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ROUGH MAXIMAL FUNCTIONS SUPPORTED BY SUBVARIETIES

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ABSTRACT. In this paper, we establish the L^p boundedness of a class of maximal functions with rough kernels supported by subvarieties. Moreover, several L^p estimates of singular integrals, Marcinkiewicz integrals, and parametric Marcinkiewicz integrals will be studied.

KEYWORDS: *Singular integrals, Marcinkiewicz integrals, parametric Marcinkiewicz integrals, square functions, maximal functions, rough kernels.*

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1. INTRODUCTION AND STATEMENT OF RESULTS

In recent years, a considerable amount of attention has been given to the study of singular integral operators with rough kernels ([7], [14], [17], among others). The main purpose of this paper is establishing L^p estimates of certain maximal functions related to a class of singular Radon transforms defined by translates of a subvariety determined by polynomial mappings. For background information on such singular Radon transforms, we refer the reader to Stein's survey article ([23], see also [7], [17]).

Let $\mathbb{R}^n$, $n \geqslant 2$ be the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $\mathrm{d}\sigma$. For non zero $y \in \mathbb{R}^n$, we shall let $y' = |y|^{-1}y$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ that satisfies the cancellation property

$$\int_{\mathbb{S}^{n-1}} \Omega(y')\mathrm{d}\sigma(y') = 0. \tag{1.1}$$

Let $L^2(\mathbb{R}_+, \mathrm{d}r/r)$ be the space of measurable functions $h : \mathbb{R}_+ \to \mathbb{R}$ that satisfy

$$\|h\|_{L^2(\mathbb{R}_+,\mathrm{d}r/r)} = \Big(\int_0^\infty |h(r)|^2 r^{-1}\mathrm{d}r\Big)^{1/2} < \infty. \tag{1.2}$$

We let $\overline{U}(L^2(\mathbb{R}_+, \mathrm{d}r/r), 1)$ denote the closed unit ball in $L^2(\mathbb{R}_+, \mathrm{d}r/r)$.

Let $\mathcal{P}(y) = (P_1, P_2, \ldots, P_d)$be a mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$ $(d \geqslant 1)$, with each P_j being a polynomial. For $x \in \mathbb{R}^d$, define the maximal function $\mathcal{M}_{\Omega,\mathcal{P}}$ by

$$\mathcal{M}_{\Omega,\mathcal{P}}(f)(x) = \sup_{h \in \overline{U}(L^2(\mathbb{R}^+, \mathrm{d}r/r))} |T_{\Omega,\mathcal{P},h}(f)(x)|, \tag{1.3}$$

where $T_{\Omega,\mathcal{P},h}$ is the singular integral operator given by

$$T_{\Omega,\mathcal{P},h}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \mathcal{P}(y))|y|^{-n} h(|y|)\Omega(y')\mathrm{d}y. \tag{1.4}$$

If $n = d$ and $\mathcal{P}(y) = (y_1, y_2, \ldots, y_d)$, we shall simply denote the operator $\mathcal{M}_{\Omega,\mathcal{P}}$ by $\mathcal{M}_{\Omega}$ and the operator $T_{\Omega,\mathcal{P},h}$ by $T_{\Omega,h}$.

The study of integral operators in the form (1.3) is motivated by the early work of R. Fefferman on singular integral operators with kernels multiplied by bounded radial functions [18]. More precisely, R. Fefferman showed that the operator $T_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n)$for $1 < p < \infty$ provided that $h \in L^\infty(\mathbb{R}_+)$ and Ω satisfies a Lipschitz condition of positive order on $\mathbb{S}^{n-1}$ [18]. Since Fefferman's result, the L^p boundedness of integral operators with kernels multiplied by radial functions has been investigated by many authors ([7], [9], [14], [17], [20]).

The classical operator $\mathcal{M}_\Omega$ was introduced by Chen and Lin in [11]. In [11], Chen and Lin showed that if Ω is continuous in $\mathcal{C}(\mathbb{S}^{n-1})$and satisfies (1.1), then $\mathcal{M}_\Omega$is bounded on $L^p(\mathbb{R}^n)$for any $p > 2n/(2n-1)$. In a very recent paper [5], we were able to establish an L^p boundedness result for $\mathcal{M}_\Omega$ in which the $\mathcal{C}(\mathbb{S}^{n-1})$ condition was substantially weakened. The precise statement of the result in [5] is the following:

THEOREM 1.1 ([5]). *If $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$is a homogeneous function of degree zero on $\mathbb{R}^n$that satisfies* (1.1), *then, for $2 \leqslant p < \infty$,*

$$\|\mathcal{M}_\Omega(f)\|_p \leqslant C_p \|f\|_p \, .$$

Also, in [5] the condition $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$ was shown to be nearly optimal in the sense that if the exponent $1/2$ in $L(\log L)^{1/2}(\mathbb{S}^{n-1})$ is replaced by any smaller number, then $\mathcal{M}_\Omega$ may fail to be bounded on $L^2(\mathbb{R}^n)$.

