)L_{n-1}P_n + \phi(P_n)^\perp L_{n-1}P_n^\perp.$$

As before, either $L_n = \phi(P_n)L_{n-1}P_n$ or $L_n = \phi(P_n)^\perp L_{n-1}P_n^\perp$ for all $n \in \mathbb{N}$.

We conclude that there exist sequences of projections $(Q_n) \subset \mathcal{D}_2, (R_n) \subset \mathcal{D}_1$ such that $L_n = \Big(\bigwedge\limits_{i=1}^{n} Q_i\Big) T \Big(\bigwedge\limits_{i=1}^{n} R_i\Big), n \in \mathbb{N}$. We observe that $T = L_n + M_n$ where $M_n = \phi(P_1)TP_1^\perp + \phi(P_2)L_1P_2^\perp + \cdots + \phi(P_n)L_{n-1}P_n^\perp, n \in \mathbb{N}$.

Since $\bigwedge\limits_{i=1}^{n} Q_i \overset{\text{sot}}{\to} \bigwedge\limits_{i=1}^{\infty} Q_i$, $\bigwedge\limits_{i=1}^{n} R_i \overset{\text{sot}}{\to} \bigwedge\limits_{i=1}^{\infty} R_i$ and T has rank 1,

$$L_n \overset{\|\cdot\|_1}{\to} \Big(\bigwedge_{i=1}^{\infty} Q_i\Big) T \Big(\bigwedge_{i=1}^{\infty} R_i\Big) = L, \text{ say.}$$

Now $\phi(P_i)^\perp L_n P_i = \phi(P_i)L_nP_i^\perp = 0, i = 1,2,\ldots,n$ for all $n \in \mathbb{N}$, therefore $\phi(P_i)^\perp LP_i = \phi(P_i)LP_i^\perp = 0$ for all $i \in \mathbb{N}$. Thus $L \in R_1(\Delta(\mathcal{U}))$.

We have $M_n = T - L_n \overset{\|\cdot\|_1}{\to} T - L = S \in [R_1(\mathcal{U}_0)]^{-\|\cdot\|_1}$. ∎

PROPOSITION 6.2. $\mathcal{U}_0 \subset (R_1(\Delta(\mathcal{U}))^*)^0$.

Proof. For every $T \in \mathcal{U}$, $P \in \mathcal{S}_{1,\phi}$ and $R \in R_1(\Delta(\mathcal{U}))$ we have

$$\operatorname{tr}(\phi(P)TP^\perp R^*) = \operatorname{tr}(T(\phi(P)RP^\perp)^*) = \operatorname{tr}(T0) = 0.$$

Taking the w^*-closed linear span we get $\operatorname{tr}(SR^*) = 0$ for all $S \in \mathcal{U}_0$. ∎

PROPOSITION 6.3. (i) $R_1(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})(I - Q_1)$.
(ii) $\Delta(\mathcal{U}) \cap \mathcal{K} = [R_1(\Delta(\mathcal{U}))]^{-\|\cdot\|} \subset \Delta(\mathcal{U})(I - Q_1)$.

Proof. Let $R \in R_1(\Delta(\mathcal{U}))$, as in Theorem 3.4, then $RQ_1 \in \Delta(\mathcal{U})Q_1 = \mathcal{U}_0 \cap \Delta(\mathcal{U}) \subset \mathcal{U}_0$. By the previous proposition we have: $\operatorname{tr}(RQ_1R^*) = 0 \Rightarrow \operatorname{tr}(R^*RQ_1) = 0 \Rightarrow RQ_1 = 0 \Rightarrow R = R(I - Q_1)$. We conclude that $R_1(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})(I - Q_1)$.

For part (ii), observe that if $K \in \Delta(\mathcal{U}) \cap \mathcal{K}$ then K can be approximated in the norm topology by sums of rank 1 operators in $\Delta(\mathcal{U})$ (Proposition 3.4 in [9]). ∎

REMARK 6.4. We will see below that if $\mathcal{U}$ is a strongly reflexive masa bimodule then $[R_1(\Delta(\mathcal{U}))]^{-w^*} = \Delta(\mathcal{U})(I - Q_1)$. This is not true in general. For example take $\mathcal{U}$ to be a TRO which is not strongly reflexive. Then $[R_1(\Delta(\mathcal{U}))]^{-w^*}$ is strictly contained in $\Delta(\mathcal{U})(I - Q_1) = \mathcal{U}$.

PROPOSITION 6.5. $\Delta(\mathcal{U}) \subset (R_1(\mathcal{U}_0)^*)^0$.

