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WEIGHTED INEQUALITIES INVOLVING TWO HARDY OPERATORS WITH APPLICATIONS TO EMBEDDINGS OF FUNCTION SPACES

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ABSTRACT. We find necessary and sufficient conditions for the two-operator weighted inequality $\Big(\int\limits_0^\infty\Big(\frac{1}{t}\int\limits_0^t f(s)\mathrm{d}s\Big)^q w(t)\mathrm{d}t\Big)^{1/q}\leqslant C\Big(\int\limits_0^\infty\Big(\int\limits_t^\infty \frac{f(s)}{s}\mathrm{d}s\Big)^p v(t)\mathrm{d}t\Big)^{1/p}$. We use this inequality to study embedding properties between the function spaces $S^p(u)$ equipped with the norm $\|f\|_{S^p(u)}=\Big(\int\limits_0^\infty [f^{**}(t)-f^*(t)]^p u(t)\mathrm{d}t\Big)^{1/p}$ and the classical Lorentz spaces $\Lambda^p(v)$ and $\Gamma^q(w)$. Moreover, we solve the only missing open case of the embedding $\Lambda^p(v)\hookrightarrow\Gamma^q(w)$, where $0<q<p\leqslant 1$.

KEYWORDS: *Two-operator weighted inequality, spaces $S^p(w)$, classical Lorentz spaces, average operator, dual average operator, embeddings, weighted inequalities.*

MSC (2000): 26D10, 46E20.

1. PROLOGUE

We study function spaces whose norms are defined in terms of the functional $f^{**}-f^*$, where f^* is the *non-increasing rearrangement* of a measurable function f on $(0,\infty)$, defined by

$$f^*(t)=\inf\{s>0 : f_*(s)\leqslant t\},\quad t\in[0,\infty),$$

with $f_*(t)=\mu(\{x\in(0,\infty) : |f(x)|>t\})$, $t>0$ being the *distribution function* of f, and $f^{**}(t)=t^{-1}\int\limits_0^t f^*(s)\,\mathrm{d}s$.

The functional $f^{**}-f^*$ has been shown to be useful in various parts of analysis including the Interpolation Theory (see [7] for some history and references). In [3], the functional

$$\Big(\int\limits_0^\infty t^{1/p}[f^{**}(t)-f^*(t)]^q\frac{\mathrm{d}t}{t}\Big)^{1/q}$$

was introduced (for $1 < p < \infty$ and $0 < q \leqslant \infty$) and interesting applications were found. In particular, the celebrated Weak-L^∞ space, determined by the seminorm $\sup\limits_{t\in(0,\infty)} (f^{**}(t) - f^*(t))$ (corresponding formally to the case $q = \infty$) was created and proved useful in situations in which the classical L^∞ fails.

It should be noticed that in the study of function spaces defined in terms of the functional $f^{**} - f^*$ certain care must be exercised. In particular, this functional vanishes on constant functions and, moreover, the operation $f \to f^{**} - f^*$ is not subadditive. Therefore, quantities involving $f^{**} - f^*$ do not have norm properties, which makes the study of the corresponding function spaces difficult.

Recently, various structures involving the quantity $f^{**} - f^*$ appear quite regularly as natural function spaces in various situations. For example, they play an important role in the problem of characterization of the optimal partner norms in Sobolev-type embeddings [17], [20], [2], in the duality problem for classical Lorentz spaces of type Γ [25], in the investigation of the boundedness of maximal Calderón-Zygmund singular integral operators on classical Lorentz spaces [5], and so on. For more detailed history and more references we refer the reader to our previous paper [7].

The main object of study in this paper will be the fairly general class of weighted function spaces denoted by $S^p(v)$, which was introduced in [7].

Let $0 < p < \infty$ and let v be a weight on $(0,\infty)$, that is, a measurable non-negative function. Then, the space $S^p(v)$ is the collection of all measurable functions on $(0,\infty)$ such that $\|f\|_{S^p(v)} < \infty$, where

$$\|f\|_{S^p(v)} := \Big(\int_0^\infty [f^{**}(t) - f^*(t)]^p v(t)\,\mathrm{d}t\Big)^{1/p}.$$

As already noted above, the functional $\|f\|_{S^p(v)}$ is not a norm because it vanishes on constant functions. To overcome this problem, one can either factor out constants or assume that $f^*(\infty) = 0$. Even then, however, it is not necessarily a norm. It is therefore desirable, first, to know when $\|f\|_{S^p(v)}$ is at least equivalent to a norm, and, second, to carry out a thorough research of relations of the spaces $S^p(v)$ to other, more familiar function spaces.

In [7] we studied some basic properties of the spaces $S^p(v)$ such as linearity, the lattice property and normability, and we also characterized their associate spaces. In the present paper we shall concentrate on their embedding relations.

Our principal objective is to study embedding relations between the spaces $S^p(v)$ and the so-called *classical Lorentz spaces* of type Λ and Γ.

(Recall that if two spaces X, Y are endowed with certain appropriate norm-like functionals $\|\cdot\|_X$, $\|\cdot\|_Y$, then we say that X *is continuously embedded into* Y, written $X \hookrightarrow Y$, if $X \subset Y$ in the set-theoretic sense and moreover $\|f\|_Y \leqslant C\|f\|_X$ for some $C > 0$ and all functions $f \in X$.)

The classical Lorentz spaces $\Lambda^p(v)$ and $\Gamma^p(v)$ are related to the spaces $S^p(v)$, but their norms involve f^* or f^{**} rather than $f^{**} - f^*$. More precisely, $\Lambda^p(v)$ and $\Gamma^p(v)$ are families of all measurable functions such that $\|f\|_{\Lambda^p(v)} < \infty$ or $\|f\|_{\Gamma^p(v)} < \infty$, respectively, where

$$\|f\|_{\Lambda^p(v)} := \Big(\int_0^\infty f^*(t)^p v(t)\mathrm{d}t\Big)^{1/p} < \infty \quad \text{and} \quad \|f\|_{\Gamma^p(v)} := \Big(\int_0^\infty f^{**}(t)^p v(t)\mathrm{d}t\Big)^{1/p} < \infty.$$

The spaces $\Lambda^p(v)$ were introduced by Lorentz in 1951 in [18]. The spaces $\Gamma^p(v)$ with $0 < p < \infty$ were first defined by Sawyer in [23].

Again, there is a plenty of motivation for such research. For example, the well-known inequality (cf. [4])

$$t^{-1/n}(f^{**}(t) - f^*(t)) \leqslant C(\nabla f)^{**}(t),$$

which is valid for every smooth f and every $t > 0$, leads, for $p > 1$, to

$$\|t^{-1/n}(f^{**}(t) - f^*(t))\|_p \leqslant C\|\nabla f(t)\|_p,$$

which can be regarded as an embedding of certain Sobolev space into the space $S^p(t^{-p/n})$. Since quite a lot is known about the embeddings of Sobolev spaces into other types of spaces including Lorentz spaces and classical Lorentz spaces, the knowledge about relations between the spaces $S^p(v)$ and other function spaces would be quite useful.

As we shall see, the characterization of one of the embeddings below in this paper requires a necessary and sufficient condition for the two-operator weighted norm inequality

$$\|Ph\|_{L^q(w)} \lesssim \|Qh\|_{L^p(v)} \tag{1.1}$$

for every positive measurable h on $(0,\infty)$, where P is the *average Hardy operator*,

$$(Ph)(t) := \frac{1}{t}\int_0^t h(s)\,\mathrm{d}s, \quad h \geqslant 0, \tag{1.2}$$

and Q is its dual operator (under the pairing $\int_0^\infty fg$),

$$(Qh)(t) := \int_t^\infty \frac{h(s)}{s}\,\mathrm{d}s, \quad h \geqslant 0. \tag{1.3}$$

The inequality (1.1) is clearly of independent interest, and it will have many other important consequences apart from the one we have in mind. It has not been studied in the required generality; apparently the only result available in literature seems to be its characterization when $1 < p = q < \infty$ and $v = w$ due to Neugebauer [21].

The study of the inequality (1.1) constitutes the second main goal of this paper.

In fact, we will deal with this second problem first. In Section 2, we give necessary and sufficient conditions for (1.1) to hold. In order not to make this paper too long, we restrict ourselves to the case $1 \leqslant q \leqslant \infty$ (the technical reason for this restriction is the use of duality in our methods). Other cases have to be treated in a different way and we will study them in a future paper.

Since 1990, an enormous effort has been spent by many authors in the hunt of a characterization of embeddings between classical Lorentz spaces of both types Λ and Γ. Thanks to this extensive research powerful new techniques and methods were developed, and most of the desired characterizations have been obtained ([1], [23], [28], [8], [9], [24], [27], [6], [15], [10], [11], [25], [26], [13]). In fact, at this moment, there is only one "missing case" left open, namely the embedding of $\Lambda^p(v)$ into $\Gamma^q(w)$ with $0 < q < p \leqslant 1$, whose characterization has not been known (for certain related results see [14]).

The third main goal of this paper is to solve this open problem. Namely, in Section 3 we establish necessary and sufficient conditions on weights v, w such that the inequality

$$\Big(\int_0^\infty f^{**}(t)^q w(t)\,\mathrm{d}t\Big)^{1/q} \lesssim \Big(\int_0^\infty f^*(t)^p v(t)\,\mathrm{d}t\Big)^{1/p}$$

holds when $0 < q < p \leqslant 1$. This result will be then found useful in the proofs of our other main results.

