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COORDINATE SYSTEMS AND BOUNDED ISOMORPHISMS

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ABSTRACT. For a Banach $\mathcal{D}$-bimodule $\mathcal{M}$ over an abelian unital C^*-algebra $\mathcal{D}$, we define $\mathcal{E}^1(\mathcal{M})$ as the collection of norm-one eigenvectors for the dual action of $\mathcal{D}$ on the Banach space dual $\mathcal{M}^\#$. Equip $\mathcal{E}^1(\mathcal{M})$ with the weak*-topology. We develop general properties of $\mathcal{E}^1(\mathcal{M})$. It is properly viewed as a coordinate system for $\mathcal{M}$ when $\mathcal{M} \subseteq \mathcal{C}$, where $\mathcal{C}$ is a unital C^*-algebra containing $\mathcal{D}$ as a regular MASA with the extension property; moreover, $\mathcal{E}^1(\mathcal{C})$ coincides with Kumjian's twist in the context of C^*-diagonals. We identify the C^*-envelope of a subalgebra $\mathcal{A}$ of a C^*-diagonal when $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$. For triangular subalgebras, each containing the MASA, a bounded isomorphism induces an algebraic isomorphism of the coordinate systems which can be shown to be continuous in certain cases. For subalgebras, each containing the MASA, a bounded isomorphism that maps one MASA to the other MASA induces an isomorphism of the coordinate systems. We show that the weak operator closure of the image of a triangular algebra in an appropriate representation is a CSL algebra and that a bounded isomorphism of triangular algebras extends to an isomorphism of these CSL algebras. We prove that for triangular algebras in our context, any bounded isomorphism is completely bounded. Our methods simplify and extend various known results; for example, isometric isomorphisms of the triangular algebras extend to isometric isomorphisms of the C^*-envelopes, and the conditional expectation $E : \mathcal{C} \to \mathcal{D}$ is multiplicative when restricted to a triangular subalgebra. Also, we use our methods to prove that the inductive limit of C^*-diagonals with regular connecting maps is again a C^*-diagonal.

KEYWORDS: *Coordinates, C^*-diagonal, triangular operator algebra, bounded isomorphism.*

MSC (2000): Primary 46L05; Secondary 46H25, 47L55, 47L40.

1. INTRODUCTION

This paper presents the results of our study of bounded isomorphisms of coordinatized (nonselfadjoint) operator algebras. Isometric isomorphisms have been extensively studied (see, for example, [18], [25], [34]) and are quite natural, as they include restrictions of $*$-isomorphisms. Isometric isomorphisms of

C^*-algebras preserve adjoints. Bounded isomorphisms, in contrast, need not preserve adjoints or map $*$-subalgebras to $*$-subalgebras. Nonetheless, we obtain structural results, most notably, that bounded isomorphisms of triangular subalgebras of C^*-diagonals are completely bounded and also that they factor into diagonal-fixing, spatial, and isometric parts, analogous to Arveson-Josephson's factorization of bounded isomorphisms of analytic crossed products.

Coordinates have been used in the categories of von Neumann algebras and C^*-algebras with $*$-homomorphisms for decades, going back at least to the work of Feldman and Moore ([14], [15]) for von Neumann algebras and Renault's construction of C^*-algebras for a wide range of topological groupoids [38]. Of particular interest to us is Kumjian's construction, in [21], of a certain $\mathbb{T}$-groupoid which he called a *twist*, which he showed is a classifying invariant for a *diagonal pair*, a separable C^*-algebra with a distinguished MASA satisfying various properties. Renault describes twists in terms of the dual groupoid [39] and this perspective is often helpful, see for example the work of Thomsen in [42]. Twists have been used by various authors, most notably Muhly, Qiu, and Solel, and Muhly and Solel, to study varied categories of subalgebras and submodules of groupoid C^*-algebras with isometric morphisms [24], [27].

To apply coordinate methods to bounded isomorphisms, we found it necessary to revisit these coordinate constructions, eliminating as much as possible the use of adjoints and clarifying the role of the extension property. We define coordinates for bimodules over an abelian C^*-algebra which are intrinsic to the bimodule structure and not *a priori* closely tied to the $*$-structure. These definitions allow us to simplify and extend some of the structural results in the literature. In particular, we obtain a number of results for algebras containing the abelian C^*-algebra: e.g. isomorphism of the coordinates is equivalent to diagonal-preserving isomorphism of the algebras. This analysis of coordinate systems will be useful, we expect, in applying coordinate constructions to more general settings.

Bounded isomorphisms play a role in the study of norm-closed operator algebras which is parallel to similarity transforms in the study of weakly-closed subalgebras of $\mathcal{B}(\mathcal{H})$. During the mid-1980's and early 1990's, there was considerable interest in the structural analysis of such algebras via their similarity theory. This was particularly successful with the class of nest algebras (see, for example, [6], [7],[22], [23], [29]) and, to a lesser degree, the CSL algebras. Interestingly, by using certain faithful representations of C^*-diagonals, we can employ similarity theory for atomic CSL algebras to obtain structural results for bounded isomorphisms between triangular subalgebras of C^*-diagonals.

We turn now to a more detailed outline of the paper. Throughout the paper, we consider bimodules over an abelian unital C^*-algebra $\mathcal{D}$. Our view is that the set of coordinates for such a bimodule $\mathcal{M}$ is the collection $\mathcal{E}^1(\mathcal{M})$ of norm one elements of the Banach space dual $\mathcal{M}^\#$ which are eigenvectors for the bimodule action of $\mathcal{D}$ on $\mathcal{M}^\#$. We use *eigenfunctional* for such elements, and we topologize

$\mathcal{E}^1(\mathcal{M})$ using the weak* topology. With this structure, we call $\mathcal{E}^1(\mathcal{M})$ the coordinate system for $\mathcal{M}$. In Section 2, we establish some very general, but useful, properties. For example, Theorem 2.6 shows that when $\mathcal{M}_1$ is a submodule of $\mathcal{M}$, then an element of $\mathcal{E}^1(\mathcal{M}_1)$ can be extended (but not necessarily uniquely) to an element of $\mathcal{E}^1(\mathcal{M})$.

A key step in minimizing the use of adjoints is replacing normalizers with intertwiners, that is, elements $m \in \mathcal{M}$ so that $m\mathcal{D} = \mathcal{D}m$. Section 3 starts by showing that intertwiners and normalizers are closely related, at least when $\mathcal{D}$ is a MASA in a unital C^*-algebra $\mathcal{C}$ containing $\mathcal{M}$ (Propositions 3.3 and 3.4). The extension property is used to construct intertwiners and to slightly strengthen a key technical result of Kumjian. In this generality, $\mathcal{E}^1(\mathcal{M})$ need not separate points of $\mathcal{M}$.

In Section 4 we work in the context of $\mathcal{D}$-bimodules $\mathcal{M} \subseteq \mathcal{C}$, where $\mathcal{C}$ is a unital C^*-algebra and $\mathcal{D} \subseteq \mathcal{C}$ is a regular MASA with the extension property. Such a pair $(\mathcal{C}, \mathcal{D})$ we term a *regular C^*-inclusion*. Here the coordinates are better behaved: eigenfunctionals on submodules of $\mathcal{M}$ extend uniquely to $\mathcal{M}$. Nevertheless, the coordinates for regular C^*-inclusions again are not sufficiently rich to separate points. However, the failure to separate points is intimately related to a certain ideal of $\mathfrak{N} \subseteq \mathcal{C}$, and Theorem 4.8 shows that the quotient of $\mathcal{C}$ by $\mathfrak{N}$ is a C^*-diagonal, which is a mild generalization of Kumjian's notion of diagonal pair due to Renault [39]. Essentially, a C^*-diagonal is a regular C^*-inclusion where the conditional expectation $E : \mathcal{C} \to \mathcal{D}$ arising from the extension property is faithful. There are an abundance of C^*-diagonals: crossed products of abelian C^*algebras by freely acting amenable groups are C^*-diagonals, and Theorem 4.23 shows that inductive limits of C^*-diagonals are again C^*-diagonals when the connecting maps satisfy a regularity condition. In particular, AF-algebras and circle algebras can be viewed as C^*-diagonals. While our primary interest in this paper is the use of coordinate methods to study nonselfadjoint algebras, Theorem 4.8 and Theorem 4.23 are examples of results in the theory of C^*-algebras obtained using our perspective.