It is natural to ask whether the operator $\mathcal{M}_{\Omega,\mathcal{P}}$ has the same mapping properties as that of the classical operator $\mathcal{M}_\Omega$. This problem is resolved by our next result:

THEOREM 1.2. *If $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$ and satisfies* (1.1), *then*

$$\|\mathcal{M}_{\Omega,\mathcal{P}}(f)\|_p \leqslant C_p \|f\|_p \tag{1.5}$$

for $2 \leqslant p < \infty$, where C_p is a constant that may depend on the degrees of the polynomials $P_1, \ldots, P_d$ but it is independent of their coefficients.

By observing that $|T_{\Omega,\mathcal{P},h} f(x)| \leqslant \|h\|_{L^2(\mathbb{R}_+, \mathrm{d}r/r)} \, \mathcal{M}_{\Omega,\mathcal{P}}(f)(x)$ whenever $h \in L^2(\mathbb{R}_+, \mathrm{d}r/r)$, we immediately obtain the following result on singular integrals:

COROLLARY 1.3. *Suppose that $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying* (1.1) *and $h \in L^2(\mathbb{R}_+, \mathrm{d}r/r)$. Then the singular integral operator $T_{\Omega,\mathcal{P},h}$ is bounded on L^p for $1 < p < \infty$ with L^p bounds independent of the coefficients of the polynomials $P_1, \ldots, P_d$.*

The proof of Theorem 1.2, to be presented in Sections 2–3, can be adapted to treat other classes of maximal functions. A sample of such maximal functions will be presented in Section 4. Also, we shall present in Section 4 several results on singular integrals, Marcinkiewicz integrals, and parametric Marcinkiewicz integrals that follow from the corresponding results on maximal functions.

Throughout this paper the letter C will stand for a constant that may vary at each occurrence, but it is independent of the essential variables.

2. A GENERAL THEORY

For a family of measures $\sigma = \{\sigma_r : r \in \mathbb{R}_+\}$, define the square function S_σ by

$$S_\sigma(f)(x) = \Big(\int_0^\infty |\sigma_r * f(x)|^2 r^{-1}\mathrm{d}r\Big)^{1/2}. \tag{2.1}$$

Also for any $a \geqslant 1$, introduce the maximal function $M_{\sigma,a}$ which is defined by

$$M_{\sigma,a}(f)(x) = \sup_{k\in\mathbb{Z}} \int_{2^{ak}}^{2^{a(k+1)}} |\sigma_r * f(x)| \frac{\mathrm{d}r}{r}. \tag{2.2}$$

LEMMA 2.1. *Let $d, a \in \mathbb{N}$, and $\alpha, \beta > 0$. Let $L : \mathbb{R}^d \to \mathbb{R}^m$ be a linear transformation. Suppose that $\sigma = \{\sigma_r : r \in \mathbb{R}_+\}$ is a family of measures that satisfy:*

(i) $\sup_{r\in\mathbb{R}} \|\sigma_r\| \leqslant C$.

(ii) $\int_{2^{ak}}^{2^{a(k+1)}} |\widehat{\sigma}_r(\xi)|^2 \frac{\mathrm{d}r}{r} \leqslant aC|2^{ak}L(\xi)|^{-\alpha/a}$ *for* $\xi \in \mathbb{R}^d$.

(iii) $\int_{2^{ak}}^{2^{a(k+1)}} |\widehat{\sigma}_r(\xi)|^2 \frac{\mathrm{d}r}{r} \leqslant aC|2^{ak}L(\xi)|^{\beta/a}$ *for* $\xi \in \mathbb{R}^d$.

(iv) *For $1 < p < \infty$, there exists a constant $C_p > 0$ that is independent of the parameter a and the linear transformation L such that the maximal function $M_{\sigma,a}$ satisfies*

$$\|M_{\sigma,a}(f)\|_p \leqslant aC_p\|f\|_p.$$

Then the square function S_σ satisfies

$$\|S_\sigma(f)\|_p \leqslant \sqrt{a}C\|f\|_p \tag{2.3}$$

for all $2 \leqslant p < \infty$ with L^p bounds independent of the linear transformation L and the parameter a.

We should point out here that when $d = n$ and $L(\xi) = \xi$, the inequality (2.3) was proved in Theorem 2.1 in [3] under the conditions (i)–(iii) above, and the following condition

(iv′) For $1 < p < \infty$, there exists a constant $C_p > 0$ such that the maximal function $\sigma^*(f)(x) = \sup_{t\in\mathbb{R}} |\sigma_r * f(x)|$ satisfies $\|\sigma^*(f)\|_p \leqslant C\|f\|_p$.

Clearly, condition (iv′) above is stronger than the corresponding condition (iv) in Lemma 2.1. The distinction in Lemma 2.1 between the condition (iv) and the condition (iv′) is supported by the fact that sometimes (as in this paper), the former is available while the latter is not. Examples of families of measures for which (iv) is satisfied while (iv′) is not can be found easily. For instance, if σ_r is such that $\widehat{\sigma}_r(\xi) = \int_{\mathbb{S}^{n-1}} \mathrm{e}^{-\mathrm{i}(\xi\cdot y')r} \mathrm{d}\sigma(y')$, then (iv) is satisfied while (iv′) is not. This can be easily seen by observing that $M_{\sigma,a}(f)(x) \leqslant aHL(f)(x)$ where HL is the Hardy-Littlewood maximal function which is bounded on L^p for all $1 < p < \infty$, while σ^* is the spherical maximal function that is bounded only for $p > n/(n-1)$.