Having the theory of embeddings of the spaces of type Λ and Γ complete, it is time for the spaces of type S to come into the play. The rest of the paper is devoted to the study of embeddings between the spaces of type S, Λ and Γ. However, as mentioned above, the functional $f^{**}(t) - f^*(t)$ is zero when f is constant on $(0, \infty)$. Hence, constant functions always belong to any $S^p(v)$ whereas they do not necessarily belong to analogous spaces of type Λ and Γ. For this reason, we will study the appropriate embeddings restricted to the set

$$\mathbb{A} = \Big\{f \in \mathcal{M}_+(0,\infty) : \lim_{t\to\infty} f^*(t) = 0\Big\}$$

(here and in the sequel, we denote by $\mathcal{M}(0,\infty)$ the family of all measurable functions on $(0,\infty)$, and by $\mathcal{M}_+(0,\infty)$ the set of nonnegative functions from $\mathcal{M}(0,\infty)$). We will denote this restriction by writing, for example,

$$S^p(v) \hookrightarrow \Lambda^q(w), \quad f \in \mathbb{A},$$

meaning

$$\Big(\int_0^\infty f^*(t)^q w(t)\,\mathrm{d}t\Big)^{1/q} \leqslant C\Big(\int_0^\infty (f^{**}(t) - f^*(t))^p v(t)\,\mathrm{d}t\Big)^{1/p} \quad \text{for all } f \in \mathbb{A},$$

and so on.

In Section 4 we observe some general relations between the three types of spaces. In Section 5 we characterize pairs of weights v, w such that the embedding $S^p(v) \hookrightarrow S^q(w)$ holds. In Section 6 we give necessary and sufficient conditions for embeddings $S^p(v) \hookrightarrow \Gamma^q(w)$ and $\Gamma^p(v) \hookrightarrow S^q(w)$. Finally, in Section 7 we study the analogous problem with Γ replaced by Λ. The results of this last section lead to the inequality (1.1) and therefore are, naturally, restricted to $1 \leqslant q < \infty$.

Let us recall that one of the principal achievements of [7] was the introduction of the operator T, which is defined for every positive non-increasing function f on $(0,\infty)$ such that $\lim_{t\to\infty} f(t) = 0$ by

$$(Tf)(t) := \frac{1}{t}((Pf)(1/t) - f(1/t)). \tag{1.4}$$

As observed in [7],

$$T \circ T = \mathrm{id} \tag{1.5}$$

and, for $f \in \mathbb{A}$,

$$f^{**}(t) - f^*(t) = \frac{1}{t}(Tf^*)(1/t). \tag{1.6}$$

Note that, for $f \in \mathbb{A}$, Tf^* is non-increasing, and also that given $p \in (0,\infty)$, we have

$$\|f\|_{S^p(v)} = \|Tf^*\|_{\Lambda^p(\widetilde{v}_p)}. \tag{1.7}$$

Here, and throughout the paper, we denote

$$\widetilde{v}_p(t) := v(1/t)t^{p-2} \quad \text{and} \quad \widetilde{w}_q(t) := w(1/t)t^{q-2}. \tag{1.8}$$

We also write

$$V(t) = \int_0^t v(s)\,\mathrm{d}s, \quad W(t) = \int_0^t w(s)\,\mathrm{d}s.$$

For the benefit of the reader, the constants in inequalities are denoted as $A_{(5.3)}$, $A_{(5.4)}$ and so on. The subscript indicates the label of the formula in which the corresponding constant is introduced.

Constants, whose precise value is immaterial, are throughout the paper denoted by C. We write $A \lesssim B$ if $A \leqslant CB$, where C does not depend on appropriate quantities in A and B; if both $A \lesssim B$ and $B \lesssim A$ are true, we write $A \approx B$.

When $1 \leqslant p \leqslant \infty$, we set

$$p' = \begin{cases} \infty & \text{if } p = 1, \\ \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ 1 & \text{if } p = \infty. \end{cases}$$

2. WEIGHTED NORM INEQUALITIES INVOLVING TWO HARDY-TYPE OPERATORS

In this section we study the inequality (1.1). Our ultimate aim is to characterize the quantity

$$A := \sup_{h \geqslant 0} \frac{\|Ph\|_{L^q(w)}}{\|Qh\|_{L^p(v)}}, \tag{2.1}$$

where $0 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, v, w are weights on $(0,\infty)$ and

$$\|f\|_{L^p(v)} := \begin{cases} \Big(\int\limits_0^\infty |f(x)|^p v(x)\,\mathrm{d}x\Big)^{1/p} & \text{if } 0 < p < \infty, \\ \operatorname*{ess\,sup}_{x\in(0,\infty)} |f(x)|v(x) & \text{if } p = \infty. \end{cases}$$

We start with recalling a useful inequality, which is just a particular case of the general result in Theorem 3.2 of [9] (cf. also [27]). Let $0 < p \leqslant 1$ and let v be a weight on $(0,\infty)$. Then,

$$\Big(\int\limits_0^t f^*(s)^{1/p} v(s)\,\mathrm{d}s\Big)^p \leqslant p \int\limits_0^t f^*(s) V(s)^{p-1} v(s)\,\mathrm{d}s, \quad t \in (0,\infty). \tag{2.2}$$

We will now find a necessary and sufficient condition for the reverse Hardy inequality involving the operator Q. Given a function $f \in \mathcal{M}_+(0,\infty)$, a weight v and $p \in (0,\infty]$, we define

$$B(f) := \sup_{g \geqslant 0} \frac{\int_0^\infty f(t)g(t)\,\mathrm{d}t}{\|Qg\|_{L^p(v)}}. \tag{2.3}$$

THEOREM 2.1. *Let f be a non-negative measurable function on $(0,\infty)$. Let v be a weight on $(0,\infty)$ and let $B(f)$ be given by* (2.3)*.*

(i) *Assume that $1 < p < \infty$. Then*

$$B(f) \approx \Big(\int\limits_0^\infty \Big[\operatorname*{ess\,sup}_{0<s\leqslant t} sf(s)\Big]^{p'} \frac{v(t)}{V^{p'}(t)}\,\mathrm{d}t\Big)^{1/p'} + \frac{\operatorname{ess\,sup}_{0<s<\infty} sf(s)}{V(\infty)^{1/p}}. \tag{2.4}$$

(ii) *Assume that $0 < p \leqslant 1$. Then $B(f) \approx \operatorname*{ess\,sup}_{0<s<\infty} \frac{sf(s)}{V(s)^{1/p}}$.*

(iii) *Assume that $p = \infty$. Then*

$$B(f) \approx \int\limits_0^\infty \operatorname*{ess\,sup}_{0<s\leqslant t} sf(s)\,\mathrm{d}\Big(\frac{-1}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)}\Big) + \frac{\operatorname{ess\,sup}_{0<s\leqslant\infty} sf(s)}{\operatorname{ess\,sup}_{0<s\leqslant\infty} v(s)}.$$

Proof. (i) The assertion is a simple modification of a recent result of Sinnamon ([26], Corollary 3.8) (for various related results see also [16]).

(ii) We recall the inequality (cf. [9], [27] and [28])

$$\int_0^\infty h^*(t)V(t)^{1/p-1}v(t)\,\mathrm{d}t \lesssim \Big(\int_0^\infty h^*(t)^p v(t)\,\mathrm{d}t\Big)^{1/p}, \quad h \in \mathcal{M}_+(0,\infty). \tag{2.5}$$

Using (2.5) and the Fubini theorem,

$$\begin{aligned} B(f) &\leqslant \operatorname*{ess\,sup}_{0<t<\infty} \frac{tf(t)}{V(t)^{1/p}} \sup_{g\geqslant 0} \frac{\int_0^\infty \frac{g(s)}{s} V(s)^{1/p}\,\mathrm{d}s}{\|Qg\|_{L^p(v)}} \\ &\approx \operatorname*{ess\,sup}_{0<t<\infty} \frac{tf(t)}{V(t)^{1/p}} \sup_{g\geqslant 0} \frac{\int_0^\infty \frac{g(s)}{s} \int_0^s V(y)^{1/p-1}v(y)\,\mathrm{d}y\,\mathrm{d}s}{\|Qg\|_{L^p(v)}} \\ &= \operatorname*{ess\,sup}_{0<t<\infty} \frac{tf(t)}{V(t)^{1/p}} \sup_{g\geqslant 0} \frac{\int_0^\infty (Qg)(s)V(s)^{1/p-1}v(s)\,\mathrm{d}s}{\|Qg\|_{L^p(v)}} \lesssim \operatorname*{ess\,sup}_{0<t<\infty} \frac{tf(t)}{V(t)^{1/p}}. \end{aligned}$$

(The last inequality is (2.5) applied to $h^* = Qg$.)

Conversely, given $\varepsilon > 0$ and $x \in (0,\infty)$, set

$$g_{\varepsilon,x}(t) := \frac{t}{\varepsilon}\chi_{(x-\varepsilon,x)}(t), \quad t \in (0,\infty).$$

Then, $(Qg_{\varepsilon,x})(t) \leqslant \chi_{(0,x)}(t)$, whence we have the following proving the claim:

$$B(f) \geqslant \sup_{x,\varepsilon} \frac{\int_0^\infty f(t)g_{\varepsilon,x}(t)\,\mathrm{d}t}{\|Qg_{\varepsilon,x}\|_{L^p(v)}} \geqslant \sup_{x,\varepsilon} \frac{\frac{1}{\varepsilon}\int_{x-\varepsilon}^x tf(t)\,\mathrm{d}t}{\|\chi_{(0,x)}\|_{L^p(v)}} \geqslant \operatorname*{ess\,sup}_{0<x<\infty} \frac{xf(x)}{V(x)^{1/p}}.$$

(iii) Let $p = \infty$. We claim that then

$$B(f) = \sup_{g\geqslant 0} \frac{\int_0^\infty g(t)\operatorname{ess\,sup}_{0<s\leqslant t} sf(s)\,\mathrm{d}t}{\operatorname{ess\,sup}_{0<t<\infty} v(t)\int_t^\infty g(s)\,\mathrm{d}s}. \tag{2.6}$$

By Theorem 2.1, (2.3) of [26], we have

$$\begin{aligned} \sup_{g\geqslant 0} \frac{\int_0^\infty g(t)\operatorname{ess\,sup}_{0<s\leqslant t} sf(s)\,\mathrm{d}t}{\operatorname{ess\,sup}_{0<t<\infty} v(t)\int_t^\infty g(s)\,\mathrm{d}s} &= \sup_{g\geqslant 0}\ \sup_{\int_t^\infty h\leqslant \int_t^\infty g} \frac{\int_0^\infty h(t)tf(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{0<t<\infty} v(t)\int_t^\infty g(s)\,\mathrm{d}s} \\ &\leqslant \sup_{g\geqslant 0}\ \sup_{\int_t^\infty h\leqslant \int_t^\infty g} \frac{\int_0^\infty h(t)tf(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{0<t<\infty} v(t)\int_t^\infty h(s)\,\mathrm{d}s} \\ &\leqslant \sup_{h\geqslant 0} \frac{\int_0^\infty h(t)tf(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{0<t<\infty} v(t)\int_t^\infty h(s)\,\mathrm{d}s}. \end{aligned}$$

Since the converse inequality is trivial, this proves (2.6).

