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HEREDITARY SUBALGEBRAS OF OPERATOR ALGEBRAS

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ABSTRACT. In recent work of the second author, a technical result was proved establishing a bijective correspondence between certain open projections in a C^*-algebra containing an operator algebra A, and certain one-sided ideals of A. Here we give several remarkable consequences of this result. These include a generalization of the basic theory of hereditary subalgebras of a C^*-algebra, and the solution of a ten year old problem concerning the Morita equivalence of operator algebras. In particular, the latter gives a very clean generalization of the notion of Hilbert C^*-modules to nonselfadjoint algebras. We show that an "ideal" of a general operator space X is the intersection of X with an "ideal" in any containing C^*-algebra or C^*-module. Finally, we discuss the noncommutative variant of the classical theory of "peak sets".

KEYWORDS: *Hereditary subalgebra, open projection, approximate identity, faces, state spaces, ideals, nonselfadjoint operator algebras, Hilbert C^*-modules, M-ideals, peak set.*

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1. INTRODUCTION

In [26], [27], a technical result was proved establishing a bijective correspondence between certain open projections in a C^*-algebra containing an operator algebra A, and certain one-sided ideals of A. Here we give several remarkable consequences of this result. These include a generalization of the theory of hereditary subalgebras of a C^*-algebra, and the solution of a ten year old problem concerning the Morita equivalence of operator algebras. In particular, the latter yields the conceptually cleanest generalization of the notion of Hilbert C^*-modules to nonselfadjoint algebras. We show that an "ideal" of a general operator space X is the intersection of X with an "ideal" in any containing C^*-algebra or C^*-module. Finally, we discuss the noncommutative variant of the classical theory of "peak sets". If A is a function algebra on a compact space X, then a *p-set* may be characterized as a closed subset E of X such that for any open set U containing E there is a function in Ball(A) which is 1 on E, and $< \varepsilon$ in modulus outside of U. We prove a noncommutative version of this result.

An *operator algebra* is a closed algebra of operators on a Hilbert space; or equivalently a closed subalgebra of a C^*-algebra. We refer the reader to [11] for the basic theory of operator algebras which we shall need. We say that an operator algebra A is *unital* if it has an identity of norm 1, and *approximately unital* if it has a contractive approximate identity (cai). A *unital-subalgebra* of a C^*-algebra B is a closed subalgebra containing 1_B. In this paper we will often work with closed right ideals J of an operator algebra A possessing a contractive left approximate identity (or left cai) for J. For brevity we will call these *r-ideals*. The matching class of left ideals with right cai will be called *l-ideals*, but these will not need to be mentioned much for reasons of symmetry. In fact r-ideals are exactly the *right M-ideals* of A if A is approximately unital [10]. For C^*-algebras r-ideals are precisely the closed right ideals, and there is an obvious bijective correspondence between r-ideals and l-ideals, namely $J \mapsto J^*$. For nonselfadjoint operator algebras it is not at all clear that there is a bijective correspondence between r-ideals and l-ideals. In fact there is, but this seems at present to be a deep result, as we shall see. It is easy to see that there is a bijective correspondence between r-ideals J and certain projections p in the second dual A^{**} (we recall that A^{**} is also an operator algebra ([11], Section 2.5)). This bijection takes J to its *left support projection*, namely the weak* limit of a left cai for J; and conversely takes p to the right ideal $pA^{**} \cap A$. The main theorem from [27], which for brevity we will refer to as *Hay's theorem*, states that if A is a unital-subalgebra of a C^*-algebra B then the projections p here may be characterized as the projections in $A^{\perp\perp}$ which are open in B^{**} in the sense of e.g. [1], [33]. Although this result sounds innocuous, its proof is presently quite technical and lengthy, and uses the noncommutative Urysohn lemma [2] and various nonselfadjoint analogues of it. One advantage of this condition is that it has a left/right symmetry, and thus it leads naturally into a theory of hereditary subalgebras (*HSA's* for short) of general operator algebras. For commutative C^*-algebras of course HSA's are precisely the closed two-sided ideals. For noncommutative C^*-algebras the hereditary subalgebras are the intersections of a right ideal with its canonically associated left ideal [19], [33]. They are also the selfadjoint "inner ideals". (In this paper, we say that a subspace J of an algebra A is an inner ideal if $JAJ \subset J$. Inner ideals in this sense are sometimes called "hereditary subalgebras" in the literature, but we will reserve the latter term for something more specific.) The fact that HSA's of C^*-algebras are the selfadjoint inner ideals follows quickly from Proposition 2.2 below and its proof, or it can be deduced from [17]. HSA's play some of the role that two-sided ideals play in the commutative theory. Also, their usefulness stems in large part because many important properties of the algebra pass to hereditary subalgebras (for example, primeness or primitivity).

We now summarize the content of our paper. In Section 2, we use Hay's theorem to generalize some of the C^*-algebraic theory of HSA's. Also in Section 2 we use our results to give a solution to a problem raised in [9]. An earlier incomplete attempt to solve this problem was made in [30]. In [9] an operator algebra A was said to have *property* $(\mathcal{L})$ if it has a left cai (e_t) such that $e_s e_t \to e_s$ with t for each s. It was asked if every operator algebra with a left cai has property $(\mathcal{L})$? As an application of this, in Section 3 we settle a problem going back to the early days of the theory of strong Morita equivalence of nonselfadjoint operator algebras. This gives a very clean generalization of the notion of Hilbert C^*-module to such algebras. In Section 4, we generalize to nonselfadjoint algebras the connections between HSA's, weak* closed faces of the state space, and lowersemicontinuity. We remark that facial structure in the algebra itself has been looked at in the nonselfadjoint literature, for example in [31] and references therein. In Section 5 we show that every right M-ideal in any operator space X is an intersection of X with a canonical right submodule of any C^*-module (or "TRO") containing X. Similar results hold for two-sided, or "quasi-", M-ideals. This generalizes to arbitrary operator spaces the theme from e.g. Theorem 2.9 below, and from [27], that r-ideals (respectively HSA's) are very tightly related to matching right ideals (respectively HSA's) in a containing C^*-algebra. In the final Section 6 we discuss connections with the peak and p-projections introduced in [26], [27]. The motivation for looking at these objects is to attempt to generalize the tools of peak sets and "peak interpolation" from the classical theory of function algebras (due to Bishop, Glicksberg, Gamelin, and others). In particular, we reduce the main open question posed in [27], namely whether the p-projections coincide with the support projections of r-ideals, to a simple sounding question about approximate identities: If A is an approximately unital operator algebra then does A have an approximate identity of form $(1 - x_t)$ with $x_t \in \mathrm{Ball}(A^1)$? Here 1 is the identity of the unitization A^1 of A. We imagine that the answer to this is in the negative. We also show that p-projections are exactly the closed projections satisfying the "nonselfadjoint Urysohn lemma" or "peaking" property discussed at the beginning of this introduction. Thus even if the question above turns out in the negative, these projections should play an important role in future "nonselfadjoint interpolation theory".

Hereditary subalgebras of not necessarily selfadjoint unital operator algebras have previously been considered in the papers [32], [35] on inner ideals. We thank Lunchuan Zhang for sending us a copy of these papers. Another work that has a point of contact with our paper is the unpublished note [29]. Here *quasi-M-ideals*, an interesting variant of the one-sided M-ideals of Blecher, Effros, and Zarikian [10] were defined. Kaneda showed that the product RL of an r-ideal and an l-ideal in an approximately unital operator algebra A is an inner ideal (inner ideals are called "quasi-ideals" there), and is a quasi-M-ideal. It is also noted there that in a C^*-algebra A, the following three are the same: quasi-M-ideals in A, products RL of an r-ideal and an l-ideal, and inner ideals (see also [17], particularly Corollary 2.6 there). Hereditary subalgebras in the sense of our paper were not considered in [29]. We thank Kaneda for permission to describe his work here and in Section 5.

Some notations: In this paper, all projections are orthogonal projections. If X and Y are sets (in an operator algebra say) then we write XY for the *norm closure* of the span of terms of the form xy, for $x \in X, y \in Y$. The second dual A^{**} of an operator algebra A is again an operator algebra, and the first dual A^* is a bimodule over A^{**} via the actions described, for example, on the bottom of p. 78 of [11]. A projection p in the second dual of a C^*-algebra B is called *open* if it is the sup of an increasing net of positive elements of B. Such projections p are in a bijective correspondence with the right ideals J of B, or with the HSA's (see [33]). It is well known, and easy to see, that p is open if and only if there is a net (x_t) in B with $x_t \to p$ weak*, and $px_t = x_t$. We recall that TRO's are essentially the same thing as Hilbert C^*-modules, and may be viewed as closed subspaces Z of C^*-algebras with the property that $ZZ^*Z \subset Z$. See e.g. Section 8.3 of [11]. Every operator space X has a "noncommutative Shilov boundary" or "ternary envelope" (Z, j) consisting of a TRO Z and a complete isometry $j : X \to Z$ whose range "generates" Z. This ternary envelope has a universal property which may be found in [24], [11]: For any complete isometry $i : X \to Y$ into a TRO Y, whose range "generates" Y, there exists a (necessarily unique and surjective) "ternary morphism" $\theta : Y \to Z$ such that $\theta \circ i = j$. If A is an approximately unital operator algebra then the noncommutative Shilov boundary is written as $C^*_e(A)$ (see e.g. Section 4.3 of [11]), and was first introduced by Arveson [5].