In order to prove Lemma 2.1, we first establish the following analogy of Lemma 6.4 in [17]:

PROPOSITION 2.2. *Let $d, a \in \mathbb{N}$. Let $\sigma = \{\sigma_r : r \in \mathbb{R}_+\}$ be a suitable family of measures. For $s \leqslant d$, let $G : \mathbb{R}^d \to \mathbb{R}^d$ and $H : \mathbb{R}^s \to \mathbb{R}^s$ be nonsingular linear transformations and let $\varphi \in \mathcal{S}(\mathbb{R}^s)$. Define J and $X_{k,r}$ by:*

$$J(f)(x) = f(G^t(H^t \otimes \mathrm{id}_{\mathbb{R}^{d-s}})x), \quad X_{k,r}(f)(x) = J^{-1}((|\Phi_{a,k}| \otimes \delta_{\mathbb{R}^{d-s}}) * J(\sigma_r * f))(x),$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{Z}$, and $\Phi_{a,k} \in \mathcal{S}(\mathbb{R}^s)$ satisfies $\widehat{\Phi}_{a,k}(\xi) = \varphi(2^{ak} H \pi^d_s G\xi)$ with π^d_s is the usual projection. Then for $1 < p < \infty$, there exists a positive constant C_p independent of a and the linear transformations G and H such that the maximal function

$$M_{X,a}(f)(x) = \sup_{k\in\mathbb{Z}} \int_{2^{ak}}^{2^{a(k+1)}} |X_{k,r}(f)(x)| \frac{\mathrm{d}r}{r}$$

satisfies

$$\|M_{X,a}(f)\|_p \leqslant C_p \|M_{\sigma,a}(f)\|_p. \tag{2.4}$$

Proof. Let $M^{(s)}$ and $I_{\mathbb{R}^{d-s}}$ be the Hardy-Littlewood maximal function on $\mathbb{R}^s$ and the identity operator on $\mathbb{R}^{d-s}$ respectively. Then, it is straightforward to see that

$$\int_{2^{ak}}^{2^{a(k+1)}} |X_{k,r}(f)(x)| \frac{\mathrm{d}r}{r} \leqslant J^{-1}((|\Phi_{a,k}| \otimes \delta_{\mathbb{R}^{d-s}}) * J(M_{\sigma,a}(f)))(x). \tag{2.5}$$

On the other hand, it is obvious that $\sup_{k\in\mathbb{Z}}((|\Phi_{a,k}| \otimes \delta_{\mathbb{R}^{d-s}}) * J(M_{\sigma,a}(f)))(x) \leqslant C(M^{(s)} \otimes I_{\mathbb{R}^{d-s}})(J(M_{\sigma,a}(f)))(x)$ with constant C independent of a. Thus by (2.5) and the previous, we obtain

$$M_{X,a}(f)(x) \leqslant C[J^{-1} \circ (M^{(s)} \otimes I_{\mathbb{R}^{d-s}}) \circ J](M_{\sigma,a}(f))(x). \tag{2.6}$$

Hence (2.4) follows by (2.6) and the boundedness of $M^{(s)}$ and $I_{\mathbb{R}^{d-s}}$ on L^p for all $1 < p < \infty$. This completes the proof. ∎

Now, we are ready to prove Lemma 2.1. For reader's convenience, we prove the following theorem which is more general than Lemma 2.1:

THEOREM 2.3. *Let* $M, d \in \mathbb{N}, \{m_s : 1 \leqslant s \leqslant M\} \subset N$, $\{l_s : 1 \leqslant s \leqslant M\} \subset (0,\infty)$, *and* $\{\varepsilon_{s,j} : 1 \leqslant s \leqslant M \text{ and } 1 \leqslant j \leqslant 2\} \subset (0,\infty)$. *Let* $L_s : \mathbb{R}^d \to \mathbb{R}^{m_s}$ *be linear transformations for* $1 \leqslant s \leqslant M$. *Suppose that* $\sigma_s = \{\sigma_{r,s} : r \in \mathbb{R}_+\}$, $0 \leqslant s \leqslant M$ *are families of measures that satisfy:*

(i) $\sup_{r\in\mathbb{R}} \|\sigma_{r,s}\| \leqslant C_s$ *for* $1 \leqslant s \leqslant M$.

(ii) $\sigma_0 = \{0\}$, *i.e.*, $\sigma_{r,0} = 0$ *for all* $r \in \mathbb{R}$.

(iii) $\int_{2^{ak}}^{2^{a(k+1)}} |\widehat{\sigma}_{r,s}(\xi)|^2 \frac{dr}{r} \leqslant aC_s|2^{al_sk}L_s(\xi)|^{-\varepsilon_{s,1}/a}$ *for* $\xi \in \mathbb{R}^d$ *and* $1 \leqslant s \leqslant M$.