Proof. Let $T \in R_1(\mathcal{U}_0)$. As in Proposition 6.1 we decompose $T = L + M$ where $L \in R_1(\Delta(\mathcal{U}))$ and $M \in [R_1(\phi(P_n)\mathcal{U}P_n^\perp) : n \in \mathbb{N}]^{-\|\cdot\|_1} \subset \mathcal{U}_0$. So $L = T - M \in \mathcal{U}_0 \cap R_1(\Delta(\mathcal{U}))$.

Using Proposition 6.3, $\mathcal{U}_0 \cap R_1(\Delta(\mathcal{U})) \subset \mathcal{U}_0 \cap \Delta(\mathcal{U})(I - Q_1)$ which vanishes by Theorem 3.4 so $L = 0$ and hence $T = M$.

We conclude that

$$R_1(\mathcal{U}_0) \subset [R_1(\phi(P_n)\mathcal{U}P_n^\perp) : n \in \mathbb{N}]^{-\|\cdot\|_1}. \tag{6.1}$$

Let $A \in \Delta(\mathcal{U})$. We want to show that $\mathrm{tr}(A^*R) = 0$ for every $R \in R_1(\mathcal{U}_0)$. Using (6.1) it suffices to show that $\mathrm{tr}(A^*R) = 0$ for every $R \in R_1(\phi(P_n)\mathcal{U}P_n^\perp)$ and $n \in \mathbb{N}$. If R is a rank 1 operator such that $R = \phi(P_n)RP_n^\perp$ then

$$\begin{aligned}\mathrm{tr}(A^*R) &= \mathrm{tr}(A^*\phi(P_n)RP_n^\perp) = \mathrm{tr}(P_n^\perp A^*\phi(P_n)R) \\ &= \mathrm{tr}((\phi(P_n)AP_n^\perp)^*R) = \mathrm{tr}(0R) = 0. \quad \blacksquare\end{aligned}$$

Let $P \in \mathcal{S}_{1,\phi}$. We suppose that $\bigvee\{\phi(L) : L \in \mathcal{S}_{1,\phi}, \phi(L) < \phi(P)\} < \phi(P)$. Since $\mathcal{S}_{2,\phi}$ is join complete there exists $P_0 \in \mathcal{S}_{1,\phi}$ such that

$$\phi(P_0) = \bigvee\{\phi(L) : L \in \mathcal{S}_{1,\phi}, \phi(L) < \phi(P)\}.$$

We call the projection $P - P_0$ an *atom* of $\mathcal{U}$ and we denote the projection $\phi(P) - \phi(P_0)$ by $\delta(P - P_0)$.

PROPOSITION 6.6. *Let F be an atom of $\mathcal{U}$.*

(i) *The projection F is minimal in the algebra $(\mathcal{S}_{1,\phi})''$.*

(ii) *The projection $\chi(I)\delta(F)$ is minimal in the algebra $\chi(I)(\mathcal{S}_{2,\phi})''$.*

(iii) $\chi(I)\delta(F)B(H_1, H_2)F \subset \Delta(\mathcal{U})$.

(iv) $\chi(I)^\perp\delta(F)B(H_1, H_2)F \subset \mathcal{U}_0$.

Proof. (i) Let $P, P_0 \in \mathcal{S}_{1,\phi}$ be such that $\phi(P_0) = \bigvee\{\phi(L) : L \in \mathcal{S}_{1,\phi}, \phi(L) < \phi(P)\} < \phi(P)$ and $F = P - P_0$. If $Q \in \mathcal{S}_{1,\phi}$ either $P \leqslant Q$ or $QP < P$. If $P \leqslant Q$ then $QF = F$. If $QP < P$ then (since $QP \in \mathcal{S}_{1,\phi}$ and ϕ is 1-1 on $\mathcal{S}_{1,\phi}$) $\phi(QP) < \phi(P) \Rightarrow \phi(QP) \leqslant \phi(P_0) \Rightarrow QP \leqslant P_0$, so $QF = 0$.

We conclude that $QFB(H_1)F = FB(H_1)QF$ for all $Q \in \mathcal{S}_{1,\phi}$, therefore $FB(H_1)F \subset (\mathcal{S}_{1,\phi})'$, hence F is a minimal projection in $(\mathcal{S}_{1,\phi})''$.