2. HEREDITARY SUBALGEBRAS

Throughout this section A is an operator algebra (possibly not approximately unital). Then A^{**} is an operator algebra. We shall say that a projection p in A^{**} is *open in* A^{**} if $p \in (pA^{**}p \cap A)^{\perp\perp}$. In this case we also say that $p^\perp$ is *closed in* A^{**}, or is an *approximate p-projection* (this notation was used in [27] since these projections have properties analogous to the *p-sets* in the theory of uniform algebras; see e.g. Theorem 5.12 of [27]). Clearly these notions are independent of any particular C^*-algebra containing A. If A is a C^*-algebra then these concepts coincide with the usual notion of open and closed projections (see e.g. [1], [33]).

EXAMPLE 2.1. Any projection p in the multiplier algebra $M(A) \subset A^{**}$ is open in A^{**}, if A is approximately unital. Indeed $pA^{**}p \cap A = pAp$, and if (e_t) is a cai for A, then $pe_tp \to p$ weak*.

If p is open in A^{**} then clearly $D = pA^{**}p \cap A$ is a closed subalgebra of A, and it has a cai by Proposition 2.5.8 of [11]. We call such a subalgebra a *hereditary subalgebra* of A (or for brevity, a *HSA*). Perhaps more properly (in view of the next result) we should call these "approximately unital HSA's", but for convenience we use the shorter term. We say that p is the *support projection* of the HSA D; and it follows by routine arguments that p is the weak* limit of any cai from D.

PROPOSITION 2.2 ([32], [35]). *A subspace of an operator algebra A is a* HSA *if and only if it is an approximately unital inner ideal.*

Proof. We have already said that HSA's are approximately unital, and clearly they are inner ideals.

If J is an approximately unital inner ideal then by Proposition 2.5.8 of [11] we have that $J^{\perp\perp}$ is an algebra with identity e say. Clearly $J^{\perp\perp} \subset eA^{**}e$. Conversely, by a routine weak* density argument $J^{\perp\perp}$ is an inner ideal, and so $J^{\perp\perp} = eA^{**}e$. Thus $J = eA^{**}e \cap A$, and e is open. ∎

We can often assume that the containing algebra A above is unital, simply by adjoining a unit to A (see Section 2.1 of [11]). Indeed it follows from the last proposition that a subalgebra D of A will be hereditary in the unitization A^1 if and only if it is hereditary in A.

The following is a second (of many) characterization of HSA's. We leave the proof to the reader.

COROLLARY 2.3. *Let A be an operator algebra and suppose that (e_t) is a net in* $\mathrm{Ball}(A)$ *such that $e_t e_s \to e_s$ and $e_s e_t \to e_s$ with t. Then*

$$\{x \in A : xe_t \to x, e_t x \to x\}$$

is a HSA *of A. Conversely, every* HSA *of A arises in this way.*

Note that this implies that any approximately unital subalgebra D of A is contained in a HSA.

We next refine Hay's theorem from [27].

THEOREM 2.4. *Suppose that A is an operator algebra (possibly not approximately unital), and that p is a projection in A^{**}. The following are equivalent:*

(i) *p is open in A^{**}.*

(ii) *p is open as a projection in B^{**}, if B is a C^*-algebra containing A as a subalgebra.*

(iii) *p is the left support projection of an* r*-ideal of A (or, equivalently, p is contained in $(pA^{**} \cap A)^{\perp\perp}$).*

(iv) *p is the right support projection of an* l*-ideal of A.*

(v) *p is the support projection of a hereditary subalgebra of A.*

Proof. That (v) is equivalent to (i) is just the definition of being open in A^{**}. Also (i) implies (ii) by facts about open projections mentioned in the introduction. Supposing (ii), consider A^1 as a unital-subalgebra of B^1. Then p is open as a projection in $(B^1)^{**}$. Since $p \in A^{\perp\perp}$ it follows from Hay's theorem that $J = p(A^1)^{**} \cap A^1$ is an r-ideal of A^1 with left support projection p. If $x \in J$ then

$$x = px \in (A^{\perp\perp}A^1) \cap A^1 \subset A^{\perp\perp} \cap A^1 = A.$$

Thus $J = pA^{**} \cap A$, and we have proved (iii). Thus to complete the proof it suffices to show that (iii) implies (i) (the equivalence with (iv) following by symmetry).

(iii) $\Rightarrow$ (i) First assume that A is unital, in which case (iii) is equivalent to (ii) by Hay's theorem. We work in A^*. As stated in the Introduction, A^* is a right A^{**}-module via the action $(\psi\eta)(a) = \langle \eta a, \psi \rangle$ for $\psi \in A^*, \eta \in A^{**}, a \in A$. Similarly it is a left A^{**} module. Let $q = p^{\perp}$, a closed projection in B^{**} for any C^*-algebra B generated by A. We first claim that $A^*q = J^{\perp}$, where J is the right ideal of A corresponding to p, and so A^*q is weak* closed. To see that $A^*q = J^{\perp}$, note that clearly $A^*q \subset J^{\perp}$, since $J = pA^{**} \cap A$. Thus if $\psi \in J^{\perp}$, then $\psi q \in J^{\perp}$, and so $\psi p \in J^{\perp}$ since $\psi = \psi p + \psi q$. However, if $\psi p \in J^{\perp} = (pA^{**})_{\perp}$, then $\psi p \in (A^{**})_{\perp} = \{0\}$. Thus $\psi = \psi q \in A^* q$.

Similarly, using the equivalence with (ii) here, we have that qA^* is weak* closed. Now $qA^* + A^*q$ is the kernel of the projection $\psi \to p\psi p$ on A^*, and hence it is norm closed. By Lemma I.1.14 of [25], $qA^* + A^*q$ is weak* closed. Claim: $(qA^* + A^*q)^{\perp} = pA^{**}p$. Assuming this claim, note that $(qA^* + A^*q)_{\perp} \subset pA^{**}p \cap A$; and $pA^{**}p \cap A \subset (qA^* + A^*q)^{\perp}$, so that $pA^{**}p \cap A = (qA^* + A^*q)_{\perp}$. Thus $(pA^{**}p \cap A)^{\perp\perp} = pA^{**}p$, and the proof is complete.

In order to prove the claim, first note that it is clear that $pA^{**}p \subset (qA^* + A^*q)^{\perp}$. On the other hand, if $\eta \in (qA^* + A^*q)^{\perp}$ then write $\eta = p\eta p + p\eta q + q\eta p + q\eta q$. Thus $p\eta q + q\eta p + q\eta q \in (qA^* + A^*q)^{\perp}$. In particular, applying this element to a functional $q\psi \in qA^*$ gives

$$0 = \langle p\eta q + q\eta q, q\psi \rangle = \langle p\eta q + q\eta q, \psi \rangle, \quad \psi \in A^*.$$

Thus $p\eta q + q\eta q = 0$, and left multiplying by p shows that $p\eta q = q\eta q = 0$. Similarly $q\eta p = 0$. Thus $\eta \in pA^{**}p$.

Now assume that A is nonunital. If J is the r-ideal, then J is an r-ideal in A^1. Thus by the earlier part, $p \in (p(A^1)^{**}p \cap A^1)^{\perp\perp}$. If (e_t) is the cai for $p(A^1)^{**}p \cap A^1$, then $e_t \to p$ weak*. Since $p(A^1)^{**} \cap A^1 = J$ we have $p(A^1)^{**}p \cap A^1 \subset J \subset A$. Thus $e_t \in pA^{**}p \cap A$, and so $p \in (pA^{**}p \cap A)^{\perp\perp}$. Note too that the above shows that $p(A^1)^{**}p \cap A^1 = pA^{**}p \cap A$. ∎

REMARK 2.5. (i) It is clear from the above that a sup of open projections in A^{**} is open in A^{**}. From this remark, it is easy to give an alternative proof of a result from [13] which states that the closure of the span of a family of r-ideals, again is an r-ideal.

(ii) If A is approximately unital then one can add to the characterization of open projections in the theorem, the condition that $A^*p^{\perp}$ is weak* closed in A^*. The second paragraph of the proof above shows one direction of this. Conversely, if $A^*p^{\perp}$ is weak* closed, then $A^*p^{\perp} = J^{\perp}$ for a subspace J of A such that $J^{\perp\perp} = (A^*p^{\perp})^{\perp} = pA^{**}$. Thus p is the support projection of the r-ideal $A \cap pA^{**} = J$.

(iii) A modification of part of the proof of the theorem shows that if A is approximately unital and if p, r are open projections in A^{**} then $(pA^{**}r \cap A)^{\perp\perp} = pA^{**}r$.

Note that $pA^{**}r \cap A$ is an inner ideal of A. Such subspaces are precisely the intersection of an r-ideal and an l-ideal.

COROLLARY 2.6. *Every operator algebra with a left* cai *has property* $(\mathcal{L})$.

Proof. Let C be an operator algebra with a left cai, and let A be its unitization. Then C is an r-ideal in A, and the left support projection p of C in A^{**} is a weak* limit of the left cai. Also, $C = pA^{**} \cap A$. By Theorem 2.4, we have $p \in (pA^{**}p \cap A)^{\perp\perp}$, and $pA^{**}p \cap A$ is a closed subalgebra of C containing a cai (x_t) with $x_t \to p$ weak*. If $J = \{a \in A : x_t a \to a\}$ then J is a right ideal of A with support projection p, so that $J = C$. Hence C has property $(\mathcal{L})$. ∎

Some implications of this result are mentioned in [9], however our main application appears in the next section.

In the following, we use some notation introduced in [9]. Namely, if J is an operator algebra with a left cai (e_t) such that $e_s e_t \to e_s$ with t, then we set $\mathcal{L}(J) = \{a \in J : ae_t \to a\}$. This latter space does not depend on the particular (e_t), as is shown in [9].