When $\mathcal{M}$ is a $\mathcal{D}$-bimodule contained in a C^*-diagonal, the elements of $\mathcal{E}^1(\mathcal{M})$ do separate points, and when the bimodule is an algebra, also have a continuous product. In Section 4, we use the extension property to show that the coordinate system $\mathcal{E}^1(\mathcal{C})$ for a C^*-diagonal agrees with Kumjian's twist. Our methods provide some simplifications and generalizations of Kumjian's results. One of the interesting features of our approach is that it allows us to show that the coordinate systems for bimodules $\mathcal{M} \subseteq \mathcal{C}$ are intrinsic to the bimodule alone, and not dependent on the choice of the embedding into the particular C^*-diagonal. This and the agreement of our construction with Kumjian's is achieved in Theorems 4.13, Corollary 4.15, and Proposition 4.17. An interesting application of our coordinate methods is Theorem 4.21, which shows that if the pair $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal, and $\mathcal{A}$ is a norm closed algebra with $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$, then the C^*-envelope of $\mathcal{A}$ coincides with the C^*-subalgebra of $\mathcal{C}$ generated by $\mathcal{A}$.

We study the representation theory of C^*-diagonals in Section 5, obtaining a faithful atomic representation compatible with the C^*-diagonal structure. Our methods are reminiscent of Gardner's work on isomorphisms of C^*-algebras in [16]. The significance to us of these representations is that they carry subalgebras containing the diagonal to algebras weakly dense in a CSL-algebra, see Theorem 5.9. This theorem enables us to prove, with fewer hypotheses than previously needed, that the conditional expectation is multiplicative when restricted to a triangular subalgebra, Theorem 5.10.

Section 6 considers diagonal-preserving bounded isomorphisms, those that map the diagonal of one algebra onto the diagonal of the other. Theorem 6.3 shows that in this case there is an isomorphism between coordinate systems arising naturally from the algebra isomorphism. Consequently, we are able to prove several results, such as Theorem 6.7, which shows that an automorphism of a triangular algebra which fixes the diagonal pointwise arises from a cocycle, and Theorems 6.11 and 6.14, which show that coordinates are invariant under diagonal-preserving bounded isomorphisms, extending previous results for isometric isomorphisms.

We then turn to bounded isomorphisms of triangular algebras which do not preserve the diagonal. A main result, Theorem 7.7, shows that a given bounded isomorphism of triangular subalgebras induces an *algebraic* isomorphism γ of their coordinate systems, but our methods are not strong enough to show that this isomorphism is continuous everywhere. Nevertheless, this does show that the algebraic structure of coordinate systems for triangular algebras is invariant under bounded isomorphism, a fact we believe would be difficult to show using previously existing methods.

While we are not able to prove continuity of the map γ on coordinate systems in general, we can prove it in various special cases. A bounded isomorphism of triangular algebras induces a canonical $*$-isomorphism of the diagonals. If this $*$-isomorphism extends to a $*$-isomorphism of the C^*-envelopes, then Corollary 7.8 shows that the product of γ with an appropriate cocycle yields a continuous isomorphism. Also, Theorem 7.11 shows that bounded isomorphism of triangular algebras implies an isomorphism of their coordinate systems when the triangular algebras are generated by their *algebra-preserving normalizers*. This is a new class of algebras which contains a variety of known classes, including those generated by order-preserving normalizers or those generated by monotone G-sets.

Another of our main results, Theorem 8.2, shows that if boundedly isomorphic triangular subalgebras are represented in the faithful representation constructed in Theorem 5.9, then the isomorphism extends to an isomorphism of the weak closures of the triangular algebras. A crucial ingredient in proving this is Theorem 7.7. Theorem 8.2 allows us to use known results about isomorphisms of CSL algebras to prove another main result, Theorem 8.8, which asserts that every bounded isomorphism of triangular subalgebras of C^*-diagonals is completely bounded. Consequently, we are able to prove that every isometric isomorphism of triangular algebras extends to an isometric isomorphism of the corresponding C^*-envelopes. Another application of Theorem 8.2 is Theorem 8.7, which gives the factorization into diagonal-fixing, spatial, and isometric parts mentioned earlier.

Together with Power, we asserted ([12], Theorem 4.1) that two limit algebras are isomorphic if and only if a certain type of coordinates for the algebras, namely their *spectra*, are isomorphic. Unfortunately, there is a serious gap in the proof, and another of our motivations for the work in this paper was an attempt to provide a correct proof. While we have not yet done this, our results provide evidence that Theorem 4.1 of [12] is true. The main result of Section 9, Theorem 9.9, shows that an (algebraic) isomorphism of a limit algebra $\mathcal{A}_1$ onto another limit algebra $\mathcal{A}_2$ implies the existence of a $*$-isomorphism τ of their C^*-envelopes. If we could choose τ so that $\tau(\mathcal{A}_1) = \mathcal{A}_2$, then Theorem 4.1 of [12] would follow easily, but we do not know this. However, any isomorphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$ induces a $*$-isomorphism α of $\mathcal{A}_1 \cap \mathcal{A}_1^*$ onto $\mathcal{A}_2 \cap \mathcal{A}_2^*$. If τ can be chosen so that τ extends α, Corollary 7.8 shows that τ carries $\mathcal{A}_1$ onto $\mathcal{A}_2$, and moreover, there is an isomorphism of the corresponding coordinate systems. It is somewhat encouraging that Theorem 9.9 shows that there is no K-theoretic obstruction to the existence of a τ which extends α. This is as close as we have been able to come in our efforts to provide a correct proof of Theorem 4.1 of [12].

2. INTERTWINERS AND EIGENFUNCTIONALS

In this section we provide a very general discussion of coordinates. Although our focus in this paper is on C^*-diagonals and regular C^*-inclusions, defined in Section 4, we start in a more general framework, with a view to extending coordinate methods beyond our focus here. Indeed, there are several useful general results, most notably Theorem 2.6, which shows that eigenfunctionals can be extended from one bimodule to another bimodule containing the first. From this result, we characterize when an eigenfunctional with a given range and source exists, using a suitable seminorm.

2.1. NOTATIONAL CONVENTION. Given a Banach space X, we denote its Banach space dual by $X^\#$, to minimize confusion with adjoints.

Throughout this section, $\mathcal{D}$ will be a unital, abelian C^*-algebra, and $\mathcal{M}$ will be a Banach space which is also a bounded $\mathcal{D}$-bimodule, that is, there exists a constant $K > 0$ such that for every $d, f \in \mathcal{D}$ and $m \in \mathcal{M}$,

$$\|d \cdot m \cdot f\| \leqslant K\|d\|\|m\|\|f\|.$$

As usual, $\mathcal{M}^\#$ becomes a Banach $\mathcal{D}$-module with the action,

$$\langle m, f \cdot \phi \cdot d\rangle = \langle d \cdot m \cdot f, \phi\rangle \quad d, f \in \mathcal{D}, m \in \mathcal{M}, \text{ and } \phi \in \mathcal{M}^\#.$$

DEFINITION 2.1. A nonzero element $m \in \mathcal{M}$ is a $\mathcal{D}$-*intertwiner*, or more simply, an *intertwiner* if

$$m \cdot \mathcal{D} = \mathcal{D} \cdot m.$$

If $m \in \mathcal{M}$ is an intertwiner such that for every $d \in \mathcal{D}$, $d \cdot m \in \mathbb{C}m$, we call m a *minimal intertwiner*.