(ii) Since $P, P_0 \in \mathcal{S}_{1,\phi}$ we have that $\phi(P)\Delta(\mathcal{U}) = \Delta(\mathcal{U})P$ and $\phi(P_0)\Delta(\mathcal{U}) = \Delta(\mathcal{U})P_0$ hence

$$\delta(F)\Delta(\mathcal{U}) = \Delta(\mathcal{U})F \quad \text{and so } \chi(I)\delta(F) = \chi(F).$$

Let $Q \in \mathcal{S}_{1,\phi}$.

$$\text{If } QF = 0 \text{ then } \chi(I)\delta(F)\phi(Q) = 0. \tag{6.2}$$

Indeed, since $\delta(F)\Delta(\mathcal{U}) = \Delta(\mathcal{U})F$ we obtain $\delta(F)\Delta(\mathcal{U})Q = 0$. We have that $\delta(F)\chi(Q) = 0$ hence $\chi(I)\delta(F)\phi(Q) = 0$.

$$\text{If } QF = F \text{ then } \chi(I)\delta(F)\phi(Q) = \chi(I)\delta(F). \tag{6.3}$$

Indeed, $\delta(F)\Delta(\mathcal{U}) = \Delta(\mathcal{U})F$ hence $\delta(F)\Delta(\mathcal{U})Q = \Delta(\mathcal{U})F$. We have that $\delta(F)\chi(Q) = \chi(F)$ hence $\chi(I)\delta(F)\phi(Q) = \chi(I)\delta(F)$.

Using equations (6.2), (6.3) as in (i) we have that $\chi(I)\delta(F)$ is a minimal projection in $\chi(I)(\mathcal{S}_{2,\phi})''$.

(iii) Let $T \in B(H_1, H_2)$ and $Q \in \mathcal{S}_{1,\phi}$. From equations (6.2), (6.3) it follows that $\phi(Q)\chi(I)\delta(F)TF = \chi(I)\delta(F)TFQ$, so $\chi(I)\delta(F)TF \in \Delta(\mathcal{U})$.

(iv) If $T \in \mathcal{U}$ then $\chi(I)^{\perp}T \in \mathcal{U}_0$. Indeed, by Theorem 3.2 there exist $T_1 \in \mathcal{U}_0$, $T_2 \in \Delta(\mathcal{U})$ so that $T = T_1 + T_2$. But $T_2 = \chi(I)T_2$ so $\chi(I)^{\perp}T = \chi(I)^{\perp}T_1 \in \mathcal{U}_0$.

Now it suffices to show that $\delta(F)B(H_1, H_2)F \subset \mathcal{U}$. Let $T \in B(H_1, H_2)$ and $Q \in \mathcal{S}_{1,\phi}$. If $FQ = 0$ then $\phi(Q)^{\perp}\delta(F)TFQ = 0$. If $FQ = F$ then $P - P_0 \leqslant Q$ hence $\delta(F) = \phi(P) - \phi(P_0) \leqslant \phi(P - P_0) \leqslant \phi(Q)$ so $\phi(Q)^{\perp}\delta(F)TFQ = 0$. We conclude that $\delta(F)TF \in \mathcal{U}$. ∎

REMARK 6.7. There exists a simple example of a reflexive masa bimodule $\mathcal{U}$ so that $\delta(F)B(H_1, H_2)F \subset \mathcal{U}_0$ for any atom F in $\mathcal{U}$. Take $\mathcal{U}$ to be the set of 3×3 matrixes with zero diagonal. This is an instance of the different behaviour of algebras and bimodules: it is known that if $\mathcal{U}$ is a CSL algebra in a Hilbert space H and F is an atom in $\mathcal{U}$ then $FB(H)F \subset \Delta(\mathcal{U})$.

We thank Dr. I. Todorov for suggesting the "atomic decomposition"in the theorem below.

THEOREM 6.8. *Suppose that* $\{F_n : n \in \mathbb{N}\} = \{F : F \text{ atom of } \mathcal{U}\}$, *then*

$$[R_1(\Delta(\mathcal{U}))]^{-w^*} = \sum_{n=1}^{\infty} \bigoplus \chi(I)\delta(F_n)B(H_1, H_2)F_n.$$

Proof. By the previous proposition it follows that

$$[R_1(\Delta(\mathcal{U}))]^{-w^*} \supset \sum_{n=1}^{\infty} \bigoplus \chi(I)\delta(F_n)B(H_1, H_2)F_n.$$

Let $R = x \otimes y^* \in \Delta(\mathcal{U})$. For every $Q \in \mathcal{S}_{1,\phi}$ we have that $x \otimes (Qy)^* = (\phi(Q)x) \otimes y^*$ so $\phi(Q)x \neq 0 \Leftrightarrow Qy \neq 0 \Leftrightarrow \phi(Q)x = x \Leftrightarrow Qy = y$.

