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NON-WEAKLY SUPERCYCLIC OPERATORS

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ABSTRACT. Several methods based on an easy geometric argument are provided to prove that a given operator is not weakly supercyclic. The methods apply to different kinds of operators like composition operators or bilateral weighted shifts. In particular, it is shown that the classical Volterra operator is not weakly supercyclic on any of the $L^p[0,1]$ spaces, $1 \leqslant p < \infty$. This is in contrast with the fact that the Volterra operator, extended in a natural way to certain Hilbert spaces, is hypercyclic. With the help of Gaussian measures, a general theorem of non-weak supercyclicity is proved, which can be applied to bilateral shifts or analytic functions of the Volterra operator. For instance, it is shown that a weighted bilateral shift acting on $\ell^p(\mathbb{Z})$, $1 \leqslant p < 2$, is weakly supercyclic if and only if it is supercyclic.

KEYWORDS: *Weakly supercyclic operators, cyclic operators, Volterra operator, bilateral weighted shifts, Gaussian measures, composition operators.*

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1. INTRODUCTION

Let $\mathbb{C}$ denote the field of complex numbers. A bounded linear operator T on a complex Banach space $\mathcal{B}$ is said to be (norm) supercyclic if there is a vector x in $\mathcal{B}$ such that $\{\lambda T^n x : \lambda \in \mathbb{C}$ and $n = 0, 1, 2, \dots\}$ is (norm) dense in $\mathcal{B}$ (when the scalar multiples are not needed, the operator is called hypercyclic). Supercyclic operators have been studied intensely in the last decade, see [21] for a survey on the subject.

Most recently, weakly supercyclic operators have come into the stage, see [2], [22] and [27] for instance. The operator is said to be weakly supercyclic when the norm density is replaced by the density with respect to the weak topology. Of course, this is not a mere generalization because weakly supercyclic operators are still cyclic. The latter is a consequence of a theorem of Mazur, [7], that asserts the weak closure and the norm closure of convex sets coincide. Weakly supercyclic operators is a broader class of operators that share many properties with the supercyclic ones. For instance, all the powers of a supercyclic operator are again supercyclic, see [1], and the same proof works for weakly supercyclic operators. Another instance of this fact is that the operator $\alpha I \oplus T : \mathbb{C} \oplus \mathcal{B}$, where $\mathcal{B}$ is a Banach space and $\alpha \neq 0$, is supercyclic if and only if $(1/\alpha)T$ is hypercyclic, see Theorem 5.1 of [11]. The proof of the latter result also works if the norm topology is replaced by the weak one (the corresponding result has been stated in the final remarks in [20] and observed in [27]). In any case, it can be said that this weak supercyclicity is somewhat artificial, since it comes from the fact that $(1/\alpha)T$ is weakly hypercyclic. Sometimes proving a given property for weakly supercyclic operators is much more involved than for supercyclic ones.

In this work we provide some methods to prove that a given operator is not weakly supercyclic. Mainly, this is achieved with a geometric argument in which we select weakly open sets which absorb the scalar multiples. The method applies to several kind of operators like the classical Volterra operator, composition operators or bilateral weighted shifts. Of course, one cannot expect that if an operator is not supercyclic, then it is not weakly supercyclic either. For instance, Bayart and Matheron [2] show that even certain unitary operators on Hilbert spaces are weakly supercyclic. Since such operators cannot be supercyclic, this provides natural examples of weakly supercyclic, non-supercyclic operators. Weak supercyclicity can also be considered for real Banach spaces and the methods we use here can also be applied.

Section 2 is devoted to the classical Volterra operator. We will prove that the Volterra operator is not weakly supercyclic. This is in contrast with two facts. The first one is that, the derivative operator, the unbounded left inverse of the Volterra operator is hypercyclic, ([10], Theorem 2.1). It is known that an invertible bounded operator is supercyclic (or hypercyclic) if and only if its inverse is, see [13] (the corresponding result for weakly supercyclic operators is not known and probably false). The second one, which will be proved in the next section, is that the Volterra operator can be extended in a natural way to certain Hilbert spaces in which it is hypercyclic.

In Section 3, we deal with composition operators. One of the difficulties to prove that the Volterra operator is not supercyclic is that the spectrum of the Volterra operator is just one point set. We show that linear fractional composition operators, that have much bigger spectrum, are much easier to handle in connection with the weak supercyclicity.

In Section 4 we study the orbits of nuclear, self-adjoint and positive operators. It will be proved that such operators cannot compress the space too much in all directions and there is certain control on the orbits. This will be achieved by introducing a suitable Gaussian measure. The results of this section are the key to apply the basic method to other operators.

In Section 5, we deal with bounded linear operators acting on Banach spaces in a complete general situation. In an extensive work in preparation, Bermudo and the authors [3] have used the latter result to show that $\varphi(V)$, where φ is holomorphic at 0 and V is the Volterra operator, is not weakly supercyclic.

In Section 6, we include an application to bilateral weighted shifts. In particular, it is proved that a weighted bilateral shift acting on $\ell^p(\mathbb{Z})$, with $1 \leqslant p < 2$, is weakly supercyclic if and only if it is supercyclic.

In 1982, Deddens (unpublished) proved that if T is a bounded linear operator on a Hilbert space, whose matrix with respect to some orthonormal basis consists of real entries, then $T \oplus T^\star$ is not cyclic. In Section 7, we will provide a very short proof of the latter result for bounded linear operators T defined on any Banach space. In particular, we use the result to prove that $V \oplus \lambda V$, where V is the Volterra operator, is cyclic if and only if λ belongs to $\mathbb{C} \setminus [0, \infty)$. Thus the direct sum of the Volterra operator with itself is not cyclic. The same is true for any operator quasisimilar to its dual (Banach space adjoint) operator. Thus there exist cyclic operators, whose adjoints have empty point spectrum, such that the direct sum with itself is not cyclic. This proves that there are cyclic operators whose point spectrum is the empty set which do not satisfy the Cyclicity Criterion introduced in [12].

2. THE VOLTERRA OPERATOR IS NOT WEAKLY SUPERCYCLIC

The aim of this section is to get a better understanding of the cyclic properties of the Volterra operator. Recall that for $1 \leqslant p \leqslant \infty$, we may consider the complex Banach space $L^p[0,1]$. The Volterra operator is defined as

$$(Vf)(x) = \int_0^x f(t)\,\mathrm{d}t, \quad f \in L^p[0,1].$$

It is well known and easy to show that V acts boundedly on $L^p[0,1]$, $1 \leqslant p \leqslant \infty$. In addition, V acting on $L^p[0,1]$, $1 \leqslant p < \infty$, is cyclic with cyclic vector the constant function 1, that is, the linear span of the orbit $\{V^n 1 : n = 0, 1, 2, \dots\}$ is dense in $L^p[0,1]$. Indeed, V is unicellular, see [28] and [18]. In a recent work, [10], Gallardo and the first named author, solving a question posed by Salas [26], have shown that V is not supercyclic on any of the $L^p[0,1]$ spaces, $1 \leqslant p < \infty$. We remark here that there is another published article that claims to prove the same result, but there is a gap in the proof. By Ansari's result mentioned in the introduction, one idea to prove that a given operator is not weakly supercyclic is to check that there is some power which is not cyclic. This does not apply to the Volterra operator, since, by the Müntz-Szasz Theorem [25], all positive powers V^n are also cyclic with cyclic vector the constant function 1.

To prove that V is not weakly supercyclic, we begin with a lemma that provides lower estimates on the orbits under V of certain functions.

LEMMA 2.1. *Suppose that f is a continuous function on $[0,1]$ non-vanishing at $1/2$. Then there is a positive constant c, depending only on f, such that*

$$\|V^n f\|_2 \geqslant \frac{c}{4^n n!}.$$

Proof. For each non-negative integer n consider $f_n(x) = x^n(1-x)^n$. The n-th derivative $p_n = f_n^{(n)}$ is the Legendre polynomial of degree n, see [23], p. 162. An easy computation yields $\int_0^1 f_n(s)\,\mathrm{d}s = (n!)^2/(2n+1)!$. As in [10], it follows that $h_n = ((2n+1)!/(n!)^2)f_n$, where $n = 0,1,2,\dots$ is a positive summability kernel at $1/2$, see [15], pp. 9–10. Thus, for any g in $L^1[0,1]$ continuous at $1/2$ we have $\langle g, h_n\rangle \to g(1/2)$ as $n \to \infty$.

Since $V^{\star n} p_n = (-1)^n f_n$ and $\|p_n\|_2 = n!/\sqrt{2n+1}$, we have

$$\|V^n f\|_2 \geqslant \frac{|\langle V^n f, p_n\rangle|}{\|p_n\|_2} = \frac{|\langle f, V^{\star n} p_n\rangle|}{\|p_n\|_2} = \frac{|\langle f, f_n\rangle|}{\|p_n\|_2} = \frac{n!\sqrt{2n+1}}{(2n+1)!}|\langle f, h_n\rangle|.$$

Thus using Stirling's formula one easily sees that

$$\|V^n f\|_2 \geqslant \sqrt{\frac{\pi}{2}}\frac{1}{4^n n!}|f(1/2)|(1+o(1)) \quad \text{as } n \to \infty.$$

Since $f(1/2) \neq 0$, there is a constant $c > 0$ satisfying the requirement of the lemma. The result is proved. ∎

To prove our next lemma we need the following one that can be found in [8], see also Lemma 6.5, where the result is strengthened.