A minimal intertwiner of $\mathcal{M}^{\#}$ will be called a *eigenfunctional*; when necessary for clarity, we use $\mathcal{D}$-*eigenfunctional*. That is, an eigenfunctional is a nonzero linear functional $\phi : \mathcal{M} \to \mathbb{C}$ so that, for all $d \in \mathcal{D}$, $x \mapsto \phi(dx)$, $x \mapsto \phi(xd)$ are multiples of ϕ.

Denote the set of all $\mathcal{D}$-eigenfunctionals by $\mathcal{E}_{\mathcal{D}}(\mathcal{M})$ (or $\mathcal{E}(\mathcal{M})$, if the context is clear). We equip $\mathcal{E}(\mathcal{M})$ with the relative weak*-topology (i.e. the relative $\sigma(\mathcal{M}^{\#}, \mathcal{M})$-topology).

Denote the set of all *norm-one* $\mathcal{D}$-eigenfunctionals by $\mathcal{E}^1_{\mathcal{D}}(\mathcal{M})$ or $\mathcal{E}^1(\mathcal{M})$.

Given an eigenfunctional $\phi \in \mathcal{E}_{\mathcal{D}}(\mathcal{M})$, the associativity of the maps $d \in \mathcal{D} \mapsto d \cdot \phi$ and $d \in \mathcal{D} \mapsto \phi \cdot d$ yields the existence of unique multiplicative linear functionals $s(\phi)$ and $r(\phi)$ on $\mathcal{D}$ satisfying $s(\phi)(d)\phi = d \cdot \phi$ and $r(\phi)(d)\phi = \phi \cdot d$, that is,

$$\phi(xd) = \phi(x)[s(\phi)(d)], \quad \phi(dx) = [r(\phi)(d)]\phi(x). \tag{2.1}$$

DEFINITION 2.2. We call $s(\phi)$ and $r(\phi)$ the *source* and *range* of ϕ, respectively.

There is a natural action of the nonzero complex numbers on $\mathcal{E}(\mathcal{M})$, sending (λ, ϕ) to the functional $m \mapsto \lambda\phi(m)$; clearly $s(\lambda\phi) = s(\phi)$ and $r(\lambda\phi) = r(\phi)$.

We next record a few basic properties of eigenfunctionals.

PROPOSITION 2.3. *With the weak*-topology, $\mathcal{E}(\mathcal{M}) \cup \{0\}$ is closed. Furthermore, $r : \mathcal{E}(\mathcal{M}) \to \widehat{\mathcal{D}}$ and $s : \mathcal{E}(\mathcal{M}) \to \widehat{\mathcal{D}}$ are continuous.*

Proof. Suppose $(\phi_\lambda)_{\lambda \in \Lambda}$ is a net in $\mathcal{E}(\mathcal{M}) \cup \{0\}$ and $\phi_\lambda \overset{w*}{\to} \phi \in \mathcal{M}^{\#}$.

If $\phi = 0$, there is nothing to do, so we assume that $\phi \neq 0$. Choose $m \in \mathcal{M}$ such that $\phi(m) \neq 0$. Then $\phi_\lambda(m) \to \phi(m)$, so there exists $\lambda_0 \in \Lambda$ such that $\phi_\lambda(m) \neq 0$ for every $\lambda \succeq \lambda_0$. For any $d \in \mathcal{D}$, and $\lambda \succeq \lambda_0$, we have

$$s(\phi_\lambda)(d) = \frac{\phi_\lambda(md)}{\phi_\lambda(m)} \to \frac{\phi(md)}{\phi(m)}.$$

Thus, $s(\phi_\lambda)$ converges weak* to the functional $\sigma \in \widehat{\mathcal{D}}$ given by $d \mapsto \phi(md)/\phi(m)$. Similarly, $r(\phi_\lambda)$ converges weak* to $\rho \in \widehat{\mathcal{D}}$ given by $\rho(d) = \phi(dm)/\phi(m)$. It now follows that ϕ is a eigenfunctional with $r(\phi) = \rho$ and $s(\phi) = \sigma$. Thus, $\mathcal{E}(\mathcal{M}) \cup \{0\}$ is closed.

In particular, if $\phi_\lambda \to \phi \in \mathcal{E}(\mathcal{M})$, then $s(\phi_\lambda) \to s(\phi)$ and similarly for the ranges, so s and r are continuous. ∎

To be useful, there should be many eigenfunctionals. This need not occur for arbitrary bimodules, as Example 2.8 shows. However, for bimodules of C^*-diagonals, which are the bimodules of principal interest in the present paper, Proposition 4.20 below will show that eigenfunctionals exist in abundance.

We need the following seminorm to extend eigenfunctionals and to characterize when eigenfunctionals exist.

DEFINITION 2.4. For $\rho, \sigma \in \widehat{\mathcal{D}}$, define $B_{\rho,\sigma} : \mathcal{M} \to \mathbb{R}$ by

$$B_{\rho,\sigma}(m) := \inf\{\|dmf\| : d, f \in \mathcal{D}, \rho(d) = \sigma(f) = 1\}.$$

These infima do not increase if we restrict to elements d or f of norm one. Indeed, for any elements d and f as above, since $|\rho(d)| \leqslant \|d\|$ and $|\sigma(f)| \leqslant \|f\|$, we can replace them with $d/\|d\|$ and $f/\|f\|$ and this will only decrease the norm of $\|dmf\|$. Thus,

$$B_{\rho,\sigma}(m) = \inf\{\|dmf\| : d, f \in \mathcal{D}, \rho(d) = \sigma(f) = 1 = \|d\| = \|f\|\}.$$

In particular, we have $B_{\rho,\sigma}(m) \leqslant \|m\|$.

A variant of this seminorm was used by Steve Power in [35] to distinguish families of limit algebras associated to singular MASAs.

PROPOSITION 2.5. *For $\rho, \sigma \in \widehat{\mathcal{D}}$, we have the following:*

(i) *$B_{\rho,\sigma}$ is a seminorm.*

(ii) *For $m \in \mathcal{M}$ and $\phi \in \mathcal{E}(\mathcal{M})$, $|\phi(m)| \leqslant \|\phi\| B_{r(\phi),s(\phi)}(m)$.*

(iii) *For $m \in \mathcal{M}$, $d, f \in \mathcal{D}$, $B_{\rho,\sigma}(dmf) = |\rho(d)| B_{\rho,\sigma}(m) |\sigma(f)|$.*

(iv) *If $f \in \mathcal{M}^\# \setminus \{0\}$ satisfies $|f(m)| \leqslant B_{\rho,\sigma}(m)$ for all $m \in \mathcal{M}$, then $f \in \mathcal{E}(\mathcal{M})$ with $s(f) = \sigma$, $r(f) = \rho$, and $\|f\| \leqslant 1$.*

Proof. For (i), it is immediate that $B_{\rho,\sigma}(\lambda m) = |\lambda| B_{\rho,\sigma}(m)$, for $\lambda \in \mathbb{C}$ and $m \in \mathcal{M}$.

To show subadditivity, let $a, b \in \mathcal{M}$ and choose $\varepsilon > 0$. Pick norm one elements d_1, d_2, f_1, f_2 of $\mathcal{D}$, satisfying $\rho(d_1) = \rho(d_2) = 1 = \sigma(f_1) = \sigma(f_2)$, $\|d_1 a f_1\| < B_{\rho,\sigma}(a) + \varepsilon$ and $\|d_2 b f_2\| < B_{\rho,\sigma}(b) + \varepsilon$. Then

$$\|d_1 d_2 (a+b) f_1 f_2\| \leqslant \|d_2 d_1 a f_1 f_2\| + \|d_1 d_2 b f_2 f_1\| \leqslant \|d_1 a f_1\| + \|d_2 b f_2\| < B_{\rho,\sigma}(a) + B_{\rho,\sigma}(b) + 2\varepsilon,$$

whence $B_{\rho,\sigma}(a+b) \leqslant B_{\rho,\sigma}(a) + B_{\rho,\sigma}(b)$.

