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# SPECTRAL PROPERTIES OF JACOBI MATRICES OF CERTAIN BIRTH AND DEATH PROCESSES

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ABSTRACT. We show the absolute continuity of the spectrum of Jacobi matrices associated to certain birth and death processes. This is done by the conjugate operator theory of Mourre.

KEYWORDS: *Mourre estimate, Jacobi matrix, absolutely continuous spectrum.*

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## 1. INTRODUCTION

1.1. OVERVIEW. Recently, important efforts have been expended to the spectral analysis of unbounded Jacobi matrices see [7], [8], [9], [13], [14], [15], [18], [19] and references therein. One meets there various methods of investigation which fall mainly into two categories. The first one is based on the Gilbert-Kahn-Pearson theory of subordinate solution, see [13], [14], [15], [16], [18]. The second one, is based on the positive commutator method of Putnam-Kato, see for example [7], [8], [9], [13], [19].

Our main goal in this paper is to show that spectral properties of unbounded Jacobi matrices can be studied quite efficiently by the conjugate operator theory of Mourre. It is a powerful tool for the spectral and scattering theory of self-adjoint operators [2], [5], [20]. Its field of application is large enough to include Schrödinger operators [2], acoustic propagator [4], quantum field Hamiltonians models [6] and the list is obviously far from being exhaustive. Surprisingly, it has never been used in the spectral analysis of unbounded Jacobi matrices.

We focus here on Jacobi matrices coming from certain birth and death processes [12], [17]. As we will see, in this case Mourre's method applies in a rather simple and elegant manner. We insist on this point since it is one of our motivations to consider this model. Notice for example that such models are studied in [16] by asymptotic analysis which turned out to be especially difficult. Indeed, their approach involves transfer matrices which are in this case asymptotically similar to a Jordan box, and that makes harder the estimation of products of a large number of them.

In addition, we obtain not only spectral properties of $H$ but also various informations on the asymptotic behavior of the resolvent of $H$ around the real axis. This plays a key role for the analysis of birth and death processes [12], [17].

1.2. NOTATIONS AND MAIN RESULTS. Let $\mathcal{H}$ be the Hilbert space of square summable sequences $(\psi_n)_{n\geqslant 1}$ equipped with the scalar product $\langle \phi, \psi \rangle = \sum\limits_{n\geqslant 1} \overline{\phi}_n \psi_n$. Consider a sequence $(\eta_n)_{n\geqslant 1}$ of real numbers such that

$$\lim_{n\to+\infty} \frac{\eta_n}{n} = 0. \tag{1.1}$$

Define for any $n \geqslant 1$

$$a_n = n + \eta_n, \tag{1.2}$$

$$b_n = -(a_n + a_{n-1}), \tag{1.3}$$

where for the calculation of $b_1$ we set $a_0 = \eta_0 = 0$.

In this paper we consider the operator $H$ defined in $\mathcal{H}$ by

$$(H\psi)_n = a_{n-1}\psi_{n-1} + b_n\psi_n + a_n\psi_{n+1} \quad \text{for all } n \geqslant 1. \tag{1.4}$$

For the definition of $(H\psi)_1$ we set $\psi_0 = 0$. Clearly (cf. [3]), the operator $H$ is essentially self-adjoint on the subspace $l_0^2$ of sequences with only finitely many nonzero coordinates. It is the infinitesimal matrix of the well-known birth and death process [12], [17].

Let $U$ be the unitary operator defined in $\mathcal{H}$ by $(U\psi)_n = (-1)^{n+1}\psi_n$. It is obvious that $U^{-1}HU = H'$ is the operator obtained from $H$ by replacing $b_n$ by $-b_n = a_n + a_{n-1}$. In particular, the minus in the expression (1.3) is not so important in our purposes and we choose it since its physical significance.

On the other hand, one has $\lim\limits_{n\to\infty} b_n/a_n = l$ and $|l| = 2$. This is of interest, since the family $H_\lambda$ of Jacobi operators obtained form $H$ by replacing $b_n$ by $\lambda b_n$ presents a spectral transition from a purely discrete spectrum if $|\lambda| > 1$ to a purely absolutely continuous if $|\lambda| < 1$, see [15]. The first property follows easily from the smallness of the off-diagonal component of $H$ with respect to its diagonal one, while the second one follows from Theorem 3.1 of [15] based on Gilbert-Kahn-Pearson theory of subordinate solution. Thus the models under consideration are examples of the borderline case $|\lambda| = 1$ where a spectral transition happens.

We start by studying the operator $H_0$ corresponding to the case where $\eta_n = 0$, which we call the unperturbed operator. Let $N$ be the positive operator acting in $\mathcal{H}$ by

$$(N\psi)_n = n\psi_n.$$

Denote by $B(\mathcal{H})$ the Banach algebra of bounded operators in $\mathcal{H}$.