(iv) $\int_{2^{ak}}^{2^{a(k+1)}} |\widehat{\sigma}_{r,s}(\xi) - \widehat{\sigma}_{r,s-1}(\xi)|^2 \frac{dr}{r} \leqslant aC_s|2^{al_sk}L_s(\xi)|^{\varepsilon_{s,2}/a}\}$ *for* $\xi \in \mathbb{R}^d$ *and* $1 \leqslant s \leqslant M$.

(v) *For* $1 < p < \infty$ *and* $1 \leqslant s \leqslant M$, *there exists a constant* $C_{p,s} > 0$ *that is independent of a and the linear transformation* L_s *such that the maximal function* $M_{\sigma_s,a}$ *satisfies*

$$\|M_{\sigma_s,a}(f)\|_p \leqslant aC_{p,s}\|f\|_p.$$

Then the square function S_{σ_M} *satisfies*

$$\|S_{\sigma_M}(f)\|_p \leqslant \sqrt{a}C\|f\|_p \tag{2.7}$$

for all $2 \leqslant p < \infty$ *with* L^p *bounds independent of the linear transformations* $\{L_s : 1 \leqslant s \leqslant M\}$ *and the parameter a.*

Proof. We shall combine the method used in the proof of Lemma 5.2 in [16] with ideas from [3], [5], [11] and Proposition 2.2. By a similar argument as in the proof of Lemma 5.2 in [16], see also [17] and Proposition 2.2, there exist families of measures $\lambda_s = \{\lambda_{r,s} : r \in \mathbb{R}_+\}$, $1 \leqslant s \leqslant M$ such that for $\xi \in \mathbb{R}^d$ and $1 \leqslant s \leqslant M$, we have

$$\sup_{r\in\mathbb{R}} \|\lambda_{r,s}\| \leqslant C_s \ , \tag{2.8}$$

$$\int_{2^{ak}}^{2^{a(k+1)}} |\widehat{\lambda}_{r,s}(\xi)|^2 \frac{dr}{r} \leqslant aC_s \min\{|2^{al_sk}L_s(\xi)|^{-\varepsilon_{s,1}/a}, |2^{al_sk}L(\xi)|^{\varepsilon_{s,2}/a}\}, \tag{2.9}$$

$$\|M_{\lambda_s,a}(f)\|_p \leqslant aC_{p,s}\|f\|_p \quad \text{for } 1 < p < \infty \, , \tag{2.10}$$

$$\sigma_{r,M}=\sum_{s=1}^{M}\lambda_{r,s}\,, \tag{2.11}$$

where C_s and $C_{p,s}$ are positive constants independent of a and the linear transformations L_s.

By (2.11) and Minkowski's inequality, we have $S_{\sigma_M}(f)(x)\leqslant\sum_{s=1}^{M}S_{\lambda_s}(f)(x)$. Thus, to prove (2.7) it suffices to show that for any $1\leqslant s\leqslant M$, the inequality

$$\|S_{\lambda_s}(f)\|_p\leqslant\sqrt{a}C_s\|f\|_p \tag{2.12}$$

holds for all $2\leqslant p<\infty$ where C_s is a constant independent of the linear transformation L_s and the parameter a.

Given $1\leqslant s\leqslant M$ by the same argument in [17], we may assume that $d\leqslant m_s$ and $\mathbb{L}=\pi^d_{m_s}$ is the usual projection. Choose a collection of $\mathcal{C}^\infty$ functions $\{\omega_{k,a}\}_{k\in\mathbb{Z}}$ on $(0,\infty)$ such that $\operatorname{supp}(\widehat{\omega}_{k,a})\subseteq[2^{-ak-a},2^{-ak+a}],0\leqslant\widehat{\omega}_{k,a}\leqslant 1$, $\sum_{k\in\mathbb{Z}}\widehat{\omega}_{k,a}(u)=1$, and $\left|\frac{d^s\widehat{\omega}_{k,a}}{du^s}(u)\right|\leqslant C_l u^{-l}$, where C_l are constants independent of a and k. For $j\in\mathbb{Z}$, define the square function $S^{(a)}_{\lambda_s,j}$ by

$$S^{(a)}_{\lambda_s,j}(f)(x)=\Big(\sum_{k\in\mathbb{Z}}\int_{2^{ak}}^{2^{a(k+1)}}|\lambda_{s,r}*\omega_{k+j,a}*f(x)|^2r^{-1}\mathrm{d}r\Big)^{1/2}. \tag{2.13}$$

Thus we have

$$S_{\lambda_s}(f)(x)\leqslant\sum_{j\in\mathbb{Z}}S^{(a)}_{\lambda_s,j}(f)(x). \tag{2.14}$$