The projection $P = \bigwedge\{Q \in \mathcal{S}_{1,\phi} : Qy = y\}$ belongs to $\mathcal{S}_{1,\phi}$. If $Q \in \mathcal{S}_{1,\phi}$ so that $\phi(Q) < \phi(P)$ then $\phi(Q)x = 0$. (If $\phi(Q)x = x$ then $Qy = y$ so $Q \geqslant P$.)

Let $P_0 \in \mathcal{S}_{1,\phi}$ with $\phi(P_0) = \bigvee\{\phi(L) : L \in \mathcal{S}_{1,\phi}, \phi(L) < \phi(P)\}$. We observe that $\phi(P_0)x = 0$ and $\phi(P)x = x$, hence $\phi(P_0) < \phi(P)$. We conclude that $F = P - P_0$ is an atom of $\mathcal{U}$. The equalities $(P - P_0)y = y$ and $(\phi(P) - \phi(P_0))x = x$ imply that $R = \delta(F)RF$. But $R = \chi(I)R$ so $R = \chi(I)\delta(F)RF$. The proof is complete. ∎

Every strongly reflexive TRO is a masa bimodule [9]. So using the previous theorem we have a new proof of the following result in [9].

COROLLARY 6.9. *If* $\mathcal{M}$ *is a strongly reflexive* TRO, $\zeta = \mathrm{Map}(\mathcal{M})$ *and* $\{A_n : n \in \mathbb{N}\} = \{A : A \text{ atom of } \mathcal{M}\}$, *then*

$$\mathcal{M} = \sum_{n=1}^{\infty} \bigoplus \zeta(A_n)B(H_1, H_2)A_n.$$

Let $(P_n) \subset \mathcal{S}_{1,\phi}$ be a sequence such that

$$\mathcal{U} = \{T \in B(H_1, H_2) : \phi(P_n)^{\perp}TP_n = 0 \text{ for all } n \in \mathbb{N}\}.$$

Let $V_n, U_n : B(H_1, H_2) \longrightarrow B(H_1, H_2), n \in \mathbb{N}$ be as in the proof of Theorem 3.1. By Theorem 6.8

$$[R_1(\Delta(\mathcal{U}))]^{-w^*} = \sum_{n=1}^{\infty} \bigoplus E_n B(H_1, H_2) F_n,$$

where $E_n = \chi(I)\delta(F_n)$ for all $n \in \mathbb{N}$.

Thus $[R_1(\Delta(\mathcal{U}))]^{-w^*}$ is the range of the contractive projection D defined by

$$D : B(H_1, H_2) \longrightarrow B(H_1, H_2) : D(T) = \sum_{n=1}^{\infty} E_n T F_n.$$

PROPOSITION 6.10. *If $K \in \mathcal{K}$, then the sequence $(U_n(K))$ converges to $D(K)$ in norm.*

Proof. We observe that $(V_n|_{C_2})$ is a commuting sequence of orthogonal projections in the Hilbert space C_2. Hence $(U_n|_{C_2})$ is a decreasing sequence of orthogonal projections. Therefore if $T \in C_2$ the sequence $(U_n(T))$ converges in the Hilbert-Schmidt norm $\|\cdot\|_2$.

Let $K \in \mathcal{K}$. Given $\varepsilon > 0$ there exist $K_\varepsilon \in C_2$ such that $\|K - K_\varepsilon\| < \varepsilon/3$ and $n_0 \in \mathbb{N}$ such that $\|U_n(K_\varepsilon) - U_m(K_\varepsilon)\|_2 < \varepsilon/3$ for every $n, m \geqslant n_0$. Then

$$\begin{aligned}\|U_n(K) - U_m(K)\| &\leqslant \|U_n(K) - U_n(K_\varepsilon)\| + \|U_n(K_\varepsilon) - U_m(K_\varepsilon)\| + \|U_m(K_\varepsilon) - U_m(K)\| \\ &\leqslant \|K - K_\varepsilon\| + \|U_n(K_\varepsilon) - U_m(K_\varepsilon)\|_2 + \|K - K_\varepsilon\| < \varepsilon\end{aligned}$$

for every $n, m \geqslant n_0$. Thus $(U_n(K))$ converges in operator norm. Let $D_1(K)$ be its norm limit.

Since $\phi(P_i)^\perp U_n(K) P_i = \phi(P_i) U_n(K) P_i^\perp = 0$ for every $i = 1, 2, \ldots, n$, the limit $D_1(K)$ belongs to the diagonal $\Delta(\mathcal{U})$. Since $\|U_n(K)\| \leqslant \|K\|$ for all $n \in \mathbb{N}$, D_1 is a contraction. We observe that if $K \in \Delta(\mathcal{U}) \cap \mathcal{K}$ then $U_n(K) = K$ for all $n \in \mathbb{N}$ hence D_1 projects onto $\Delta(\mathcal{U}) \cap \mathcal{K}$. Now $D_1|_{C_2}$ is the orthogonal projection onto $\Delta(\mathcal{U}) \cap C_2$, being the infimum of the sequence $(U_n|_{C_2})$.