THEOREM 1.1. *The spectrum of $H_0$ is purely absolutely continuous and coincides with $(-\infty, 0]$. Moreover, for any $s > 1/2$ the limits*

$$R_{0,s}(\lambda \pm \mathrm{i}0) := \lim_{\mu \to 0+} N^{-s}(H_0 - \lambda \mp \mathrm{i}\mu)^{-1} N^{-s}$$

*exist locally uniformly on $(-\infty, 0)$ for the norm topology of $B(\mathcal{H})$.*

In particular, we recover the results of [7], [16], [18]. The existence of the boundary values of the resolvent is however a new result.

Let $(E, \|\cdot\|)$ be a Banach space and $f : \mathbb{R} \to E$ a bounded continuous function. For an integer $m \geqslant 1$ let $w_m$ be the *modulus of continuity of order m* of $f$ defined on $(0, 1)$ by

$$w_m(\varepsilon) = \sup_{x \in \mathbb{R}} \Big\| \sum_{j=1}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} f(x + j\varepsilon) \Big\|.$$

We say that $f$ is of class $\Lambda^\sigma$, $\sigma > 0$, if there is an integer $m > \sigma$ such that

$$\sup_{0<\varepsilon<1} \varepsilon^{-\sigma} w_m(\varepsilon) < \infty.$$

The following theorem describes some continuity properties of the boundary values of the resolvent of $H_0$ and propagation consequences of them:

THEOREM 1.2. *Let $\varphi \in C_0^\infty((-\infty, 0))$ and $\sigma > 0$. Then the map*

$$\lambda \longmapsto R_{0,\sigma+\frac{1}{2}}(\lambda \pm \mathrm{i}0) \in B(\mathcal{H})$$

*is locally of class $\Lambda^\sigma$ on $(-\infty, 0)$. Moreover for some $C > 0$:*

$$\|N^{-\sigma} \mathrm{e}^{-\mathrm{i}H_0 t} \varphi(H_0) N^{-\sigma}\| \leqslant C(1 + |t|)^{-\sigma}.$$

Technically speaking, we will exhibit a self-adjoint operator $A$ such that $H_0$ is $A$-homogeneous, i.e. $[H_0, \mathrm{i}A] = H_0$ or equivalently

$$\mathrm{e}^{-\mathrm{i}At} H_0 \mathrm{e}^{\mathrm{i}At} = \mathrm{e}^t H_0 \quad \text{for all } t \in \mathbb{R}.$$

This $A$-homogeneity property is one of our motivations to choose this model, since in this framework the proofs are simple and elegant, without speaking about the optimality of the results, see [5].

Notice that we do not have this $A$-homogeneity property in the general case where the perturbation $\eta_n$ is nonzero, see Section 5. We have, however,

THEOREM 1.3. *Assume that*

$$\lim_{n \to \infty} n(\eta_n - \eta_{n-1}) = 0, \tag{1.5}$$

$$\sup_{n \geqslant 1} n^2 |\eta_{n+1} - 2\eta_n + \eta_{n-1}| < \infty. \tag{1.6}$$

*Then the operator $H$ has no singular continuous spectrum and the possible eigenvalues of $H$ can only accumulate to 0.*

It should be clear that (see our proofs) we get also a similar result to Theorem 1.2 for the perturbed operator $H$ but we will not state it separately.

The paper is organized as follows. In Section 2 we give a brief review on the conjugate operator theory and in Section 3 we study the critical value 0 of the spectrum. Section 4 is devoted to the study of the unperturbed operator $H_0$ while Section 5 is dedicated to the general case where $\eta_n \neq 0$. Section 6 contains two examples.

## 2. THE CONJUGATE OPERATOR METHOD

The following brief review on the conjugate operator theory is based on [2], [5], [20].

Let $H, A$ be two self-adjoint operators in a Hilbert space $\mathcal{H}$. For a complex number $z$ in the resolvent set $\rho(H)$of $H$ we set $R(z) = (H - z)^{-1}$.

DEFINITION 2.1. Let $k$ be a positive integer and $\sigma > 0$ a real number. We say that $H$ is of class $C^k(A)$ (respectively $\mathcal{C}^\sigma(A)$) if the map $t \longmapsto \mathrm{e}^{-\mathrm{i}At}R(z)\mathrm{e}^{\mathrm{i}At} \in B(\mathcal{H})$ is strongly of class $C^k$ (respectively $\Lambda^\sigma$) on $\mathbb{R}$ for some $z \in \rho(H)$.

Note that if $\sigma > 1$ then clearly $C^1(A) \subset \mathcal{C}^\sigma(A)$.

Let us equip $D(H)$ with the graph norm $\|f\|_H = \|f\| + \|Hf\|$. One has (see Theorem 6.2.10 of [2]):

PROPOSITION 2.2. *The operator $H$ is of class $C^1(A)$ if and only if:*

(i) *the set $D(z) := \{\psi \in D(A) : R(z)\psi$ and $R(\overline{z})\psi \in D(A)\}$ is dense in $D(A)$ for some $z \in \rho(H)$ and*

(ii) *there is a constant $c > 0$ such that for all $\psi \in D(A) \cap D(H)$:*

$$|\langle \psi, [H, \mathrm{i}A]\psi \rangle| = |2\Re\langle H\psi, \mathrm{i}A\psi\rangle| \leqslant c\|\psi\|_H^2.$$

From now on, we assume that $H$ is at least of class $C^1(A)$. Then $[H, iA]$ can be defined as a bounded operator from $D(H)$ into its adjoint $D(H)^*$.