For (ii), suppose $d, f \in \mathcal{D}$ with $s(\phi)(d) = r(\phi)(f) = 1$. Then $|\phi(m)| = |\phi(dmf)| \leqslant \|\phi\| \|dmf\|$. Taking the infimum over all such d and f gives the inequality.

For (iii), we show that $B_{\rho,\sigma}(dm) = |\rho(d)| B_{\rho,\sigma}(m)$; the proof for σ and f is similar.

For any $a, b \in \mathcal{D}$ with $\|a\| = \|b\| = 1 = \rho(a) = \sigma(b)$ we have

$$\|admb\| \leqslant \|\rho(d)amb\| + \|(ad - \rho(d)a)mb\| \leqslant |\rho(d)| \|amb\| + \|ad - \rho(d)a\| \|m\|.$$

Hence $B_{\rho,\sigma}(dm) \leqslant |\rho(d)| \|amb\| + \|ad - \rho(d)a\| \|m\|$. The definition of $B_{\rho,\sigma}$ and the fact that $\inf\{\|ad - \rho(d)a\| : a \in \mathcal{D}, \|a\| = 1 = \rho(d)\} = 0$, imply that given $\varepsilon > 0$, we may find norm one elements a_1, b_1 and a_2 of $\mathcal{D}$ such that $\rho(a_1) = \rho(a_2) = \sigma(b_1) = 1$ and

$$\|a_1 m b_1\| < B_{\rho,\sigma}(m) + \varepsilon \quad \text{and} \quad \|a_2 d - \rho(d) a_2\| \|m\| < \varepsilon.$$

Then

$$\begin{aligned} B_{\rho,\sigma}(dm) &\leqslant |\rho(d)| \|a_1 a_2 m b_1\| + \|a_1 a_2 d - \rho(d) a_1 a_2\| \|m\| \\ &\leqslant |\rho(d)| \|a_1 m b_1\| + \|a_2 d - \rho(d) a_2\| \|m\| \leqslant |\rho(d)| (B_{\rho,\sigma}(m) + \varepsilon) + \varepsilon, \end{aligned}$$

whence $B_{\rho,\sigma}(dm) \leqslant |\rho(d)| B_{\rho,\sigma}(m)$.

To obtain $|\rho(d)| B_{\rho,\sigma}(m) \leqslant B_{\rho,\sigma}(dm)$, observe that for $a, b \in \mathcal{D}$ and $\|a\| = \|b\| = 1 = \rho(a) = \sigma(b)$,

$$|\rho(d)| B_{\rho,\sigma}(m) \leqslant \|\rho(d) a m b\| \leqslant \|\rho(d) a - a d\| \|m\| + \|a d m b\|,$$

and argue as above.

Finally, suppose f is a nonzero linear functional on $\mathcal{M}$ satisfying $|f(m)| \leqslant B_{\rho,\sigma}(m)$ for every $m \in \mathcal{M}$. As $B_{\rho,\sigma}(x) \leqslant \|x\|$, we see that f is bounded and $\|f\| \leqslant 1$.

Suppose $d \in \mathcal{D}$, and let $k = d - \sigma(d) I$. Clearly $\sigma(k) = 0$, and so, for $x \in \mathcal{M}$, $|f(xk)| \leqslant B_{\sigma,\rho}(xk) = 0$. Therefore $f(xd) = f(x)\sigma(d) + f(xk) = f(x)\sigma(d)$. Similarly, $f(dx) = \rho(d) f(x)$. Thus, f is an eigenfunctional with range ρ and source σ. ∎

THEOREM 2.6. *Suppose $\mathcal{M}$ is a norm-closed $\mathcal{D}$-bimodule and $\mathcal{N} \subseteq \mathcal{M}$ is a norm-closed sub-bimodule. Given $\phi \in \mathcal{E}(\mathcal{N})$, there is $\psi \in \mathcal{E}(\mathcal{M})$ with $\psi|_{\mathcal{N}} = \phi$ and $\|\phi\| = \|\psi\|$.*

Necessarily, ψ has the same range and source as ϕ.

Proof. Let $\sigma = s(\phi)$ and $\rho = r(\phi)$. From Proposition 2.5 (ii), $|\phi(n)| \leqslant \|\phi\| B_{\rho,\sigma}(n)$ for all $n \in \mathcal{N}$. By the Hahn-Banach Theorem, there exists an extension of ϕ to a linear functional ψ on $\mathcal{M}$ satisfying $|\psi(x)| \leqslant \|\phi\| B_{\rho,\sigma}(x)$ for all $x \in \mathcal{M}$. Now apply the last part of Proposition 2.5. ∎

We would like to be able to say that the extension in Theorem 2.6 is unique, but this need not be true. For example, if $\mathcal{N} = \mathcal{D}$ and $\mathcal{M} = \mathcal{C}$, then we are considering extensions of pure states, which need not be unique (see [1], [20], for example). However, in the context of regular C^*-inclusions the extension is unique, as we show in Section 4.

We now characterize the existence of eigenfunctionals in terms of the $B_{\rho,\sigma}$ seminorms.

THEOREM 2.7. *Suppose $\rho, \sigma \in \widehat{\mathcal{D}}$. There is $\phi \in \mathcal{E}(\mathcal{M})$ with $r(\phi) = \rho$ and $s(\phi) = \sigma$ if and only if $B_{\rho,\sigma}|_{\mathcal{M}} \neq 0$.*

Proof. If $\phi \in \mathcal{E}(\mathcal{M})$ with $s(\phi) = \sigma$ and $r(\phi) = \rho$, then Proposition 2.5 (ii) implies $B_{\rho,\sigma}|_{\mathcal{M}} \neq 0$.

Conversely, suppose $B_{\rho,\sigma}(m) \neq 0$. Define a linear functional f on $\mathbb{C}m$ by $f(\lambda m) = \lambda B_{\rho,\sigma}(m)$. Now use the Hahn-Banach Theorem to extend f to a linear functional on all of $\mathcal{M}$ which satisfies $|f(m)| \leqslant B_{\rho,\sigma}(m)$, and apply Proposition 2.5. ∎

EXAMPLE 2.8. Here is an example where $\mathcal{E}(\mathcal{M})$ is $\{0\}$. In $\mathcal{B}(L^2[0,1])$, let $\mathcal{D}$ be the operators of multiplication by elements of $C[0,1]$, and let $\mathcal{M}$ be the compact operators. If $\rho \in [0,1]$ and $\Lambda = \{d \in \mathcal{D} : \widehat{d}(\rho) = 1 \text{ and } 0 \leqslant d \leqslant I\}$, then Λ becomes a directed set with the direction $d \preceq e$ if and only if $d - e \geqslant 0$. Viewing Λ as a net indexed by itself, then Λ is a bounded net converging strongly to zero. Hence given any compact operator K, the net $\{dK\}_{d\in\Lambda}$ converges to zero in norm. It follows that $B_{\rho,\sigma}(K) = 0$ for all $\rho, \sigma \in \widehat{\mathcal{D}}$, so by Theorem 2.7, the set of eigenfunctionals is $\{0\}$.

Not surprisingly, eigenfunctionals behave appropriately under bimodule maps.

For $i = 1, 2$, let $\mathcal{M}_i$ be $\mathcal{D}$-bimodules and let $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ be a bounded $\mathcal{D}$-bimodule map. Recall the Banach adjoint map $\theta^\# : \mathcal{M}_2^\# \to \mathcal{M}_1^\#$, given by $\theta^\#(\phi) = \phi \circ \theta$. If $\phi \in \mathcal{E}(\mathcal{M}_2)$, then $\theta^\#\phi \in \mathcal{E}(\mathcal{M}_1)$, and we have $s(\phi \circ \theta) = s(\phi)$, and $r(\phi \circ \theta) = r(\phi)$.