Now, by Plancherel's theorem and Fubini's theorem, we have the following for $I_{k,a}=\{x\in\mathbb{R}^d:2^{-ak-a}\leqslant|\pi^d_{m_s}(x)|<2^{-ak+a}\}$:

$$\|S^{(a)}_{\lambda_s,j}(f)\|_2^2\leqslant\sum_{k\in\mathbb{Z}}\int_{I_{k+j,a}}|\widehat{f}(\xi)|^2\Big(\int_{2^{ak}}^{2^{a(k+1)}}|\widehat{\lambda}_{s,r}(\xi)|^2\frac{\mathrm{d}r}{r}\Big)\mathrm{d}\xi. \tag{2.15}$$

By (2.15), (2.9), and the properties of $\{\omega_{k,a}\}_{k\in\mathbb{Z}}$, we have $\|S^{(a)}_{\lambda_s,j}(f)\|_2^2\leqslant aC2^{-\alpha|j|}\|f\|_2^2$ for some $\alpha>0$. Thus

$$\|S^{(a)}_{\lambda_s,j}(f)\|_2\leqslant\sqrt{a}C2^{-\alpha|j|}\|f\|_2. \tag{2.16}$$

Next, for $p\geqslant 2$, there exists $g\in L^{(p/2)'}$ with $\|g\|_{(p/2)'}=1$ such that

$$\|S^{(a)}_{\lambda_s,j}(f)\|_p^2=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{ak}}^{2^{a(k+1)}}|\lambda_{s,r}*\omega_{k+j,a}*f(x)|^2r^{-1}\mathrm{d}r|g(x)|\mathrm{d}x \tag{2.17}$$

$$\leqslant \Big\{ \sup_{t\in\mathbb{R}} \|\sigma_t\| \Big\} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\omega_{k+j,a} * f(x)|^2 \int_{2^{ak}}^{2^{a(k+1)}} |\lambda_{s,r}| * g(x) \frac{\mathrm{d}r\mathrm{d}x}{r}$$

$$\leqslant \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\omega_{k+j,a} * f(z)|^2 M_{\lambda_s,a}(\widetilde{g})(z)\mathrm{d}z$$

$$\leqslant C\Big\| \Big(\sum_{k\in\mathbb{Z}} |\omega_{k+j,a} * f|^2 \Big)^{1/2} \Big\|_p^2 \|M_{\lambda_s,a}(\widetilde{g})\|_{(p/2)'} \leqslant aC\|f\|_p^2,$$

where $\widetilde{g}(z) = g(-z)$, the third inequality follows by Hölder's inequality, and the last inequality follows by (iii) and the Littlewood-Paley theory. The constant C may depend on the underlying dimension n, the constants C_s, and p, but it is independent of k and a (for details see [7]). Thus, we obtained for all $2 < p < \infty$

$$\|S_{\lambda_{s,j}}(f)\|_p \leqslant \sqrt{a}C\|f\|_p\,. \tag{2.18}$$

By interpolation between (2.16) and (2.18), we obtain

$$\|S_{\lambda_{s,j}}(f)\|_p \leqslant C\sqrt{a}2^{-\beta|j|}\|f\|_p, \tag{2.19}$$

for all $2 < p < \infty$. Hence (2.12) follows by (2.14), (2.16), (2.19), and Minkowski's inequality. This completes the proof. ∎

We end this section by the following special case of Proposition 5.1 in [17].

LEMMA 2.4. *Let l and n be positive integers. Let $V_l(n)$ be the space of real-valued homogenous polynomials of degree l on $\mathbb{R}^n$. Let $U_l(n)$ be a subspace of $V_l(n)$ with $|x|^l \notin U_l(n)$. Let $\Omega \in L^2(\mathbb{S}^{n-1})$. Then there exists a positive constant A independent of Ω such that*

$$\int_{2^k}^{2^{k+1}} \Big| \int_{\mathbb{S}^{n-1}} \mathrm{e}^{\mathrm{i}F(ry')}\Omega(y')\mathrm{d}\sigma(y')\Big| \frac{\mathrm{d}r}{r} \leqslant A\|\Omega\|_{L^2}(2^{kl}\|P_l\|)^{-1/8l}$$

for all $k \in \mathbb{Z}$ and functions $F : \mathbb{R}^n \to \mathbb{R}$ of the form

$$F(x) = \sum_{j=0}^{l} P_j + W(|x|)$$

where P_j is a homogenous polynomial of degree j, $0 \leqslant j \leqslant m$, $P_l \in U_l(n)$, and W is an arbitrary function. The constant A may depend on the subspace $U_l(n)$ if l is even, but it is independent of $U_l(n)$ if l is odd. Here, $\|P_l\| = \sum_{|\alpha|=l} |a_\alpha|$ where $P_l(y) = \sum_{|\alpha|=l} a_\alpha y^\alpha$.