COROLLARY 2.7. *A subalgebra of an operator algebra A is hereditary if and only if it equals $\mathcal{L}(J)$ for an* r*-ideal J of A. Moreover the correspondence $J \mapsto \mathcal{L}(J)$ is a bijection from the set of* r*-ideals of A onto the set of* HSA*'s of A. The inverse of this bijection is the map $D \mapsto DA$. Similar results hold for the* l*-ideals of A.*

Proof. If D is a HSA of A then by Corollary 2.3 we have $D = \{x \in A : xe_t \to x, e_t x \to x\}$, and (e_t) is the cai for D. Set $J = \{x \in A : e_t x \to x\}$, an r-ideal with $D = \mathcal{L}(J)$.

Conversely, if J is an r-ideal then by Corollary 2.6, we can choose a left cai (e_t) of J with the property that $e_s e_t \to e_s$ with t. Then $D = \{x \in A : xe_t \to x, e_t x \to x\}$ is an HSA by Corollary 2.3, and $D = \mathcal{L}(J)$. Note that $\mathcal{L}(J)A \subset J$, and conversely if $x \in J$ then $x = \lim_t e_t x \in \mathcal{L}(J)A$. Thus $J = \mathcal{L}(J)A$. This shows that $J \mapsto \mathcal{L}(J)$ is one-to-one. The last paragraph shows that it is onto. ∎

COROLLARY 2.8. *If D is a hereditary subalgebra of an operator algebra A, and if $J = DA$ and $K = AD$, then $JK = J \cap K = D$.*

Proof. Clearly $JK \subset J \cap K$. Conversely, if $x \in J \cap K$ and (e_t) is the cai for D then $x = \lim_t xe_t \in JK$. So $JK = J \cap K$ (see also e.g. Proposition 6.2 of [3] and Lemma 1.4.1 of [28]). Clearly $JK \subset D$ since D is an inner ideal. Conversely, $D = D^4 \subset JK$. ∎

THEOREM 2.9. *If A is a closed subalgebra of a C^*-algebra B then there is a bijective correspondence between* r*-ideals of A and right ideals of B with left support in $A^{\perp\perp}$. Similarly, there is a bijective correspondence between* HSA*'s of A and* HSA*'s of B with support in $A^{\perp\perp}$. The correspondence takes an* r*-ideal (respectively* HSA*) J of A to JB (respectively JBJ^*). The inverse bijection is simply intersecting with A.*

Proof. We leave the proof of this to the reader, using the ideas above (and, in particular, Hay's theorem). At some point an appeal to Lemma 2.1.6 of [11] might be necessary. ∎

In the C^*-algebra case the correspondence between r-ideals and l-ideals has a simple formula: $J \mapsto J^*$. For nonselfadjoint algebras A, one formula setting up the same correspondence is $J \mapsto A\mathcal{L}(J)$. It is easy to see from the last theorem that, for subalgebras A of a C^*-algebra B, this correspondence becomes $J \mapsto BJ^* \cap A$. Here J is an r-ideal; and notice that $BJ^* \cap A$ also equals $BD^* \cap A$, where D is the associated HSA of A (we remark that by Lemma 2.1.6 of [11] it is easy to see that $BD^* = BD$). This allows us to give another description of $\mathcal{L}(J)$ as $J \cap BJ^*$.

THEOREM 2.10. *Suppose D is a hereditary subalgebra of an approximately unital operator algebra A. Then every $f \in D^*$ has a unique Hahn-Banach extension to a functional in A^* (of the same norm).*

Proof. Let g and h be two such extensions. Since $D = pA^{**}p \cap A$ for an open projection p, it is easy to see that $pgp = php$. Since $\|g\| = \|pgp\| = \|php\| = \|h\|$, we need only show that $g = pgp$ and similarly $h = php$. Consider A^{**} as a unital-subalgebra of a W*-algebra B. Since the canonical projection from B onto $pBp + (1-p)B(1-p)$ is contractive, and since $\|pbp + (1-p)b(1-p)\| = \max\{\|pbp\|, \|(1-p)b(1-p)\|\}$ for $b \in B$, it is easy to argue that

$$\|g\| \geqslant \|pgp + (1-p)g(1-p)\| = \|pgp\| + \|(1-p)g(1-p)\| \geqslant \|g\|.$$

Hence, $(1-p)g(1-p) = 0$. Since $g = pgp + pg(1-p) + pg(1-p) + (1-p)g(1-p)$, it suffices to show that $pg(1-p) + pg(1-p) = 0$. To this end, we follow the proof in Proposition 1 of [22], which proves the analogous result for JB*-triples. For the readers convenience, we will reproduce this pretty argument in our setting, adding a few more details. Of course B is a JB*-triple. We will use the notation $pBp = B_2(p)$, $pB(1-p) + (1-p)Bp = B_1(p)$, and $(1-p)B(1-p) = B_0(p)$. For this proof only, we will write x^{2n+1} for $x(x^*x)^n$ (this unusual notation is used in the JB*-triple literature). In Lemma 1.4 of [22], it is proved that, for $x \in B_2(p) \cup B_0(p)$, $y \in B_1(p)$, and $t > 0$,

$$(x+ty)^{3^n} = x^{3^n} + t2^n D(x^{3^{n-1}}, x^{3^{n-1}}) \cdots D(x^3, x^3) D(x,x)y + O(t^2). \tag{2.1}$$

where, in our setting, $D(w,w)$ is the operator $D(w,w)z = (ww^*z + zw^*w)/2$ on B. Here, $O(t^2)$ denotes a polynomial in x, y, and t, with all terms at least quadratic in t. This polynomial has a certain number of terms that depends only on n, and the coefficients of the monomials in x, y and t also depend only on n.

Choose $y \in B_1(p) \cap A^{**}$. We may assume that $\|g\| = 1$, $g(y) \geqslant 0$, and $\|y\| \leqslant 1$. Given $\varepsilon > 0$, we choose $x \in D$ with $\|x\| = 1$ and $g(x) \geqslant 1 - \varepsilon$. Then, for $t > 0$, we have

$$\|x + ty\| \geqslant g(x+ty) = g(x) + tg(y) \geqslant 1 - \varepsilon + tg(y).$$

Thus, by (2.1) above, and the fact that $\|x\| \leqslant 1$,

$$(1-\varepsilon+tg(y))^{3^n} \leqslant \|x+ty\|^{3^n} = \|(x+ty)^{3^n}\|$$
$$\leqslant \|x^{3^n}\| + t2^n\|y\| + \|O(t^2)\| \leqslant 1 + t2^n\|y\| + p(t),$$

where $p(t)$ is a polynomial in t with all terms at least degree 2, and coefficients which depend only on n and $\|y\|$. Letting $\varepsilon \to 0$ we have $(1+tg(y))^{3^n} \leqslant 1 + t2^n\|y\| + p(t)$, and so

$$1 + 3^n tg(y) \leqslant 1 + t2^n\|y\| + r(t),$$

where $r(t)$ is a polynomial with the same properties as p, and in particular has all terms at least degree 2. Dividing by $3^n t$, we obtain

$$g(y) \leqslant \left(\frac{2}{3}\right)^n \|y\| + \frac{r(t)}{t3^n}.$$

Letting $t \to 0$ and then $n \to \infty$, we see that $g(y) = 0$. Hence $pg(1-p) + (1-p)gp = 0$ as desired. ∎

One might hope to improve the previous theorem to address extensions of completely bounded maps from D into $B(H)$. Unfortunately, simple examples such as the one-dimensional HSA D in ℓ^∞_2 which is supported in the first entry, with $f : D \to M_2$ taking $(1,0)$ to E_{11}, shows that one needs to impose strong restrictions on the extensions. This two dimensional example contradicts several theorems on unique completely contractive extensions in the literature. We found the following positive result after reading [35]. Although some part of it is somewhat tautological, it may be the best that one could hope for. To explain the notation in (iii), if A is an approximately unital operator algebra and B is a unital weak* closed operator algebra, then we say that a bounded map $T : A \to B$ is *weakly nondegenerate* if the canonical weak* continuous extension $\widetilde{T} : A^{**} \to B$ is unital. By 1.4.8 in [11] for example, this is equivalent to: $T(e_t) \to 1_B$ weak* for some contractive approximate identity (e_t) of A; and is also equivalent to the same statement with "some" replaced by "every".

PROPOSITION 2.11. *Let D be an approximately unital subalgebra of an approximately unital operator algebra A. The following are equivalent:*

(i) *D is a hereditary subalgebra of A.*

(ii) *Every completely contractive unital map from D^{**} into a unital operator algebra B, has a unique completely contractive unital extension from A^{**} into B.*

(iii) *Every completely contractive weakly nondegenerate map from D into a unital weak* closed operator algebra B has a unique completely contractive weakly nondegenerate extension from A into B.*

Proof. We are identifying D^{**} with $D^{\perp\perp} \subset A^{**}$. Let e be the identity of D^{**}.

(ii) $\Rightarrow$ (i) If (ii) holds, then the identity map on D^{**} extends to a unital complete contraction $S : A^{**} \to D^{**} \subset eA^{**}e$. The map $x \mapsto exe$ on A^{**} is also a completely contractive unital extension of the inclusion map $D^{**} \to eA^{**}e$. It follows from the hypothesis that these maps coincide, and so $eA^{**}e = D^{**}$, which implies that D is a HSA.

(i) $\Rightarrow$ (ii) If D is a HSA, then extensions of the desired kind exist by virtue of the canonical projection from A^{**} onto $D^{\perp\perp}$. For the uniqueness, suppose that Φ is such an extension of a completely contractive unital map $T : D^{\perp\perp} \to B$. Since e is an orthogonal projection in A^{**}, it follows from the last remark in 2.6.16 of [11] that

$$T(exe) = \Phi(exe) = \Phi(e)\Phi(x)\Phi(e) = \Phi(x), \quad x \in A^{**}.$$

Hence (ii) holds.