We now claim that

$$\int_0^\infty g(t)\operatorname*{ess\,sup}_{0<s\leqslant t} sf(s)\,\mathrm{d}t = \sup_{\int_0^t h\leqslant \operatorname{ess\,sup}_{0<s\leqslant t} sf(s)} \int_0^\infty g(t)\int_0^t h(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Indeed, we first observe that the inequality "$\geqslant$" is obvious. To get the converse one, note that, for every non-decreasing function Φ on $(0,\infty)$, there is a sequence $\{H_n\}_{n=1}^{\infty}$ of smooth increasing functions such that $H_n \nearrow \Phi$ as $n \to \infty$. By the Fatou Lemma,

$$\int_0^\infty g(t)\Phi(t)\,\mathrm{d}t \leqslant \limsup_{n\to\infty} \int_0^\infty g(t)H_n(t)\,\mathrm{d}t.$$

Applying this to the (non-decreasing) function $\Phi(t) = \operatorname*{ess\,sup}_{0<s\leqslant t} sf(s)$ and noting that the functions H_n, being smooth, can be represented as $H_n(t) = \int_0^t h_n(s)\,\mathrm{d}s$ for some positive measurable functions h_n on $(0,\infty)$, we obtain

$$\int_0^\infty g(t) \operatorname*{ess\,sup}_{0<s\leqslant t} sf(s)\,\mathrm{d}t \leqslant \sup_{\int_0^t h \leqslant \operatorname{ess\,sup}_{0<s\leqslant t} sf(s)} \int_0^\infty g(t) \int_0^t h(s)\,\mathrm{d}s\,\mathrm{d}t,$$

proving our claim. Thus, we have

$$\begin{aligned}\int_0^\infty g(t) \operatorname*{ess\,sup}_{0<s\leqslant t} sf(s)\,\mathrm{d}t &= \sup_{\int_0^t h \leqslant \operatorname{ess\,sup}_{0<s\leqslant t} sf(s)} \int_0^\infty g(t) \int_0^t h(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= \sup_{\int_0^t h \leqslant \operatorname{ess\,sup}_{0<s\leqslant t} sf(s)} \int_0^\infty h(t) \int_t^\infty g(s)\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Inserting this into (2.6), we have

$$B(f) = \sup_{g\geqslant 0} \sup_{\int_0^t h \leqslant \operatorname{ess\,sup}_{0<s\leqslant t} sf(s)} \frac{\int_0^\infty h(t) \int_t^\infty g(s)\,\mathrm{d}s\,\mathrm{d}t}{\operatorname{ess\,sup}_{0<t<\infty} v(t) \int_t^\infty g(s)\,\mathrm{d}s}.$$

Thus, by the monotonicity of $\int_t^\infty g(s)\,\mathrm{d}s$, the Fubini theorem, and Theorem 3.3 or Theorem 9.1 (ii) of [11], we have

$$\begin{aligned}B(f) &= \sup_{\int_0^t h \leqslant \operatorname{ess\,sup}_{0<s\leqslant t} sf(s)} \sup_{k\in\mathcal{M}_+(0,\infty)} \frac{\int_0^\infty h(t)k^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{0<t<\infty} v(t)k^*(t)} \\ &= \sup_{\int_0^t h \leqslant \operatorname{ess\,sup}_{0<s\leqslant t} sf(s)} \sup_{k\in\mathcal{M}_+(0,\infty)} \frac{\int_0^\infty h(t)k^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{0<t<\infty} (\operatorname{ess\,sup}_{0<s\leqslant t} v(s))k^*(t)} \\ &= \sup_{\int_0^t h \leqslant \operatorname{ess\,sup}_{0<s\leqslant t} sf(s)} \int_0^\infty \frac{h(t)}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)}\,\mathrm{d}t.\end{aligned}$$

Finally, the integration by parts for the Stieltjes integral and the Fatou lemma give

$$B(f) = \sup_{\int_0^t h \leqslant \operatorname{ess\,sup}_{0<s\leqslant t} sf(s)} \Big[\int_0^\infty \Big(\int_0^t h(s) \mathrm{d}s \Big) \mathrm{d} \Big(\frac{-1}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)} \Big) + \frac{\int_0^\infty h(t) \mathrm{d}t}{\operatorname{ess\,sup}_{0<s<\infty} v(s)} \Big]$$
$$= \Big[\int_0^\infty \operatorname*{ess\,sup}_{0<s\leqslant t} sf(s) \, \mathrm{d} \Big(\frac{-1}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)} \Big) + \frac{\operatorname{ess\,sup}_{0<s<\infty} sf(s)}{\operatorname{ess\,sup}_{0<s<\infty} v(s)} \Big].$$

Of course, the last summand disappears if $\operatorname*{ess\,sup}_{0<s<\infty} v(s) = \infty$. ∎

Given weights u, v and $t \in (0, \infty)$, we define

$$\overline{u}(t) := \operatorname*{ess\,sup}_{0<s<t} u(s) \quad \text{and} \quad \sigma_p(t) := \begin{cases} \Big(\int_t^\infty \frac{v(s)^{1-p'}}{s^{p'}} \, \mathrm{d}s \Big)^{1/p'} & \text{if} \quad p > 1, \\ \operatorname*{ess\,sup}_{t\leqslant s\leqslant\infty} \frac{1}{sv(s)} & \text{if} \quad p = 1. \end{cases}$$

PROPOSITION 2.2. *Let $0 < p < \infty$, $1 \leqslant q < \infty$, $0 < \alpha < \infty$ and let v, w be weights on $(0, \infty)$. If $q < p$, set $r = \frac{pq}{p-q}$. Then the inequality*

$$\Big(\int_0^\infty \Big[\sup_{0<s\leqslant t} s^\alpha (Qg)(s) \Big]^q w(t) \, \mathrm{d}t \Big)^{1/q} \lesssim \Big(\int_0^\infty g(t)^p v(t) \, \mathrm{d}t \Big)^{1/p} \tag{2.7}$$

holds for all $g \geqslant 0$ if and only if one of the following conditions holds:

(i) $1 \leqslant p \leqslant q < \infty$ *and*

$$A_{(2.8)} := \sup_{0<t<\infty} \Big(t^{\alpha q} \int_t^\infty w(s) \, \mathrm{d}s + \int_0^t s^{\alpha q} w(s) \, \mathrm{d}s \Big)^{1/q} \sigma_p(t) < \infty; \tag{2.8}$$

(ii) $0 < q < p$, $1 \leqslant p < \infty$,

$$A_{(2.9)} := \Big(\int_0^\infty \Big(\int_0^t s^{\alpha q} w(s) \, \mathrm{d}s \Big)^{r/q} w(t) t^{\alpha q} [\sigma_p(t)]^r \, \mathrm{d}t \Big)^{1/r} < \infty, \tag{2.9}$$

$$A_{(2.10)} := \Big(\int_0^\infty \Big(\int_t^\infty w(s) \, \mathrm{d}s \Big)^{r/p} w(t) \sup_{0<s\leqslant t} \big(s^{\alpha r} [\sigma_p(s)]^r \big) \, \mathrm{d}t \Big)^{1/r} < \infty. \tag{2.10}$$

Moreover, the best constant in (2.7) *is comparable to $A_{(2.8)}$ in the case* (i) *and to $A_{(2.9)} + A_{(2.10)}$ in the case* (ii).

Proof. We will prove both assertions for the case when $\alpha = 1$ (the general case can be then obtained by a simple change of variables).