LEMMA 2.2. *Let $\{x_n\}$ be a sequence in a Banach space $\mathcal{B}$ such that there exists a constant $c > 2$ with $\|x_n\| \geqslant c^n$ for each non-negative integer n. Then there exists a bounded functional g in $\mathcal{B}^\star$ such that $|\langle g, x_n\rangle| > 1$ for each non-negative integer n.*

LEMMA 2.3. *Let f be a continuous function on $[0,1]$ non-vanishing at $1/2$. Then there are g and h in $L^2[0,1]$, with h non-identically zero, such that, for each non-negative integer n*

$$|\langle V^n f, g\rangle| \geqslant \frac{1}{12^n n!} \quad \textit{and} \tag{2.1}$$

$$|\langle V^n f, h\rangle| \leqslant \frac{1}{13^n n!}. \tag{2.2}$$

Proof. To find g, we set $x_n = 12^n n! V^n f$. By Lemma 2.1, there is a constant $c > 0$ such that $\|V^n f\|_2 \geqslant c/4^n n!$, for $n = 0,1,\dots$. Therefore, $\|x_n\| \geqslant c3^n$. By Lemma 2.2, it follows that there is g in $L^2[0,1]$ such that $|\langle x_n, g\rangle| > 1$ for each non-negative integer n. Thus

$$|\langle V^n f, g\rangle| \geqslant \frac{1}{12^n n!}, \quad \text{for } n = 0,1,2\dots.$$

To find h, we consider $h_a = a\chi_{[0,13^{-1}]}$, where $\chi_{[0,13^{-1}]}$ denotes the characteristic function of $[0,13^{-1}]$ and $a > 0$. We have

$$\langle V^n f, h_a\rangle = a\int_0^{13^{-1}} V^n f(x)\,\mathrm{d}x = \frac{a}{(n-1)!}\int_0^{13^{-1}}\int_0^{x} f(t)(x-t)^{n-1}\,\mathrm{d}t\,\mathrm{d}x.$$

Let $\|f\|_\infty$ denote the supremum norm of f. Then

$$|\langle V^n f, h_a\rangle| \leqslant \frac{a\|f\|_\infty}{(n-1)!}\int_0^{13^{-1}}\int_0^{x}(x-t)^{n-1}\,\mathrm{d}t\,\mathrm{d}x = \frac{a\|f\|_\infty}{n!}\int_0^{13^{-1}} x^n\,\mathrm{d}x = \frac{a\|f\|_\infty}{13^n(n+1)!}.$$

Hence for $h = h_a$ with a small enough, we also find that (2.2) is true for each non-negative integer. Of course, $h \neq 0$. The result is proved. ∎

Now we can prove the main result in this section.

THEOREM 2.4. *The Volterra operator is not weakly supercyclic on any of the* $L^p[0,1]$ *spaces*, $1 \leqslant p < \infty$.

Proof. Since the weak topology of $L^1[0,1]$ is weaker than the weak topology of $L^p[0,1]$, $1 \leqslant p < \infty$, and the latter spaces are weakly dense in $L^1[0,1]$, it is enough to prove the result for $L^1[0,1]$. Suppose that f in $L^1[0,1]$ is weakly supercyclic for V acting on $L^1[0,1]$. Since $V : L^1[0,1] \to L^2[0,1]$ is weak-to-weak continuous, has dense range, and the image of a weakly dense set under an operator with dense range is weakly dense, we find that $\{\lambda V^n f : \lambda \in \mathbb{C}$ and $n = 1,2,\ldots\}$ is weakly dense in $L^2[0,1]$. Thus, Vf is weakly supercyclic for V acting on $L^2[0,1]$. We conclude that it is enough to show that V is not weakly supercyclic on $L^2[0,1]$.

Proceeding by contradiction, suppose that the operator V, acting on $L^2[0,1]$, is weakly supercyclic. Then the set $\mathcal{S}$ of weakly supercyclic vectors for V is weakly dense in $L^2[0,1]$ and its image under V is contained in $\mathcal{S}$. Let $\mathcal{C}_0[0,1]$ denote the Banach space of continuous functions that vanish at 0 endowed with the supremum norm. Since $V : L^2[0,1] \to \mathcal{C}_0[0,1]$ is norm-to-norm continuous and with dense range, it is also weak-to-weak continuous, and has weakly dense range, see [7], pp. 11–12.

Therefore, the image of $\mathcal{S}$ under V is weakly dense in $\mathcal{C}_0[0,1]$. Since the set of functions of $\mathcal{C}_0[0,1]$ that vanish at $1/2$ is weakly nowhere dense in $\mathcal{C}_0[0,1]$, there is a weakly supercyclic function f in $\mathcal{C}_0[0,1]$ such that $f(1/2) \neq 0$. Let g and h be the functions furnished by Lemma 2.3. In particular, since h is different from zero, the functions g and h are linearly independent. Thus the following is a non-empty, weakly-open set:

$$U = \{G : |\langle G, h\rangle| > |\langle G, g\rangle|\}.$$

Finally, set $F = \lambda V^n f$, where λ is in $\mathbb{C}$ and n is any non-negative integer. From Lemma 2.3, it follows $|\langle F, h\rangle| \leqslant |\langle F, g\rangle|$ and, therefore, F is not in U. In other words, $\{\lambda V^n f : \lambda \in \mathbb{C} \text{ and } n = 0, 1, \dots\}$ does not meet U, a contradiction with the weak supercyclicity of f. The proof is complete. ∎

Apart of Lemma 2.3, one of the points in the proof above is to consider a weakly open set in which the scalar multiples are absorbed.

REMARK 2.5. Similarly, assume that the Volterra operator acts boundedly on a Banach space which is densely contained in $L^1[0,1]$. If the natural embedding from $\mathcal{B}$ into $L^1[0,1]$ is continuous, then V is not weakly supercyclic on $\mathcal{B}$.

Next we show that the Volterra operator can be extended to a hypercyclic operator. By *natural extension of an operator* defined on a Banach space $\mathcal{B}$, we mean that the operator can be boundedly defined on a bigger space, where $\mathcal{B}$ is dense and in such a way that the restriction of the extension to $\mathcal{B}$ coincides with the original operator. For $1 \leqslant p < \infty$ and q the conjugate exponent, the dual pairing we use in the remainder of this section is

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, \mathrm{d}x, \quad \text{where } f \in L^p[0,1] \text{ and } g \in L^q[0,1].$$

Observe that there is no complex conjugation above.

A word about notation, when we mean Banach space adjoint, we write T^* instead of $T^\star$.

PROPOSITION 2.6. *The Volterra operator extends, in a natural way, to a Hilbert space where it is hypercyclic.*

Proof. For each $a > 0$, consider the Hilbert space $\mathcal{H}_a$ consisting of entire functions

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{a^n n!} (z-1)^n \quad \text{for which } \|g\|^2_{\mathcal{H}_a} = \sum_{n=0}^{\infty} |a_n|^2$$

is finite. Clearly, $\mathcal{H}_a$ is densely contained in $L^p[0,1]$, $1 \leqslant p < \infty$. For each bounded linear functional $\widetilde{f}$ in $\mathcal{H}_a^\star$, the dual space of $\mathcal{H}_a$, let $\langle \widetilde{f}, g \rangle$ denote $\widetilde{f}$ acting on g in $\mathcal{H}_a$. Observe that we do not perform the usual identification of $\mathcal{H}_a^\star$ with $\mathcal{H}_a$. Since each f in $L^1[0,1]$ defines on $\mathcal{H}_a$ a unique bounded linear functional

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, \mathrm{d}x, \quad \text{for each } g \in \mathcal{H}_a,$$

we may identify f with an element $\widetilde{f}$ in $\mathcal{H}_a^\star$. In particular, $\langle \widetilde{f}, g \rangle = \langle f, g \rangle$ for each g in $\mathcal{H}_a$. In this way, $\mathcal{H}_a \subset L^1[0,1] \subset \mathcal{H}_a^\star$.

Let V^* denote the dual of the Volterra operator V acting from $L^\infty[0,1]$ into itself with respect to the above dual pairing. One can easily verify that V^* acts according to $(V^* f)(x) = \int_x^1 f(t)\, \mathrm{d}t$. Then, the image of $\mathcal{H}_a$ under V^* is contained in $\mathcal{H}_a$. Indeed, for a function $g(z) = \sum_{n=0}^{\infty} a_n (z-1)^n/(a^n n!) \in \mathcal{H}_a$, we have

$$(V^*g)(z) = \sum_{n=0}^{\infty} \frac{a_n}{a^n n!} \int_z^1 (t-1)^n \, dt = -\sum_{n=1}^{\infty} \frac{a a_{n-1}}{a^n n!} (z-1)^n \in \mathcal{H}_a.$$

This allows us to extend the Volterra operator to $\mathcal{H}_a^\star$. Indeed, for $\widetilde{f} \in \mathcal{H}_a^\star$, we define $\widetilde{V}\widetilde{f}$ to be the unique element in $\mathcal{H}_a^\star$ such that

$$\langle \widetilde{V}\widetilde{f}, g \rangle = \langle \widetilde{f}, V^* g \rangle, \quad \text{for each } g \in \mathcal{H}_a.$$

If f in $L^1[0,1]$ is identified with $\widetilde{f}$ in $\mathcal{H}_a^\star$, then, for each $g \in \mathcal{H}_a$, we have $\langle \widetilde{V}\widetilde{f}, g \rangle = \langle \widetilde{f}, V^* g \rangle = \langle f, V^* g \rangle = \langle Vf, g \rangle$. Thus $\widetilde{V}$ coincides with V on $L^1[0,1]$.