Inspecting the proof above shows that (i) is equivalent to the variant of (ii) where B is weak* closed and all maps are also weak* continuous. Then the equivalence with (iii) is easy to see using the facts immediately above the proposition statement, and also the bijective correspondence between complete contractions $A \to B$ and weak* continuous complete contractions $A^{**} \to B$ (see 1.4.8 in [11]). ∎

3. APPLICATION: A GENERALIZATION OF C^*-MODULES

In the early 1990's, the first author together with Muhly and Paulsen generalized Rieffel's strong Morita equivalence to nonselfadjoint operator algebras [12]. This study was extended to include a generalization of Hilbert C^*-modules to nonselfadjoint algebras, which were called *rigged modules* in [6], and *(P)-modules* in [12]. See Section 11 of [7] for a survey. There are very many equivalent definitions of these objects in these papers. The main purpose of this section is to settle a problem going back to the early days of this theory. This results in the conceptually clearest definition of rigged modules; and also tidies up one of the characterizations of strong Morita equivalence. The key tool we will use is the Corollary 2.6 to our main theorem from Section 2.

Throughout this section, A is an approximately unital operator algebra. For a positive integer n we write $C_n(A)$ for the $n \times 1$ matrices with entries in A, which may be thought of as the first column of the operator algebra $M_n(A)$. In our earlier work mentioned above $C_n(A)$ plays the role of the prototypical right A-rigged module, out of which all others may be built via "asymptotic factorizations" similar to the kind considered next.

DEFINITION 3.1. An operator space Y which is also a right A-module is *A-Hilbertian* if there exists a net of positive integers n_α, and completely contractive A-module maps $\varphi_\alpha : Y \to C_{n_\alpha}(A)$ and $\psi_\alpha : C_{n_\alpha}(A) \to Y$, such that $\psi_\alpha \varphi_\alpha \to I_Y$ strongly on Y.

The name "A-Hilbertian" is due to Paulsen around 1992, who suggested that these modules should play an important role in the Morita theory. A few years later the question became whether they coincide with the rigged modules/(P)-modules from [6], [12]. This question appears explicitly in Section 11 of [7] for example, and was discussed also several times in Chapter 4 of [12] in terms of the necessity of adding further conditions to what we called the "approximate identity property". Assuming for simplicity that A is unital, one of the many equivalent definitions of rigged modules is that they are the modules satisfying Definition 3.1, but that in addition $\varphi_\beta \psi_\alpha \varphi_\alpha \to \varphi_\beta$ in norm for each fixed β in the directed set. We were not able to get the theory going without this extra condition. Thus the open question referred to above may be restated as follows: can one always replace the given nets in Definition 3.1 with ones which satisfy this additional condition? The first author proved this if A is a C^*-algebra in [8]; indeed A-Hilbertian modules coincide with C^*-modules if A is a C^*-algebra. A simpler proof of this result due to Kirchberg is included in Theorem 4.24 of [12] (there is a typo there, in Lemma 4.25, x has norm 1).

Although the "asymptotic factorization" in the definition above is clean, it can sometimes be clumsy to work with, as is somewhat illustrated by the proof of the next result.

PROPOSITION 3.2. *Let Y be an operator space and right A-module, such that there exists a net of positive integers n_α, A-Hilbertian modules Y_α, and completely contractive A-module maps $\varphi_\alpha : Y \to Y_\alpha$ and $\psi_\alpha : Y_\alpha \to Y$, such that $\psi_\alpha \varphi_\alpha \to I_Y$ strongly. Then Y is A-Hilbertian.*

Proof. We use a net reindexing argument based on Lemma 2.1 of [6]. Suppose that $\sigma^\alpha_\beta : Y_\alpha \to Z^\alpha_\beta$ and $\tau^\alpha_\beta : Z^\alpha_\beta \to Y_\alpha$, are the "asymptotic factorization" nets corresponding to Y_α. We define a new directed set Γ consisting of 4-tuples $\gamma = (\alpha, \beta, V, \varepsilon)$, where V is a finite subset of Y, $\varepsilon > 0$, and such that

$$\|\psi_\alpha \tau^\alpha_\beta \sigma^\alpha_\beta \varphi_\alpha(y) - \psi_\alpha \varphi_\alpha(y)\| < \varepsilon, \quad y \in V.$$

This is a directed set with ordering $(\alpha, \beta, V, \varepsilon) \leqslant (\alpha', \beta', V', \varepsilon')$ if and only if $\alpha \leqslant \alpha', V \subset V'$ and $\varepsilon' \leqslant \varepsilon$. (We recall that directed sets for nets make no essential use of the "antisymmetry" condition for the ordering, and we follow many authors in not requiring this.) Define $\varphi^\gamma = \sigma^\alpha_\beta \varphi_\alpha$ and $\psi^\gamma = \psi_\alpha \tau^\alpha_\beta$, if $\gamma = (\alpha, \beta, V, \varepsilon)$. Given $y \in Y$ and $\varepsilon > 0$, choose α_0 such that $\|\psi_\alpha \varphi_\alpha(y) - y\| < \varepsilon$ whenever $\alpha \geqslant \alpha_0$. Choose β_0 such that $\gamma_0 = (\alpha_0, \beta_0, \{y\}, \varepsilon) \in \Gamma$. If $\gamma \geqslant \gamma_0$ in Γ then

$$\|\psi^\gamma \varphi^\gamma(y) - y\| \leqslant \|\psi_\alpha \tau^\alpha_\beta \sigma^\alpha_\beta \varphi_\alpha(y) - \psi_\alpha \varphi_\alpha(y)\| + \|\psi_\alpha \varphi_\alpha(y) - y\| < \varepsilon' + \varepsilon \leqslant 2\varepsilon.$$

Thus $\psi^\gamma \varphi^\gamma(y) \to y$, and so Y is A-Hilbertian. ∎

REMARK 3.3. If desired, the appearance of the integers n_α in Definition 3.1 may be avoided by the following trick. Let $C_\infty(A)$ be the space of columns $[x_k]_{k \in \mathbb{N}}$, with $x_k \in A$, such that $\sum_k x_k^* x_k$ converges in A. It is easy to see that $C_\infty(A)$ is A-Hilbertian, and for any $m \in \mathbb{N}$ there is an obvious factorization of the identity map on $C_m(A)$ through $C_\infty(A)$. It follows from this, and from the last proposition, that Definition 3.1 will be unchanged if all occurrences of n_α there are replaced by ∞.

THEOREM 3.4. *An operator space Y which is also a right A-module is a rigged A-module if and only if it is A-Hilbertian. This is also equivalent to Y having the "approximate identity property" of Definition* 4.6 *in* [12].

Proof. Suppose that Y is an operator space and a right A-module which is A-Hilbertian. It is easy to see that Y is an operator A-module, since the $C_{n_\alpha}(A)$ are, and since

$$\|[y_{ij}]\| = \sup_\alpha \|[\varphi_\alpha(y_{ij})]\| = \lim_\alpha \|[\varphi_\alpha(y_{ij})]\|, \quad [y_{ij}] \in M_n(Y).$$

If (e_t) is a cai for A then the triangle inequality easily yields that for any α,

$$\begin{aligned} \|y - ye_t\| &= \|y - \psi_\alpha\varphi_\alpha(y) + \psi_\alpha(\varphi_\alpha(y) - \varphi_\alpha(y)e_t) + (\psi_\alpha\varphi_\alpha(y) - y)e_t\| \\ &\leqslant 2\|y - \psi_\alpha\varphi_\alpha(y)\| + \|\varphi_\alpha(y) - \varphi_\alpha(y)e_t\|, \end{aligned}$$

from which the nondegeneracy is easily seen. Next, we reduce to the unital case. Let $B = A^1$, the unitization of A. Note that A is B-Hilbertian: the maps $A \to B$ and $B \to A$ being respectively the inclusion, and left multiplication by elements in the cai (e_t). Tensoring these maps with the identity map on C_{n_α}, we see that $C_{n_\alpha}(A)$ is B-Hilbertian. By Proposition 3.2, Y is B-Hilbertian. By Proposition 2.5 of [6] it is easy to see that Y satisfies (the right module variant of) Definition 4.6 (ii) of [12]. By the results following that definition we have that $C = Y \otimes_{hB} CB_B(Y, B)$ is a closed right ideal in $CB_B(Y)$ which has a left cai. By Theorem 2.7 of [6] or Theorem 4.9 of [12] we know that $CB_B(Y)$ is a unital operator algebra.

By Corollary 2.6, C possesses a left cai (v_β) such that $v_\gamma v_\beta \to v_\gamma$ with β for each γ. Let $D = \{a \in C : av_\beta \to a\}$, which is an operator algebra with cai. Since the uncompleted algebraic tensor product J of Y with $CB_B(Y, B)$ is a dense ideal in C, and since $Jv_\gamma \subset D$ for each γ, it is easy to see by the triangle inequality that $J \cap D$ is a dense ideal in D. Thus we can rechoose a cai (u_ν) for D from this ideal, if necessary by using Lemma 2.1 of [6]. This cai will be a left cai for C (e.g. see proof of Corollary 2.6). This implies that Y satisfies (a), and hence also (b), of Definition 4.12 in [12]. That is, Y is a (P)-module, or equivalently a rigged module, over B. It is known that this implies that Y is A-rigged. One way to see this is to observe that by an application of Cohen's factorization theorem as in Lemma 8.5.2 of [11], we have $\mathbb{B}_B(Y, B) = \mathbb{B}_A(Y, A)$. It follows that Y satisfies Definition 4.12 of [12] as an A-module, and hence Y is an A-rigged module. That every rigged module is A-Hilbertian follows from Definition 3.1 of [6]. The equivalence with the "approximate identity property" is essentially contained in the above argument. ∎

This theorem impacts only a small portion of [6]. Namely, that paper may now be improved by replacing Definition 3.1 there by the modules in Definition 3.1 above; and by tidying up some of the surrounding exposition. One may also now give alternative constructions of, for example, the interior tensor product of rigged modules, by following the idea in Theorem 8.2.11 of [11].