We include the following simple result for reference purposes; the proof is left to the reader.

PROPOSITION 2.9. *For $i = 1, 2$, let $\mathcal{M}_i$ be $\mathcal{D}$-bimodules and suppose $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ is a bounded linear map which is also a $\mathcal{D}$-bimodule map. Then $\theta^\#|_{\mathcal{E}(\mathcal{M}_2)}$ is a continuous map of $\mathcal{E}(\mathcal{M}_2)$ into $\mathcal{E}(\mathcal{M}_1) \cup \{0\}$.*

If θ is bijective, then $\theta^\#|_{\mathcal{E}(\mathcal{M}_2)}$ is a homeomorphism of $\mathcal{E}(\mathcal{M}_2)$ onto $\mathcal{E}(\mathcal{M}_1)$. If θ is isometric, $\theta^\#|_{\mathcal{E}^1(\mathcal{M}_2)}$ is a homeomorphism of $\mathcal{E}^1(\mathcal{M}_2)$ onto $\mathcal{E}^1(\mathcal{M}_1)$.

3. NORMALIZERS AND THE EXTENSION PROPERTY

We begin this section with a discussion of the relationship between normalizers and intertwiners. Together, Propositions 3.3 and 3.4 show that all intertwiners of a maximal abelian C^*-algebra are normalizers and every normalizer can be approximated as closely as desired by intertwiners. This shows that for our purposes, there is no disadvantage in using intertwiners instead of normalizers; moreover, the fact that intertwiners behave well under bounded isomorphism is a considerable advantage.

In the second part of the section, we consider a C^*-algebra $\mathcal{C}$ containing a MASA $\mathcal{D}$ which has the extension property (see Definition 3.6) and a $\mathcal{D}$-bimodule $\mathcal{M} \subseteq \mathcal{C}$. We use the extension property and a theorem from [1] to construct intertwiners in $\mathcal{M}$ from intertwiners in $\mathcal{C}$, and strengthen a key technical result of

Kumjian, Proposition 3.12. We see these results as a step towards extending coordinates from C^*-diagonals to more general settings.

3.1. CONTEXT FOR SECTION 3. Throughout this section, $\mathcal{C}$ will be a unital C^*-algebra and $\mathcal{D} \subseteq \mathcal{C}$ will be an abelian C^* subalgebra containing the unit of $\mathcal{C}$. Bimodules considered in this section will be closed subspaces $\mathcal{M}$ of $\mathcal{C}$ which are $\mathcal{D}$-bimodules under multiplication in $\mathcal{C}$.

DEFINITION 3.1. An element $v \in \mathcal{M}$ is a *normalizer* of $\mathcal{D}$ if $v\mathcal{D}v^* \cup v^*\mathcal{D}v \subseteq \mathcal{D}$. The set of all such elements is denoted $\mathcal{N}_{\mathcal{D}}(\mathcal{M})$ or, if $\mathcal{D}$ is clear, $\mathcal{N}(\mathcal{M})$. Recall that $\mathcal{M}$ is said to be *regular* if the closed span of $\mathcal{N}_{\mathcal{D}}(\mathcal{M})$ equals $\mathcal{M}$.

REMARK 3.2. The notion of normalizer in Definition 3.1 is the same as that used by Kumjian in [21].

Typically, normalizers play a major role in constructing coordinates for operator algebras. Since normalizers depend on the involution, it can be difficult to determine if isomorphisms that are not $*$-extendible or isometric preserve normalizers. Intertwiners (Definition 2.1) are not defined in terms of the involution and so it can be easier to decide if they are preserved by such isomorphisms. We begin with a comparison of normalizers and intertwiners.

It is easy to find examples of intertwiners of abelian C^*-algebras which are not normalizers: for a simple example, observe that every element of $M_2(\mathbb{C})$ is an intertwiner for $\mathbb{C}I_2$. However, the next proposition shows that when the abelian algebra is a MASA, intertwiners are normalizers.

PROPOSITION 3.3. *If $v \in \mathcal{C}$ is an intertwiner for $\mathcal{D}$, then $v^*v, vv^* \in \mathcal{D}' \cap \mathcal{C}$. If $\mathcal{D}$ is maximal abelian in $\mathcal{C}$, then v is a normalizer of $\mathcal{D}$.*

Proof. Let v be an intertwiner. Let

$$J_s := \{d \in \mathcal{D} : vd = 0\} \quad \text{and} \quad J_r := \{d \in \mathcal{D} : dv = 0\}.$$

Then J_s and J_r are norm-closed ideals in $\mathcal{D}$. Define a mapping α_v between $\mathcal{D}/J_s$ and $\mathcal{D}/J_r$ by $\alpha_v(d + J_s) = d' + J_r$, where $d' \in \mathcal{D}$ is chosen so that $vd = d'v$. It is easy to check that α_v is a well-defined $*$-isomorphism of $\mathcal{D}/J_s$ onto $\mathcal{D}/J_r$.

Let $d = d^* \in \mathcal{D}$. Then $\alpha_v(d + J_s) = d' + J_r$ where d' is chosen so that $d' = d'^* \in \mathcal{D}$. Thus, we have the equality of sets,

$$\begin{aligned}\{v^*vd\} = v^*v(d + J_s) = v^*(d' + J_r)v = [(d' + J_r)v]^*v \\ = [v(d + J_s)]^*v = (d + J_s)v^*v = \{dv^*v\}.\end{aligned}$$

Hence v^*v commutes with the selfadjoint elements of $\mathcal{D}$ and so commutes with $\mathcal{D}$. Since v^* is also an intertwiner, we conclude similarly that $vv^* \in \mathcal{D}'$.

If $\mathcal{D}$ is maximal abelian, then $v\mathcal{D}v^* = \mathcal{D}vv^* \subseteq \mathcal{D}$ and $v^*\mathcal{D}v = v^*v\mathcal{D} \subseteq \mathcal{D}$, and v is a normalizer as desired. ∎

For $v \in \mathcal{N}(\mathcal{C})$, let $S(v) := \{\phi \in \widehat{\mathcal{D}} : \phi(v^*v) > 0\}$; note this is an open set in $\widehat{\mathcal{D}}$. As observed by Kumjian (see Proposition 6 of [21]), there is a homeomorphism

$\beta_v : S(v) \to S(v^*)$ given by

$$\beta_v(\phi)(d) = \frac{\phi(v^* d v)}{\phi(v^* v)}.$$

It is easy to show that $\beta_v^{-1} = \beta_{v^*}$.

PROPOSITION 3.4. *For $v \in \mathcal{N}(\mathcal{C})$, if β_{v^*} extends to a homeomorphism of $\overline{S(v^*)}$ onto $\overline{S(v)}$, then v is an intertwiner. Moreover, if $\mathcal{I} := \{w \in \mathcal{C} : w\mathcal{D} = \mathcal{D}w\}$ is the set of intertwiners, then $\mathcal{N}(\mathcal{C})$ is contained in the norm-closure of $\mathcal{I}$, and when $\mathcal{D}$ is a* MASA *in $\mathcal{C}$, $\mathcal{N}(\mathcal{C}) = \overline{\mathcal{I}}$.*

Proof. It is clear that the set of normalizers is norm-closed.

Regard $\mathcal{C}$ as sitting inside its double dual $\mathcal{C}^{\#\#}$. Let $v = u|v| = |v^*|u$ be the polar decomposition for v. Since u is the strong* limit of $u_n := v(1/n + |v|)^{-1}$, we find that u also normalizes $\mathcal{D}^{\#\#}$.