Now, for each non-negative integer n, let $\delta^{(n)}$ denote the linear functional, which is in $\mathcal{H}_a^\star$, that to each $f \in \mathcal{H}_a$ assigns the value $(-1)^n f^{(n)}(1)$. As usual, the 0-th derivative of f is f itself. Easy considerations, involving the Riesz Representation Theorem, show that $\{e_n = a^n \delta^{(n)}\}_{n \geqslant 0}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_a^\star$. It is easy to see that $\widetilde{V}\delta^{(n)} = -\delta^{(n-1)}$ for $n \geqslant 1$ and $\widetilde{V}\delta^{(0)} = 0$. It follows that $\widetilde{V}^n e_n = -a e_{n-1}$, that is, $\widetilde{V}$ acting on $\mathcal{H}_a^\star$ is a scalar multiple of the unweighted unilateral backward shift. Now, Rolewicz [24] proved that this operator is hypercyclic as soon as $a > 1$. The proof is complete. ∎

If $a > 0$, then $\widetilde{V}$ is supercyclic because $\lambda\widetilde{V}$ is hypercyclic for $|\lambda| > a^{-1}$.

REMARK 2.7. The idea of the proof of Proposition 2.6 has a nice application. Let f be a cyclic vector for the Volterra operator. Then, since the topology of $L^1[0,1]$ is stronger than the one it inherits from $\mathcal{H}_a^\star$ and $L^1[0,1]$ is dense in $\mathcal{H}_a^\star$, we find that $\widetilde{f}$ is also cyclic for the backward shift. Although cyclic vectors for the backward shift are characterized in terms of analytic pseudo-continuations, see Chapter V in [5], the characterization is not very useful to determine whether a given function is cyclic for the backward shift or not.

REMARK 2.8. If only supercyclicity is required, instead of the spaces of entire functions in the proof of Proposition 2.6, one can consider spaces of analytic functions on the disk $D(1,1)$ of the complex plane centered at 1 and of radius 1. For instance, for any real ν, one can consider the weighted Dirichlet space $\mathcal{S}_\nu$ of functions $\sum_{n=0}^{\infty} a_n (z-1)^n$ analytic on $D(1,1)$ for which the norm $\|f\|_\nu^2 = \sum_{n=0}^{\infty} (n+1)^{2\nu} |a_n|^2$ is finite. The Volterra operator can be extended to $\mathcal{S}_\nu^\star$ in a similar fashion. The extended V's become weighted backward unilateral shifts, which are always supercyclic, see [14]. Indeed, they satisfy the Supercyclicity Criterion, see [20]. It is easily seen that if $\nu > 1/4$, then $\mathcal{S}_\nu^\star$ contains $L^1[0,1]$.

3. LINEAR FRACTIONAL COMPOSITION OPERATORS

One of the problems in order to prove Theorem 2.4 is the fact that the Volterra operator has the smallest possible spectrum. When the operator has bigger spectrum it is much easier to prove that a given operator is not weakly supercyclic. To exemplify this, we will check that linear fractional composition operators, which are not supercyclic, are not weakly supercyclic either. Our setting will be the weighted Dirichlet spaces $\mathcal{S}_\nu$, as defined in the Remark 2.7, with the disk centered at 1 and of radius 1 replaced by the unit disk $\mathbb{D}$ of the complex plane.

For special choices of the parameter ν the space $\mathcal{S}_\nu$ turns out to be the Hardy space, the Bergman space or the Dirichlet space, see [6]. Let φ be a linear fractional map that takes $\mathbb{D}$ into itself. Then C_φ, defined by $C_\varphi f = f \circ \varphi$ with $f \in \mathcal{S}_\nu$, is a bounded linear operator on $\mathcal{S}_\nu$. A complete description of the cyclicity, supercyclicity and hypercyclicity properties of C_φ is given in [9]. The proof of the next proposition shows four different ways to prove that a given operator is not weakly supercyclic.

PROPOSITION 3.1. *If a linear fractional composition operator is not supercyclic on $\mathcal{S}_\nu$, then it is not weakly supercyclic either.*

Proof. Since non-cyclic operators are not weakly supercyclic, an analysis of Table I in [9] shows that we need only consider four cases:

(i) The map φ is an elliptic automorphism conjugate to a rotation by an irrational multiple of π.

(ii) The map φ has an interior and an exterior fixed point.

(iii) The map φ is a hyperbolic non-automorphism.

(iv) The map φ is a parabolic non-automorphism.

In Case (i), C_φ cannot be weakly supercyclic because its adjoint has many eigenvalues and, as for supercyclic operators, the adjoint of a weakly supercyclic operator has at most one eigenvalue.

In Case (ii), φ has a fixed point p in $\mathbb{D}$. Let φ_0 denote the identity map and $\varphi_n = \varphi \circ \varphi_{n-1}$. Let f be in $\mathcal{S}_\nu$. Then $(C_\varphi^n f)(p) = (f \circ \varphi_n)(p) = f(p)$. If $f(p) = 0$, then f cannot be weakly supercyclic because the set of functions that vanish at p is weakly nowhere dense. If $f(p) \neq 0$, we may suppose that $f(p) = 1$. In this case, see [9], p. 15, $f \circ \varphi_n \to 1$ in the norm of $\mathcal{S}_\nu$ as $n \to \infty$. Then, taking the functions $g = 1$ and $h = 1/2 + z$, we have that $|\langle C_\varphi^n f, g\rangle| \geqslant 4/5$ and $|\langle C_\varphi^n f, h\rangle| \leqslant 3/4$ for large enough n. A similar argument to the one of the proof of Theorem 2.4 shows that f is not weakly supercyclic for C_φ.

In Case (iii), C_φ is hypercyclic for $\nu < 1/2$ and supercyclic for $\nu = 1/2$, see Table I in [9]. Therefore, we need only to consider the case in which $\nu > 1/2$. In [9] it is proved that this operator is not supercyclic using a result of Herrero that if for an operator T there is $r > 0$ such that each component of the spectrum of T does not meet the circle of radius r, then T is not supercyclic. The same argument shows that C_φ is not weakly supercyclic, since Herrero's theorem holds true for supercyclicity replaced by weak supercyclicity, see Lemma 4.3 in [2] or Theorem 4.2 in [22].

In Case (iv), it follows from Chapter V in [9] that if C_φ is weakly supercyclic on $\mathcal{S}_\nu$, then there is an operator $\widehat{C}_\varphi$ defined on $\widehat{S}^1_\nu \oplus \widehat{S}^2_\nu$ which is weakly supercyclic and such that $\widehat{S}^i_\nu$, $1 = 1, 2$, are reducing subspaces for $\widehat{C}_\varphi$. In addition, the decomposition can be done in such a way that, for some $\lambda > 0$, the spectrum of $\lambda\widehat{C}^1_\varphi$ is contained in $\mathbb{C} \setminus \mathbb{D}$ and the spectrum of $\lambda\widehat{C}^2_\varphi$ is contained in $\mathbb{D}$. Then, the same arguments of the proof of Lemma 4.3 in [2] show that $\widehat{C}_\varphi$ is not weakly supercyclic. The proof is complete. ∎

REMARK 3.2. As for the Volterra operator, in the four cases in the proof of Proposition 3.1 any positive power of C_φ is again cyclic.

REMARK 3.3. We remark here that Lemma 4.3 in [2] or Theorem 4.2 in [22] has a further consequence. T. Miller and V. Miller [19] proved that if T is supercyclic and decomposable, then the spectrum of T must be contained in a circle. It follows from Lemma 4.3 in [2] that the result is also true for weakly supercyclic operators.

4. ORBITS OF NUCLEAR SELF-ADJOINT, POSITIVE OPERATORS AND GAUSSIAN MEASURES

To prove non-weakly supercyclicity of more general operators we need to analyze the orbits of nuclear, self-adjoint, positive operators. In this section, $\mathbb{K}$ stands for the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Recall that an operator A acting on a Hilbert space $\mathcal{H}$ is positive if $\langle Ax, x\rangle \geqslant 0$ for every $x \in \mathcal{H}$. An operator A is said to be nuclear if there are sequences $\{x_n\}$ and $\{z_n\}$ in $\mathcal{H}$ such that

$$\sum_{n=0}^{\infty} \|x_n\|\|z_n\| < \infty \quad \text{and} \quad Ax = \sum_{n=0}^{\infty} \langle x, x_n\rangle z_n, \quad \text{for each } x \in \mathcal{H}.$$

The next theorem is the key for most of what follows.

THEOREM 4.1. *Let A be a nuclear, self-adjoint, positive, bounded linear operator on a separable infinite-dimensional Hilbert space $\mathcal{H}$ over the field $\mathbb{K}$. Let $\{y_n\}_{n\geqslant 0}$ be in $\mathcal{H}$ satisfying $\langle Ay_n, y_n\rangle = 1$ for each $n \geqslant 0$ and let $\{\alpha_n\}_{n\geqslant 0}$ be a sequence of positive numbers such that $\sum\limits_{n=0}^{\infty} \alpha_n^{dk} < \infty$ for some positive integer k, where $d = 1$ if $\mathbb{K} = \mathbb{R}$ or $d = 2$ if $\mathbb{K} = \mathbb{C}$. Then there exist $g_1, \dots, g_k \in \mathcal{H}$ such that*

$$\max_{1\leqslant j\leqslant k} |\langle g_j, y_n\rangle| \geqslant \alpha_n, \quad \text{for each } n \geqslant 0. \tag{4.1}$$

The idea of the proof of Theorem 4.1 is to introduce an appropriate measure on $\mathcal{H}^k$. Then one shows that the set of g's in $\mathcal{H}^k$, satisfying the statement of Theorem 4.1 has positive measure. In order to do this, we need the sigma-additivity criterion for Gaussian cylindrical measures.