Similarly, one may now tidy up one of the characterizations of strong Morita equivalence. By the above, what was called the "approximate identity property" in Chapter 4 of [12] implies that the module is a (P)-module, and so in Theorems 4.21 and 4.23 in [12] one may replace conditions from Definition 4.12 with those in 4.6. That is, we have the following improved characterization of the strong Morita equivalence of [12]. (The reader needing further details is referred to that source.)

THEOREM 3.5. *If Y is a right A-Hilbertian module with the "dual approximate identity property" of Definition* 4.18 *in* [12], *then Y implements a strong Morita equivalence between A and the algebra $\mathbb{K}_A(Y)$ of so-called "compact" operators on Y. Conversely, every strong Morita equivalence arises in such a way.*

REMARK 3.6. The "dual approximate identity property" mentioned in the theorem may also be phrased in terms of "asymptotic factorization" of I_A through spaces of the form $C_m(Y)$ — this is mentioned in p. 416 of [6] with a mistake that is discussed in Remark 4.20 of [12].

We refer the reader to [6] for the theory of rigged modules. It is easy to see using Corollary 2.8 or Theorem 3.4, that any hereditary subalgebra D of an approximately unital operator algebra A gives rise to a rigged module. Indeed, if $J = DA$, then J is a right rigged A-module, the canonical dual rigged module $\widetilde{J}$ is just the matching l-ideal AD, and the operator algebra $\mathbb{K}_A(J)$ of "compact operators" on J is just D completely isometrically isomorphically. From the theory of rigged modules [6] we know for example that any completely contractive representation of A induces a completely contractive representation of D, and vice versa. More generally, any left operator A-module will give rise to a left operator D-module by left tensoring with J, and vice versa by left tensoring with $\widetilde{J}$. Since $J \otimes_{hA} \widetilde{J} = D$ it follows that there is an "injective" (but not in general "surjective") functor from D-modules to A-modules.

If A and B are approximately unital operator algebras which are strongly Morita equivalent in the sense of [12], then A and B will clearly be hereditary subalgebras of the "linking operator algebra" associated with the Morita equivalence [12]. Unfortunately, unlike the C^*-algebra case, not every HSA D of an operator algebra A need be strongly Morita equivalent to ADA. One would also need a condition similar to that of Definition 5.10 in [12]. Assuming the presence of such an extra condition, it follows that the representation theory for the algebra A is "the same" as the representation theory of D; as is always the case if one has a Morita equivalence.

EXAMPLE 3.7. If $a \in \mathrm{Ball}(A)$, for an operator algebra A, let D be the closure of $(1-a)A(1-a)$. Then it follows from the later Lemma 6.8 that D is a hereditary subalgebra of A. The associated r-ideal is J, the closure of $(1-a)A$. The dual rigged module $\widetilde{J}$ is equal to the closure of $A(1-a)$, and $\mathbb{K}_A(J) \cong D$. It is easy to check that even for examples of this kind, the C^*-algebra $C^*(D)$ generated by D need not be a hereditary C^*-subalgebra of $C^*(A)$ or $C^*_e(A)$. For example, take A to be the subalgebra of $M_2(B(H))$ consisting of all matrices whose 1-1 and 2-2 entries are scalar multiples of I_H, and whose 2-1 entry is 0. Let $a = 0_H \oplus I_H$. In this case $D = (1-a)A(1-a)$ is one dimensional, and it is not a HSA of $C^*(A)$. Also D is not strongly Morita equivalent to ADA.

4. CLOSED FACES AND LOWERSEMICONTINUITY

Suppose that A is an approximately unital operator algebra. The state space $S(A)$ is the set of functionals $\varphi \in A^*$ such that $\|\varphi\| = \lim_t \varphi(e_t) = 1$, if (e_t) is a cai for A. These are all restrictions to A of states on any C^*-algebra generated by A. If p is a projection in A^{**}, then any $\varphi \in S(A)$ may be thought of as a state on A^{**}, and hence $p(\varphi) \geqslant 0$. Thus p gives a nonnegative scalar function on $S(A)$, or on the quasistate space (that is, $\{\alpha\varphi : 0 \leqslant \alpha \leqslant 1, \varphi \in S(A)\}$). We shall see that this function is lowersemicontinuous if and only if p is open in A^{**}.

In the following generalization of a well known result from the C^*-algebra theory [33], we assume for simplicity that A is unital. If A is only approximately unital then a similar result holds with a similar proof, but one must use the quasistate space in place of $S(A)$; this is weak* compact.

THEOREM 4.1. *Suppose that A is a unital-subalgebra of a C^*-algebra B. If p is a nontrivial projection in $A^{\perp\perp} \cong A^{**}$, then the following are equivalent:*

(i) *p is open as a projection in B^{**} (or, equivalently, in A^{**}).*

(ii) *The set $F_p = \{\varphi \in S(A) : \varphi(p) = 0\}$ is a weak* closed face in $S(A)$.*

(iii) *p is lowersemicontinuous on $S(A)$.*

Proof. (i) $\Rightarrow$ (ii) For any nontrivial projection $p \in A^{\perp\perp}$, the set F_p is a face in $S(A)$. For if $\psi_i \in S(A)$, $t \in (0,1)$, and $t\psi_1 + (1-t)\psi_2 \in F_p$ then $t\psi_1(p) + (1-t)\psi_2(p) = 0$ which forces $\psi_1(p) = \psi_2(p) = 0$, and $\psi_i \in F_p$. If p is open then $G_p = \{\varphi \in S(B) : \varphi(p) = 0\}$ is a weak* compact face in $S(B)$ by 3.11.9 in [33]. The restriction map $r : \varphi \in S(B) \mapsto \varphi_{|A} \in S(A)$ is weak* continuous, and maps G_p into F_p. On the other hand, if $\varphi \in F_p$ and $\widehat{\varphi}$ is a Hahn-Banach extension of φ to B then one can show that $\langle p, \varphi \rangle = \langle p, \widehat{\varphi} \rangle$, and so the map r above maps G_p onto F_p. Hence F_p is weak* closed.

(ii) $\Rightarrow$ (i) We use the notation of the last paragraph. If F_p is weak* closed, then the inverse image of F_p under r is weak* closed. But this inverse image is G_p, since if $\varphi \in S(B)$ then $\langle p, \varphi \rangle = \langle p, r(\varphi) \rangle$ by a fact in the last paragraph. Thus by 3.11.9 in [33] we have (i).

(i) $\Rightarrow$ (iii) If p is open, then p is a lowersemicontinuous function on $S(B)$. Thus $\{\varphi \in S(B) : \langle p, \varphi \rangle \leqslant t\}$ is weak* compact for any $t \geqslant 0$. Hence its image under the map r above, is weak* closed in $S(A)$. However, as in the above, this image is $\{\varphi \in S(A) : \langle p, \varphi \rangle \leqslant t\}$. Thus p is lowersemicontinuous on $S(A)$.

(iii) $\Rightarrow$ (i) If p gives a lowersemicontinuous function on $S(A)$, then the composition of this function with $r : S(B) \to S(A)$ is lowersemicontinuous on $S(B)$. By facts in p. 77 of [33], we have that p is open. ∎

REMARK 4.2. Not all weak* closed faces of $S(A)$ are of the form in (ii) above. For example, let A be the algebra of 2×2 upper triangular matrices with constant diagonal entries. In this case $S(A)$ may be parametrized by complex numbers z in a closed disk of a certain radius centered at the origin. Indeed states are determined precisely by the assignment $e_{12} \mapsto z$. The faces of $S(A)$ thus include the faces corresponding to singleton sets of points on the boundary circle; and none of these faces equal F_p for a projection $p \in A = A^{\perp\perp}$.

In view of the classical situation, it is natural to ask about the relation between minimal closed projections in B^{**} which lie in $A^{\perp\perp}$ and the noncommutative Shilov boundary mentioned in the introduction. By the universal property of the latter object, if B is generated as a C^*-algebra by its subalgebra A, then there is a canonical $*$-epimorphism θ from B onto the noncommutative Shilov boundary of A, which in this case is a C^*-algebra. The kernel of θ is called (Arveson's) *Shilov boundary ideal* for A. See e.g. [5] and the third remark in 4.3.2 of [11].

PROPOSITION 4.3. *If B is generated as a C^*-algebra by a closed unital-subalgebra A, let p be the open central projection in B^{**} corresponding to the Shilov ideal for A. Then $p^\perp$ dominates all minimal projections in B^{**} which lie in $A^{\perp\perp}$.*

Proof. Suppose that q is a minimal projection in B^{**} which lies in $A^{\perp\perp}$. Then either $qp = 0$ or $qp = q$. Suppose that $qp = q$. If θ is as above, then since θ annihilates the Shilov ideal we have

$$\theta^{**}(q) = \theta^{**}(qp) = \theta^{**}(q)\theta^{**}(p) = 0.$$

On the other hand, θ is a complete isometry from the copy of A in B to the copy of A in $\theta(B)$, and so θ^{**} restricts to a complete isometry on $A^{\perp\perp}$. Thus $qp = 0$, so that $q = qp^\perp$ and $q \leqslant p^\perp$. ∎

EXAMPLE 4.4. The sup of closed projections in A^{**} which are also minimal projections in B^{**} need not give the "noncommutative Shilov boundary". Indeed if A is the 2×2 upper triangular matrices with constant main diagonal entries, then there are no nonzero minimal projections in M_2 which lie in A.