Define $\widetilde{\mu}^A(H)$ to be the open set of real numbers $\lambda$ such that

$$E(\Delta)[H, \mathrm{i}A]E(\Delta) \geqslant aE(\Delta) + K \tag{2.1}$$

for some constant $a > 0$, a neighborhood $\Delta$ of $\lambda$ and a compact operator $K$ in $\mathcal{H}$. Here-above $E$ denotes the spectral measure of $H$.

The inequality (2.1) is called the *Mourre estimate* and the set of points $\lambda \in \widetilde{\mu}^A(H)$ where it holds with $K = 0$ will be denoted by $\mu^A(H)$.

THEOREM 2.3. *The set $\widetilde{\mu}^A(H)$ contains at most a discrete set of eigenvalues of $H$ and all these eigenvalues are finitely degenerate. Moreover, if $\sigma_p(H)$ denotes the set of eigenvalues of $H$ then $\mu^A(H) = \widetilde{\mu}^A(H) \setminus \sigma_p(H)$.*

Set $\langle x\rangle^2 = 1 + x^2$. The following theorem is a particular case of the main result of [20] (see [2] for different versions):

THEOREM 2.4. *Assume that $H$ is of class $\mathcal{C}^{s+\frac{1}{2}}(A)$ for some $s > 1/2$. Then the following boundary values of the resolvent of $H$*

$$R_s(\lambda \pm \mathrm{i}0) := \lim_{\mu \to 0+} \langle A \rangle^{-s} R(\lambda \pm \mathrm{i}\mu) \langle A \rangle^{-s}$$

*exist locally uniformly on $\mu^A(H)$ for the norm topology of $B(\mathcal{H})$. In particular, $H$ has no singular continuous spectrum in $\mu^A(H)$.*

For the following theorem we will assume that $H$ has a spectral gap, i.e. there exists a real number $\lambda_0$ which belongs to $\rho(H)$. If such assumption is not satisfied then the theorem is proved only for $s < 3/2$ (see [20]):

THEOREM 2.5. *If $H$ is of class $\mathcal{C}^{s+\frac{1}{2}}(A)$ for some $s > 1/2$ then the maps*

$$\lambda \longmapsto R_s(\lambda \pm \mathrm{i}0) \in B(\mathcal{H})$$

*are locally of class $\Lambda^{s-\frac{1}{2}}$ on $\mu^A(H)$. Moreover, for every $\varphi \in C_0^\infty(\mu^A(H))$ we have*

$$\| \langle A \rangle^{-s} \mathrm{e}^{-\mathrm{i}Ht} \varphi(H) \langle A \rangle^{-s} \| \leqslant C \langle t \rangle^{-(s-\frac{1}{2})}.$$

EXAMPLE 2.6. In this example we describe a class of Hamiltonians $H$ for which the preceding results hold with simple proofs and where even the Mourre estimate is not needed explicitly. Assume that $H$ is of class $C^1(A)$ and that $[H, \mathrm{i}A] = aH + b$ for $(a, b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then $H$ has a purely absolutely continuous spectrum outside the critical value $b/a$ (of course if $a = 0$ there is no critical value). Moreover, for all $s > 1/2$ one has the following:

(i) the boundary values of the resolvent $\langle A \rangle^{-s} R(\lambda \pm \mathrm{i}0) \langle A \rangle^{-s}$ exist locally uniformly on $\mathbb{R} \setminus \{b/a\}$ for the norm topology of $B(\mathcal{H})$ and they are locally of class $\Lambda^{s-\frac{1}{2}}$ as a function of the spectral parameter $\lambda \in \mathbb{R} \setminus \{0\}$;

(ii) for any $\varphi \in C_0^\infty(\mathbb{R} \setminus \{b/a\})$ and for any $\sigma > 0$ (see [5]):

$$\| \langle A \rangle^{-\sigma} \mathrm{e}^{-\mathrm{i}Ht} \varphi(H) \langle A \rangle^{-\sigma} \| \leqslant C \langle t \rangle^{-\sigma}.$$

We mention the very special case where $(a, b) = (1, 0)$. In this case $[H, \mathrm{i}A] = H$ and we say that $H$ is $A$-homogeneous.