4.1. BASIC FACTS FROM THE THEORY OF CYLINDRICAL MEASURES. In order to make this work more self-contained, we collect a few basic facts on cylindrical measures needed to prove Theorem 4.1. The contents of this subsection can be found in [16], for instance.

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space over the field $\mathbb{K}$. Let $\mathcal{F}$ denote the set of linearly independent finite subsets $Y = \{y_1, \dots, y_n\}$ of $\mathcal{H}$. Let B denote any Borel subset of $\mathbb{K}^n$ and B_Y denote the family of sets of the form

$$\{x \in \mathcal{H} \text{ such that } (\langle x, y_1\rangle, \dots, \langle x, y_n\rangle) \in B\}.$$

Obviously, B_Y is a sub-sigma-algebra of the Borel sigma-algebra $\mathcal{B}(\mathcal{H})$. A cylindric set is any element of

$$\mathcal{R}(\mathcal{H}) = \bigcup_{Y\in\mathcal{F}} B_Y.$$

Although $\mathcal{R}(\mathcal{H})$ is not a sigma-algebra, it is an algebra of subsets of $\mathcal{H}$.

A cylindrical measure on $\mathcal{H}$ is a finite finitely-additive, non-negative measure μ on the algebra $\mathcal{R}(\mathcal{H})$ such that for each Y in $\mathcal{F}$, the restriction $\mu|_{B_Y}$ is sigma-additive. In what follows, we shall always assume that $\mu(\mathcal{H}) = 1$. The Fourier transform of μ is the function $\widehat{\mu} : \mathcal{H} \to \mathbb{C}$ defined by

$$\widehat{\mu}(y) = \int_{\mathcal{H}} \mathrm{e}^{-\mathrm{i}\Re\langle x,y\rangle}\, \mathrm{d}\mu(x).$$

Indeed, the integral above is a standard one with respect to a sigma-additive measure, since the function $x \to \mathrm{e}^{-\mathrm{i}\Re\langle x,y\rangle}$ is bounded and $B_{\{y\}}$-measurable and the restriction $\mu|_{B_{\{y\}}}$ is sigma-additive.

One of the many equivalent definitions of Gaussian cylindrical measure is a cylindrical measure whose Fourier transform is of the form

$$\widehat{\mu}(x) = \mathrm{e}^{-(1/2)\langle Ax,x\rangle + \mathrm{i}\Re\langle u,x\rangle},$$

where A is a self-adjoint, positive, bounded linear operator and u belongs to $\mathcal{H}$. Note also that any function as in the above display is the Fourier transform of some cylindrical measure, that is, there is a one-to-one correspondence between the set of Gaussian cylindrical measures and the set of pairs (A, u). The operator A is called the *covariance operator* of μ.

We need the following theorem of Prohorov, see [16], p. 29.

THEOREM GM. *A Gaussian cylindrical measure is sigma-additive if and only if its covariance operator is nuclear.*

4.2. PROOF OF THEOREM 4.1. Without loss of generality, we may assume that the operator A is one-to-one. Indeed, for any self-adjoint positive nuclear operator A on $\mathcal{H}$ there is a self-adjoint positive one-to-one, nuclear operator $\widetilde{A}$ such that $\widetilde{A} - A$ is positive and we can replace A by $\widetilde{A}$.

Consider the space $\mathcal{H}^k = \bigoplus\limits_{j=1}^{k} \mathcal{H}$ endowed with the inner product

$$\langle x, y \rangle_{\mathcal{H}^k} = \sum_{j=1}^{k} \langle x_j, y_j \rangle_{\mathcal{H}}$$

and the projections $P_j : \mathcal{H}^k \to \mathcal{H}$ defined by $P_j x = x_j$. Let $T : \mathcal{H}^k \to \mathcal{H}^k$ be the operator defined by

$$Tx = (Ax_1, \dots, Ax_k) = \sum_{j=1}^{k} P_j^\star A P_j x.$$

Clearly, $T : \mathcal{H}^k \to \mathcal{H}^k$ is a self-adjoint positive nuclear operator. Let also $y_{n,j} = P_j^\star y_n \in \mathcal{H}^k$. One can easily verify, where $\delta_{j,l}$ is the Kronecker delta, that

$$\langle Ty_{n,j}, y_{n,l} \rangle = \delta_{j,l}, \quad \text{for } 1 \leqslant j, l \leqslant k \text{ and } n = 0, 1, \dots. \tag{4.2}$$

Consider the Gaussian cylindrical measure μ on $\mathcal{H}^k$, whose Fourier transform is

$$\widehat{\mu}(x) = \mathrm{e}^{-(1/2)\langle Tx, x \rangle}. \tag{4.3}$$

Upon applying Theorem GM, it follows that μ is sigma-additive and, therefore, it admits a unique sigma-additive extension to the Borel sigma-algebra. In particular, μ is a usual Borel probability measure.

We take $c > 0$ and consider

$$B_{n,c} = \Big\{ x \in \mathcal{H}^k \text{ such that } \sum_{j=1}^{k} |\langle x_j, y_n \rangle|^2 \leqslant c^2 \alpha_n^2 \Big\}.$$

We need to estimate the value of $\mu(B_{n,c})$. Consider the Borel measure ν defined on each Borel set B of $\mathbb{K}^k$ by

$$\nu(B) = \mu\{x \in \mathcal{H}^k \text{ such that } (\langle x_1, y_n \rangle, \dots, \langle x_k, y_n \rangle) \in B\}.$$

From (4.2), (4.3) and the definition of ν, it follows that the Fourier transform of ν is $\widehat{\nu}(t) = \mathrm{e}^{-|t|^2/2}$. Hence, ν has the density $\rho_\nu(s) = (2\pi)^{-dk/2} \mathrm{e}^{-|s|^2/2}$. Denote $D_a^k = \{x \in \mathbb{K}^k : |x| \leqslant a\}$. Then

$$\mu(B_{n,c}) = \nu(D^k_{c\alpha_n}) = (2\pi)^{-dk/2} \int\limits_{D^k_{c\alpha_n}} \mathrm{e}^{-|s|^2/2}\,\mathrm{d}s < (2\pi)^{-dk/2} \lambda(D^k_{c\alpha_n}) = v_k c^{dk} \alpha_n^{dk},$$

where λ is the Lebesgue measure on $\mathbb{K}^k$ and $v_k = (2\pi)^{-dk/2}\lambda(D_1^k)$. Hence, $\mu\Big(\bigcup\limits_{n=0}^{\infty} B_{n,c}\Big) \leqslant \sum\limits_{n=0}^{\infty} \mu(B_{n,c}) < v_k c^{dk} \sum\limits_{n=0}^{\infty} \alpha_n^{dk}$. Thus, by taking c small enough we can ensure that $\mu(\Lambda_c) < 1 = \mu(\mathcal{H})$, where $\Lambda_c = \bigcup\limits_{n=0}^{\infty} B_{n,c}$. Therefore, there must be $f = (f_1, \dots, f_k)$ in $\mathcal{H}^k \setminus \Lambda_c$. Clearly,

$$\sum_{j=1}^{k} |\langle f_j, y_n\rangle|^2 > c^2\,\alpha_n^2, \quad \text{for } n = 0, 1, 2, \dots.$$

Hence $\max\limits_{1\leqslant j\leqslant k} |\langle g_j, y_n\rangle| > \alpha_n$, where $g = (\sqrt{k}/c)f$. Thus $g = (g_1, \dots, g_k)$ satisfies the statement of the theorem. The proof is complete.

REMARK 4.2. Theorem 4.1 is sharp in the sense that if A is a compact positive self-adjoint operator on a real Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} \geqslant 2$, then there is a sequence $\{y_n\}$ in $\mathcal{H}$ with $\langle Ay_n, y_n\rangle = 1$ for all n and a sequence $\{\alpha_n\}$ of positive numbers such that $\sum\limits_{n=0}^{\infty} \alpha_n^{\rho}$ is finite for each $\rho > 1$ in such a way that for each $g \in \mathcal{H}$, there is n such that $|\langle g, y_n\rangle| < \alpha_n$.

5. IN THE GENERAL SITUATION

In this section, it will be shown how the ideas to prove that the Volterra operator is not weakly supercyclic along with Theorem 4.1 can be used in the complete general situation of a bounded linear operator T acting on a Banach space $\mathcal{B}$. The dual space of $\mathcal{B}$ will be denoted by $\mathcal{B}^\star$ and the Banach space adjoint of T, called also the dual of T, that acts on $\mathcal{B}^\star$ will be denoted by T^*. As usual, $\langle x, y\rangle$ will denote both the linear functional y in $\mathcal{B}^\star$ acting on x in $\mathcal{B}$ and, when x and y are vectors in a Hilbert space, their Hilbert inner product. It worths mentioning that although we run all the proofs for complex spaces, the same arguments works as well for the real case. We begin with

LEMMA 5.1. *Let T be a bounded linear operator on a Banach space $\mathcal{B}$. Let f be in $\mathcal{B}$ such that zero does not belong to the orbit $\{T^n f\}_{n\geqslant 0}$. Assume also that there exist a bounded operator R from $\mathcal{B}$ into a separable Hilbert space $\mathcal{H}$, a Hilbert-Schmidt operator S acting on $\mathcal{H}$ and non-zero $x_1, \dots, x_m$ in $\mathcal{B}^\star$ such that $SRT^n f \neq 0$ for each non-negative integer and for some $\rho > 0$ the following holds*

$$\sum_{n=0}^{\infty} \Big(\frac{\min\limits_{1\leqslant l\leqslant m} \|T^{*n} x_l\|_{\mathcal{B}^\star}}{\|SRT^n f\|_{\mathcal{H}}}\Big)^{\rho} < \infty. \tag{5.1}$$

Then f is not weakly supercyclic for T.