Therefore, given any $d \in \mathcal{D}$, $vdv^* = u|v|du^*|v^*| = udu^*vv^*$. Hence for $\phi \in S(v^*)$, we have

$$\beta_{v^*}(\phi)(d) = \phi(udu^*). \tag{3.1}$$

Suppose now that β_{v^*} extends to a homeomorphism of $\overline{S(v^*)}$ onto $\overline{S(v)}$. By Tietze's Extension Theorem, we may then find an element $d_1 \in \mathcal{D}$ such that for every $\phi \in S(v^*)$, $\phi(d_1) = \beta_{v^*}(\phi)(d)$. Thus, for every $\phi \in \widehat{\mathcal{D}}$, $\phi(vdv^*) = \phi(udu^*)\phi(vv^*) = \phi(d_1 vv^*)$, so that

$$(udu^* - d_1)vv^*(ud^*u^* - d_1^*) = 0.$$

This shows that $udu^*v = d_1 v$ and so

$$vd = u|v|d = uu^*ud|v| = udu^*u|v| = udu^*v = d_1 v.$$

Hence $v\mathcal{D} \subseteq \mathcal{D}v$. Since the adjoint of a normalizer is again a normalizer and $\beta_v = (\beta_{v^*})^{-1}$, we may repeat this argument to obtain $v^*\mathcal{D} \subseteq \mathcal{D}v^*$. Taking adjoints yields $\mathcal{D}v \subseteq v\mathcal{D}$. Hence v is an intertwiner.

Given a general normalizer v, let $\varepsilon > 0$ and let $K = \{\phi \in \widehat{\mathcal{D}} : \phi(vv^*) \geqslant \varepsilon^2\}$. Then K is a compact subset of $S(v^*)$. Choose $d_0 \in \mathcal{D}$ so that $0 \leqslant d_0 \leqslant I$ and $\widehat{d_0}$ is compactly supported in $S(v^*)$ and $\widehat{d_0}|_K = 1$. Since $\beta_{v^*d_0} = \beta_{v^*}|_{S(v^*d_0)}$ and β_{v^*} is a homeomorphism, it extends to a homeomorphism of $\overline{S(v^*d_0)}$ onto $\overline{S(d_0 v)}$. Thus $d_0 v$ is an intertwiner. Further, $\|d_0 v - v\| = \|(d_0 - 1)vv^*(d_0 - 1)\|^{1/2} < \varepsilon$ so $d_0 v$ approximates v to within ε. Thus, $\mathcal{N}(\mathcal{C}) \subseteq \overline{\mathcal{I}}$. When $\mathcal{D}$ is a MASA in $\mathcal{C}$, Proposition 3.3 shows every intertwiner of $\mathcal{D}$ is a normalizer, so $\mathcal{N}(\mathcal{C}) = \overline{\mathcal{I}}$. ∎

REMARK 3.5. Taken together, Proposition 3.3 and Proposition 3.4 show that for a MASA $\mathcal{D}$ in a C^*-algebra $\mathcal{C}$, a *partial isometry* v is a normalizer if and only if it is an intertwiner. Related results for partial isometries are known ([30], Lemma 3.2).

The $*$-isomorphism $\alpha_v : \mathcal{D}/J_s \to \mathcal{D}/J_r$ appearing in the proof of Proposition 3.3 induces a homeomorphism h from the zero set $Z_r := \{\sigma \in \widehat{\mathcal{D}} : \sigma|_{J_r} = 0\}$

onto the zero set $Z_s := \{\sigma \in \widehat{\mathcal{D}} : \sigma|_{J_s} = 0\}$. It is not hard to show that if v is an intertwiner and $\mathcal{D}$ is a MASA, then $Z_s = \overline{S(v)}$, $Z_r = \overline{S(v^*)}$ and h is the extension of $\beta_{v^*} = \beta_v^{-1}$ to Z_r. Thus, it is possible to describe β_v without explicit reference to the $*$-structure.

We turn to constructing intertwiners in a module using the next property.

DEFINITION 3.6. Let $\mathcal{C}$ be a unital C^*-algebra. A C^*-subalgebra $\mathcal{D} \subseteq \mathcal{C}$ is said to have the *extension property* if every pure state of $\mathcal{D}$ has a unique extension to a state on $\mathcal{C}$ and no pure state of $\mathcal{C}$ annihilates $\mathcal{D}$.

If $\mathcal{D} \subseteq \mathcal{C}$ is abelian and $\mathcal{D}$ has the extension property relative to $\mathcal{C}$, then the Stone-Weierstrass Theorem implies that $\mathcal{D}$ is a MASA ([20], p. 385). We shall make essential use of the following result characterizing the extension property for abelian algebras.

THEOREM 3.7 ([1], Corollary 2.7). *Let $\mathcal{C}$ be a unital C^*-algebraand let $\mathcal{D}$ be an abelian C^*-subalgebra of $\mathcal{C}$ which contains the unit of $\mathcal{C}$. Then $\mathcal{D}$ has the extension property if and only if*

$$\overline{\mathrm{co}}\{gxg^{-1} : g \in \mathcal{D} \text{ and } g \text{ is unitary}\} \cap \mathcal{D} \neq \emptyset.$$

Furthermore, when this occurs, $\mathcal{D}$ is a MASA *and there exists a conditional expectation $E : \mathcal{C} \to \mathcal{D}$ such that*

$$\overline{\mathrm{co}}\{gxg^{-1} : g \in \mathcal{D} \text{ and } g \text{ is unitary}\} \cap \mathcal{D} = \{E(x)\}.$$

REMARK 3.8. Archibold, Bunce and Gregson in [1] also show that the condition

$$\mathcal{C} = \mathcal{D} + \overline{\mathrm{span}}\{cd - dc : c \in \mathcal{C}, d \in \mathcal{D}\}$$

also characterizes the extension property for an abelian subalgebra $\mathcal{D} \subseteq \mathcal{C}$. This characterization was important in Kumjian's work on C^*-diagonals.

DEFINITION 3.9. For $v \in \mathcal{N}(\mathcal{C})$, define $E_v : \mathcal{C} \to \mathcal{N}(\mathcal{C})$ by

$$E_v(x) = vE(v^*x).$$

Our first application of Theorem 3.7 is essentially contained in the proof of Proposition 4.4 in [24], but Theorem 3.7 provides a different (and simpler) proof. The crucial implication of Proposition 3.10 is that bimodules contain many normalizers.

PROPOSITION 3.10. *Suppose $\mathcal{D}$ is a commutative C^*-subalgebra of the unital C^*-algebra $\mathcal{C}$ which has the extension property, and $\mathcal{M} \subseteq \mathcal{C}$ is a $\mathcal{D}$-bimodule. If $v \in \mathcal{C}$ is a normalizer (respectively intertwiner) and $x \in \mathcal{M}$, then $E_v(x) \in \mathcal{M}$ and is a normalizer (respectively intertwiner).*

Proof. Fix $x \in \mathcal{M}$ and let G be the unitary group of $\mathcal{D}$. Since $v \in N_{\mathcal{D}}(\mathcal{C})$, for every $g \in G$, $vgv^* \in \mathcal{D}$, so that $(vgv^*)xg^{-1} \in \mathcal{M}$. Thus the norm-closed convex hull,

$$H := \overline{\mathrm{co}}\{(vgv^*)xg^{-1} : g \in G\} \subseteq \mathcal{M}.$$

By Theorem 3.7, $E(v^*x)$ belongs to $K := \overline{\text{co}}\{gv^*xg^{-1} : g \in G\}$. Since $vK \subseteq H$, we conclude that $vE(v^*x) = E_v(x) \in \mathcal{M}$. ∎

Although we have no application for it, the following result, which is a corollary of Proposition 3.10, provides another means of constructing normalizers in a bimodule.