5. HEREDITARY *M*-IDEALS

A *left M-projection* of an operator space X is a projection in the C^*-algebra of (left) adjointable maps on X; and the latter may be viewed as the restrictions of adjointable right module maps on a C^*-module containing X (see e.g. Theorem 4.5.15 and Section 8.4 in [11]). This C^*-module can be taken to be the ternary envelope of X. The range of a left M-projection is a *right M-summand* of X. A *right M-ideal* of an operator space X is a subspace J such that $J^{\perp\perp}$ is a right M-summand of X^{**}. The following result from [10] has been sharpened in the summand case:

PROPOSITION 5.1. *If A is an approximately unital operator algebra, then the left M-projections on A are precisely "left multiplications" by projections in the multiplier algebra $M(A)$. Such projections are all open in A^{**}. The right M-summands of A are thus the spaces pA for a projection $p \in M(A)$. The right M-ideals of A coincide with the* r*-ideals of A.*

Proof. We claim that if p is a projection (or more generally, any hermitian) in the left multiplier algebra $LM(A)$, then $p \in M(A)$. Suppose that B is a C^*-algebra generated by A, and view $LM(A) \subset A^{\perp\perp} \subset B^{**}$. If $a \in A$ and if (e_t) is a cai for A, then by Lemma 2.1.6 of [11] we have $pa^* = \lim_t pe_ta^* \in B$. Thus p is a selfadjoint element of $LM(B)$, and so $p \in M(B)$. Thus $Ap \subset B \cap A^{\perp\perp} = A$, and so $p \in M(A)$. Hence p is open as remarked early in Section 2. The remaining assertions follow from Proposition 6.4 of [10]. ∎

The M-ideals of a unital operator algebra are the approximately unital two-sided ideals [20]. In this case these coincide with the *complete M-ideals* of [21], which are shown in [10] to be just the right M-ideals which are also left M-ideals. See e.g. Section 7 of [13] for more information on these. The HSA's of C^*-algebras are just the selfadjoint inner ideals as remarked in the introduction; or equivalently as we shall see below, they are the selfadjoint "quasi-M-ideals". With the above facts in mind, it is tempting to try to extend some of our results for ideals and hereditary algebras to general M-ideals, be they one-sided, two-sided, or "quasi". A first step along these lines is motivated by the fact, which we have explored in Theorem 2.9 and in [27], that r-ideals in an operator algebra A are closely tied to a matching right ideal in a C^*-algebra B containing A. We will show that a general (one-sided, two-sided, or "quasi") M-ideal in an arbitrary operator space X is the intersection of X with the same variety of M-ideal in any C^*-algebra or TRO containing X. This generalizes a well known fact about M-ideals in subspaces of $C(K)$ spaces (see Proposition I.1.18 of [25]).

For an operator space X, Kaneda proposed in [29] a *quasi-M-ideal* of X to be a subspace $J \subset X$ such that $J^{\perp\perp} = pX^{**}q$ for respectively left and right M-projections p and q of X^{**}. Right (respectively two-sided, "quasi") M-ideals of a TRO or C^*-module are exactly the right submodules (respectively subbimodules, inner ideals). See e.g. p. 339 of [11] and [15], [16]. Here, by an inner ideal of a TRO Z we mean a subspace J with $JZ^*J \subset J$. The assertion here that they coincide with the quasi M-ideals of Z follows immediately from Edwards and Rüttimann's characterization of weak* closed inner ideals. Indeed if J is an inner ideal of Z, then so is $J^{\perp\perp}$; hence [16] gives that $J^{\perp\perp}$ is of the desired form $pZ^{**}q$. The other direction follows by reversing this argument (it also may be seen as a trivial case of Theorem 5.4 below). In fact Kaneda has considered the quasi-M-ideals of an approximately unital operator algebra A in this unpublished work [29]. What we will need from this is the following argument: If $J \subset A$ is a quasi-M-ideal, then by Proposition 5.1 it is clear that there exist projections $p, q \in A^{**}$ such that $J^{\perp\perp} = pA^{**}q$. Thus J is the algebra $pA^{**}q \cap A$.

PROPOSITION 5.2. *The hereditary subalgebras of an approximately unital operator algebra A are precisely the approximately unital quasi-M-ideals.*

Proof. If $J \subset A$ is a quasi-M-ideal, then as we stated above, there exist projections $p, q \in A^{**}$ such that $J^{\perp\perp} = pA^{**}q$, and $J = pA^{**}q \cap A$. If this is approximately unital then by Proposition 2.5.8 of [11] $pA^{**}q$ contains a projection e which is the identity of $pA^{**}q$. Since $e = peq$ we have $e \leqslant p$ and $e \leqslant q$. So $pA^{**}q = epA^{**}qe = eA^{**}e$. Thus $J = eA^{**}e \cap A$, which is a HSA. Conversely, if D is a HSA then $J^{\perp\perp} = pA^{**}p$, and so J is a quasi-M-ideal. ∎

If $\mathcal{S}$ is a subset of a TRO we write $\langle \mathcal{S} \rangle$ for the subTRO generated by $\mathcal{S}$. We write $\widehat{\ }$ for the canonical map from a space into its second dual.

LEMMA 5.3. *If X is an operator space, and if $(\mathcal{T}(X^{**}), j)$ is a ternary envelope of X^{**}, then $\langle j(\widehat{X}) \rangle$ is a ternary envelope of X.*

Proof. This follows from a diagram chase. Suppose that $i : X \to W$ is a complete isometry into a TRO, such that $\langle i(X) \rangle = W$. Then $i^{**} : X^{**} \to W^{**}$ is a complete isometry. By the universal property of the ternary envelope, there is a ternary morphism $\theta : \langle i^{**}(X^{**}) \rangle \to \mathcal{T}(X^{**})$ such that $\theta \circ i^{**} = j$. Now W may also be regarded as the subTRO of W^{**} generated by $i(X)$, and the restriction π of θ to $W = \langle i(X) \rangle$ is a ternary morphism into $\mathcal{T}(X^{**})$ which has the property that $\pi(i(x)) = j(\widehat{x})$. Thus $\langle j(\widehat{X}) \rangle$ has the universal property of the ternary envelope. ∎

THEOREM 5.4. *Suppose that X is a subspace of a* TRO *Z and that J is a right M-ideal (respectively quasi M-ideal, complete M-ideal) of X. In the "complete M-ideal" case we also assume that $\langle X \rangle = Z$. Then J is the intersection of X with the right M-ideal (respectively quasi M-ideal, complete M-ideal) JZ^*Z (respectively JZ^*J, ZJ^*Z) of Z.*

Proof. There are three steps. We will also use the fact that in a TRO Z, for any $z \in Z$ we have that z lies in the closure of $z\langle z \rangle^* z$. This follows by considering the polar decomposition $z = u|z|$, which implies that $zz^*z = u|z|^3$, for example. Then use the functional calculus for $|z|$, and the fact that one may approximate the monomial t by polynomials in t with only odd powers and degree $\geqslant 3$. Similarly, z lies in the closure of $\langle z \rangle z^* \langle z \rangle$.

First, suppose that Z is the ternary envelope $\mathcal{T}(X)$ of X. Suppose that J is a right M-ideal (respectively quasi-M-ideal, complete M-ideal) in X. If $(\mathcal{T}(X^{**}), j)$

is a ternary envelope of X^{**}, then $j(J^{\perp\perp}) = pj(X^{**})$ for a left adjointable projection p (respectively $j(J^{\perp\perp}) = pj(X^{**})q$ for left/right adjointable projections p, q) on $\mathcal{T}(X^{**})$. In the complete M-ideal case we have $pw = wq$ for all $w \in \mathcal{T}(X^{**})$; this follows from e.g. Theorem 7.4 (vi) of [13] and its proof. We view $\mathcal{T}(X) \subset \mathcal{T}(X^{**})$ as above. Let $\widetilde{J}$ be the set of $z \in \mathcal{T}(X)$ such that $pz = z$ (respectively $z = pzq$). Then $\widetilde{J} \cap j(\widehat{X}) = j(\widehat{J})$, since $J = J^{\perp\perp} \cap X$. Next, define $\overline{J} = j(\widehat{J})\mathcal{T}(X)^*\mathcal{T}(X)$ (respectively $= j(\widehat{J})\mathcal{T}(X)^*j(\widehat{J}), = \mathcal{T}(X)j(\widehat{J})^*\mathcal{T}(X)$). This is a right M-ideal (respectively inner ideal, M-ideal) in $\mathcal{T}(X)$, and it is clear, using the fact in the first paragraph of the proof, that

$$j(\widehat{J}) \subset \overline{J} \cap j(\widehat{X}) \subset \widetilde{J} \cap j(\widehat{X}) = j(\widehat{J}).$$

Thus $J = \overline{J} \cap X$.

In the rest of the proof we consider only the quasi M-ideal case, the others are similar.

Second, suppose that X generates Z as a TRO. Let $j : X \to \mathcal{T}(X)$ be the Shilov embedding. If $x \in (JZ^*J) \cap X$ then applying the universal property of $\mathcal{T}(X)$ there exists a ternary morphism $\theta : Z \to \mathcal{T}(X)$ with

$$j(x) = \theta(x) \in j(X) \cap \theta(J)\theta(Z)^*\theta(J) \subset j(X) \cap j(J)\mathcal{T}(X)^*j(J) = j(X) \cap \overline{J} = j(J),$$

by the last paragraph. Hence $x \in J$.