(i) Assume that $1 \leqslant p \leqslant q < \infty$. Then, by a consecutive change of variables,

$$\begin{aligned}\int_0^\infty \Big[\sup_{0<s\leqslant t} s(Qg)(s)\Big]^q w(t)\,\mathrm{d}t &= \int_0^\infty \Big[\sup_{0<s\leqslant t} s\int_0^{1/s} \frac{g(1/y)}{y}\,\mathrm{d}y\Big]^q w(t)\,\mathrm{d}t\\ &= \int_0^\infty \Big[\sup_{1/t\leqslant s<\infty} \frac{1}{s}\int_0^{s} \frac{g(1/y)}{y}\,\mathrm{d}y\Big]^q w(t)\,\mathrm{d}t\\ &= \int_0^\infty \Big[\sup_{t\leqslant s<\infty} \frac{1}{s}\int_0^{s} \frac{g(1/y)}{y}\,\mathrm{d}y\Big]^q \frac{w(1/t)}{t^2}\,\mathrm{d}t,\end{aligned}$$

while

$$\int_0^\infty g(t)^p v(t)\,\mathrm{d}t = \int_0^\infty g(1/t)^p t^{-p}\widetilde{v}_p(t)\,\mathrm{d}t.$$

Thus, putting $h(y) := g(1/y)\frac{1}{y}$, we obtain that (2.7) is equivalent to

$$\Big(\int_0^\infty \Big[\sup_{t\leqslant s<\infty} \frac{1}{s}\int_0^{s} h(y)\,\mathrm{d}y\Big]^q \frac{w(1/t)}{t^2}\,\mathrm{d}t\Big)^{1/q} \lesssim \Big(\int_0^\infty h(t)^p\widetilde{v}_p(t)\,\mathrm{d}t\Big)^{1/p}. \tag{2.11}$$

Then, by Theorem 4.1 of [12], (2.11) holds if and only if

$$\sup_{0<t<\infty}\Big(\frac{1}{t^q}\int_0^t \frac{w(1/s)}{s^2}\,\mathrm{d}s + \int_t^\infty \frac{w(1/s)}{s^{q+2}}\,\mathrm{d}s\Big)^{1/q}\sigma_p(1/t) < \infty. \tag{2.12}$$

By calculation, we have, for every $t \in (0,\infty)$,

$$\int_0^t \frac{w(1/s)}{s^2}\,\mathrm{d}s = \int_{1/t}^\infty w(s)\,\mathrm{d}s, \tag{2.13}$$

$$\int_t^\infty \frac{w(1/s)}{s^{q+2}}\,\mathrm{d}s = \int_0^{1/t} s^q w(s)\,\mathrm{d}s. \tag{2.14}$$

Altogether, (2.7) holds if and only if

$$\sup_{0<t<\infty}\Big(\frac{1}{t^q}\int_{1/t}^\infty w(s)\,\mathrm{d}s + \int_0^{1/t} s^q w(s)\,\mathrm{d}s\Big)^{1/q}\sigma_p(1/t) < \infty,$$

which is clearly equivalent to (2.8). The best-constant relation follows from the argument.

(ii) Again, (2.7) is equivalent to (2.11). By Theorem 4.4 of [12], (2.11) holds if and only if

$$\Big(\int_0^\infty \Big(\int_t^\infty \frac{w(1/s)}{s^{q+2}}\,\mathrm{d}s\Big)^{r/p} \sigma_p(1/t)^r \frac{w(1/t)}{t^{q+2}}\,\mathrm{d}t\Big)^{1/r} < \infty, \tag{2.15}$$

$$\Big(\int_0^\infty \Big(\int_0^t \frac{w(1/s)}{s^2}\,\mathrm{d}s\Big)^{r/p} \operatorname*{ess\,sup}_{t\leqslant s<\infty} s^{-q}\sigma_p(1/s)^r \frac{w(1/t)}{t^2}\,\mathrm{d}t\Big)^{1/r} < \infty. \tag{2.16}$$

Using (2.13)–(2.14) and changing variables, we get (2.15) and (2.16) equivalent to (2.9) and (2.10), respectively. ∎

We are now in a position to characterize (1.1).

THEOREM 2.3. *Let* $0 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. *When* $0 < p < q < \infty$, *we set* $r = \frac{pq}{p-q}$. *Let* v, w *be weights on* $(0,\infty)$. *Then* (1.1) *holds if and only if one of the following conditions holds:*

(i) $1 < p \leqslant q < \infty$ *and* $\displaystyle\sup_{0<t<\infty} \Big(\int_t^\infty \frac{w(s)}{s^q}\,\mathrm{d}s\Big)^{1/q} \Big(\int_0^t \frac{s^{p'}v(s)}{V(s)^{p'}}\,\mathrm{d}s + \frac{t^{p'}}{V(t)^{p'-1}}\Big)^{1/p'} < \infty;$

(ii) $1 < p < \infty$, $q = \infty$ *and* $\displaystyle\sup_{0<t<\infty} \Big(\int_0^t \frac{s^{p'}v(s)}{V(s)^{p'}}\,\mathrm{d}s + \frac{t^{p'}}{V(t)^{p'-1}}\Big)^{1/p'} \operatorname*{ess\,sup}_{t\leqslant s<\infty} \frac{w(s)}{s} < \infty;$

(iii) $p=q=\infty$ *and* $\displaystyle\sup_{0<t<\infty}\Big(\frac{t}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)} + \int_0^t s\,\mathrm{d}\Big(\frac{-1}{\operatorname{ess\,sup}_{0<y\leqslant s} v(y)}\Big)\Big) \operatorname*{ess\,sup}_{t\leqslant s<\infty} \frac{w(s)}{s} < \infty;$

(iv) $1 < q < p < \infty$,

$$\Big(\int_0^\infty \Big(\sup_{0<s\leqslant t} s^q \int_s^\infty \frac{w(y)}{y^q}\,\mathrm{d}y\Big)^{r/q} \frac{v(t)}{V(t)^{r/q}}\,\mathrm{d}t\Big)^{1/r} < \infty, \quad \frac{\sup_{0<t<\infty} t\big(\int_t^\infty \frac{w(s)}{s^q}\,\mathrm{d}s\big)^{1/q}}{V(\infty)^{1/p}} < \infty,$$

and $$\Big(\int_0^\infty \Big(\int_0^t \frac{s^{p'}v(s)}{V(s)^{p'}}\,\mathrm{d}s\Big)^{r/p'} \frac{t^{p'}v(t)}{V(t)^{p'}} \Big(\int_t^\infty \frac{w(s)}{s^q}\,\mathrm{d}s\Big)^{r/q}\,\mathrm{d}t\Big)^{1/r} < \infty;$$

(v) $p = \infty$, $1 < q < \infty$,

$$\int_0^\infty \Big(\int_0^t s\,\mathrm{d}\Big(\frac{-1}{\operatorname{ess\,sup}_{0<y\leqslant s} v(y)}\Big)\Big)^q t\Big(\int_t^\infty \frac{w(s)}{s^q}\,\mathrm{d}s\Big)^{1/q} \mathrm{d}\Big(\frac{-1}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)}\Big) < \infty,$$

$$\frac{\operatorname{ess\,sup}_{0<s<\infty} w(s)}{\operatorname{ess\,sup}_{0<s<\infty} v(s)} < \infty, \text{ and}$$

$$\int_0^\infty \Big(\int_t^\infty \mathrm{d}\Big(\frac{-1}{\operatorname{ess\,sup}_{0<y\leqslant s} v(y)}\Big)\Big)^{q-1} \sup_{0<s\leqslant t} s\Big(\int_s^\infty \frac{w(y)}{y^q}\mathrm{d}y\Big)^{1/q} \mathrm{d}\Big(\frac{-1}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)}\Big) < \infty;$$

(vi) $0 < p \leqslant 1 < q < \infty$ *and* $\sup\limits_{0<t<\infty} \frac{t}{V(t)^{1/p}} \Big(\int\limits_t^\infty \frac{w(s)}{s^q}\, ds \Big)^{1/q} < \infty$;

(vii) $0 < p \leqslant 1$, $q = \infty$ *and* $\sup\limits_{0<t<\infty} \Big(\sup\limits_{0<s\leqslant t} \frac{s}{V(s)^{1/p}} \Big) \operatorname{ess\,sup}\limits_{t\leqslant s<\infty} \frac{w(s)}{s} < \infty$;

(viii) $q = 1 < p < \infty$,

$$\Big(\int\limits_0^\infty \Big(\sup_{0<s\leqslant t} s \int\limits_s^\infty \frac{w(y)}{y} dy \Big)^{p'} \frac{v(t)}{V(t)^{p'}} dt \Big)^{1/p'} < \infty \quad \textit{and} \quad \frac{\sup_{0<t<\infty} t \int_t^\infty \frac{w(s)}{s}\, ds}{V(\infty)^{1/p}} < \infty;$$

(ix) $0 < p \leqslant 1 = q$ *and* $\sup\limits_{0<t<\infty} \frac{t}{V(t)^{1/p}} \int\limits_t^\infty \frac{w(s)}{s}\, ds < \infty$;

(x) $p = \infty$, $q = 1$,

$$\int\limits_0^\infty \sup_{0<s\leqslant t} s \int\limits_s^\infty \frac{w(s)}{s}\, d\Big(\frac{-1}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)} \Big) < \infty \quad \textit{and} \quad \frac{\sup_{0<t<\infty} t \int_t^\infty \frac{w(s)}{s}\, ds}{\operatorname{ess\,sup}_{0<s<\infty} v(s)} < \infty;$$

(xi) $p = 1$, $0 < q < 1$ *and*

$$\Big(\int\limits_0^\infty \Big(\int\limits_t^\infty \frac{w(s)}{s^q} \Big)^{q/(1-q)} \Big[\operatorname*{ess\,sup}_{0<s\leqslant t} \frac{1}{(Pv)(s)} \Big]^{q/(1-q)} \frac{w(t)}{t^q}\, dt \Big)^{q/(1-q)} < \infty.$$

Proof. By the standard duality argument, when $1 < q < \infty$, we have for A from (2.1)

$$A = \sup_{h\geqslant 0} \sup_{g\geqslant 0} \frac{\int_0^\infty (Ph)(t) g(t)\, dt}{\|g\|_{L^{q'}(w^{1-q'})} \|Qh\|_{L^p(v)}} = \sup_{g\geqslant 0} \frac{1}{\|g\|_{L^{q'}(w^{1-q'})}} \sup_{h\geqslant 0} \frac{\int_0^\infty h(t)(Qg)(t)\, dt}{\|Qh\|_{L^p(v)}}.$$