We also observe that $D|_{C_2}$ is an orthogonal projection in the Hilbert space C_2. If $T \in \Delta(\mathcal{U}) \cap C_2$ then by Proposition 6.3 $T = \sum\limits_{n=1}^{\infty} E_n T F_n = D(T)$.

Thus $D|_{C_2}$ and $D_1|_{C_2}$ are both orthogonal projections onto $\Delta(\mathcal{U}) \cap C_2$, hence $D|_{C_2} = D_1|_{C_2}$. Since C_2 is norm dense in $\mathcal{K}$ and $D|_{\mathcal{K}}, D_1$ are norm continuous, $D|_{\mathcal{K}} = D_1$. ∎

PROPOSITION 6.11. *Suppose that $\bigvee\limits_n F_n = F$. Then the sequence $(U_n(T)F)$ converges strongly to the operator $D(T)$ for every $T \in B(H_1, H_2)$.*

Proof. First we observe that if $x \in F_m(H_1), m \in \mathbb{N}$, then the operator $x \otimes x^*$ is in $(\mathcal{S}_{1,\phi})'$. Indeed, if $y \in E_m(H_2)$ then $R = y \otimes x^* \in \Delta(\mathcal{U})$. It follows that $R^* R = \|y\|^2 x \otimes x^* \in \Delta(\mathcal{U})^* \Delta(\mathcal{U}) \subset (\mathcal{S}_{1,\phi})'$.

Let $T \in B(H_1, H_2)$ and $x \in F_m(H_1), m \in \mathbb{N}, \|x\| = 1$. By Proposition 6.10

$$U_i(Tx \otimes x^*) \xrightarrow{\|\cdot\|} D(Tx \otimes x^*) \quad i \to \infty,$$

hence

$$U_i(Tx \otimes x^*)(x) \xrightarrow{\|\cdot\|} D(Tx \otimes x^*)(x) \quad i \to \infty, \tag{6.4}$$

$$D(Tx \otimes x^*)(x) = \sum_{n=1}^{\infty} E_n(Tx \otimes x^*)F_n(x) = E_m T(x), \tag{6.5}$$

$$D(T)(x) = \sum_{n=1}^{\infty} E_n T F_n(x) = E_m T(x). \tag{6.6}$$

We have that

$$V_i(Tx \otimes x^*) = \phi(P_i)\ (T\ x \otimes x^*)\ P_i + \phi(P_i)^{\perp}(T\ x \otimes x^*)\ P_i^{\perp} \quad i \in \mathbb{N},$$

since $x \otimes x^* \in (\mathcal{S}_{1,\phi})'$,

$$V_i(Tx \otimes x^*) = (\phi(P_i)\ T\ P_i)\ (x \otimes x^*) + (\phi(P_i)^{\perp} T\ P_i^{\perp})\ (x \otimes x^*) \quad i \in \mathbb{N},$$

hence

$$U_i(Tx \otimes x^*) = U_i(T)x \otimes x^* \Rightarrow U_i(Tx \otimes x^*)(x) = U_i(T)(x) \quad i \in \mathbb{N}. \tag{6.7}$$

Using (6.4), (6.5), (6.6), (6.7)

$$U_i(T)(x) \xrightarrow{\|\cdot\|} D(T)(x), \quad i \to \infty \text{ for all } x \in \Big[\bigcup_{m=1}^{\infty} F_m(H_1)\Big].$$

Since the U_i are contractions $U_i(T)(x) \xrightarrow{\|\cdot\|} D(T)(x)$, for all $x \in F(H_1)$. Since $D(T)F = D(T)$, the proof is complete. ∎

REMARK 6.12. The sequence $(U_n(T))$ has similar properties to the net of finite diagonal sums in the case of nest algebras. Propositions 6.10, 6.11 are analogous to Propositions 4.3, 4.4 in [2].

THEOREM 6.13. *For every compact operator $K \in \mathcal{U}$ there exist unique compact operators $K_1 \in \mathcal{U}_0, K_2 \in \Delta(\mathcal{U})$ such that $K = K_1 + K_2$. Moreover $K_2 = D(K)$.*

Proof. Let $K_2 = D(K)$ and $K_1 = K - K_2$. Now $K_1 = \lim(K - U_n(K))$ by Proposition 6.10. As in Theorem 3.2 $K - U_n(K) \in \mathcal{U}_0$ for all $n \in \mathbb{N}$. Hence $K_1 \in \mathcal{U}_0$.