Third, suppose that $X \subset Z$, and that the subTRO W generated by X in Z is not Z. We claim that $X \cap (JZ^*J) = X \cap (JW^*J)$. To see this, we set $J' = JW^*J$. This is an inner ideal in W. Moreover, $J \subset J'$ by the fact at the start of the proof. We claim that for any inner ideal K in W, we have $(KZ^*K) \cap W = K$. Indeed if e and f are the support projections for K, then

$$(KZ^*K) \cap W \subset (eZf) \cap W \subset (eWf) \cap W = K.$$

This implies

$$X \cap (JZ^*J) \subset X \cap (J'Z^*J') \subset X \cap J' = X \cap (JW^*J) = J,$$

as required. ∎

6. REMARKS ON PEAK AND p-PROJECTIONS

Let A be a unital-subalgebra of a C^*-algebra B. We recall from [27] that a *peak projection* q for A is a closed projection in B^{**}, such that there exists an $a \in \mathrm{Ball}(A)$ with $qa = q$ and satisfying any one of a long list of equivalent conditions; for example $\|ar\| < 1$ for every closed projection $r \leqslant q^{\perp}$. We say that a peaks at q. A *p-projection* is an infimum of peak projections; and this is equivalent to it being a weak* limit of a decreasing net of peak projections by Proposition 5.6 of [27]. Every p-projection is an *approximate p-projection*, where the latter term means a closed projection in A^{**}. The most glaring problem concerning these projections is that it is currently unknown whether the converse of this is true, as is the case in the classical setting of function algebras [23]. Motivated partly by this question, in this section we offer several results concerning these projections. Our next result implies that this question is equivalent to the following simple-sounding question:

Question: Does every approximately unital operator algebra A have an approximate identity of form $(1 - x_t)$ with $x_t \in \mathrm{Ball}(A^1)$? Here 1 is the identity of the unitization A^1 of A.

Equivalently, does every operator algebra A with a left cai have a left cai of the form $(1 - x_t)$ for $x_t \in \mathrm{Ball}(A^1)$?

By a routine argument, these are also equivalent to: If A is an approximately unital operator algebra and $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$, does there exist $x \in \mathrm{Ball}(A^1)$ with $1 - x \in A$ and $\|xa_k\| < \varepsilon$ for all $k = 1, \ldots, n$?

Note that if these were true, and if A does not have an identity, then necessarily $\|x_t\| = 1$. For if $\|x_t\| < 1$ then $1 - x_t$ is invertible in A^1, so that $1 \in A$.

THEOREM 6.1. *If J is a closed subspace of a unital operator algebra A, then the following are equivalent:*

(i) *J is a right ideal with a left approximate identity (respectively a* HSA *with approximate identity) of the form $(1 - x_t)$ for $x_t \in \mathrm{Ball}(A)$.*

(ii) *J is an r-ideal (respectively* HSA*) for whose support projection p we have that $p^\perp$ is a p-projection for A.*

Proof. Suppose that $J = \{a \in A : q^\perp a = a\}$ for a p-projection q for A in B^{**}. We may suppose that q is a decreasing weak* limit of a net of peak projections (q_t) for A. If $a \in A$ peaks at q_t, then by a result in [27] we have that $a^n \to q_t$ weak*. Next let $\mathcal{C} = \{1 - x : x \in \mathrm{Ball}(A)\} \cap J$, a convex subset of J containing the elements $1 - a^n$ above. Thus $q_t^\perp \in \overline{\mathcal{C}}^{\mathrm{w}*}$ and therefore $q^\perp \in \overline{\mathcal{C}}^{\mathrm{w}*}$. Let $e_t \in \mathcal{C}$ with $e_t \to q^\perp$ w*. Then $e_t x \to q^\perp x = x$ weak* for all $x \in J$. Thus $e_t x \to x$ weakly. Next, for fixed $x_1, \ldots, x_m \in J$ consider the convex set $F = \{(x_1 - ux_1, x_2 - ux_2, \ldots, x_m - ux_m) : u \in \mathcal{C}\}$. (In the HSA case one also has to include coordinates $x_k - x_k u$ here.) Since $(0, 0, \ldots, 0)$ is in the weak closure of F it is in the norm closure. Given $\varepsilon > 0$, there exists $u \in \mathcal{C}$ such that $\|x_k - ux_k\| < \varepsilon$ for all $k = 1, \ldots, m$. From this it is clear (see the end of the proof of Proposition 2.5.8 in [11]) that there is a left approximate identity for J in $\mathcal{C}$, which shows (i).

Suppose that J is a right ideal with a left approximate identity (e_t) of the stated form $e_t = 1 - x_t$. If (x_{t_μ}) is any w*-convergent subnet of (x_t), with limit r, then $\|r\| \leqslant 1$. Also $1 - x_{t_\mu} \to 1 - r$. On the other hand, $(1 - x_{t_\mu})x \to x$ for any $x \in J$, so that $(1 - r)x = x$. Hence $(1 - r)\eta = \eta$ for any $\eta \in J^{\perp\perp}$, so that $1 - r$ is the (unique) left identity p for $J^{\perp\perp}$. Hence $1 - r$ is idempotent, so that r is idempotent. Hence r is an orthogonal projection, and therefore so also is $p = 1 - r$. Also, $e_t \to p$ w*, by a fact in topology about nets with unique accumulation points. We have $J = pA^{**} \cap A = \{a \in A : pa = a\}$. Since p has norm 1, J has a left cai. Since $pe_t = e_t$, p is an open projection in B^{**}, so that

$q = 1 - p$ is closed. If $a = e_t$ then we have that $l(a)^\perp(1-a) = l(a)^\perp$, where $l(a)$ is the left support projection for a. Thus by a result in [27] there is a peak projection q_a which is a peak for $a_0 = 1 - a/2 \in A$ such that $l(a)^\perp \leqslant q_a$. Since $a_0^n \to q_a$ weak*, and since $(1-p)a_0^n = 1 - p$, we have $(1-p)q_a = 1 - p$. That is, $q \leqslant q_a$. Let $J_a = \{x \in A : q_a^\perp x = x\}$. By the last paragraph, J_a is an r-ideal, and since $q \leqslant q_a$ we have that $J_a \subset J$. The closed span of all the J_a for $a = e_t$ equals J, since $e_t \in J_{e_t}$ and any $x \in J$ is a limit of $e_t x \in J_{e_t}$. By the proof of Theorem 4.8.6 in [11] we deduce that the supremum of the $q_a^\perp$ equals $q^\perp$. Thus q is a p-projection. The HSA case follows easily from this and Corollary 2.8. ∎

COROLLARY 6.2. *Let A be a unital-subalgebra of a C^*-algebra B. A projection $q \in B^{**}$ is a p-projection for A in B^{**}, if and only if there exists a net (x_t) in* $\mathrm{Ball}(A)$ *with $qx_t = q$, and $x_t \to q$ weak*.*

Proof. Supposing that q is a p-projection, we have by the last result that $J = \{a \in A : q^\perp a = a\}$ has a left approximate identity $(1 - x_t)$ with $x_t \in \mathrm{Ball}(A)$, and by the proof of that result $q^\perp$ is the support projection, so that $1 - x_t \to q^\perp$ weak*.

Conversely, supposing the existence of such a net, let $J = \{a \in A : q^\perp a = a\}$. This is a right ideal. Moreover $J^{\perp\perp} \subset q^\perp A^{**}$. If $a \in A$ then $q^\perp a = \lim_t (1 - x_t)a \in J^{\perp\perp}$. By a similar argument, $q^\perp \eta \in J^{\perp\perp}$ for any $\eta \in A^{**}$. Thus $J^{\perp\perp} = q^\perp A^{**}$, and so $q^\perp$ is the support projection for J, and J has a left cai. By a slight variation of the argument at the end of the first paragraph of the proof of the last result, J satisfies (i) of that result, and hence by that result q is a p-projection. ∎

The following known result (see e.g. [4], [14]) is quite interesting in light of the question just above Theorem 6.1.

PROPOSITION 6.3. *If J is an nonunital operator algebra with a cai (respectively left cai), then J has an (respectively a left) approximate identity of the form $(1 - x_t)$, where $x_t \in J^1$ and $\lim_t \|x_t\| = 1$ and $\lim_t \|1 - x_t\| = 1$. Here J^1 is the unitization of J.*

Proof. We just sketch the proof in the left cai case, following the proof of Theorem 3.1 in [4]. Let $A = J^1$. Thus J is an r-ideal in the unital operator algebra A. Suppose that the support projection is $p = q^\perp \in A^{**}$, and that (u_t) is the left cai in J. If B is a C^*-algebra generated by A, then there is an increasing net in $\mathrm{Ball}(B)$ with weak* limit p. We can assume that the increasing net is indexed by the same directed set. Call it (e_t). Since $e_t - u_t \to 0$ weakly, new nets of convex combinations $(\widetilde{e}_s)$ and $(\widetilde{u}_s)$ will satisfy $\|\widetilde{e}_s - \widetilde{u}_s\| \to 0$. We can assume that $(\widetilde{u}_s)$ is a left cai for J. We have

$$\|1 - \widetilde{u}_s\| \leqslant \|1 - \widetilde{e}_s\| + \|\widetilde{e}_s - \widetilde{u}_s\| \leqslant 1 + \|\widetilde{e}_s - \widetilde{u}_s\| \to 1.$$

The result follows easily from this. ∎

We are also able to give another characterization of p-projections, which is of "nonselfadjoint Urysohn lemma" or "peaking" flavor, and therefore should be useful in future applications of "nonselfadjoint peak interpolation". This result should be compared with Theorem 5.12 of [27].