Similarly, when $q = 1$,

$$A = \sup_{h\geqslant 0} \frac{\int_0^\infty (Ph)(t) w(t)\, dt}{\|Qh\|_{L^p(v)}} = \sup_{h\geqslant 0} \frac{\int_0^\infty h(t)(Qw)(t)\, dt}{\|Qh\|_{L^p(v)}}$$

and when $q = \infty$,

$$A = \sup_{h\geqslant 0} \sup_{g\geqslant 0} \frac{\int_0^\infty (Ph)(t) g(t)\, dt}{\|\frac{g}{w}\|_{L^1} \|Qh\|_{L^p(v)}} = \sup_{g\geqslant 0} \frac{1}{\|\frac{g}{w}\|_{L^1}} \sup_{h\geqslant 0} \frac{\int_0^\infty h(t)(Qg)(t)\, dt}{\|Qh\|_{L^p(v)}}.$$

Using Theorem 2.1 and (2.2), we get the following characterization of the quantity A: if $1 < p, q < \infty$, then

$$A \approx \sup_{g\geqslant 0} \frac{1}{\|g\|_{L^{q'}(w^{1-q'})}} \Big(\int\limits_0^\infty \Big[\sup_{0<s<t} s(Qg)(s) \Big]^{p'} \frac{v(t)}{V(t)^{p'}}\, dt \Big)^{1/p'} + \sup_{g\geqslant 0} \frac{\sup_{0<t<\infty} t(Qg)(t)}{\|g\|_{L^{q'}(w^{1-q'})} V(\infty)^{1/p}}; \tag{2.17}$$

if $1 < p < q = \infty$, then

$$A \approx \sup_{g \geqslant 0} \frac{1}{\|\frac{g}{w}\|_{L^1}} \Big(\int_0^\infty \Big[\sup_{0<s<t} s(Qg)(s) \Big]^{p'} \frac{v(t)}{V(t)^{p'}} \mathrm{d}t \Big)^{1/p'} + \sup_{g \geqslant 0} \frac{\sup_{0<t<\infty} t(Qg)(t)}{\|\frac{g}{w}\|_{L^1} V(\infty)^{1/p}}; \tag{2.18}$$

if $p = q = \infty$, then

$$A \approx \sup_{g \geqslant 0} \frac{1}{\|\frac{g}{w}\|_{L^1}} \int_0^\infty \Big[\sup_{0<s<t} s(Qg)(s) \Big] \mathrm{d}\Big(\frac{-1}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)} \Big) + \sup_{g \geqslant 0} \frac{\sup_{0<t<\infty} t(Qg)(t)}{\|\frac{g}{w}\|_{L^1} \operatorname{ess\,sup}_{0<s<\infty} v(s)}; \tag{2.19}$$

if $p = \infty$ and $1 < q < \infty$, then

$$A \approx \sup_{g \geqslant 0} \frac{1}{\|g\|_{L^{q'}(w^{1-q'})}} \int_0^\infty \Big[\sup_{0<s<t} s(Qg)(s) \Big] \mathrm{d}\Big(\frac{-1}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)} \Big) + \sup_{g \geqslant 0} \frac{\sup_{0<t<\infty} t(Qg)(t)}{\|g\|_{L^{q'}(w^{1-q'})} \operatorname{ess\,sup}_{0<s<\infty} v(s)}; \tag{2.20}$$

if $0 < p \leqslant 1 < q < \infty$, then

$$A \approx \sup_{g \geqslant 0} \frac{1}{\|g\|_{L^{q'}(w^{1-q'})}} \sup_{0<s<\infty} \frac{s(Qg)(s)}{V(s)^{1/p}}; \tag{2.21}$$

if $0 < p \leqslant 1$ and $q = \infty$, then

$$A \approx \sup_{g \geqslant 0} \frac{1}{\|\frac{g}{w}\|_{L^1}} \sup_{0<s<\infty} \frac{s(Qg)(s)}{V(s)^{1/p}}; \tag{2.22}$$

if $q = 1 < p < \infty$, then

$$A \approx \Big(\int_0^\infty \Big[\sup_{0<s<t} s(Qw)(s) \Big]^{p'} \frac{v(t)}{V(t)^{p'}} \mathrm{d}t \Big)^{1/p'} + \frac{\sup_{0<t<\infty} t(Qw)(t)}{V(\infty)^{1/p}}; \tag{2.23}$$

if $0 < p \leqslant q = 1$, then

$$A \approx \sup_{0<s<\infty} \frac{s(Qw)(s)}{V(s)^{1/p}}; \tag{2.24}$$

if $p = \infty$ and $q = 1$, then

$$A \approx \int_0^\infty \sup_{0<s\leqslant t} s(Qw)(s) \, \mathrm{d}\Big(\frac{-1}{\operatorname{ess\,sup}_{0<s\leqslant t} v(s)} \Big) + \frac{\sup_{0<t<\infty} t(Qw)(t)}{\operatorname{ess\,sup}_{0<s<\infty} v(s)} \tag{2.25}$$

and if $p = 1$ and $0 < q \leqslant 1$, then

$$A = \sup_{h \geqslant 0} \frac{\|Ph\|_{L^q(w)}}{\|h\|_{L^1(Pv)}}. \tag{2.26}$$

Let us note that by a standard duality argument, we have for $1 < q < \infty$

$$\sup_{g\geqslant 0} \frac{\sup_{0<t<\infty} t(Qg)(t)}{\|g\|_{L^{q'}(w^{1-q'})}} = \sup_{0<t<\infty} t\Big(\int_t^\infty \frac{w(s)}{s^q}\,\mathrm{d}s\Big)^{1/q} \tag{2.27}$$

and, by the inequality due to Sinnamon and Stepanov [27], we have

$$\sup_{g\geqslant 0} \frac{\sup_{0<t<\infty} t(Qg)(t)}{\big\|\frac{g}{w}\big\|_{L^1}} = \operatorname*{ess\,sup}_{0<s<\infty} w(s). \tag{2.28}$$

Now, taking into account (2.27) and (2.28), we can establish all the statements of the theorem. Precisely, the assertions (i), (ii) and (iii) follow from Proposition 2.2 (i) combined with (2.17), (2.18) and (2.19), respectively. The assertions (iv) and (v) follow from Proposition 2.2 (ii) combined with (2.17) and (2.20), respectively. The assertions (vi) and (vii) follow from the classical Hardy inequality (cf. e.g. [22]) applied to (2.21) and (2.22), respectively. The assertions (viii), (ix) and (x) are immediate consequences of (2.23), (2.24) and (2.25), respectively. Finally, (xi) follows from (2.26) and the characterization of the appropriate weighted Hardy inequality due to Sinnamon and Stepanov ([27], Theorem 3.3). ∎

REMARK 2.4. The case (xi) is added to the theorem despite its different nature. Indeed, it is the only case in which we allow $q < 1$. In this particular case (i.e. when $p = 1$), the characterization is very easy. All the other cases ($0 < q < 1$ and $p \neq 1$) will be treated in our forthcoming paper.

3. THE EMBEDDING $\Lambda^p(v) \hookrightarrow \Gamma^q(w)$ WHEN $0 < q < p \leqslant 1$

Here we will find necessary and sufficient conditions for the "missing case" of embedding between classical Lorentz spaces.

THEOREM 3.1. *Let* $0 < q < p \leqslant 1$ *and let* $r = \frac{pq}{p-q}$. *Let* v, w *be weights on* $(0,\infty)$. *Then the inequality*

$$\Big(\int_0^\infty f^{**}(t)^q w(t)\,\mathrm{d}t\Big)^{1/q} \lesssim \Big(\int_0^\infty f^*(t)^p v(t)\,\mathrm{d}t\Big)^{1/p} \tag{3.1}$$

holds if and only if

$$A_{(3.2)} := \Big(\int_0^\infty \Big[\frac{W(t)}{V(t)}\Big]^{r/p} w(t)\,\mathrm{d}t\Big)^{1/r} < \infty, \tag{3.2}$$

$$A_{(3.3)} := \Big(\int_0^\infty \sup_{0<s\leqslant t} \frac{s^r}{V(s)^{r/p}} \Big(\int_t^\infty \frac{w(s)}{s^q}\,\mathrm{d}s\Big)^{r/p} \frac{w(t)}{t^q}\,\mathrm{d}t\Big)^{1/r} < \infty, \tag{3.3}$$

and the best constant in (3.1) *is comparable to* $A_{(3.2)} + A_{(3.3)}$.

Proof. First assume that (3.1) is true. Then, since $\sup_{0<s\leqslant t} sf^*(s) \leqslant \int_0^t f^*(s)\,\mathrm{d}s$, $t \in (0,\infty)$, we obtain

$$\left(\int_0^\infty \Big[\sup_{0<s\leqslant t} sf^*(s)\Big]^q \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{1/q} \lesssim \left(\int_0^\infty f^*(t)^p v(t)\,\mathrm{d}t\right)^{1/p}. \tag{3.4}$$

Assume that $f^*(t)^p = \int_t^\infty h(s)\,\mathrm{d}s$ for some $h \in \mathcal{M}_+(0,\infty)$. Then, $\sup_{0<s\leqslant t} sf^*(s) = \Big[\sup_{0<s\leqslant t} s^p \int_t^\infty h(y)\,\mathrm{d}y\Big]^{1/p}$, hence, by the Fubini theorem, (3.4) yields

$$\left(\int_0^\infty \Big[\sup_{0<s\leqslant t} s^p \int_s^\infty h(y)\,\mathrm{d}y\Big]^{q/p} \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{1/q} \lesssim \left(\int_0^\infty h(t)V(t)\,\mathrm{d}t\right)^{1/p} \tag{3.5}$$

for all $h \in \mathcal{M}_+(0,\infty)$. Substituting $h(t) \to \frac{g(t)}{t}$, we get

$$\left(\int_0^\infty \Big[\sup_{0<s\leqslant t} s^p (Qg)(s)\Big]^{q/p} \frac{w(t)}{t^q}\,\mathrm{d}t\right)^{p/q} \lesssim \int_0^\infty g(t)\frac{V(t)}{t}\,\mathrm{d}t \tag{3.6}$$

for all $g \in \mathcal{M}_+(0,\infty)$. By Proposition 2.2 (ii), applied to the choice of parameters $q = \frac{q}{p}$, $p = 1$, $\alpha = p$, $w(t) = w(t)t^{-q}$ and $v(t) = \frac{V(t)}{t}$, (3.6) holds if and only if both (3.2) and (3.3) are satisfied.