The decomposition $K = K_1 + K_2$ in $\mathcal{U}_0 + (\Delta(\mathcal{U}) \cap \mathcal{K})$ is unique because by Proposition 6.3, $\Delta(\mathcal{U}) \cap \mathcal{K} \subset \Delta(\mathcal{U})(I - Q_1)$, while by Theorem 3.4, $\mathcal{U} = \mathcal{U}_0 \oplus \Delta(\mathcal{U})(I - Q_1)$. ∎

COROLLARY 6.14. *For every finite rank operator $F \in \mathcal{U}$ there exist unique finite rank operators $F_1 \in \mathcal{U}_0, F_2 \in \Delta(\mathcal{U})$ such that $F = F_1 + F_2$. Moreover* $\operatorname{rank} F_2 \leqslant \operatorname{rank} F$ *and $F_2 = D(F)$.*

Proof. It can be shown that for each $n \in \mathbb{N}$ we have $\operatorname{rank}(U_n(F)) \leqslant \operatorname{rank}(F)$. Therefore if $F_2 = \|\cdot\|\text{-}\lim U_n(F)$ then $\operatorname{rank}(F_2) \leqslant \operatorname{rank}(F)$ and $F_2 = D(F)$. Setting $F_1 = F - F_2$ we obtain the desired decomposition. ∎

COROLLARY 6.15. *Let $1 \leqslant p < \infty$ and $K \in \mathcal{U} \cap C_p$. There exist unique operators $K_1 \in \mathcal{U}_0 \cap C_p, K_2 \in \Delta(\mathcal{U}) \cap C_p$ such that $K = K_1 + K_2$. Moreover $\|K_2\|_p \leqslant \|K\|_p$.*

Proof. As in Theorem 6.13 $K = K_1 + D(K)$ where $K_1 \in \mathcal{U}_0$. We observe that $D(K) \in C_p$ and $\|D(K)\|_p \leqslant \|K\|_p$. ∎

7. DECOMPOSITION OF A STRONGLY REFLEXIVE MASA BIMODULE

Let $\mathcal{U}$, $\mathcal{U}_0$, $\Delta(\mathcal{U})$, ϕ, $\mathcal{D}_1$, $\mathcal{D}_2$ be as in Section 3 and $\chi = \operatorname{Map}(\Delta(\mathcal{U}))$. We now assume that $\mathcal{U}$ is a strongly reflexive masa bimodule.

PROPOSITION 7.1. *The space $\mathcal{U}_0$ is strongly reflexive.*

Proof. Let $T \in \mathcal{U}$, $P \in \mathcal{S}_{1,\phi}$. Since $\mathcal{U}$ is a strongly reflexive masa bimodule there exists a net $(R_i) \subset [R_1(\mathcal{U})]$ such that $R_i \overset{wot}{\to} T$ (Corollary 2.5 in [6]). So we have that $\phi(P)R_iP^{\perp} \overset{wot}{\to} \phi(P)TP^{\perp}$. Since $(\phi(P)R_iP^{\perp}) \subset [R_1(\mathcal{U}_0)]$ we conclude that $\phi(P)TP^{\perp} \in [R_1(\mathcal{U}_0)]^{-wot}$. We proved that $\phi(P)\mathcal{U}P^{\perp} \subset [R_1(\mathcal{U}_0)]^{-wot}$ for all $P \in \mathcal{S}_{1,\phi}$. Hence $\mathcal{U}_0 = [R_1(\mathcal{U}_0)]^{-wot}$. ∎

REMARK 7.2. The diagonal of a strongly reflexive masa bimodule is not necessarily strongly reflexive. For example if $\mathcal{U}$ is a nonatomic nest algebra, then $\Delta(\mathcal{U})$ does not contain rank 1 operators.

PROPOSITION 7.3. (i) $\mathcal{U}_0 = \mathcal{U} \cap (R_1(\Delta(\mathcal{U}))^*)^0$.
(ii) $\mathcal{U}_0 \cap \Delta(\mathcal{U}) = \Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U}))^*)^0$.

Proof. By Proposition 6.2 we have $\mathcal{U}_0 \subset (R_1(\Delta(\mathcal{U}))^*)^0$. It suffices to show that $\mathcal{U} \cap (R_1(\Delta(\mathcal{U}))^*)^0 \subset \mathcal{U}_0$.