## 3. THE CRITICAL VALUE 0

Let $H$ be the self-adjoint operator defined in $\mathcal{H}$ by (1.4) and (1.3) with arbitrary $a_n > 0$ for every $n \geqslant 1$. One has

PROPOSITION 3.1. *For any $\psi \in D(H)$ we have*

$$\langle \psi, H\psi \rangle = -\sum_{n \geqslant 1} a_n |\psi_{n+1} - \psi_n|^2.$$

*In particular, $\sigma(H) \subset (-\infty, 0]$ and 0 is not an eigenvalue of $H$.*

*Proof. Step 1.* According to (1.3), we have $b_n = -(a_n + a_{n+1})$ for all $n \geqslant 1$. Then straightforward computation shows that for any $\psi \in l_0^2$:

$$\langle \psi, H\psi \rangle = \sum_{n \geqslant 1} \overline{\psi}_n (a_{n-1}\psi_{n-1} - (a_{n-1} + a_n)\psi_n + a_n \psi_{n+1}) = -\sum_{n \geqslant 1} a_n |\psi_{n+1} - \psi_n|^2.$$

This identity extends very easily to $D(H)$ (see [7] for more details). As $a_n > 0$ for all $n \geqslant 1$, $\langle \psi, H\psi \rangle \leqslant 0$ for any $\psi \in D(H)$ and therefore $\sigma(H) \subset (-\infty, 0]$.

*Step 2.* Assume that 0 is an eigenvalue of $H$ and let $\psi$ be a corresponding eigenvector. Then

$$0 = \langle \psi, H\psi \rangle = -\sum_{n \geqslant 1} a_n |\psi_{n+1} - \psi_n|^2.$$

Since $a_n > 0$ for all $n \geqslant 1$ we clearly have

$$a_n |\psi_{n+1} - \psi_n| = 0,$$

which implies that for all $n \geqslant 1$

$$\psi_{n+1} - \psi_n = 0.$$

But $\psi$ is square summable and then necessarily

$$\psi_n = 0 \quad \text{for all } \ n \geqslant 1.$$

Hence 0 is not an eigenvalue of $H$ and the proof is finished. ∎

REMARK 3.2. Notice that the assertion $\sigma(H) \subset (-\infty, 0]$ is not true anymore if the condition $a_n > 0$ only holds for $n$ greater to some given integer $N_0$. In such case one can decompose our operator $H$ in a sum of a main part $H_1$ covered by the proposition and a suitable operator $V$ of finite rank. In particular, by Weyl criterion the essential spectrum of $H$ coincides with that of $H_1$ which is included in $(-\infty, 0]$ by the proposition. In addition, $H$ will have at most a finite number of eigenvalues.

## 4. THE UNPERTURBED MODEL $H_0$

This section is entirely devoted to the analysis of the unperturbed operator $H_0$ corresponding to the case where $a_n = n$. To apply Mourre's theory we need a conjugate operator $A$ which we define here in $\mathcal{H}$ by

$$(\mathrm{i}A\psi)_n = -\frac{1}{2}(a_{n-1}\psi_{n-1} - a_n \psi_{n+1}). \tag{4.1}$$

Since $a_n = n$ for $n \geqslant 1$, $A$ is clearly an essentially self-adjoint operator in $l_0^2$.

THEOREM 4.1. *The operator $H_0$ is $A$-homogeneous, i.e. $H_0$ is of class $C^1(A)$ and $[H_0, \mathrm{i}A] = H_0$.*

*Proof. Step 1.* Direct computation shows that for all $\varphi, \psi \in l_0^2$:

$$\langle \varphi, [H_0, \mathrm{i}A]\psi\rangle = \langle H_0\varphi, \mathrm{i}A\psi\rangle + \langle \mathrm{i}A\varphi, H_0\psi\rangle = \frac{1}{2}\sum_{n\geqslant 1} \overline{\varphi}_n(c_{n-1}\psi_{n-1} + 2d_n\psi_n + c_n\psi_{n+1}),$$

where for any $n \geqslant 1$:

$$d_n = -(a_n^2 - a_{n-1}^2) = -(a_n + a_{n+1}) = b_n, \quad c_n = -a_n(b_{n+1} - b_n) = 2a_n.$$

Here-above we only used the definition of $a_n, b_n$ for $n \geqslant 1$ and the convention $a_0 = 0$. Hence for all $\varphi, \psi \in l_0^2$ one has:

$$\langle \varphi, [H_0, \mathrm{i}A]\psi\rangle = \langle \varphi, H_0\psi\rangle.$$

In particular, for all $\varphi, \psi \in l_0^2$ we have

$$|\langle \varphi, [H_0, \mathrm{i}A]\psi\rangle| = |\langle \varphi, H_0\psi\rangle| \leqslant \|\varphi\| \cdot \|H_0\psi\| \leqslant \|\varphi\|_{H_0}\|\psi\|_{H_0}.$$

In particular, the sesquilinear form $[H_0, \mathrm{i}A]$ can be extended uniquely to a continuous sesquilinear form on $D(H_0)$, and the associated operator, which we denote by the same symbol, is $[H_0, \mathrm{i}A] = H_0$. One might think that this commutation rule implies the $C^1(A)$-property of $H_0$. This is not true, since one needs some domain regularity as it is required in the point (i) of the Proposition 2.2. A counterexample is available in [11].