Conversely, assume that (3.2) and (3.3) hold. We will be done if we can show

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f^*(s)^{1/p}\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{p/q} \lesssim \int_0^\infty f^*(t)v(t)\,\mathrm{d}t, \quad f \in \mathcal{M}(0,\infty).$$

By (2.2), this will follow if we prove, for all $f \in \mathcal{M}(0,\infty)$,

$$\left(\int_0^\infty \left(\frac{1}{t^p}\int_0^t f^*(s)s^{p-1}\,\mathrm{d}s\right)^{q/p} w(t)\,\mathrm{d}t\right)^{p/q} \lesssim \int_0^\infty f^*(t)v(t)\,\mathrm{d}t. \tag{3.7}$$

Given $f \in \mathbb{A}$, there is a sequence $\{h_n\}$ of positive functions such that

$$F_n(t) := \int_t^\infty h_n(s)\,\mathrm{d}s \nearrow f, \quad t \in (0,\infty).$$

Moreover, we have $\frac{1}{t^p}\int_0^t F_n^*(s)s^{p-1}\,\mathrm{d}s \approx \int_t^\infty h_n(s)\,\mathrm{d}s + \frac{1}{t^p}\int_0^t h_n(s)s^p\,\mathrm{d}s$ and, by the Fubini theorem, $\int_0^\infty F_n^*(t)v(t)\mathrm{d}t = \int_0^\infty h_n(t)V(t)\mathrm{d}t$. Summarizing, by the Fatou lemma, we see that (3.7) holds if and only if both the inequalities

$$\Big(\int_0^\infty\Big(\int_t^\infty h(s)\,\mathrm{d}s\Big)^{q/p} w(t)\,\mathrm{d}t\Big)^{p/q} \lesssim \int_0^\infty h(t)V(t)\,\mathrm{d}t, \tag{3.8}$$

$$\Big(\int_0^\infty\Big(\frac{1}{t^p}\int_0^t h(s)s^p\,\mathrm{d}s\Big)^{q/p} w(t)\,\mathrm{d}t\Big)^{p/q} \lesssim \int_0^\infty h(t)V(t)\,\mathrm{d}t, \tag{3.9}$$

are satisfied for all $h \in \mathcal{M}(0,\infty)$. Now, by Theorem 3.3 of [27] and by its analogue for integral $\int_t^\infty$ in place of $\int_0^t$, necessary and sufficient conditions for (3.8) and (3.9) are (3.2) and (3.3), respectively. ∎

4. GENERAL RELATIONS BETWEEN THE SPACES OF TYPE S, Λ, Γ. THE SINGLE-WEIGHT CASE

We start with a simple but important relation.

PROPOSITION 4.1. *Let $0 < p < \infty$ and let v be a weight on $(0,\infty)$. Then*

$$\Gamma^p(v) = \Lambda^p(v) \cap S^p(v). \tag{4.1}$$

Proof. Since both the quantities f^* and $f^{**} - f^*$ are majorized by f^{**}, we clearly have

$$\Gamma^p(v) \subset \Lambda^p(v) \cap S^p(v).$$

The converse inclusion follows at once from

$$\|f\|_{\Gamma^p(v)} = \|f^{**}-f^*+f^*\|_{L^p(v)} \leqslant \|f^{**}-f^*\|_{L^p(v)} + \|f^*\|_{L^p(v)} = \|f\|_{S^p(v)} + \|f\|_{\Lambda^p(v)}.$$ ∎

Following [1], we say that a weight v on $(0,\infty)$ belongs to the class B_p if

$$\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s \lesssim \frac{1}{t^p}\int_0^t v(s)\,\mathrm{d}s, \quad t \in (0,\infty).$$

We say that a weight v on $(0,\infty)$ belongs to the reverse class B_p, written $w \in \mathrm{R}B_p$ (cf. [1], [7]), if

$$\frac{1}{t^p}\int_0^t v(s)\,\mathrm{d}s \lesssim \int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s, \quad t \in (0,\infty).$$

From Proposition 4.1 we immediately have

COROLLARY 4.2. *Let $0 < p < \infty$ and let v be a weight on $(0,\infty)$. Then, the following statements are equivalent:*

(i) $\Lambda^p(v) \hookrightarrow S^p(v)$;

(ii) $\Lambda^p(v) = \Gamma^p(v)$;

(iii) $v \in B_p$.

Proof. The equivalence (i)⇔(ii) follows at once from (4.1). The equivalence (ii)⇔(iii) was proved in [1] for $1 \leqslant p < \infty$ and in [28] for $0 < p < 1$. ∎

COROLLARY 4.3. *Let* $1 \leqslant p < \infty$ *and let* v *be a weight on* $(0,\infty)$. *Then, the following statements are equivalent:*

(i) $S^p(v) \hookrightarrow \Lambda^p(v)$;

(ii) $S^p(v) = \Gamma^p(v)$;

(iii) $v \in \mathrm{R}B_p$.

Proof. Again, the equivalence (i)⇔(ii) follows from (4.1). A similar argument to that in the proof of (3.7) now yields (ii) equivalent to

$$\int_0^\infty \Big(\int_t^\infty h(s)\,\mathrm{d}s\Big)^p v(t)\,\mathrm{d}t \lesssim \int_0^\infty \Big(\frac{1}{t}\int_0^t sh(s)\,\mathrm{d}s\Big)^p v(t)\,\mathrm{d}t \tag{4.2}$$

for all $h \in \mathcal{M}_+(0,\infty)$. However, (4.2) was shown in Theorem 4.1 of [21] to be equivalent to $v \in \mathrm{R}B_p$. ∎

REMARK 4.4. The equivalence in the preceding corollary holds in fact when $p \in (0,\infty)$.

5. EMBEDDINGS OF TYPE $S \hookrightarrow S$

Our aim in this section is to characterize pairs of weights v, w such that the following embedding holds:

$$S^p(v) \hookrightarrow S^q(w). \tag{5.1}$$

REMARK 5.1. We note that every $f \in \mathcal{M}_+(0,\infty)$ can be written in the form $f = g + c$, where $g \in \mathbb{A}$ and $c \geqslant 0$.

THEOREM 5.2. *Let* $0 < p, q < \infty$ *and let* v, w *be weights on* $(0,\infty)$. *Then,* (5.1) *is equivalent to the following, with* $\widetilde{v}_p$ *and* $\widetilde{w}_q$ *from* (1.8)*:*

$$\Lambda^p(\widetilde{v}_p) \hookrightarrow \Lambda^q(\widetilde{w}_q). \tag{5.2}$$

Proof. Assume first that (5.2) holds, and let $f \in \mathbb{A}$. Then, by (5.2) and a double use of (1.7),

$$\|f\|_{S^q(w)} = \|Tf^*\|_{\Lambda^q(\widetilde{w}_q)} \leqslant C\|Tf^*\|_{\Lambda^p(\widetilde{v}_p)} = \|f\|_{S^p(v)},$$

proving (5.1) for $f \in \mathbb{A}$. Since (5.1) is trivial for constant functions, it follows from Remark 5.1 that (5.1) holds for every $f \in \mathcal{M}_+(0,\infty)$.

Conversely, assume that (5.1) is satisfied. For every $f \in \mathbb{A}$, there exists a $g \in \mathbb{A}$ such that $f^* = Tg^*$. Thus, using (1.7) again,

$$\|f\|_{\Lambda^q(\widetilde{w}_q)} = \|Tg^*\|_{\Lambda^q(\widetilde{w}_q)} = \|g^*\|_{S^q(w)} \leqslant C\|g^*\|_{S^p(v)} = C\|Tg^*\|_{\Lambda^p(\widetilde{v}_p)} = C\|f\|_{\Lambda^p(\widetilde{v}_p)},$$

proving (5.2) for $f \in \mathbb{A}$.

In view of Remark 5.1, it only remains to show that (5.2) holds for constant functions, that is, we have to prove the inequality

$$\Big(\int_0^\infty \widetilde{w}_q(t)\,\mathrm{d}t\Big)^{1/q} \leqslant C\Big(\int_0^\infty \widetilde{v}_p(t)\,\mathrm{d}t\Big)^{1/p}.$$

Assuming that the right-hand side is finite (otherwise there is nothing to prove) and given $T \in (0,\infty)$, we apply the result just established to the function $f = \chi_{(0,T)}$, which obviously belongs to $\mathbb{A}$. We get

$$\Big(\int_0^T \widetilde{w}_q(t)\,\mathrm{d}t\Big)^{1/q} \leqslant C\Big(\int_0^T \widetilde{v}_p(t)\,\mathrm{d}t\Big)^{1/p}$$

with C independent of T. The result now follows by letting $T \to \infty$. ∎

COROLLARY 5.3. (i) *If* $0<p\leqslant q<\infty$, *then the embedding* (5.1) *holds if and only if*

$$A_{(5.3)} := \sup_{t\in(0,\infty)} \frac{\big(\int_t^\infty s^{-q}w(s)\,\mathrm{d}s\big)^{1/q}}{\big(\int_t^\infty s^{-p}v(s)\,\mathrm{d}s\big)^{1/p}} < \infty, \tag{5.3}$$

and the optimal constant C of the embedding (5.1) *satisfies* $C \approx A_{(5.3)}$.