Since $\mathcal{U} \cap (R_1(\Delta(\mathcal{U}))^*)^0$ is a masa bimodule, as in Theorem 3.2 we can decompose it in the following sum:

$$\mathcal{U} \cap (R_1(\Delta(\mathcal{U}))^*)^0 = \mathcal{U}_0 \cap (R_1(\Delta(\mathcal{U}))^*)^0 + \Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U}))^*)^0.$$

Now we must prove that $\Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U}))^*)^0 \subset \mathcal{U}_0$. Using Theorem 2.2, there exist projections $P_1 \in \mathcal{D}_1, P_2 \in \mathcal{D}_2$ such that $[R_1(\Delta(\mathcal{U}))]^{-w^*} = P_2\Delta(\mathcal{U})P_1$ and $\Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U}))^*)^0 = P_2^{\perp}\Delta(\mathcal{U})P_1^{\perp}$.

Let $T \in \Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U}))^*)^0$. Since $\mathcal{U}$ is a strongly reflexive masa bimodule there exists a net $(R_i) \subset [R_1(\mathcal{U})]$ such that $R_i \overset{wot}{\to} T$ [6]. By Proposition 6.1 there exist $M_i \in [R_1(\Delta(\mathcal{U}))], L_i \in \mathcal{U}_0$ such that $R_i = M_i + L_i$. Thus $M_i + L_i \overset{wot}{\to} T$ so $P_2^{\perp}M_iP_1^{\perp} + P_2^{\perp}L_iP_1^{\perp} \overset{wot}{\to} P_2^{\perp}TP_1^{\perp}$ and thus $P_2^{\perp}L_iP_1^{\perp} \overset{wot}{\to} T$. It follows that $T \in \mathcal{U}_0$. ∎

THEOREM 7.4. $\mathcal{U} = \mathcal{U}_0 \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}$.

Proof. By Theorem 2.2,

$$\Delta(\mathcal{U}) = \Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U})^*)^0 \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}$$

so by Proposition 7.3 $\Delta(\mathcal{U}) = \mathcal{U}_0 \cap \Delta(\mathcal{U}) + [R_1(\Delta(\mathcal{U}))]^{-w^*}$. Since $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})$ we have that $\mathcal{U} = \mathcal{U}_0 + [R_1(\Delta(\mathcal{U}))]^{-w^*}$. By Proposition 6.3 and Theorem 3.4 the last sum is direct. ∎

Propositions 3.7 and 3.8 have the following consequences:

COROLLARY 7.5. (i) *The following are equivalent:*
(a) $R_1(\Delta(\mathcal{U})) = 0$.
(b) $\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{L}_1 \cap \mathcal{A}_1$.
(c) $\Delta(\mathcal{U})\Delta(\mathcal{U})^* \subset \mathcal{L}_2 \cap \mathcal{A}_2$.
(ii) *The following are equivalent:*
(a) $\Delta(\mathcal{U})$ *is strongly reflexive.*
(b) $\Delta(\mathcal{U})\ (\mathcal{L}_1 \cap \mathcal{A}_1) = 0$.
(c) $(\mathcal{L}_2 \cap \mathcal{A}_2)\ \Delta(\mathcal{U}) = 0$.

Theorems 6.8, 7.4 and Corollary 5.3 give the following form of the decomposition of $\mathcal{U}$ when it is a strongly reflexive CSL algebra.

COROLLARY 7.6. *If $\mathcal{S}$ is a completely distributive* CSL *in a Hilbert space H and $\{A_n : n \in \mathbb{N}\} = \{A : A$ atom of $\mathcal{S}\}$ then:*

$$\mathrm{Alg}(\mathcal{S}) = \mathrm{Rad}(\mathrm{Alg}(\mathcal{S}))^{-w^*} \oplus \sum_{n=1}^{\infty} \bigoplus A_n B(H) A_n.$$

Recall the notation $[R_1(\Delta(\mathcal{U}))]^{-w^*} = \sum_{n=1}^{\infty} \bigoplus \chi(I)\delta(F_n)B(H_1,H_2)F_n$, where $\{F_n : n \in \mathbb{N}\} = \{F : F \text{ atom of } \mathcal{U}\}$ and

$$D : B(H_1,H_2) \longrightarrow B(H_1,H_2) : D(T) = \sum_{n=1}^{\infty} \chi(I)\delta(F_n)TF_n.$$

PROPOSITION 7.7. *Let $\theta : \mathcal{U} \to \mathcal{U}$ be the projection onto $[R_1(\Delta(\mathcal{U}))]^{-w^*}$ defined by the decomposition in Theorem* 7.4. *Then $\theta = D|_{\mathcal{U}}$.*