PROPOSITION 3.11. *With $\mathcal{C}, \mathcal{D}$, and $\mathcal{M}$ as in Proposition 3.10 and $\mathcal{C}$ regular in $\mathcal{D}$, if $v \in \mathcal{N}(\mathcal{C})$, $m \in \mathcal{M}$ and $|\phi(v)| \leqslant |\phi(m)|$ for all $\phi \in \mathcal{E}^1(\mathcal{C})$, then $v \in \mathcal{M}$.*

Proof. We show in Proposition 4.7 below that $\mathcal{E}^1(\mathcal{C})$ consists precisely of functionals of the form $[v, \sigma]$, described in Definition 4.3. It is convenient to use this description for the proof of the present result; the proof of Proposition 4.7 does not use Proposition 3.11, so there is no circular reasoning.

For $\sigma \in \widehat{\mathcal{D}}$ with $\sigma(v^*v) \neq 0$, $\sigma(v^*v)$ equals

$$\sigma(v^*v)^{1/2}[v,\sigma](v) \leqslant |\sigma(v^*v)^{1/2}[v,\sigma](m)| = |\sigma(v^*m)| = |\sigma(E(v^*m))|,$$

and so, for $\sigma \in \widehat{\mathcal{D}}$, $\sigma(v^*v) \leqslant |\sigma(E(v^*m))|$. Thus, there exists $d \in \mathcal{D}$ with $v^*v = E(v^*m)d$. Hence for $n \in \mathbb{N}$, $(v^*v)^{1/n}$ belongs to the closed ideal of $\mathcal{D}$ generated by $E(v^*m)$. By Proposition 3.10, $vE(v^*m) \in \mathcal{M}$ and we conclude $v(v^*v)^{1/n} \in \mathcal{M}$. But $v = \lim_{n\to\infty} v(v^*v)^{1/n}$, whence $v \in \mathcal{M}$. ∎

Our second application of Theorem 3.7 is to provide an alternate proof of, and slightly strengthen, a result of Kumjian.

PROPOSITION 3.12 ([21], Lemma 9, p. 972). *Let $\mathcal{D}$ be an abelian C^*-subalgebra of the unital C^*-algebra $\mathcal{C}$ with the extension property. For $v \in \mathcal{N}(\mathcal{C})$, $v^*E(v)$ and $vE(v^*)$ both belong to $\mathcal{D}$. If $\sigma \in S(v)$, then the following are equivalent:*

(i) $\sigma(v) \neq 0$;
(ii) $\beta_v(\sigma) = \sigma$;
(iii) $\sigma(E(v^*)) \neq 0$.

Proof. Taking $x = I$ and $\mathcal{M} = \mathcal{D}$ in Proposition 3.10 we see that $v^*E(v)$ and $vE(v^*)$ both belong to $\mathcal{D}$.

Suppose $\sigma(v) \neq 0$. An easy calculation shows that when $d \in \mathcal{D}$ and $\sigma(d) \neq 0$, then $\beta_{vd} = \beta_v$. Let $d = v^*E(v)$. By hypothesis, we find $\sigma(d) = |\sigma(v)|^2 \neq 0$, and another calculation shows that $\beta_{vd} = \beta_v$, so (i) implies (ii).

Assume (ii) holds. Letting G again be the unitary group of $\mathcal{D}$, we have, for all $g \in G$,

$$\sigma(v^*gvg^{-1}) = \sigma(v^*gv)\sigma(g^{-1}) = \beta_v(\sigma)(g)\sigma(g^{-1})\sigma(v^*v) = \sigma(v^*v).$$

Thus, $\sigma(v^*\overline{\text{co}}\{gvg^{-1}\, g \in G\}) \subseteq \{\sigma(v^*v)\}$, and therefore by Theorem 3.7 we obtain $\sigma(v^*)\sigma(v) = \sigma(v^*E(v)) \neq 0$. Thus $\sigma(E(v^*)) \neq 0$ and by our earlier remarks, $vE(v^*) \in \mathcal{D}$.

Finally, it is evident that (iii) implies (i). ∎

4. REGULAR C^*-INCLUSIONS AND C^*-DIAGONALS

We now turn to the principal context for our subsequent work, that of C^*-diagonals and regular C^*-inclusions. After recalling Kumjian's twist, we show that the elements of the twist are eigenfunctionals on the C^*-algebra and conversely (Proposition 4.7 and Theorem 4.9). This allows us to show that every regular C^*-inclusion has a quotient which is a C^*-diagonal with the same coordinate system, Theorem 4.8, as well as strengthening various results from Section 2 in this context. The crucial result for subsequent sections is Theorem 4.13, which shows that for a bimodule $\mathcal{M}$, the intrinsically defined eigenfunctionals on $\mathcal{M}$ and the restriction of the twist are the same.

DEFINITION 4.1. The pair $(\mathcal{C}, \mathcal{D})$ will be called a *regular C^*-inclusion* if $\mathcal{D}$ is a maximal abelian C^*-subalgebra of the unital C^*-algebra $\mathcal{C}$ such that

(i) $\mathcal{D}$ has the extension property in $\mathcal{C}$;

(ii) $\mathcal{C}$ is regular (as a $\mathcal{D}$-bimodule).

Always, E denotes the (unique) conditional expectation of $\mathcal{C}$ onto $\mathcal{D}$. We call $(\mathcal{C}, \mathcal{D})$ a *C^*-diagonal* if, in addition,

(iii) E is faithful.

REMARKS 4.2. A few comments on the definition are appropriate.

(i) Renault [39] gives, without proof, an example of an algebra satisfying only the first two conditions but not the third, i.e., a regular C^*-inclusion that is not a C^*-diagonal.

(ii) By Propositions 3.3 and 3.4, regularity of a bimodule $\mathcal{M}$ is equivalent to norm-density of the $\mathcal{D}$-intertwiners.

(iii) As observed by Renault [39], this definition of C^*-diagonal is equivalent to Kumjian's original definition, namely,

(a) there is a faithful conditional expectation $E : \mathcal{C} \to \mathcal{D}$;

(b) the closed span of the free normalizers in $\mathcal{C}$ is $\ker E$.

A normalizer v of $\mathcal{D}$ is *free* if $v^2 = 0$. Kumjian also required that $\mathcal{C}$ is separable and $\widehat{\mathcal{D}}$ is second countable, but [39] shows this is not necessary. It is often easier to verify Kumjian's axioms when working with particular examples.

(iv) Renault ([38], Definition II.4.13) defines a Cartan subalgebra $\mathcal{D}$ of a C^*-algebra $\mathcal{C}$. Using a partition of unity and the characterization of the extension property of [1] (described in Remark 3.8 above), one can show that if $\mathcal{D} \subseteq \mathcal{C}$ is a Cartan subalgebra in Renault's sense, then (assuming $\mathcal{C}$ is unital) $(\mathcal{C}, \mathcal{D})$ is a regular C^*-inclusion. Combining Theorem II.4.15 of [38] with Theorem 3.3 of [24], one observes that when $\mathcal{C}$ is nuclear and $\mathcal{D} \subseteq \mathcal{C}$ is Cartan, then $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal. Finally, we note that the class of regular C^*-inclusions properly contains the family of pairs $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D}$ is Cartan in $\mathcal{C}$. One reason for this is that the class of normalizers Renault uses to define a Cartan subalgebra is more restrictive than that appearing in Definition 3.1.

Unless explicitly stated otherwise, for the remainder of this section, we work in the following setting.

4.1. CONTEXT FOR SECTION 4. Let $(\mathcal{C}, \mathcal{D})$ be a regular C^*-inclusion and $\mathcal{M} \subseteq \mathcal{C}$ be a norm-closed $\mathcal{D}$-bimodule, where the module action is multiplication. That is, for $d, e \in \mathcal{D}$ and $m \in \mathcal{M}$, $d \cdot m \cdot e := dme \in \mathcal{M}$.