Proof. We may assume that T^* is one-to-one, otherwise T can not be weakly supercyclic. Let k be a positive integer such that $k > \rho$. The proof will be accomplished by applying Theorem 4.1. We set

$$\beta_n = \frac{\min\limits_{1\leqslant l\leqslant m} \|T^{*n} x_l\|_{\mathcal{B}^\star}}{\|SRT^n f\|_{\mathcal{H}}} \quad \text{and} \quad \alpha_n = \beta_n^{\rho/k} \quad \text{for } n = 1, 2 \ldots.$$

Since T^* is one-to-one and $x_l \neq 0$ for $1 \leqslant l \leqslant m$, we see that the numbers α_n and β_n are positive. From (5.1), it follows that $\sum\limits_{n=0}^{\infty} \alpha_n^k < \infty$. In particular, $\alpha_n \to 0$ as $n \to \infty$. Consider the operator $A = S^\star S$ and, for each non-negative n, set $y_n = RT^n f / \|SRT^n f\|_{\mathcal{H}}$. Then A is a self-adjoint positive operator on $\mathcal{H}$ and

$$\langle Ay_n, y_n \rangle = \frac{\langle S^\star SRT^n f, RT^n f\rangle}{\|SRT^n f\|_{\mathcal{H}}^2} = \frac{\langle SRT^n f, SRT^n f\rangle}{\|SRT^n f\|_{\mathcal{H}}^2} = 1.$$

Since S is a Hilbert-Schmidt operator, it also follows that A is nuclear. Thus we can apply Theorem 4.1 to see that there are $g_1, \ldots, g_k$ in $\mathcal{H}$ with $\max\limits_{1\leqslant j\leqslant k} |\langle g_j, y_n\rangle| > \alpha_n$ for each non-negative integer n. Using the definitions of y_n and α_n in the second and third equality below, we have

$$\begin{aligned}
\max_{1\leqslant j\leqslant k} |\langle R^* g_j, T^n f\rangle| &= \max_{1\leqslant j\leqslant k} |\langle g_j, RT^n f\rangle| = \max_{1\leqslant j\leqslant k} |\langle g_j, y_n\rangle| \, \|SRT^n f\|_{\mathcal{H}} > \alpha_n \|SRT^n f\|_{\mathcal{H}} \\
&= \alpha_n^{(\rho-k)/\rho} \beta_n \|SRT^n f\|_{\mathcal{H}} = \alpha_n^{(\rho-k)/\rho} \min_{1\leqslant l\leqslant m} \|T^{*n} x_l\|_{\mathcal{B}^\star} \\
&\geqslant \frac{\alpha_n^{(\rho-k)/\rho}}{\|f\|_{\mathcal{B}}} \min_{1\leqslant l\leqslant m} |\langle f, T^{*n} x_l\rangle| = \frac{\alpha_n^{(\rho-k)/\rho}}{\|f\|_{\mathcal{B}}} \min_{1\leqslant l\leqslant m} |\langle x_l, T^n f\rangle|.
\end{aligned}$$

Since $\rho < k$ and $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\min_{1\leqslant l\leqslant m} |\langle x_l, T^n f\rangle| = o\Big(\max_{1\leqslant j\leqslant k} |\langle R^* g_j, T^n f\rangle| \Big) \quad \text{as } n \to \infty. \tag{5.2}$$

Since $x_l \neq 0$ for $1 \leqslant l \leqslant m$, we can choose u in $\mathcal{B}$ such that $\langle x_l, u\rangle \neq 0$ for $1 \leqslant l \leqslant m$. Taking $c > 0$ small enough, we ensure that $\min\limits_{1\leqslant l\leqslant m} |\langle x_l, u\rangle| > c \max\limits_{1\leqslant j\leqslant k} |\langle R^* g_j, u\rangle|$. Hence u belongs to the weakly open set

$$U = \Big\{ y \in \mathcal{B} \text{ such that } \min_{1\leqslant l\leqslant m} |\langle x_l, y\rangle| > c \max_{1\leqslant j\leqslant k} |\langle R^* g_j, y\rangle| \Big\},$$

which means that U is non-empty. On the other hand, from (5.2) we see that for each positive integer n large enough, $\lambda T^n f$ does not meet U for any complex number λ. Hence, f cannot be weakly supercyclic for T. The result is proved. ∎

We will see that the next definition turns to be a natural one in connection with weak supercyclicity.

DEFINITION 5.2. Let T be a bounded linear operator on a Banach space $\mathcal{B}$ and for each non-negative integer n let $\mathcal{B}_n$ denote the Banach space $T^n(\mathcal{B})$ endowed with the norm $\|y\|_{\mathcal{B}_n} = \|y\|_{\mathcal{B}} + \inf\{\|x\|_{\mathcal{B}} : x \in \mathcal{B}, T^n x = y\}$. A set X

included in $\mathcal{B}$ is said *T-big* if there exists a non-negative integer n such that X contains a non-empty, weakly-open subset of $\mathcal{B}_n$.

Note that the topology of $\mathcal{B}_n$ is stronger than the one it inherits from $\mathcal{B}$.

REMARK 5.3. It is clear that any weakly open set is T-big. Of course, the converse is not true, as for the Volterra operator.

PROPOSITION 5.4. *Let T be a bounded weakly supercyclic operator on a Banach space $\mathcal{B}$. Then each T-big subset of $\mathcal{B}$ contains a weakly supercyclic vector for T.*

Proof. Let $\mathcal{S}$ denote the set of weakly supercyclic vectors for T. Since $\mathcal{S}$ is weakly dense in $\mathcal{B}$ and the operator $T^n : \mathcal{B} \to \mathcal{B}_n$ is bounded and onto, we find that $T^n(\mathcal{S})$ is weakly dense in $\mathcal{B}_n$. Since each non-empty open subset of a topological space meets each weakly dense set, it follows that each T-big set meets $T^n(\mathcal{S}) \subset \mathcal{S}$ for some n. The result is proved. ∎

THEOREM 5.5. *Let T be a bounded linear operator on a Banach space $\mathcal{B}$. Assume that there exists a T-big set X included in $\mathcal{B}$ such that for each f in X either zero belongs to the orbit $\{T^n f\}_{n\geqslant 0}$ or there exist a bounded operator R from $\mathcal{B}$ into a separable Hilbert space $\mathcal{H}$, a Hilbert-Schmidt operator S acting on $\mathcal{H}$ and non-zero $x_1, \dots, x_m$ in $\mathcal{B}^\star$ such that $SRT^n f \neq 0$ for each non-negative integer n and for some $\rho > 0$ the following holds*

$$\sum_{n=0}^{\infty} \Big(\frac{\min\limits_{1\leqslant l\leqslant m} \|T^{*n} x_l\|_{\mathcal{B}^\star}}{\|SRT^n f\|_{\mathcal{H}}} \Big)^{\rho} < \infty.$$

Then T is not weakly supercyclic.

Proof. Suppose that T is weakly supercyclic. According to Proposition 5.4, there is f in X weakly supercyclic for T. In particular, zero is not in the orbit $\{T^n f\}_{n\geqslant 0}$. Hence, f must be under the hypotheses of Lemma 5.1 and, consequently, cannot be weakly supercyclic for T, a contradiction. The proof is complete. ∎

The next corollary in the Hilbert space setting has a simpler appearance than that of Theorem 5.5.

COROLLARY 5.6. *Let T be a Hilbert-Schmidt operator on a separable Hilbert space $\mathcal{H}$. Assume that there exists a T-big set X included in $\mathcal{H}$ such that for each f in X either zero belongs to the orbit $\{T^n f\}_{n\geqslant 0}$ or there exist $\rho > 0$ and non-zero $y_1, \dots, y_m$ in $\mathcal{H}$ such that*

$$\sum_{n=0}^{\infty} \Big(\frac{\min\limits_{1\leqslant l\leqslant m} \|T^{\star n} y_l\|}{\|T^n f\|} \Big)^{\rho} < \infty.$$

Then T is not weakly supercyclic.

Proof. Without loss of generality, we may assume that $T^\star$ is one-to-one, otherwise T cannot be weakly supercyclic. Thus the functionals $x_l = \langle \cdot, T^\star y_l \rangle$ are non-zero and it is enough to take R to be equal to the identity and $S = T$ to be under the hypotheses of Theorem 5.5. ∎

Next theorem, in the Banach space setting, does not require that T is Hilbert-Schmidt.

THEOREM 5.7. *Let T be a bounded operator on a Banach space $\mathcal{B}$. Assume that there exists a T-big set X included in $\mathcal{B}$ such that for each f in X either zero belongs to the orbit $\{T^n f\}_{n\geqslant 0}$ or there exist non-zero $x_1, \ldots, x_m \in \mathcal{B}^\star$ such that for some $0 < r < 1$ the following holds*

$$\sum_{n=0}^{\infty} \Big(\frac{\min\limits_{1\leqslant l\leqslant m} \|T^{*n} x_l\|_{\mathcal{B}^\star}}{\|T^n f\|_{\mathcal{B}}} \Big)^r < \infty. \tag{5.3}$$

Then T is not weakly supercyclic.