*Step 2.* To show that $H_0$ is of class $C^1(A)$ it remains to verify the point (i) of the Proposition 2.2. More explicitly, for some $z > 0$ that will be chosen later, we claim that the subspace

$$D(z) := \{\psi \in D(A) : (H_0 - z)^{-1}\psi \in D(A)\}$$

is dense in $D(A)$. In fact, we will prove that $l_0^2 \subset D(z)$.

Let $\{e^k\}_{k\geqslant 1}$ be the standard orthonormal basis of $\mathcal{H}$, that is, $(e^k)_n = 0$ for all $n \neq k$ and $(e^k)_k = 1$. It suffices to prove that, for any integer $k \geqslant 1$, $e^k \in D(z)$. Since $e^k \in D(A)$, we have to show that $\psi^k = (H_0 - z)^{-1}e^k \in D(A)$. In fact, let us fix an integer $k \geqslant 1$ and establish that the coordinates $(\psi^k)_n = \langle e^n, (H_0 - z)^{-1}e^k\rangle$ decay sufficiently fast as $n$ tends to infinity. More precisely, we will prove that there exists a constant $c > 0$ such that

$$|n^3\langle e^n, (H_0 - z)^{-1}e^k\rangle| \leqslant c \quad \text{for all } n \geqslant 1.$$

This is a kind of Combes-Thomas estimate that we will establish by adapting the argument given by Aizenman in the Appendix II of [1]. Let us fix $n$ and define the operator of multiplication $L_n$ by

$$(L_n\psi)_j = \begin{cases} j^3\psi_j & \text{if} \quad j \leqslant n, \\ n^3\psi_j & \text{if} \quad j \geqslant n, \end{cases}$$

which is bounded and invertible in $\mathcal{H}$, and its inverse is given by

$$(L_n^{-1}\psi)_j = \begin{cases} \frac{1}{j^3}\psi_j & \text{if} \quad j \leqslant n, \\ \frac{1}{n^3}\psi_j & \text{if} \quad j \geqslant n. \end{cases}$$

Obviously $L_n$ and $L_n^{-1}$ leave invariant $D(H_0)$, and induce two bounded operators in $D(H_0)$ endowed with $\|\cdot\|_{H_0}$ that we denote by the same symbols. Then $L_n(H_0-z)L_n^{-1} : D(H_0) \to \mathcal{H}$ is invertible and $(L_n(H_0-z)L_n^{-1})^{-1} = L_n(H_0-z)^{-1}L_n^{-1}$. Moreover, a direct computation shows that on $D(H_0)$ we have the following identity:

$$L_n(H_0-z)L_n^{-1} = H_0 + R_n - z \tag{4.2}$$

where $R_n$ acts as

$$(R_n\psi)_j = r_{j,j+1}\psi_{j+1} + r_{j,j-1}\psi_{j-1}$$

and

$$r_{j,j+1} = \begin{cases} j\left[\left(1+\frac{1}{j}\right)^{-3} - 1\right] & \text{if} \quad 1 \leqslant j \leqslant n-1, \\ 0 & \text{if} \quad j \geqslant n; \end{cases}$$

$$r_{j+1,j} = \begin{cases} j\left[\left(1+\frac{1}{j}\right)^{3} - 1\right] & \text{if} \quad 1 \leqslant j \leqslant n-1, \\ 0 & \text{if} \quad j \geqslant n. \end{cases}$$

Clearly, there exists a constant $C > 0$ independent of $n$ such that

$$\sup_j |r_{j,j+1}| + \sup_j |r_{j,j-1}| < C.$$

In particular, $\|R_n\| < C$. But then by (4.2) and the resolvent equation we get

$$(L_n(H_0-z)L_n^{-1})^{-1} = \sum_{p\geqslant 0}(H_0-z)^{-1}(-R_n(H_0-z)^{-1})^p$$

where the right side converges in norm as soon as $\|R_n(H-z)^{-1}\| < 1$. The latter is ensured by choosing $z = \text{distance}(z, \sigma(H_0)) > C$, which can be done independently of both $n$ and $k$. In this case, one has

$$\|(L_n(H_0-z)L_n^{-1})^{-1}\| \leqslant \frac{1}{z-C}.$$

Now we are able to prove our estimate. For that observe that for every $n \geqslant k$ we have

$$n^3\langle e^n, (H_0-z)^{-1}e^k\rangle = k^3\langle e^n, L_n(H_0-z)^{-1}L_n^{-1}e^k\rangle = k^3\langle e^n, (L_n(H_0-z)L_n^{-1})^{-1}e^k\rangle.$$

Thus for every $n \geqslant k$ ($k$ being fixed) we have

$$|n^3\langle e^n, (H_0-z)^{-1}e^k\rangle| \leqslant k^3\|(L_n(H_0-z)L_n^{-1})^{-1}\| \leqslant k^3\frac{1}{z-C}.$$

which is the desired property, since the constant of the right side is independent of $n$.