3. PROOF OF THEOREM 1.2

Let $\deg(\mathcal{P}) = M$. Let $\sigma_M = \{\sigma_{r,M} : r \in \mathbb{R}_+\}$ be the family of measures defined by

$$\widehat{\sigma}_{r,M}(\xi) = \int_{\mathbb{S}^{n-1}} \mathrm{e}^{-\mathrm{i}\xi\cdot\mathcal{P}(ty')}\Omega(y')\mathrm{d}\sigma(y'). \tag{3.1}$$

Then by duality it follows that

$$\mathcal{M}_{\Omega,\mathcal{P}}(f)(x) = S_{\sigma_M}(f)(x) \tag{3.2}$$

where S_{σ_M} is given by (2.1) with the family σ replaced by the family σ_M. Now, we write the function Ω (see [5] or [7]) as follows:

$$\Omega(y') = \sum_{m\in\mathbb{D}\cup\{0\}} \theta_m A_m(y'), \tag{3.3}$$

where $\mathbb{D} \subset \mathbb{N}$, $\{\theta_m : m \in \mathbb{D}\cup\{0\}\} \subset (0,\infty)$, and $\{A_m : m \in \mathbb{D}\cup\{0\}\}$ is a sequence of homogeneous functions of degree zero on $\mathbb{R}^n$ satisfying:

$$\int_{\mathbb{S}^{n-1}} A_m(y')\mathrm{d}\sigma(y') = 0, \tag{3.4}$$

$$\|A_m\|_1 \leqslant C, \quad \|A_m\|_2 \leqslant C2^{4(m+1)}, \tag{3.5}$$

$$\sum_{m\in\mathbb{D}\cup\{0\}} \sqrt{m}\theta_m \leqslant C\|\Omega\|_{L(\log L)^{1/2}(\mathbb{S}^{n-1})}. \tag{3.6}$$

Then, by (3.3), we have the following

$$S_{\sigma_M}(f)(x) \leqslant \sum_{m\in\mathbb{D}\cup\{0\}} \theta_m S_{\sigma_M,m}(f)(x), \tag{3.7}$$

where $S_{\sigma_M,m}$ has the same definition as S_{σ_M} with Ω replaced by A_m. Therefore, to prove (1.5) and hence the theorem, it suffices by (3.2) and (3.6)–(3.7) to show that

$$\|S_{\sigma_M,m}(f)\|_p \leqslant \sqrt{m}C_p\|f\|_p \tag{3.8}$$

for all $2 \leqslant p < \infty$ with constant C_p independent of m and the coefficients of the polynomials $P_1,\dots,P_d$. However, (3.8) will follow by an application of Theorem 2.3. To this end, we argue as follows:

First, we choose integers $0 < d_1 < d_2 < \cdots < d_l \leqslant M$, and polynomials $Q_j^{(s)} : \mathbb{R}^n \to \mathbb{R}$, $R_j : \mathbb{R} \to \mathbb{R}$ for $1 \leqslant j \leqslant d$ and $1 \leqslant s \leqslant l$ such that:

(i) For each $1 \leqslant s \leqslant l$, the polynomial mapping $Q^{(s)} = (Q_1^{(s)}, Q_2^{(s)},\dots,Q_d^{(s)})$ is homogenous of degree d_s.

(ii) $Q_j^{(s)} \in U_{d_s}(n)$ where $U_{d_s}(n)$ is a suitable subspace of the space of real-valued homogenous polynomials of degree d_s on $\mathbb{R}^n$ with $|x|^{d_s} \notin U_{d_s}(n)$.

(iii) $\deg(R_j) \leqslant M$ for all $1 \leqslant j \leqslant d$.

(iv) $\mathcal{P}(y) = \sum\limits_{s=1}^{l} Q^{(s)}(y) + (R_1(|y|), \ldots, R_d(|y|))$.

Next, for $1 \leqslant j \leqslant d$ and $1 \leqslant s \leqslant l$, let $Q_j^{(s)}(y) = \sum\limits_{|\beta|=d_s} a(j,\beta)y^{\beta}$. For $1 \leqslant s \leqslant l$ let N_s be the number of multi-indices $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$ with $|\beta| = \beta_1 + \cdots + \beta_n = d_s$ and define the linear transformation $L_s : \mathbb{R}^d \to \mathbb{R}^{N_s}$ by

$$L_s(\xi) = L_s(\xi_1, \ldots, \xi_d) = \Big(\sum_{j=1}^{d} \xi_j a(j,\beta) \Big)_{|\beta|=d_s}.$$

For $0 \leqslant s \leqslant l$, let

$$\mathcal{P}_s(y) = \sum_{u=1}^{s} Q^{(u)}(y) + (R_1(|y|), \ldots, R_d(|y|))$$

and let $\lambda_s^{(m)} = \{\lambda_{r,s}^{(m)} : r \in \mathbb{R}_+\}$ be a family of measures that have the same definition as the family σ_M with $\mathcal{P}$ is replaced by $\mathcal{P}_s$ and Ω replaced by A_m. Also, let $\sigma_{M,m} = \{\sigma_{r,M}^{(m)} : r \in \mathbb{R}_+\}$ be the family of measures given by (3.1) with Ω replaced by A_m. Then by (iv) above, it is clear that

$$\lambda_l^{(m)} = \sigma_{M,m}\,. \tag{3.9}$$

On the other hand, by (3.4) and the first inequality in (3.5), we have

$$\lambda_0^{(m)} = 0, \tag{3.10}$$

$$\sup_{r>0} \|\lambda_{r,s}^{(m)}\| \leqslant C_s\,, \tag{3.11}$$

where C_s is a constant independent of m and the coefficients of the polynomial components of $\mathcal{P}_s$. Next, by the second inequality in (3.5), (ii) above, and Lemma 2.4, we have for $\xi \in \mathbb{R}^d$ and $1 \leqslant s \leqslant l$:

$$\int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}} |\widehat{\lambda}_{r,s}^{(m)}(\xi)|^2 \frac{\mathrm{d}r}{r} \leqslant 2^{4m}(m+1)C_s|2^{(m+1)d_s k} L_s(\xi)|^{-1/8d_s}. \tag{3.12}$$