Proof. Let f be in X such that zero does not belong to $\{T^n f\}_{n\geqslant 0}$. Thus we may suppose that (5.3) holds. Let ℓ^2 be the Hilbert space of square summable sequences of complex numbers and $\{e_n\}_{n\geqslant 0}$ its canonical basis. Upon applying the Hahn-Banach Theorem for each non-negative integer n, there exists g_n in $\mathcal{B}^\star$ such that $\|g_n\|_{\mathcal{B}^\star} = 1$ and $\langle T^n f, g_n\rangle = \|T^n f\|_{\mathcal{B}}$. Let $R : \mathcal{B} \to \ell^2$ and $S : \ell^2 \to \ell^2$ be defined by

$$Rg = \sum_{n=0}^{\infty} c_n \langle g, g_n\rangle e_n \quad \text{and} \quad Sh = \sum_{n=0}^{\infty} c_n \langle h, e_n\rangle e_n,$$

where

$$c_n = \Big(\frac{\min\limits_{1\leqslant l\leqslant m} \|T^{*n} x_l\|_{\mathcal{B}^\star}}{\|T^n f\|_{\mathcal{B}}} \Big)^{r/2}. \tag{5.4}$$

From (5.3), it follows that R is bounded and S is a Hilbert-Schmidt operator. Obviously, $\|SRT^n f\|_{\ell^2} \geqslant |\langle SRT^n f, e_n\rangle| = |\langle T^n f, g_n\rangle|\, c_n^2 = \|T^n f\|_{\mathcal{B}} c_n^2$. We take $\rho = r/(1-r)$ and using (5.3) and (5.4), we have

$$\sum_{n=0}^{\infty}\Big(\frac{\min\limits_{1\leqslant l\leqslant m} \|T^{*n} x_l\|_{\mathcal{B}^\star}}{\|SRT^n f\|_{\mathcal{B}}} \Big)^\rho \leqslant \sum_{n=0}^{\infty}\Big(\frac{\min\limits_{1\leqslant l\leqslant m} \|T^{*n} x_l\|_{\mathcal{B}^\star}}{\|T^n f\|_{\mathcal{B}}} c_n^2\Big)^\rho = \sum_{n=0}^{\infty}\Big(\frac{\min\limits_{1\leqslant l\leqslant m} \|T^{*n} x_l\|_{\mathcal{B}^\star}}{\|T^n f\|_{\mathcal{B}}} \Big)^{\rho(1-r)} < \infty.$$

Upon applying Theorem 5.5, it follows that T is not weakly supercyclic. The proof is complete. ∎

6. BILATERAL WEIGHTED SHIFTS

In this section, we illustrate how the general theorems in the previous sections can be applied to weighted bilateral shifts. Let $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical basis of the sequence space $\ell^p(\mathbb{Z})$, $1 \leqslant p < \infty$. Given a bounded sequence $\{w_n\}_{n\in\mathbb{Z}}$ of non-zero complex numbers the bilateral weighted shift acting on $\ell^p(\mathbb{Z})$ is defined by $We_n = w_n e_{n-1}$ for each $n \in \mathbb{Z}$. Clearly, W acts boundedly on $\ell^p(\mathbb{Z})$,

$1 \leqslant p < \infty$. Since weak supercyclicity is invariant under similarity, we may suppose that the weights w_n are positive. As usual we denote

$$\beta(k,n) = \prod_{j=k}^{n} w_j \quad \text{for } k,n \in \mathbb{Z} \text{ with } k \leqslant n.$$

PROPOSITION 6.1. *Let W be a bilateral weighted shift with the weight sequence $\{w_n\}_{n\in\mathbb{Z}}$ acting on $\ell^p(\mathbb{Z})$, $1 \leqslant p < \infty$. Suppose also that there exist a positive integer m and $0 < r < 1$ such that*

$$\sum_{n=0}^{\infty} \Big(\min_{|j|,|l|\leqslant m} \frac{\beta(l,l+n)}{\beta(j-n,j)} \Big)^r < \infty. \tag{6.1}$$

Then W is not weakly supercyclic.

Proof. Consider $X = \{f \in \ell^p(\mathbb{Z}) : \langle f, e_j \rangle \neq 0 \text{ for } |j| \leqslant m\}$, where $\langle \cdot, \cdot \rangle$ denotes the usual dual pairing of $\ell^p(\mathbb{Z})$ and $\ell^q(\mathbb{Z})$ with $1/p + 1/q = 1$. Clearly, X is weakly open and, therefore, is W-big. We take f in X. As functionals, the e_l's are also bounded and non-zero. In addition, one can easily verify that

$$\min_{-m-1\leqslant l\leqslant m-1} \|W^{*n+1} e_l\| = \min_{|l|\leqslant m} \beta(l, l+n).$$

On the other hand, since f is in X, there is a constant $c = c(f) > 0$ such that

$$\|W^{n+1} f\| \geqslant c \max_{|j|\leqslant m} \|W^{n+1} e_j\| \geqslant c \max_{|j|\leqslant m} \beta(j-n, j).$$

From the last two displays we have

$$\frac{\min\limits_{-m-1\leqslant l\leqslant m-1} \|W^{*n+1} e_l\|}{\|W^{n+1} f\|} \leqslant c^{-1} \min_{|j|,|l|\leqslant m} \frac{\beta(l,l+n)}{\beta(j-n,j)}. \qquad \blacksquare$$

For the case of a symmetric weight sequence, that is, $w_n = w_{-n}$ for every n in $\mathbb{Z}$, the following simpler condition follows immediately.

COROLLARY 6.2. *Let W be a weighted bilateral shift with symmetric sequence $\{w_n\}_{n\in\mathbb{Z}}$ acting on $\ell^p(\mathbb{Z})$, with $1 \leqslant p < \infty$. Assume that there is a real number r and a positive integer k such that*

$$\sum_{n=0}^{\infty} \beta(n, n+k)^r < \infty.$$

Then W is not weakly supercyclic.

It should be noted that a bilateral weighted shift with a symmetric weight sequence is never supercyclic. However, the summability condition in Corollary 6.2 cannot be omitted.

In the case $1 \leqslant p < 2$, Theorem 4.1 allows us to obtain a much stronger result than the one of Proposition 6.1.

THEOREM 6.3. *A weighted bilateral shift acting on $\ell^p(\mathbb{Z})$, with $1 \leqslant p < 2$ is weakly supercyclic if and only if it is supercyclic.*

We need two lemmas about weak closures of sequences. The first one is fairly easy.

LEMMA 6.4. *Let $\{x_n\}_{n\geqslant 0}$ be a sequence in a Hilbert space $\mathcal{H}$ and consider $\Omega = \{\lambda x_n : \lambda \in \mathbb{C},\ n \geqslant 0\}$. If y in $\mathcal{H}$ with $\|y\| = 1$ belongs to the weak closure of Ω, then it belongs to the weak closure of*

$$\Lambda = \Big\{ \frac{x_n}{\langle x_n, y\rangle} \text{ such that } \langle x_n, y\rangle \neq 0 \text{ and } n \geqslant 0 \Big\}.$$

Proof. Let $\mathcal{H}_0 = \{u \in \mathcal{H} : \langle u, y\rangle = 0\}$ be the orthogonal complement of y and consider $\mathcal{M} = \mathcal{H} \setminus \mathcal{H}_0$, $\Omega_0 = \Omega \cap \mathcal{H}_0$ and $\Omega_1 = \Omega \setminus \mathcal{H}_0$. Clearly, $\Omega = \Omega_0 \cup \Omega_1$ and y is not in the weak closure of Ω_0, since Ω_0 is contained in $\mathcal{H}_0$. Hence, y is in the weak closure of Ω_1.

Now consider $\mathcal{N} = \{u \in \mathcal{H} : \langle u, y\rangle = 1\}$. Then the map

$$F : \mathcal{M} \to \mathcal{N}, \quad F(u) = \frac{u}{\langle u, y\rangle}$$

is weak-to-weak continuous. Since y is in the weak closure of Ω_1, we obtain that $F(y) = y$ is in the weak closure of $F(\Omega_1) = \Lambda$, as required. ∎

Observe that in order to prove the next lemma we need the full strength of Theorem 4.1. Indeed, if p approaches to 2, then the integer k in Theorem 4.1 tends to ∞.

LEMMA 6.5. *Let $\{x_n\}_{n\geqslant 0}$ be a sequence in a Hilbert space $\mathcal{H}$ and let $0 < p < 2$ be such that*

$$\sum_{n=0}^{\infty} \|x_n\|^{-p} < \infty. \tag{6.2}$$

Then zero does not belong to the weak closure of $\{x_n : n \geqslant 0\}$.

Proof. Without loss of generality we may assume that $\mathcal{H}$ is infinite-dimensional and separable and that the sequence $\|x_n\|$ is monotonically non-decreasing.