To summarize we have proved that $H_0$ is of class $C^1(A)$ and that $[H_0, \mathrm{i}A] = H_0$. The proof of the theorem is then complete. ∎

REMARK 4.2. Theorem 4.1 is equivalent to that $e^{iAt}$ leaves invariant $D(H_0)$ and

$$e^{-iAt}H_0e^{iAt} = e^tH_0 \quad \text{for all } t \in \mathbb{R}.$$

Combining this identity with the fact that $\sigma(H_0) \subset (-\infty, 0]$ we conclude that the spectrum of $H_0$ coincides with $(-\infty, 0]$.

*Proof of Theorem* 1.1. Since $H_0$ is $A$-homogeneous, the results of the example presented at the end of Section 2 apply. Thus on one hand, the spectrum of $H_0$ is purely absolutely continuous outside the critical value 0. But Proposition 3.1 shows that 0 is not an eigenvalue of $H_0$ and therefore the spectrum of $H_0$ is purely absolutely continuous.

On the other hand, for any $s > 1/2$ the limits

$$\lim_{\mu \to 0+} \langle A \rangle^{-s}(H_0 - \lambda \mp i\mu)^{-1}\langle A \rangle^{-s}$$

exist locally uniformly on $(-\infty, 0)$ for the norm topology of $B(\mathcal{H})$. But obviously for any $s > 0$ one has

$$N^{-s}\langle A \rangle^{s} \in B(\mathcal{H}).$$

This enables us to replace $\langle A \rangle$ by $N$ in the preceding assertion and thus to get the existence of the limits

$$R_{0,s}(\lambda \mp i0) := \lim_{\mu \to 0+} N^{-s}(H_0 - \lambda \mp i\mu)^{-1}N^{-s}$$

locally uniformly on $(-\infty, 0)$ for the norm topology of $B(\mathcal{H})$. This is the desired property of the theorem and the proof is complete. ∎

*Proof of Theorem* 1.2. This proof is similar to that of Theorem 1.1 and will be omitted. ∎

## 5. THE GENERAL CASE WHERE $\eta_n \neq 0$

In this section we study the general case where $\eta_n \neq 0$. More specifically, we consider the operator $H$ defined by (1.1), (1.2), (1.3) and (1.4).

Since $\eta_n/n$ tends to zero at infinity there exists an integer $N_0$ such that $a_n = n(1 + \eta_n/n) > 0$ for all $n > N_0$. In particular, the essential spectrum is always included in $(-\infty, 0]$ with at most a finite number of possible eigenvalues which could even be positive (see the remark next the proof of Proposition 3.1).

The conjugate operator $A$ will be defined here in $\mathcal{H}$ by

$$(iA\psi)_n = \frac{1}{2}(a_{n-1}\psi_{n-1} - a_n\psi_{n+1}) \tag{5.1}$$

with of course $a_n = n + \eta_n$. By (1.1) the operator $A$ is essentially self-adjoint on $l_0^2$.

PROPOSITION 5.1. *Assume that* (1.5) *holds. Then the operator $H$ is of class $C^1(A)$ and $\widetilde{\mu}^A(H) = (-\infty, 0)$. In particular, the possible eigenvalues of $H$ can only accumulate to 0.*

*Proof. Step 1.* Let $\lambda_0$ be a positive number such that $(\lambda_0, \infty) \cap \sigma(H) = \emptyset$ (such number exists according to the discussion above). Exactly as in the Step 2 of the proof of Theorem 4.1 one can show that the subspace $D(z) := \{\psi \in D(A) / (H-z)^{-1}\psi \in D(A)\}$ is dense in $D(A)$ for some $z > \lambda_0$. This completes the proof of the point (i) of Proposition 2.2 needed to the proof of the $C^1(A)$-property of $H$.

*Step 2.* It remains to examine the commutator $[H, \mathrm{i}A]$. Direct calculation shows that for all $\varphi, \psi \in l_0^2$:

(5.2)

$$(\varphi, [H, \mathrm{i}A]\psi) = (H\varphi, \mathrm{i}A\psi) + (\mathrm{i}A\varphi, H\psi) = \frac{1}{2}\sum_{n\geqslant 1} \overline{\varphi}_n(c_{n-1}\psi_{n-1} + 2d_n\psi_n + c_n\psi_{n+1}),$$

where

$$d_n = a_n^2 - a_{n-1}^2, \quad c_n = a_n(b_{n+1} - b_n).$$

On one hand, one has

$$b_{n+1} - b_n = a_{n-1} - a_{n+1} = -2 + \eta_{n-1} - \eta_{n+1},$$

which implies that

$$c_n = -2a_n + a_n(\eta_{n-1} - \eta_{n+1}) \equiv -2a_n + r_n.$$

According to our hypothesis the rest

$$r_n = a_n(\eta_{n-1} - \eta_{n+1}) = n(\eta_{n-1} - \eta_{n+1})\Big(1 + \frac{\eta_n}{n}\Big) \tag{5.3}$$

obviously tends to zero at infinity. On the other hand, we have

$$d_n = (a_n - a_{n-1})(a_n + a_{n-1}) = -b_n + (\eta_n - \eta_{n-1})(a_n + a_{n-1}) \equiv -b_n + \varepsilon_n.$$