THEOREM 6.4. *Let A be a unital-subalgebra of C^*-algebra B and let $q \in B^{**}$ be a closed projection. Then q is a p-projection for A if and only if for any open projection $u \geqslant q$, and any $\varepsilon > 0$, there exists an $a \in \mathrm{Ball}(A)$ with $aq = q$ and $\|a(1-u)\| < \varepsilon$ and $\|(1-u)a\| < \varepsilon$.*

Proof. ($\Leftarrow$) This follows by an easier variant of the proof of Theorem 4.1 in [27]. Suppose that for each open $u \geqslant q$, and positive integer n, there exists an $a_n \in \mathrm{Ball}(A)$ with $a_n q = q$ and $\|a_n(1-u)\| < 1/n$. By taking a weak* limit we find $a \in A^{\perp\perp}$ with $aq = q$ and $a(1-u) = 0$. We continue as in Theorem 4.1 of [27]. Later in the proof where q_n is defined, we appeal to Lemma 3.5 in place of Lemma 3.6, so that q_n is a peak projection. Now the proof is quickly finished: Let $Q = \bigwedge_n q_n$, a p-projection. As in the other proof we have that $q \leqslant Q \leqslant r \leqslant u$, and that this forces $q = Q$. Thus q is a p-projection.

($\Rightarrow$) Suppose that q is a p-projection, and $u \geqslant q$ with u open. By "compactness" of q (see the remark just above Proposition 2.2 of [27]), there is a peak projection q_1 with $q \leqslant q_1 \leqslant u$. Note that if $aq_1 = q_1$ then $aq = aq_1q = q_1q = q$. Thus we may assume that q is a peak projection. By the noncommutative Urysohn lemma [2], there is an $x \in B$ with $q \leqslant x \leqslant u$. Suppose that $a \in \mathrm{Ball}(A)$ peaks at q, and $a^n \to q$ weak* (see e.g. Lemma 3.4 of [27] or the results below). Then $a^n(1-x) \to q(1-x) = 0$ weak*, and hence weakly in B. Similarly, $(1-x)a^n \to 0$ weakly. By a routine convexity argument in $B \oplus B$, given $\varepsilon > 0$ there is a convex combination b of the a^n such that $\|b(1-x)\| < \varepsilon$ and $\|(1-x)b\| < \varepsilon$. Therefore $\|b(1-u)\| = \|b(1-x)(1-u)\| < \varepsilon$. Similarly $\|(1-u)b\| < \varepsilon$. ∎

We would guess that being a p-projection is also equivalent to the special case where $a = 1$ and $x \leqslant 1$ of the following definition.

If A is a unital-subalgebra of C^*-algebra B and if $q \in B^{**}$ is closed then we say that q is a *strict p-projection* if given $a \in A$ and a strictly positive $x \in B$ with $a^*qa \leqslant x$, then there exists $b \in A$ such that $qb = qa$ and $b^*b \leqslant x$. In Proposition 3.2 of [27] it is shown that if q is a closed projection in $A^{\perp\perp}$ then the conditions in the last line hold except that $b^*b \leqslant x + \varepsilon$. So being a strict p-projection is the case $\varepsilon = 0$ of that interpolation result.

COROLLARY 6.5. *Let A be a unital-subalgebra of C^*-algebra B and let $q \in B^{**}$ be a strict p-projection for A. Then q is a p-projection.*

Proof. Using the noncommutative Urysohn lemma as in the first few lines of the proof of Theorem 4.1 in [27], it is easy to see that q satisfies the condition in Theorem 6.4. ∎

The above is related to the question of whether every r-ideal J in a (unital say) operator algebra is "proximinal" (that is, whether every $x \in A$ has a closest point in J).

PROPOSITION 6.6. *If q is a strict p-projection for a unital operator algebra A, then the corresponding* r*-ideal $J = q^\perp A^{**} \cap A$ is proximinal in A.*

Proof. Let $a \in A$. By Proposition 3.1 in [27], $\|a + J\| = \|qa\|$. Also, $a^*qa \leqslant \|qa\|^2$, so by hypothesis there exists $b \in A$ such that $qb = qa$ and $b^*b \leqslant \|qa\|^2$. Thus $\|b\|^2 = \|b^*b\| \leqslant \|qa\|^2$. Then $\|a + J\| = \|qa\| \geqslant \|b\| = \|a + (b - a)\|$. However, $b - a \in J$ since $q(b - a) = 0$. So J is proximinal. ∎

Some of the results below stated for right ideals also have HSA variants which we leave to the reader.

PROPOSITION 6.7. *A p-projection q for a unital operator algebra A is a peak projection if and only if the associated right ideal is of the form $\overline{(1-a)A}$ for some $a \in$* Ball(A). *In this case, q is the peak for $(a + 1)/2$.*

Proof. Let $J = \{a \in A : q^\perp a = a\}$, for a p-projection q.

($\Rightarrow$) If q is a peak for a then $q^\perp(1 - a) = (1 - a)$, so that $(1 - a)A \subset J$. If $\varphi \in ((1 - a)A)^\perp$ then $\varphi((1 - a^{n+1})A) = \varphi((1 - a)(1 + a + \cdots + a^n)A) = 0$. In the limit we see that $\varphi \in (q^\perp A)_\perp$, so that $\varphi \in J^\perp$. Hence $J = \overline{(1-a)A}$.

($\Leftarrow$) Suppose that $J = \overline{(1-a)A}$ for some $a \in$ Ball(A). Then $qa = q$, and by a result in [27] there exists a peak projection $r \geqslant q$ which is a peak for $b = (a+1)/2$. Since $1 - b = (1 - a)/2$, it is clear that $J = \overline{(1-b)A}$. If (e_t) is the left cai for J then $r^\perp e_t = e_t$. In the limit, $r^\perp q^\perp = q^\perp$, so that $r \leqslant q$. Thus $r = q$. ∎

This class of "singly generated" right ideals has played an important role in some work of G.A. Willis (see e.g. [34]).

LEMMA 6.8. *If A is an unital operator algebra, and if $a \in$* Ball(A) *then $\overline{(1-a)A}$ is an* r*-ideal of A with a sequential left approximate identity of the form $(1 - x_n)$ for $x_n \in$* Ball(A). *Similarly, $\overline{(1-a)A(1-a)}$ is a* HSA *of A.*

Proof. Let $J = \overline{(1-a)A}$, and let $e_n = 1 - \frac{1}{n}\sum_{k=1}^{n} a^k$, which is easy to see is in $(1 - a)A$. Moreover,

$$e_n(1-a) = 1 - \frac{1}{n}\sum_{k=1}^{n} a^k - a + \frac{1}{n}\sum_{k=2}^{n+1} a^k = 1 - a - \frac{1}{n}(a - a^{n+1}) \to 1 - a.$$

Note that J is an r-ideal by Theorem 6.1. We leave the rest to the reader. ∎

COROLLARY 6.9. *If a is a contraction in a unital C^*-algebra B then:*

(i) *The Cesaro averages of a^n converge weak* to a peak projection q with $qa = q$.*

(ii) *If $a^n \to q$ weak* then q is a peak projection. Conversely, if q is a peak projection then there exists an $a \in$* Ball(B) *with $a^n \to q$ weak*.*

Also q is the peak for $(a+1)/2$.

Proof. (i) By Theorem 6.1 and Lemma 6.8 (and its proof), $J = \overline{(1-a)A} = \{a \in A : q^{\perp}a = a\}$ for a p-projection q which is a weak* limit of $e_n = 1 - \frac{1}{n}\sum\limits_{k=1}^{n} a^k$. Thus $\frac{1}{n}\sum\limits_{k=1}^{n} a^k \to q$ weak*, and clearly $qa = q$. By 6.7 and its proof, q is a peak projection which is a peak for $(a+1)/2$.

(ii) If $a^n \to q$ weak* then it is easy to check that $\frac{1}{n}\sum\limits_{k=1}^{n} a^k \to q$ weak*. Thus one direction of (ii) follows from (i), and the other direction is in [27]. ∎

REMARK 6.10. (i) In fact it is not hard to show that the Cesaro averages in (i) above converge strongly, if B is in its universal representation.

(ii) We make some remarks on support projections. We recall from [27] that if q is a projection in B^{**} and if q is a peak for a contraction $b \in B$ then $q^{\perp}$ is the right support projection $r(1-b)$. Conversely, if $b \in \mathrm{Ball}(B)$ then the complement of the right support projection $r(1-b)$ is a peak projection for $(1+b)/2$. Thus the peak projections are precisely the complements of the right support projections $r(1-b)$ for contractions $b \in B$.

It follows that q is a p-projection for a unital-subspace A of a C^*-algebra B if and only if $q = \bigwedge\limits_{x \in \mathcal{S}} r(1-x)^{\perp}$ for a nonempty subset $\mathcal{S} \subset \mathrm{Ball}(A)$.

Also, if J is a right ideal of a unital operator algebra A, and if J has a left approximate identity of the form $(1-x_t)$ with $x_t \in \mathrm{Ball}(A)$, then it is easy to see that the support projection of J is $\bigvee\limits_{t} l(1-x_t)$.

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ADDED IN PROOFS. We noticed in 2006 that there is a "partial isometry" variant of the notions of p- and peak projections, to which one may extend many of the results of [27] and the present paper. Every element x in an operator space of norm 1 has a peak u in this sense, and u can be shown to coincide with the tripotent $u(x)$ studied in papers of Edwards and Ruttiman. We have $u(x^*x) = u(x)^*u(x)$ is a peak for x^*x. We also noticed that Proposition 6.6 above can be improved to give a much tighter link between the open question studied in Section 6, and perhaps the most important remaining open question in the theory of one-sided M-ideals [10], [13] (namely, whether they are proximinal).