Set $s_n = \|x_n\|^{-p/2}$ and $\alpha_n = \|x_n\|^{p/2-1}$. Using the Gramm-Schmidt procedure, we may choose an orthonormal basis $\{e_n\}_{n\geqslant 0}$ in $\mathcal{H}$ such that

$$x_n \in \operatorname{span}\{e_0, \dots, e_n\} \quad \text{for } n = 0, 1, 2, \dots. \tag{6.3}$$

Consider the diagonal operator $A : \mathcal{H} \to \mathcal{H}$ defined by $Ae_n = s_n e_n$. Using (6.2), one immediately checks that A^2 is a positive self-adjoint nuclear operator. Now, if we set $y_n = x_n / \|Ax_n\|$, then

$$\langle A^2 y_n, y_n\rangle = \langle Ay_n, Ay_n\rangle = \|Ay_n\|^2 = \frac{\|Ax_n\|^2}{\|Ax_n\|^2} = 1.$$

Choose a positive integer $k \geqslant 2p/(2-p)$. From (6.2), we find that $\sum\limits_{n=0}^{\infty} \alpha_n^k < \infty$. Thus applying Theorem 4.1, we can ensure that there are $g_1, \dots, g_k \in \mathcal{H}$, for which

$\max\limits_{1\leqslant j\leqslant k}|\langle y_n,g_j\rangle|\geqslant \alpha_n$, for $n=0,1,\dots$. Upon substituting the values of y_n and α_n into the last display, we obtain

$$\max_{1\leqslant j\leqslant k}|\langle x_n,g_j\rangle|\geqslant \|Ax_n\|\|x_n\|^{p/2-1}\geqslant s_n\|x_n\|\|x_n\|^{p/2-1}=1,\quad \text{for } n=0,1,\dots,$$

where in the last inequality we have used that $\{s_n\}$ is monotonically non-increasing and (6.3). Therefore, none of the x_n's belongs to $\Big\{u\in\mathcal{H}:\max\limits_{1\leqslant j\leqslant k}|\langle u,g_j\rangle|<1\Big\}$, which means that zero is not in the weak closure of $\{x_n:n\geqslant 0\}$. The result is proved. ∎

COROLLARY 6.6. *Let $\{x_n\}$ be a sequence in a Hilbert space $\mathcal{H}$ and let* (6.2) *be satisfied for some* $0<p<2$. *Then $\{x_n:n\geqslant 0\}$ is weakly closed.*

Proof. If y in $\mathcal{H}$ is different from x_n for every n, then $\{x_n-y\}$ satisfies (6.2) as well. By Lemma 6.5 zero is not in the weak closure of $\{x_n-y:n\geqslant 0\}$, or equivalently, y is not in the weak closure of $\{x_n:n\geqslant 0\}$, which proves the result. ∎

For sake of completeness we provide the following Banach space analog to Corollary 6.6.

COROLLARY 6.7. *Let $\{x_n\}$ be a sequence in a Banach space $\mathcal{B}$ and let* (6.2) *be satisfied for some* $0<p<1$. *Then $\{x_n:n\geqslant 0\}$ is weakly closed.*

Proof. Applying the Hahn-Banach theorem, we see that there exist $f_n\in\mathcal{B}^\star$ such that $\|f_n\|=1$ and $\langle f_n,x_n\rangle=\|x_n\|$ for each non-negative integer n. Since $\{\|x_n\|^{-p}\}$ is summable and $\|f_n\|=1$, we find that

$$\|x\|_0=\Big(\sum_{n=0}^{\infty}\|x_n\|^{-p}|\langle f_n,x\rangle|^2\Big)^{1/2}$$

defines a continuous seminorm on $\mathcal{B}$. Moreover, taking quotient by the kernel of this seminorm, if necessary, we obtain a pre-Hilbert space. Since $\langle f_n,x_n\rangle=\|x_n\|$, we see $\|x_n\|_0\geqslant\|x_n\|^{(2-p)/2}$. From (6.2) it follows that $\sum\|x_n\|_0^{2p/(p-2)}<\infty$. Since $2p/(2-p)<2$, Corollary 6.6 implies that $\{x_n:n\geqslant 0\}$ is closed in the weak topology of $(\mathcal{B},\|\cdot\|_0)$, which is weaker than the weak topology of $\mathcal{B}$. Hence $\{x_n:n\geqslant 0\}$ is weakly closed, which is the required result. ∎

In addition to the previous lemmas, we need Salas's Theorem, see [26], that characterizes supercyclic weighted bilateral shifts in terms of the sequence of weights.

THEOREM S. *Let W be a weighted bilateral shift with weight sequence $\{w_n\}$ acting on $\ell^p(\mathbb{Z})$ with $1 \leqslant p < \infty$. Then W is supercyclic if and only if*

$$\varliminf_{n\to+\infty} \frac{\max\limits_{|j|\leqslant m} \beta(j-n,j)}{\min\limits_{|k|\leqslant m} \beta(k,k+n)} = 0, \quad \textit{for each } m = 0,1,2,\ldots.$$

Proof of Theorem 6.3. Suppose that W is a bilateral weighted shift acting on $\ell^p(\mathbb{Z})$, with $1 \leqslant p < 2$, which is weakly supercyclic and non-supercyclic. By Theorem S there are $c_1 > 0$, a positive integer m and two sequences $\{j_n\}_{n\geqslant 0}$, $\{k_n\}_{n\geqslant 0}$ of integers such that $|j_n| < m$, $|k_n| < m$ and $\beta(j_n - n, j_n) \geqslant c_1\beta(k_n, k_n + n)$ for each non-negative integer n. Let M denote the supremum of the weight sequence of W. Since

$$\beta(j_n - n, m - n - 1)\beta(m - n, m) = \beta(j_n - n, j_n)\beta(j_n + 1, m) \quad \text{and}$$
$$\beta(k_n, m - 1)\beta(m, m + n) = \beta(k_n, k_n + n)\beta(k_n + n + 1, m + n),$$

we see that

$$\frac{\beta(m-n,m)}{\beta(m,m+n)} = \frac{\beta(j_n-n,j_n)}{\beta(k_n,k_n+n)} \frac{\beta(j_n+1,m)\beta(k_n,m-1)}{\beta(j_n-n,m-n-1)\beta(k_n+n+1,m+n)}.$$

Therefore,

$$\beta(m-n,m) \geqslant c\,\beta(-m,n-m), \quad \text{for } n = 0,1,2\ldots, \tag{6.4}$$

where $c = c_1\Big(\min\limits_{-m\leqslant j\leqslant k\leqslant m} \beta(j,k)\Big)^2(\max\{1, M^{4m}\})^{-1}$. Since the set of x's in $\ell^p(\mathbb{Z})$ such that $\langle x, e_m\rangle \neq 0$ is non-empty and weakly open, it is W-big and, therefore, there is a weakly supercyclic vector x in $\ell^p(\mathbb{Z})$ for W with $\langle x, e_m\rangle \neq 0$. The comparison principle implies that x is also weakly supercyclic for W acting on $\ell^2(\mathbb{Z})$. In particular, e_{m-1} is in the weak closure of $\{\lambda W^n x : \lambda \in \mathbb{C} \text{ and } n \geqslant 0\}$ in $\ell^2(\mathbb{Z})$. Consider $\Lambda = \{n : \langle W^{n+1}x, e_{m-1}\rangle \neq 0 \text{ and } n \geqslant 0\}$ and for $n \in \Lambda$ set $u_n = W^{n+1}x/\langle W^{n+1}x, e_{m-1}\rangle$. According to Lemma 6.4, e_{m-1} is in the weak closure in $\ell^2(\mathbb{Z})$ of $\{u_n : n \in \Lambda\}$. Clearly,

$$\|u_n\|_2 = \frac{\|W^{n+1}x\|_2}{|\langle W^{n+1}x, e_{m-1}\rangle|} = \frac{\|W^{n+1}x\|_2}{|\langle x, W^{*n+1}e_{m-1}\rangle|} = \frac{\|W^{n+1}x\|_2}{\beta(m,m+n)|\langle x, e_{m+n}\rangle|}.$$

Since $\|W^{n+1}x\|_2 \geqslant |\langle x, e_m\rangle|\|W^{n+1}e_m\| = |\langle x, e_m\rangle|\beta(m-n,m)$, we obtain

$$\|u_n\|_2 \geqslant \frac{|\langle x, e_m\rangle|\beta(m-n,m)}{|\langle x, e_{m+n}\rangle|\beta(m,m+n)}.$$

Using (6.4), we have

$$\|u_n\|_2 \geqslant \frac{c\,|\langle x, e_m\rangle|}{|\langle x, e_{m+n}\rangle|}.$$

Since x is in $\ell^p(\mathbb{Z})$, we see that $\sum \|u_n\|_2^{-p} < \infty$. Thus by Corollary 6.6, we find that $\{u_n : n \in \Lambda\}$ is weakly closed in $\ell^2(\mathbb{Z})$. Therefore, the only way for e_{m-1} to be in the weak closure of $\{u_n : n \geqslant 0 \text{ and } \langle W^{n+1}x, e_{m-1}\rangle \neq 0\}$ is to be one of the u_n's.

But this implies that x is a scalar multiple of some e_j, which is a contradiction, since such a vector cannot be even cyclic for a bilateral weighted shift. The proof is complete. ∎

REMARK 6.8. In connection with Lemma 6.5, we remark that from the famous theorem of Dvoretzky about almost spherical sections one can derive that for any infinite dimensional Banach space $\mathcal{B}$ and any sequence $\{\alpha_n\}$ of positive numbers with $\sum \alpha_n^2 = \infty$ there is a sequence $\{x_n\}$ in $\mathcal{B}$ such that $\|x_n\| = \alpha_n^{-1}$ and zero is in the weak closure of $\{x_n\}$. This shows that one cannot expect anything better than $\sum \|x_n\|^{-2} < \infty$ to ensure that zero is not in the weak closure of $\{x_n\}$.

REMARK 6.9. Considering sequences with disjoint support in the sequence space c_0, one can see that (6.2) with $p > 1$ in Corollary 6.7 is not enough to ensure that zero is not in the weak closure of $\{x_n : n \geqslant 0\}$.

7. OPERATORS SIMILAR TO THEIR ADJOINTS

In this final section, it is shown that an operator similar to its adjoint cannot be cyclic. As a consequence, it follows that $V \oplus V$ is not cyclic on $L^p[0,1] \oplus L^p[0,1]$. We will also determine for which complex numbers λ the operator $V \oplus \lambda V$ is cyclic.