On the other hand, it is easy to see that

$$\int_{2^{ak}}^{2^{a(k+1)}} |\widehat{\lambda}_{r,s}^{(m)}(\xi) - \widehat{\lambda}_{r,s-1}^{(m)}(\xi)|^2 \frac{\mathrm{d}r}{r} \leqslant (m+1)C_s|2^{(m+1)d_s k} L_s(\xi)| \tag{3.13}$$

for $\xi \in \mathbb{R}^d$ and $1 \leqslant s \leqslant l$. Thus, by interpolation between (3.11) and (3.12), and between (3.11) and (3.13), we obtain for $\xi \in \mathbb{R}^d$ and $1 \leqslant s \leqslant l$:

$$\int_{2^{(m+1)k}}^{2^{(m+1)(k+1)}} |\widehat{\lambda}_{r,s}^{(m)}(\xi)|^2 \frac{\mathrm{d}r}{r} \leqslant (m+1)C_s|2^{(m+1)d_s k} L_s(\xi)|^{-1/16d_s(m+1)} \tag{3.14}$$

$$\int_{2^{ak}}^{2^{a(k+1)}} |\widehat{\lambda}_{r,s}^{(m)}(\xi) - \widehat{\lambda}_{r,s-1}^{(m)}(\xi)|^2 \frac{\mathrm{d}r}{r} \leqslant (m+1)C_s|2^{(m+1)d_s k} L_s(\xi)|^{1/2(m+1)}. \tag{3.15}$$

Finally, it is clear that

$$M_{\lambda_s^{(m)},m+1}(f)(x) \leqslant (m+1)C_s \sup_{k\in\mathbb{Z}} \int_{2^k \leqslant |y| < 2^{k+1}} |f(x - \mathcal{P}_s(y))|\,|A_m(y')| \frac{\mathrm{d}y}{|y|^n}. \tag{3.16}$$

Thus by the first inequality in (3.5), (3.16), Hölder's inequality, Proposition 1 from [24] on page 477 (see also [15]), we get

$$\|M_{\lambda_s^{(m)},m+1}(f)\|_p \leqslant (m+1)C_{s,p}\|f\|_p \tag{3.17}$$

for all $1 < p < \infty$. Hence (3.8) follows by (3.9)–(3.11), (3.14)–(3.15), (3.17), and Theorem 2.2. This completes the proof.

4. ADDITIONAL RESULTS

This section is devoted to presenting some additional results that can be obtained by adapting similar argument as those in Sections 2–3.

4.1. Maximal functions and singular integrals along surfaces.

Let Ω and h be as in Section 1. For a suitable function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, we define the singular integral $T_{\Omega,\varphi,h}$ and the maximal function $\mathcal{M}_{\Omega,\varphi}$ by:

$$T_{\Omega,\varphi,h}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \varphi(|y|)y')|y|^{-n} h(|y|)\Omega(y')\mathrm{d}y, \tag{4.1}$$

$$\mathcal{M}_{\Omega,\varphi}(f)(x) = \sup_{h\in\overline{U}(L^2(\mathbb{R}^+,\mathrm{d}r/r))} |T_{\Omega,\varphi,h}(f)(x)|. \tag{4.2}$$

We have the following result concerning the operators $\mathcal{M}_{\Omega,\varphi}$ and $T_{\Omega,\varphi,h}$:

THEOREM 4.1. *If φ is a polynomial and $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$ which satisfies* (1.1), *then:*

(i) $\|\mathcal{M}_{\Omega,\varphi}(f)\|_p \leqslant C_p\|f\|_p$ *for* $2 \leqslant p < \infty$.

(ii) $\|T_{\Omega,\varphi,h}(f)\|_p \leqslant C_p\|f\|_p$ *for* $1 < p < \infty$ *whenever* $h \in L^2(\mathbb{R}_+, \mathrm{d}r/r)$).

Here C_p is a constant that may depend on the degree of the polynomial φ, but it is independent of its coefficients.

A proof of Theorem 4.1(i) can be constructed by following exactly the same argument in the proof of Theorem 1.2. But, the argument here is very much simplified since the linear transformations L_s in Theorem 2.3 are the identity ones, i.e., $L_s(\xi) = \xi$. The proof of Theorem 4.1(ii) follows by exactly the same argument as that for Corollary 1.3.