Proof. Since $\mathcal{U}$ decomposes as the direct sum of the masa bimodules $\mathcal{U}_0$ and $[R_1(\Delta(\mathcal{U}))]^{-w^*}$, the map θ is a masa bimodule map:

$$\theta(D_2TD_1) = D_2\theta(T)D_1$$

for every $T \in \mathcal{U}, D_1 \in \mathcal{D}_1, D_2 \in \mathcal{D}_2$. Hence if $T \in \mathcal{U}$ we have

$$\theta(T) = \sum_{n=1}^{\infty} \chi(I)\delta(F_n)\theta(T)F_n = \sum_{n=1}^{\infty} \theta(\chi(I)\delta(F_n)TF_n) = \sum_{n=1}^{\infty} \chi(I)\delta(F_n)TF_n = D(T). \quad ∎$$

PROPOSITION 7.8. $\mathcal{U}_0 = \{T \in \mathcal{U} : \chi(I)\delta(F)TF = 0$ *for every atom F of $\mathcal{U}$*$\}$.

Proof. Let F be an atom of $\mathcal{U}$. If $P \in \mathcal{S}_{1,\phi}$, as in Proposition 6.6 either $PF = F \Rightarrow P^{\perp}F = 0$ or $PF = 0 \Rightarrow \chi(I)\delta(F)\phi(P) = 0$. So $\chi(I)\delta(F)\phi(P)TP^{\perp}F = 0$ for all $P \in \mathcal{S}_{1,\phi}$ and $T \in \mathcal{U}$, thus $\chi(I)\delta(F)\mathcal{U}_0F = 0$ for every atom F. It follows that $\mathcal{U}_0 \subset \{T \in \mathcal{U} : \chi(I)\delta(F)TF = 0, \text{for every atom } F \text{ in } \mathcal{U}\}$.

For the converse, let $T \in \mathcal{U}$ be such that $\chi(I)\delta(F)TF = 0$ for every atom F in $\mathcal{U}$. By the previous proposition $D(T) = 0$, hence $T \in \mathcal{U}_0$. ∎

It is known that the linear span of the rank 1 operators in a strongly reflexive masa bimodule is *wot* dense in the module. This is not true generally for the ultraweak topology [6]. For this problem we have the equivalence in Proposition 7.10. Firstly, we need the following lemma.

LEMMA 7.9. *If $\mathcal{U}$ is a reflexive masa bimodule (not necessarily strongly reflexive) then:*

$$[R_1(\mathcal{U})]^{-w^*} = [R_1(\mathcal{U}_0)]^{-w^*} \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}.$$

Proof. Since $R_1(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})(I - Q_1)$ (Proposition 6.3), by Theorem 3.4 the previous sum is direct.

Clearly

$$[R_1(\mathcal{U})]^{-w^*} \supset [R_1(\mathcal{U}_0)]^{-w^*} \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}.$$

For the converse, let $T \in [R_1(\mathcal{U})]^{-w^*}$. There is a net $(R_i) \subset [R_1(\mathcal{U})]$ with $R_i \xrightarrow{w^*} T$. As in Proposition 6.1, we may decompose $R_i = L_i + M_i$ where $L_i \in [R_1(\mathcal{U}_0)]^{-\|\cdot\|_1}$ and $M_i \in [R_1(\Delta(\mathcal{U}))]$ for all i.

Since $M_i = D(R_i)$ (Corollary 6.14) and D is w^*-continuous the net (M_i) converges to some $M \in [R_1(\Delta(\mathcal{U}))]^{-w^*}$. So $L_i = R_i - M_i \xrightarrow{w^*} T - M = L \in [R_1(\mathcal{U}_0)]^{-w^*}$. Thus $T = L + M \in [R_1(\mathcal{U}_0)]^{-w^*} \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}$. ∎

PROPOSITION 7.10. *If $\mathcal{U}$ is a strongly reflexive masa bimodule, then:*

$$\mathcal{U} = [R_1(\mathcal{U})]^{-w^*} \Leftrightarrow \mathcal{U}_0 = [R_1(\mathcal{U}_0)]^{-w^*}.$$

Proof. Suppose $\mathcal{U} = [R_1(\mathcal{U})]^{-w^*}$. Then by Theorem 7.4 we have $\mathcal{U} = \mathcal{U}_0 \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}$. Using Lemma 7.9 we obtain $\mathcal{U}_0 = [R_1(\mathcal{U}_0)]^{-w^*}$.

If conversely $\mathcal{U}_0 = [R_1(\mathcal{U}_0)]^{-w^*}$ then again by Theorem 7.4

$$\mathcal{U} = \mathcal{U}_0 \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*} = [R_1(\mathcal{U}_0)]^{-w^*} \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*} = [R_1(\mathcal{U})]^{-w^*}$$

by Lemma 7.9. ∎

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