Here again by our hypothesis we have that the rest

$$\varepsilon_n = (a_n + a_{n-1})(\eta_n - \eta_{n-1}) = n(\eta_n - \eta_{n-1})\Big(2 + \frac{\eta_n}{n} - \frac{\eta_{n-1}}{n}\Big) \tag{5.4}$$

tends to zero at infinity. Consider the operator $K$ in $\mathcal{H}$ given by

$$(2K\psi)_n = r_{n-1}\psi_{n-1} + 2\varepsilon_n\psi_n + r_n\psi_{n+1}. \tag{5.5}$$

Since the coefficients $r_n$ and $\varepsilon_n$ tend to zero at infinity, the operator $K$ is compact in $\mathcal{H}$. Hence, for all $\varphi, \psi \in l_0^2$ one has

$$|\langle\varphi, [H, \mathrm{i}A]\psi\rangle| = |\langle\varphi, (-H+K)\psi\rangle| \leqslant \|\varphi\| \cdot (\|H\psi\| + \|K\psi\|) \leqslant (1 + \|K\|)\|\varphi\|_H\|\psi\|_H.$$

Thus $H$ is of class $C^1(A)$ and

$$[H, \mathrm{i}A] = -H + K. \tag{5.6}$$

*Step 3.* Let $\lambda < 0$ and $\varepsilon > 0$ such that the interval $\Delta = (\lambda - \varepsilon, \lambda + \varepsilon) \subset (-\infty, 0)$. By functional calculus and the identity (5.6) one has

(5.7)
$$E(\Delta)[H, \mathrm{i}A]E(\Delta) = -E(\Delta)HE(\Delta) + E(\Delta)KE(\Delta) \geqslant -(\lambda+\varepsilon)E(\Delta) + E(\Delta)KE(\Delta),$$

which is the Mourre estimate, since $a = -(\lambda+\varepsilon) > 0$ and $E(\Delta)KE(\Delta)$ is a compact operator in $\mathcal{H}$. In other words, $\widetilde{\mu}^A(H) = (-\infty, 0)$ and the proof is complete. ∎

PROPOSITION 5.2. *Under the hypotheses of Theorem* 1.3 *the commutator* $[K, \mathrm{i}A]$ *defines a bounded operator in* $\mathcal{H}$. *In particular, $H$ is of class $C^2(A)$.*

*Proof. Step 1.* Computing $\mathcal{K} := 2[K, \mathrm{i}A]$ as quadratic form on $l_0^2$ we get

$$(\mathcal{K}\psi)_n = k_n\psi_{n+2} + 2h_n\psi_{n+1} + j_n\psi_n + 2h_{n-1}\psi_{n-1} + k_{n-2}\psi_{n-2}$$

where

$$k_n = a_n r_{n+1} - a_{n+1} r_n, \tag{5.8}$$

$$h_n = a_n(\varepsilon_{n+1} - \varepsilon_n), \tag{5.9}$$

$$j_n = a_n r_n - a_{n-1} r_{n-1}. \tag{5.10}$$

It is enough to show that these coefficients are bounded. Concerning (5.8)

$$\begin{aligned} k_n &= -a_n a_{n+1}(\eta_{n+2} - \eta_{n+1} - \eta_n + \eta_{n-1}) \\ &= -a_n a_{n+1}(\eta_{n+2} - 2\eta_{n+1} + \eta_n) - a_n a_{n+1}(\eta_{n+1} - 2\eta_n + \eta_{n-1}). \end{aligned}$$

Since

$$a_n a_{n+1} = n^2\Big(1 + \frac{\eta_n}{n}\Big)\Big(1 + \frac{\eta_{n+1}+1}{n}\Big),$$

we see that $k_n$ is bounded by assumptions (1.1) and (1.6). Similarly, concerning (5.10)

$$\begin{aligned} j_n &= -a_n^2(\eta_{n+1} - \eta_{n-1}) + a_{n-1}^2(\eta_n - \eta_{n-2}) \\ &= -a_n^2\Big(\eta_{n+1} - \eta_{n-1} - \frac{a_{n-1}^2}{a_n^2}(\eta_n - \eta_{n-2})\Big) \qquad (5.11) \\ &= -a_n^2(\eta_{n+1} - \eta_{n-1} - \eta_n + \eta_{n-2}) - (a_n^2 - a_{n-1}^2)(\eta_n - \eta_{n-2}). \end{aligned}$$

The first term of the right hand side is bounded as for $k_n$. The second term of the right hand side is also bounded (it is even tending to zero at infinity), since $a_n^2 - a_{n-1}^2 = (a_n - a_{n-1})(a_n + a_{n-1})$ (see the proof of Proposition 4.1).