(ii) *If* $0 < q < p < \infty$, *then* (5.1) *holds if and only if*

$$A_{(5.4)} := \Big(\int_0^\infty \Big[\frac{\int_t^\infty s^{-q}w(s)\,\mathrm{d}s}{\int_t^\infty s^{-p}v(s)\,\mathrm{d}s}\Big]^{r/p} \frac{w(t)}{t^q}\,\mathrm{d}t\Big)^{1/r} < \infty, \tag{5.4}$$

and the optimal constant C of the embedding (5.1) *satisfies* $C \approx A_{(5.4)}$.

Proof. The claim follows from Theorem 5.2, the known characterization of (5.2) (cf. [23], [28], or the survey in [11]), and, again, a change of variables in the integrals. ∎

6. EMBEDDINGS OF TYPE $\Gamma \hookrightarrow S$ AND $S \hookrightarrow \Gamma$

In this section we will characterize embeddings between spaces of type Γ and S (in both directions). We will once again employ the operator T defined in (1.4).

THEOREM 6.1. *Let $0 < p,q < \infty$ and let v,w be weights on $(0,\infty)$. Then the embedding*

$$S^p(v) \hookrightarrow \Gamma^q(w), \quad f \in \mathbb{A}, \tag{6.1}$$

is equivalent to

$$\Lambda^p(\widetilde{v}_p) \hookrightarrow \Gamma^q(\widetilde{w}_q). \tag{6.2}$$

Similarly, the embedding

$$\Gamma^p(v) \hookrightarrow S^q(w), \quad f \in \mathbb{A}, \tag{6.3}$$

is equivalent to

$$\Gamma^p(\widetilde{v}_p) \hookrightarrow \Lambda^q(\widetilde{w}_q). \tag{6.4}$$

Proof. The proof is analogous to that of Theorem 5.2. We use (1.7) and

$$\|f\|_{\Gamma^p(w)} = \|Tf\|_{\Gamma^p(\widetilde{w}_p)}, \tag{6.5}$$

the latter being easily verified for every $f \in \mathbb{A}$ by a change of variables. ∎

THEOREM 6.2. *Let $0 < p,q < \infty$ and let v,w be weights on $(0,\infty)$.*

(i) *If $1 < p \leqslant q < \infty$, then the embedding* (6.1) *holds if and only if*

$$A_{(5.3)} < \infty, \tag{6.6}$$

$$A_{(6.7)} := \sup_{t\in(0,\infty)} W(t)^{1/q}\Big(\int_t^\infty \frac{v(s)}{\big(s^p \int_{1/s}^\infty v(y)y^{-p}\,\mathrm{d}y\big)^{p'}}\,\mathrm{d}s\Big)^{1/p'} < \infty, \tag{6.7}$$

and the optimal constant C of the embedding (6.1) *satisfies $C \approx A_{(5.3)} + A_{(6.7)}$.*

(ii) *If $0 < p < 1$ and $0 < p \leqslant q < \infty$, then* (6.1) *holds if and only if $A_{(5.3)} < \infty$ and*

$$A_{(6.8)} := \sup_{t\in(0,\infty)} \frac{W(t)^{1/q}}{t\big(\int_t^\infty v(s)s^{-p}\,\mathrm{d}s\big)^{1/p}}, \tag{6.8}$$

and the optimal constant C of the embedding (6.1) *satisfies $C \approx A_{(5.3)} + A_{(6.8)}$.*

(iii) *If $1 < p < \infty$, $0 < q < p < \infty$ and $q \neq 1$, then* (6.1) *holds if and only if*

$$A_{(5.4)} < \infty, \tag{6.9}$$

$$A_{(6.10)} := \left(\int_0^\infty \frac{W(t)^{r/q}\Big(\int_t^\infty \frac{v(s)}{\big(s^p\int_s^\infty \frac{v(y)}{y^p}\mathrm{d}y\big)^{p'}}\mathrm{d}s\Big)^{r(q-1)/q}}{\big(t^p\int_s^\infty \frac{v(y)}{y^p}\mathrm{d}y\big)^{p'}} v(t)\mathrm{d}t\right)^{1/r} < \infty, \tag{6.10}$$

and the optimal constant C of the embedding (6.1) *satisfies $C \approx A_{(5.4)} + A_{(6.10)}$.*

(iv) *If $1 = q < p < \infty$, then* (6.1) *holds if and only if $A_{(5.4)} < \infty$ and*

$$A_{(6.11)} := \frac{\int_0^\infty \frac{w(s)}{s}\,\mathrm{d}s}{\big(\int_0^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\big)^{1/p}} + \Big(\int_0^\infty \Big[\frac{\frac{W(t)}{t} + \int_t^\infty \frac{w(s)}{s}\,\mathrm{d}s}{\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s}\Big]^{p'} \frac{v(t)}{t^p}\,\mathrm{d}t\Big)^{1/p'} < \infty, \tag{6.11}$$

and the optimal constant C of the embedding (6.1) *satisfies $C \approx A_{(5.4)} + A_{(6.11)}$.*

(v) *If* $0 < q < p \leqslant 1$, *then* (6.1) *holds if and only if*

$$A_{(6.12)} := \Big(\int_0^\infty \frac{\big(\int_t^\infty \frac{w(s)}{s^q}\,ds\big)^{r/p}}{\big(\int_t^\infty \frac{v(s)}{s^p}\,ds\big)^{r/p}} \frac{w(t)}{t^q}\,dt\Big)^{1/r} < \infty, \tag{6.12}$$

$$A_{(6.13)} := \Big(\int_0^\infty W(t)^{r/p} \operatorname*{ess\,sup}_{t\leqslant s<\infty} \frac{1}{s^r\big(\int_s^\infty \frac{v(y)}{y^p}\,dy\big)^{r/p}} \frac{w(t)}{t^q}\,dt\Big)^{1/r} < \infty, \tag{6.13}$$

and the optimal constant C of the embedding (6.1) *satisfies* $C \approx A_{(6.12)} + A_{(6.13)}$.

Proof. By Theorem 6.1, (6.1) is equivalent to (6.2). In cases (i)–(iv), necessary and sufficient conditions for (6.2) are known (cf. Theorem 2 of [23] in the case (i), Proposition 2.6 b of [9] or Theorem 2 of [28] in the case (ii), Theorem 2 of [23] and Theorem 3a of [28] in the case (iii) and Theorem 4.1 (ii) of [11] in the case (iv)). In the case (v), necessary and sufficient conditions are given by Theorem 3.1. The assertion then follows by a change of variables. ∎

THEOREM 6.3. *Let* $0 < p, q < \infty$ *and let* v, w *be weights on* $(0,\infty)$.

(i) *If* $0 < p \leqslant q < \infty$ *and* $1 \leqslant q < \infty$, *then the embedding* (6.3) *holds if and only if*

$$A_{(6.14)} := \sup_{t\in(0,\infty)} \frac{\big(\int_t^\infty \frac{w(s)}{s^q}\,ds\big)^{1/q}}{\big(\frac{V(t)}{t^p} + \int_t^\infty \frac{v(s)}{s^p}\,ds\big)^{1/p}} < \infty, \tag{6.14}$$

and the optimal constant C of the embedding (6.3) *satisfies* $C \approx A_{(6.14)}$.

(ii) *If* $1 \leqslant q < p < \infty$, *then* (6.3) *holds if and only if*

$$A_{(6.15)} := \Big(\int_0^\infty \frac{\sup_{s\in(0,t)} s^r\big(\int_s^\infty \frac{w(y)}{y^q}\,dy\big)^{r/q}}{t^{r+p+1}\big(\frac{V(t)}{t^p} + \int_t^\infty \frac{v(s)}{s^p}\,ds\big)^{r/p+2}} V(t) \int_t^\infty \frac{v(s)}{s^p}\,ds\,dt\Big)^{1/r} < \infty, \tag{6.15}$$

and the optimal constant C of the embedding (6.3) *satisfies* $C \approx A_{(6.15)}$.

(iii) *If* $0 < p \leqslant q < 1$, *then* (6.3) *holds if and only if*

$$A_{(6.16)} := \sup_{t\in(0,\infty)} \frac{\big(\int_t^\infty \frac{w(s)}{s^q}ds\big)^{1/q} + \frac{1}{t}\Big(\int_0^t \Big[s\int_s^\infty \frac{w(y)}{y^q}dy\Big]^{q/(1-q)} \frac{w(s)}{s^q}ds\Big)^{(1-q)/q}}{\big(\frac{V(t)}{t^p} + \int_t^\infty \frac{v(s)}{s^p}ds\big)^{1/p}} < \infty, \tag{6.16}$$

and the optimal constant C of the embedding (6.3) *satisfies* $C \approx A_{(6.16)}$.

(iv) *If* $0 < q < 1$ *and* $0 < q < p$, *then* (6.3) *holds if and only if*

$$\begin{aligned} A_{(6.17)} := \Bigg(& \frac{\Big[\big(\int_t^\infty \frac{w(s)}{s^q}ds\big)^{1/(1-q)} + \frac{1}{t^{q/(1-q)}} \int_0^t \big(s\int_s^\infty \frac{w(y)}{y^q}dy\big)^{q/(1-q)} \frac{w(s)}{s^q}ds\Big]^{r(1-q)/q-1}}{\big(\frac{V(t)}{t^p} + \int_t^\infty \frac{v(s)}{s^p}ds\big)^{1/p}} \\ & \times \Big(\int_t^\infty \frac{w(s)}{s^q}\,ds\Big)^{q/(1-q)} \frac{w(t)}{t^q}\,dt\Bigg)^{1/r}, \end{aligned} \tag{6.17}$$

and the optimal constant C of the embedding (6.3) *satisfies* $C \approx A_{(6.17)}$.