PROPOSITION 7.1. *Let T be a bounded linear operator on a Banach space $\mathcal{B}$. Then $T \oplus T^*$ acting on $\mathcal{B} \oplus \mathcal{B}^\star$ is not cyclic. Moreover, if there exists a bounded linear operator $S : \mathcal{B} \to \mathcal{B}^\star$ with dense range such that $ST = T^*S$, then $T \oplus T$ acting on $\mathcal{B} \oplus \mathcal{B}$ is not cyclic.*

Proof. Let $f \oplus g^\star \in \mathcal{B} \oplus \mathcal{B}^\star$ be different from zero. Then the continuous linear functional F on $\mathcal{B} \oplus \mathcal{B}^\star$ defined by

$$\langle x \oplus y^\star, F\rangle = \langle x, g^\star\rangle - \langle f, y^\star\rangle$$

is different from zero. We have, $\langle T^n f \oplus T^{*n} g^\star, F\rangle = \langle T^n f, g^\star\rangle - \langle f, T^{*n} g^\star\rangle = 0$ for $n = 0, 1, 2, \dots$. Thus, the orbit of any non-zero vector under $T \oplus T^*$ is contained in the kernel of a non-zero continuous linear functional. Therefore, $T \oplus T^*$ is not cyclic.

Suppose now that $S : \mathcal{B} \to \mathcal{B}^\star$ is a bounded linear operator with dense range such that $ST = T^*S$ and $x \oplus y$ is a cyclic vector for $T \oplus T$. Then

$$(T \oplus T^*)^n (x \oplus Sy) = T^n x \oplus T^{*n} Sy = T^n x \oplus ST^n y = (I \oplus S)(T \oplus T)^n (x \oplus y).$$

Since $x \oplus y$ is cyclic for $T \oplus T$ and $I \oplus S : \mathcal{B} \oplus \mathcal{B} \to \mathcal{B} \oplus \mathcal{B}^\star$ has dense range, $x \oplus Sy$ is cyclic for $T \oplus T^*$ and, therefore, $T \oplus T^*$ is cyclic, which is a contradiction. ∎

The first part of the proposition above was also proved by Deddens, with a different proof, for operators defined on Hilbert space with a matrix with real entries with respect to some orthonormal basis.

Observe that V acting on $L^1[0,1]$ cannot be quasisimilar to its Banach space adjoint. However, we have,

COROLLARY 7.2. *The direct sum of the Volterra operator with itself is not cyclic on any of the spaces* $L^p[0,1] \oplus L^q[0,1]$, $1 \leqslant p, q < \infty$.

Proof. Since V acting on $L^2[0,1]$ is unitarily similar to its Banach space adjoint V^*, it follows that $V \oplus V$ acting on $L^2[0,1] \oplus L^2[0,1]$ is similar to $V \oplus V^*$. Thus, by Proposition 7.1, the operator $V \oplus V$ is not cyclic on $L^2[0,1] \oplus L^2[0,1]$. Now, if $f \oplus g$ is a cyclic vector for $V \oplus V$ acting on $L^1[0,1] \oplus L^1[0,1]$, then $Vf \oplus Vg$ is a cyclic vector for $V \oplus V$ acting on $L^2[0,1] \oplus L^2[0,1]$, a contradiction. Finally, for $1 \leqslant p, q < \infty$, the result follows by the Comparison Principle, see [26]. The proof is complete. ∎

REMARK 7.3. Note that $p(V)$ is cyclic for any polynomial p different from constant and again $p(V) \oplus p(V)$ is not cyclic.

REMARK 7.4. Let $\sigma_{\mathrm{p}}(T)$ denote the point spectrum of T, that is, the set of eigenvalues of T. The following question was posed in [20]: does every supercyclic operator with $\sigma_{\mathrm{p}}(T^*) = \varnothing$ satisfy the Supercyclicity Criterion? There is also a corresponding Cyclicity Criterion, see [11]. In particular, one may ask whether every cyclic operator T, with $\sigma_{\mathrm{p}}(T^*) = \varnothing$, satisfies the Cyclicity Criterion. This latter question has a negative answer. Indeed, if T satisfies the Cyclicity Criterion, so does $T \oplus T$ and, therefore, $T \oplus T$ is cyclic. We can conclude that V does not satisfy the Cyclicity Criterion.

Question 1. Is there any supercyclic operator similar to its Banach space (or Hilbert space) adjoint?

Using the Volterra operator, it is possible to produce more examples to which Proposition 7.1 and the remarks above apply. A direct application of the Müntz-Szasz Theorem shows that $V \oplus \lambda V$ is always cyclic whenever $\lambda \neq 1$ is of modulus 1 and rational in π. This is also true for λ irrational in π, but the proof requires a little more work. We have

PROPOSITION 7.5. *Let V denote the Volterra operator. Then $V \oplus \lambda V$ acting on* $L^p[0,1] \oplus L^q[0,1]$, $1 \leqslant p, q < \infty$, *is cyclic if and only if λ belongs to* $\mathbb{C} \setminus [0, +\infty)$.

Proof. We only prove the result for $L^2[0,1] \oplus L^2[0,1]$, if p or q is different from 2, the result follows in a similar way or from the latter.

For $\lambda = 0$ the result is trivial. For $\lambda > 0$ we find that λV acting on $L^2[0,1]$ is similar to V acting on $L^2[0,\lambda]$. Thus $V \oplus \lambda V$ is similar to $V \oplus V$ acting on $L^2[0,1] \oplus L^2[0,\lambda]$. If $\lambda \geqslant 1$, the latter is not cyclic, since it is not on $L^2[0,1] \oplus L^2[0,1]$. If $0 < \lambda < 1$, then $V \oplus V$ is not cyclic on $L^2[0,\lambda] \oplus L^2[0,\lambda]$, therefore, nor is $V \oplus V$ on $L^2[0,1] \oplus L^2[0,\lambda]$.

Now, assume that $\lambda = \mathrm{e}^u$ with $u = a + b\mathrm{i}$, where a and b are real with $0 < |b| \leqslant \pi$. We claim that $1 \oplus 1$ is a cyclic vector for $V \oplus \lambda V$ acting on $L^2[0,1] \oplus$

$L^2[0,1]$. In fact, suppose that $f \oplus -g$, where f and g are non-null functions in $L^2[0,1]$, is orthogonal to the orbit of $1 \oplus 1$ under $V \oplus \mathrm{e}^u V$. One easily checks that this is equivalent to the fact that

$$\Phi(z) = \int_0^1 f(t)t^z \mathrm{d}t - \mathrm{e}^{uz} \int_0^1 g(t)t^z \mathrm{d}t$$

vanishes at each non-negative integer n. On the other hand, Φ is a holomorphic function on the right half plane of finite exponential type and extends continuously to the imaginary axis. In addition, since the indicator function $h_\Phi(\theta) = \limsup\limits_{r\to+\infty}(1/r)\log|\Phi(r\mathrm{e}^{\mathrm{i}\theta})|$ for $-\pi/2 \leqslant \theta \leqslant \pi/2$ satisfies $h_\Phi(-\pi/2) + h_\Phi(\pi/2) \leqslant |b| \leqslant \pi < 2\pi$, Carlson's Theorem, see in [17] (Theorem 2.9.1, p. 116), implies that Φ is identically zero. This means that $F(z) = \mathrm{e}^{uz}G(z)$, where

$$F(z) = \int_0^1 f(t)t^z \mathrm{d}t \quad \text{and} \quad G(z) = \int_0^1 g(t)t^z \mathrm{d}t.$$

Since F and G are bounded, we may apply Cartwright's Theorem, see [17] (Theorem 7, p. 243), to find that there are real numbers $\alpha = \alpha(f)$ and $\beta = \beta(g)$ such that $\limsup\limits_{r\to+\infty}(1/r)\ln|F(r\mathrm{e}^{\mathrm{i}\theta})| = \alpha\cos\theta$ and $\limsup\limits_{r\to+\infty}(1/r)\ln|G(r\mathrm{e}^{\mathrm{i}\theta})| = \beta\cos\theta$, for all θ on $(-\pi/2, \pi/2)$. Since $F(z) = \mathrm{e}^{uz}G(z)$, it also follows that

$$\alpha\cos\theta = (a+\beta)\cos\theta - b\sin\theta \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Since the equality above is only possible for $b = 0$, the result follows. The proof is complete. ∎

REMARK 7.6. Of course, there are other examples of cyclic operators, with $\sigma_{\mathrm{p}}(T^*) = \varnothing$, for which the direct sum with itself is not cyclic. For instance, the unweighted bilateral shift B defined on $\ell^2(\mathbb{Z})$ is similar to B^* and, therefore, by Proposition 7.1 $B \oplus B$ is not cyclic on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$. From Ansari's result [1] that if T is weakly supercyclic, then so is any positive power of T, it follows that B is not weakly supercyclic either. In fact, whenever T is a normal operator, the operator $T \oplus T$ is not cyclic by the Spectral Theorem.

REMARK 7.7. As for supercyclic operators, one may wonder whether weak supercyclicity with $\sigma_{\mathrm{p}}(T^*) = \varnothing$ implies that $T \oplus T$ is cyclic. The referee has pointed out to us that this question has a negative answer. Indeed, by Bayart and Matheron's result we can take a unitary operator U which is weakly supercyclic. Then $\sigma_{\mathrm{p}}(U^\star) = \sigma_{\mathrm{p}}(U^*) = \varnothing$ and $U \oplus U$ is not cyclic. Moreover U is similar to U^* and, therefore, Question 1 has an affirmative answer if one replaces supercyclicity by weak supercyclicity.

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