It remains to study (5.9). One has

$$h_n = a_n^2(\eta_{n+1} - 2\eta_n + \eta_{n-1}) + a_n a_{n+1}\Big(\eta_{n+1} - \eta_n - \frac{a_{n-1}}{a_{n+1}}(\eta_n - \eta_{n-1})\Big).$$

The first term of the right hand side is bounded as above. Concerning the second term write the following asymptotic development

$$\frac{a_{n-1}}{a_{n+1}} = 1 + O\Big(\frac{1}{n}\Big).$$

The proof can be finished as above. ∎

*Proof of Theorem* 1.3. Theorem 1.3 follows from Proposition 5.1 and Proposition 5.2 by a simple application of Theorem 2.3 and Theorem 2.4. The proof is finished. ∎

We also mention that our proof accepts some additional diagonal perturbation. More precisely, assume that $b_n = -(a_n + a_{n-1}) + v_n$ where $v_n$ satisfies similar hypotheses to those imposed to $\eta_n$ in (1.5) and (1.6).

## 6. SOME EXAMPLES

Let us end this paper by giving some examples in order to illustrate what kind of weights $a_n$ are included in our theorem.

EXAMPLE 6.1. The first example is the model studied by Janas and Naboko in [16], [18] which, somehow, corresponds to the case where $\eta_n = a$. More specifically their operator $H$ is defined by (1.4) with

$$a_n = n + a, \tag{6.1}$$

$$b_n = -2(n + a). \tag{6.2}$$

It is clear that the constant sequence $\eta_n = a$ satisfies assumptions of Theorem 4.1, since $\eta_n - \eta_{n-1} = 0$ for all $n \geqslant 2$. Then the last whole argument remains valid with one difference in the computation of the commutator $[H, \mathrm{i}A]$. More precisely, let $A$ be the operator defined in $\mathcal{H}$ by (5.1) with $a_n = n + a$ for all $n \geqslant 1$. Then straightforward computation shows that

$$[H, \mathrm{i}A] = -(H + 1) + a^2 \langle \cdot, e_1 \rangle e_1, \tag{6.3}$$

where $\langle \cdot, e_1 \rangle e_1$ is the projection on $e_1$ which is a rank one perturbation.

Then for any $\Delta = (x, y) \subset (-\infty, -1)$ one has the inequalities $-(y + 1) > 0$ and $E(\Delta)[H, \mathrm{i}A]E(\Delta) \geqslant -(y + 1)E(\Delta)$. This means that $\mu^A(H) = (-\infty, -1)$ and therefore the spectrum of $H$ is purely absolutely continuous on $(-\infty, -1)$ and this is the main result in [16]. In the latter the authors used the theory of subordinate solution of Gilbert-Pearson which turned out to be rather difficult since in this case the transfer matrices are asymptotically similar to a Jordan box, see [16].

According to (6.3) the parameter $a$ plays a special role. Indeed, we have two possibilities:

(i) If $a \geqslant 0$ then Proposition 3.1 shows that the spectrum of $H$ coincides with $(-\infty, -1]$ and therefore it is purely absolutely continuous.

(ii) If $a < 0$ then the inequality $a_n > 0$ holds only for $n$ greater to some suitable $N_0$. Thus according to Proposition 3.1 and the remark following its proof, the essential spectrum of $H$ coincides with $(-\infty, -1]$, and $H$ always has a finite number of eigenvalues belonging to $[-1, +\infty)$ (see also [16]). One can estimate their number and their sum by virial argument but we will not do it because this can be done by an elementary argument [16].

EXAMPLE 6.2. The second example appears in the representation theory of the $su(1,1)$ Lie algebra [10], [17]. More specifically, let $\alpha > 0$ and $\beta, \gamma, \delta$ be three real numbers. Consider the weights $a_n$ given by

$$a_n = \sqrt{\alpha n^2 + \beta n + \gamma}.$$

Let $H$ be the operator defined in $\mathcal{H}$ by

$$(H\psi)_n = a_{n-1}\psi_{n-1} + b_n\psi_n + a_n\psi_{n+1} \quad \text{for all } n \geqslant 1 \tag{6.4}$$

where $b_n = \delta n$. It is not hard to see that if $\delta^2 > 4\alpha > 0$ then the spectrum is purely discrete; we refer the reader to [10], [17] where this spectrum has been computed and proved to be independent of the parameter $\gamma$. In [15] the authors proved that $H$ has a purely absolutely continuous spectrum if $\delta^2 < 4\alpha$. The same result is established for the borderline case where $\delta^2 = 4\alpha$ provided that the parameter $\gamma$ satisfies the following identity

$$\gamma = \frac{\beta^2}{4\alpha} - \frac{\alpha}{16}.$$

We show that this condition is not necessary.

In the sequel we consider the borderline case where $\delta^2 = 4\alpha$. The conjugate operator $A$ is defined in $\mathcal{H}$ by (5.1) with $a_n = \sqrt{\alpha n^2 + \beta n + \gamma}$. By repeating the same argument as above we get that $H$ is of class $C^1(A)$ and

$$[H, \mathrm{i}A] = \frac{\delta}{2}H + (\beta - \alpha).$$

Here again the results of the example presented at the end of Section 2 apply for $H$ as we have done for $H_0$ in Section 4.

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