Similar results for the operators $\mathcal{M}_{\Omega,\varphi}$ and $T_{\Omega,\varphi,h}$ still hold even if φ is not a polynomial. In particular, we have the following:

THEOREM 4.2. *Suppose that* $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$ *and satisfies* (1.1). *If* φ *satisfies the following estimates*

$$|\varphi(t)| \leqslant C_1 t^d, \quad |\varphi''(t)| \leqslant C_2 t^{d-2}, \tag{4.3}$$

$$C_3 t^{d-1} \leqslant |\varphi'(t)| \leqslant C_4 t^{d-1}, \tag{4.4}$$

for some $d \neq 0$ *and* $t \in (0,\infty)$, *where* C_1, C_2, C_3, *and* C_4 *are positive constants independent of* t, *then:*

(i) $\|\mathcal{M}_{\Omega,\varphi}(f)\|_p \leqslant C_p \|f\|_p$ *for* $2 \leqslant p < \infty$.

(ii) $\|T_{\Omega,\varphi,h}(f)\|_p \leqslant C_p \|f\|_p$ *for* $1 < p < \infty$ *whenever* $h \in L^2(\mathbb{R}_+, \mathrm{d}r/r)$).

A proof of Theorem 4.2 (i) can be obtained by first repeating the steps (3.1)–(3.8) in the proof of Theorem 1.2. Then applying Lemma 2.1 with $L(\xi) = \xi$. The estimates required to apply Lemma 2.1 can be easily verified. In fact, the corresponding estimates (ii) and (iii) in Lemma 2.1 follow by an integration by parts and the assumptions (4.3)–(4.4) on φ (see [1]). Finally, the corresponding estimate (iv) follows by the assumptions on φ and simple change of variables.

We remark here that the results above concerning the singular integrals show that the classes of the operators $T_{\Omega,\varphi,h}$, $h \in L^2(\mathbb{R}^+, \mathrm{d}r/r)$ behave quite differently from the class of the classical Calderón-Zygmund singular integral operators.

4.2. INTEGRAL OPERATORS OF MARCINKIEWICZ TYPE. Let Ω be as above. For a suitable mapping $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ and $\rho > 0$, we define the operator $\mu^\rho_{\Omega,\Phi}$ by

$$\mu^\rho_{\Omega,\Phi} f(x) = \Big(\int_{-\infty}^{\infty} \Big| 2^{-\rho t} \int_{|y|\leqslant 2^t} f(x - \Phi(y)) |y|^{-n+\rho} \Omega(y) \mathrm{d}y \Big|^2 \mathrm{d}t \Big)^{1/2}. \tag{4.5}$$

If $n = d$ and $\Phi(y) = (y_1, y_2, \ldots, y_d)$, we shall simply denote the operator $\mu^\rho_{\Omega,\Phi}$ by μ^ρ_Ω.

The operator μ^ρ_Ω is the well known parametric Marcinkiewicz integral operator introduced by Hörmander [19]. On the other hand, the operator μ^1_Ω is the Marcinkiewicz integral operator introduced by E.M. Stein [21]. For a background information about Marcinkiewicz integral operators, we refer the reader to consult [3], [4], [2], [10], [13], [19], [21], and the references therein.

By simple change of variables and duality, it can be easily seen that

$$\mu^{\rho}_{\Omega,\Phi} f(x) \leqslant C(\rho)\mathcal{M}_{\Omega,\Phi}(f)(x). \tag{4.6}$$

Therefore, by Theorem 1.2 and the inequality (4.6), we immediately obtain the following:

THEOREM 4.3. *Suppose that* $\rho > 0$ *and that* $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$ *satisfying* (1.1)*. Let* $\mathcal{P}$ *be as in Theorem* 1.2*. Then* $\|\mu^{\rho}_{\Omega,\mathcal{P}}(f)\|_p \leqslant C_p\|f\|_p$ *for* $2 \leqslant p < \infty$ *with* L^p *bounds independent of the coefficients of the polynomial components of* $\mathcal{P}$*.*

It should be pointed out here that in the case of the classical Marcinkiewicz integral operator μ^1_{Ω}, the result in Theorem 4.3 was obtained in [4]. On the other hand, Ding–Lu–Yabuta established L^2 boundedness of μ^{ρ}_{Ω} under the condition that $\Omega \in L(\log L)(\mathbb{S}^{n-1})$ [13]. Therefore, Theorem 4.3 is a proper extension of the results in [4], [13] in the range $2 \leqslant p < \infty$.

Also, by the inequality (4.6) and Theorems 4.1–4.2, we have the following improvement of the corresponding results in [4] and [13] in the range $2 < p < \infty$:

THEOREM 4.4. *Suppose that* $\rho > 0$ *and that* $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$ *satisfying* (1.1)*. If* $\Phi(y) = \varphi(|y|)y'$ *where* φ *is as in Theorem* 4.1 *or Theorem* 4.2*, then* $\|\mu^{\rho}_{\Omega,\Phi}(f)\|_p \leqslant C_p\|f\|_p$ *for* $2 \leqslant p < \infty$*. Moreover, if* φ *is as in Theorem* 4.1*, then the* L^p *bounds are independent of the coefficients of* φ*.*

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