Proof. The proof follows the pattern of that of Theorem 6.2. Necessary and sufficient conditions for (6.4) can be found in p. 473 of [29] in the case (i) and in [13] in other cases. ∎

7. EMBEDDINGS OF TYPE $\Lambda \hookrightarrow S$ AND $S \hookrightarrow \Lambda$

We start with a simple observation that the embeddings $S^p(v) \hookrightarrow \Lambda^q(w)$ and $\Lambda^p(v) \hookrightarrow S^q(w)$ (restricted to $f \in \mathbb{A}$) are interchangeable (with an appropriate change of weights) and therefore it is enough to investigate only one of them.

PROPOSITION 7.1. *Let $0 < p, q < \infty$ and let v, w be weights on $(0, \infty)$. Then the embedding*

$$S^p(v) \hookrightarrow \Lambda^q(w), \quad f \in \mathbb{A}, \tag{7.1}$$

is equivalent to the embedding

$$\Lambda^p(\widetilde{v}_p) \hookrightarrow S^q(\widetilde{w}_q), \quad f \in \mathbb{A}, \tag{7.2}$$

where $\widetilde{v}_p$ and $\widetilde{w}_q$ are from (1.8).

Proof. Using the operator T defined in (1.4), we see that (7.1) reads

$$\Big(\int_0^\infty f^*(t)^q w(t)\,\mathrm{d}t\Big)^{1/q} \lesssim \Big(\int_0^\infty \Big[\frac{1}{t}(Tf^*)(1/t)\Big]^p v(t)\,\mathrm{d}t\Big)^{1/p}.$$

As already noted above, for every $f \in \mathbb{A}$, there exists a $g \in \mathbb{A}$ such that $f^* = Tg^*$. Therefore, (7.1) is equivalent to

$$\Big(\int_0^\infty (Tg^*)(t)^q w(t)\,\mathrm{d}t\Big)^{1/q} \lesssim \Big(\int_0^\infty \Big[\frac{1}{t}g^*(1/t)\Big]^p v(t)\,\mathrm{d}t\Big)^{1/p}.$$

Changing the variables, we get the following which is (7.2), as desired:

$$\Big(\int_0^\infty [g^{**}(t) - g^*(t)]^q \widetilde{w}_q(t)\,\mathrm{d}t\Big)^{1/q} \lesssim \Big(\int_0^\infty g^*(t)^p \widetilde{v}_p(t)\,\mathrm{d}t\Big)^{1/p}. \quad ∎$$

We next reduce (7.1) to (1.1).

PROPOSITION 7.2. *Let $0 < p, q < \infty$ and let v, w be weights on $(0, \infty)$. Then the embedding*

$$\Lambda^p(v) \hookrightarrow S^q(w), \quad f \in \mathbb{A}, \tag{7.3}$$

holds if and only if the inequality (1.1) *is satisfied for every non-negative function h.*

Proof. We shall proceed as in the proof of (3.7). Given $f \in \mathbb{A}$, there is a sequence $\{h_n\}$ of positive functions such that

$$F_n(t) := \int_t^\infty h_n(s)\,\mathrm{d}s \nearrow f, \quad t \in (0,\infty).$$

Moreover, $f^{**}(t)-f^{*}(t)=\lim_{n\to\infty}(F_n^{**}(t)-F_n^{*}(t)), t\in(0,\infty)$. Thus, by the Fatou lemma,

$$\|f\|_{S^q(w)} \leqslant \liminf_{n\to\infty} \|F_n\|_{S^q(w)} \lesssim \liminf_{n\to\infty} \|F_n\|_{\Lambda^p(v)} = \|f\|_{\Lambda^p(v)}.$$

On the other hand, we have

$$F_n^{**}(t) - F_n^{*}(t) = \frac{1}{t}\int_0^t \int_s^\infty h_n(y)\,\mathrm{d}y\,\mathrm{d}s - \int_t^\infty h_n(s)\,\mathrm{d}s = \frac{1}{t}\int_0^t s h_n(s)\,\mathrm{d}s.$$

Altogether, (7.3) is equivalent to

$$\Big(\int_0^\infty \Big[\frac{1}{t}\int_0^t sh(s)\,\mathrm{d}s\Big]^q w(t)\,\mathrm{d}t\Big)^{1/q} \lesssim \Big(\int_0^\infty \Big[\int_t^\infty h(s)\,\mathrm{d}s\Big]^p v(t)\,\mathrm{d}t\Big)^{1/p},$$

which is, after the substitution $sh(s) \to h(s)$, exactly the inequality (1.1). ∎

COROLLARY 7.3. *Let* $0 < p < \infty$, $1 \leqslant q < \infty$, *and let* v, w *be weights on* $(0,\infty)$. *Then, the embedding*

$$\Lambda^p(v) \hookrightarrow S^q(w), \quad f \in \mathbb{A},$$

holds if and only if one of the conditions (i)–(xi) *of Theorem* 2.3 *holds.*

REMARK 7.4. Combining Corollary 7.3 and Proposition 7.1 we can get necessary and sufficient conditions for the embedding

$$S^p(v) \hookrightarrow \Lambda^q(w), \quad f \in \mathbb{A},$$

where $0<p\leqslant\infty$, $1\leqslant q\leqslant\infty$, and let v, w be weights on $(0,\infty)$. We omit the details.

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REFERENCES

[1] M. ARIÑO, B. MUCKENHOUPT, Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for non-increasing functions, *Trans. Amer. Math. Soc.* **320**(1990), 727–735.

[2] J. BASTERO, M. MILMAN, F. RUIZ, A note on $L(\infty, q)$ spaces and Sobolev embeddings, *Indiana Univ. Math. J.* **52**(2003), 1215–1230.

[3] C. BENNETT, R.A. DE VORE, R. SHARPLEY, Weak-L^∞ and *BMO*, *Ann. of Math.* **113**(1981), 601–611.

[4] C. BENNETT, R. SHARPLEY, *Interpolation of Operators*, Pure Appl. Math., vol. 129, Academic Press, Boston 1988.

[5] S. BOZA, J. MARTÍN, Equivalent expressions for norms in classical Lorentz spaces, *Forum Math.* **17**(2005), 361–374.

[6] M. CARRO, A. GARCÍA DEL AMO, J. SORIA, Weak-type weights and normable Lorentz spaces, *Proc. Amer. Math. Soc.* **124**(1996), 849–857.

[7] M. CARRO, A. GOGATISHVILI, J. MARTÍN, L. PICK, Functional properties of rearrangement invariant spaces defined in terms of oscillations, *J. Funct. Anal.* **229**(2005), 375–404.

[8] M. CARRO, J. SORIA, Weighted Lorentz spaces and the Hardy operator, *J. Funct. Anal.* **112**(1993), 480–494.

[9] M. CARRO, J. SORIA, Boundedness of some integral operators, *Canad. J. Math.* **45**(1993), 1155–1166.

[10] M. CARRO, J. SORIA, The Hardy-Littlewood maximal function and weighted Lorentz spaces, *J. London Math. Soc.* **55**(1997), 146–158.

[11] M. CARRO, J. SORIA, L. PICK, V. STEPANOV, On embeddings between classical Lorentz spaces, *Math. Ineq. Appl.* **4**(2001), 397–428.

[12] A. GOGATISHVILI, B. OPIC, L. PICK, Weighted inequalities for Hardy-type operators involving suprema, *Collect. Math.* **57**(2006), 227–255.

[13] A. GOGATISHVILI, L. PICK, Discretization and anti-discretization of rearrangement-invariant norms, *Publ. Mat.* **47**(2003), 311–358.

[14] M.L. GOL'DMAN, Sharp estimates for the norms of Hardy-type operators on the cones of quasimonotone functions, *Proc. Steklov Inst. Math.* **232**(2001), 1–29.

[15] M.L. GOL'DMAN, H.P. HEINIG, V.D. STEPANOV, On the principle of duality in Lorentz spaces, *Canad. J. Math.* **48**(1996), 959–979.

[16] K.-G. GROSSE-ERDMANN, *The Blocking Technique, Weighted Mean Operators and Hardy's Inequality*, Lect. Notes Math., vol. 1679, Springer, Berlin 1998.

[17] R. KERMAN, L. PICK, Optimal Sobolev imbeddings, *Forum Math.* **18**(2006), 535–570.

[18] G.G. LORENTZ, On the theory of spaces Λ, *Pacific J. Math.* **1**(1951), 411–429.

[19] J. MARTÍN, M. MILMAN, Symmetrization inequalities and Sobolev embeddings, *Proc. Amer. Math. Soc.* **134**(2006), 2335–2347.

[20] M. MILMAN, E. PUSTYLNIK, On sharp higher order Sobolev embeddings, *Comm. Contemp. Math.* **6**(2004), 495–511.

[21] C.J. NEUGEBAUER, Weighted norm inequalities for averaging operators of monotone functions, *Publ. Math.* **35**(1992), 429–447.

[22] B. OPIC, A. KUFNER, *Hardy-Type Inequalities*, Pitman Res. Notes Math., Longman Sci&Tech., Harlow 1990.

[23] E. SAWYER, Boundedness of classical operators on classical Lorentz spaces, *Studia Math.* **96**(1990), 145–158.

[24] G. SINNAMON, Spaces defined by level functions and their duals, *Studia Math.* **111**(1994), 19–52.

[25] G. SINNAMON, Embeddings of concave functions and duals of Lorentz spaces, *Publ. Math.* **46**(2002), 489–515.

[26] G. SINNAMON, Transferring monotonicity in weighted norm inequalities, *Collect. Math.* **54**(2003), 181–216.

[27] G. SINNAMON, V.D. STEPANOV, The weighted Hardy inequality: new proofs and the case $p = 1$, *J. London Math. Soc.* **54**(1996), 89–101.

[28] V.D. STEPANOV, The weighted Hardy's inequality for nonincreasing functions, *Trans. Amer. Math. Soc.* **338**(1993), 173–186.

[29] V.D. STEPANOV, Integral operators on the cone of monotone functions, *J. London Math. Soc.* **48**(1993), 465–487.

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