For $\sigma \in \widehat{\mathcal{D}}$, the unique extension on $\mathcal{C}$ is $\sigma \circ E$, which we will again denote by σ. Thus, we regard σ as either a multiplicative linear functional on $\mathcal{D}$ or a pure state on $\mathcal{C}$ which satisfies $\sigma \circ E = \sigma$. In particular, for $d \in \mathcal{D}$ and $x \in \mathcal{C}$, $\sigma(dx) = \sigma(xd) = \sigma(d)\sigma(x)$.

We now summarize some results and definitions from [21] and then relate eigenfunctionals to the elements of Kumjian's twist. Note that these results from [21] hold for regular C^*-inclusions.

DEFINITION 4.3 (Kumjian). For $v \in \mathcal{N}(\mathcal{C})$ and $\sigma \in \widehat{\mathcal{D}}$ with $\sigma(v^*v) > 0$, define a linear functional on $\mathcal{C}$, $[v, \sigma]$, by

$$[v, \sigma](x) = \frac{\sigma(v^*x)}{\sigma(v^*v)^{1/2}}.$$

Kumjian denotes by Γ the collection of all such linear functionals. We shall see in Proposition 4.7 below that $\Gamma = \mathcal{E}^1(\mathcal{C})$.

We follow Kumjian ([21], p. 982) in pointing out that Proposition 3.12 implies

COROLLARY 4.4 (Kumjian). *The following are equivalent:*

(i) $[v, \sigma] = [w, \sigma]$;
(ii) $\sigma(v^*w) > 0$;
(iii) *there are* $d, e \in \mathcal{D}$ *with* $\sigma(d), \sigma(e) > 0$ *so that* $vd = we$.

REMARK 4.5. Using Theorem 3.7 and the techniques of the proof of Proposition 3.10, one can show that when $\sigma(v^*w) > 0$, we may take $d = w^*wE(v^*w)$ and $e = w^*vE(v^*w)$ in the third part of Corollary 4.4.

Kumjian shows that Γ, with a suitable operation and the relative weak*-topology, is a groupoid and admits a natural $\mathbb{T}$-action, given by $\lambda[v, \sigma] = [\overline{\lambda}v, \sigma]$. The range map for Γ is $[v, \sigma] \mapsto \beta_v(\sigma)$ and the source map for Γ is $[v, \sigma] \mapsto \sigma$. The map $[v, \sigma] \in \Gamma \mapsto (\beta_v(\sigma), \sigma)$ sends Γ to a Hausdorff equivalence relation (principal groupoid) on $\widehat{\mathcal{D}}$, denoted $\Gamma \backslash \mathbb{T}$, and Γ is a locally trivial principal $\mathbb{T}$-bundle over $\Gamma \backslash \mathbb{T}$.

While the range and source maps of this groupoid have, a priori, no connection to the range and source maps for eigenfunctionals, they turn out to be the same.

We recall the multiplication on Γ and use it to define multiplication of eigenfunctionals for regular C^*-inclusions. The partially defined multiplication on Γ

has $[v_1,\sigma_1],[v_2,\sigma_2]$ composable if $\sigma_1 = \beta_{v_2}(\sigma_2)$, in which case,

$$[v_1,\sigma_1]\cdot[v_2,\sigma_2] = [v_1v_2,\sigma_2].$$

DEFINITION 4.6 (Kumjian). A *twist* is a proper $\mathbb{T}$-groupoid Γ so that $\Gamma/\mathbb{T}$ is an r-discrete principal groupoid.

Kumjian constructs, for each twist Γ, a C^*-diagonal, $(A(\Gamma),B(\Gamma))$. The main result of [21] is that for $(\mathcal{C},\mathcal{D})$ a C^*-diagonal, there is a unique (up to isomorphism) twist Γ and an isomorphism $\Phi : A(\Gamma)\to\mathcal{C}$ such that $\Phi(B(\Gamma)) = \mathcal{D}$. Thus, every twist arises as a Γ as above, for some C^*-diagonal $(\mathcal{C},\mathcal{D})$.

From our point of view, justified by the following proposition, the twist Γ associated to $(\mathcal{C},\mathcal{D})$ is $\mathcal{E}^1(\mathcal{C})$ equipped with this groupoid operation, $\mathbb{T}$-action, and topology.

PROPOSITION 4.7. *For all $\sigma\in\widehat{\mathcal{D}}$ and $v\in\mathcal{N}(\mathcal{C})$ with $\sigma(v^*v)>0$, $[v,\sigma]\in\mathcal{E}^1(\mathcal{C})$. Moreover, the range and source maps agree, that is, viewed as an element of $\mathcal{E}^1(\mathcal{C})$, we have $\sigma = s([v,\sigma])$ and $\beta_v(\sigma) = r([v,\sigma])$.*

Conversely, if $\phi\in\mathcal{E}(\mathcal{C})$ there exists a normalizer $v\in\mathcal{C}$ such that $\phi(v)\neq 0$, and v may be taken to be an intertwiner if desired. For any normalizer (or intertwiner) v with $\phi(v)\neq 0$, we have $s(\phi)(v^*v)\neq 0$ and

$$\phi(v)[v,s(\phi)] = [v,s(\phi)](v)\phi.$$

In particular, if $\phi\in\mathcal{E}^1(\mathcal{C})$ then $\lambda := \frac{\phi(v)}{[v,s(\phi)](v)}\in\mathbb{T}$ and $\phi = \lambda[v,s(\phi)]$.

Proof. We have $[v,\sigma](xd) = [v,\sigma](x)\sigma(d)$ and, as $v^*v(v^*dx) = (v^*dv)v^*x$, we can apply σ to this equation and divide by $\sigma(v^*v)^{3/2}$ to obtain

$$[v,\sigma](dx) = \frac{\sigma(v^*dv)}{\sigma(v^*v)}[v,\sigma](x) = \beta_v(d)[v,\sigma](x),$$

so $[v,\sigma]$ is a $\mathcal{D}$-eigenfunctional with range $\beta_v(\sigma)$ and source σ. Letting $w = \sigma(v^*v)^{-1/2}v$, $[v,\sigma](w) = 1$, so $[v,\sigma]\in\mathcal{E}^1(\mathcal{C})$.

For the converse, let $\phi\in\mathcal{E}^1(\mathcal{C})$ and set $\sigma = s(\phi)$. Since the span of $N_{\mathcal{D}}(\mathcal{C})$ is norm dense in $\mathcal{C}$, there exists $v\in N_{\mathcal{D}}(\mathcal{C})$ with $\phi(v)\neq 0$, which by Proposition 3.4 may be taken to be an intertwiner if desired. Fix such a v.

For any $n\in\mathbb{N}$, $\phi(v(v^*v)^{1/n}) = \phi(v)\sigma(v^*v)^{1/n}$. Since $v(v^*v)^{1/n}$ converges to v, $\sigma(v^*v)>0$. Also, for any $d\in\mathcal{D}$ we have,

$$r(\phi)(d)\phi(v)=\phi(dv)=\frac{\phi(dv(v^*v))}{\sigma(v^*v)}=\frac{\phi(v(v^*dv))}{\sigma(v^*v)}=\frac{\phi(v)\sigma(v^*dv)}{\sigma(v^*v)}=\phi(v)\beta_v(\sigma)(d),$$

so $r(\phi) = \beta_v(\sigma)$.

For any unitary element $g\in\mathcal{D}$ we have,

$$\phi(vgv^*xg^{-1}) = r(\phi)(vgv^*)\phi(x)\sigma(g^{-1}) = \frac{\sigma(v^*vgv^*v)}{\sigma(v^*v)}\phi(x)\sigma(g)^{-1} = \sigma(v^*v)\phi(x).$$

Theorem 3.7 implies that $vE(v^*x)$ belongs to the closed convex hull of $\{vgv^*xg^{-1} : g \in \mathcal{U}(\mathcal{D})\}$, and thus $\phi(vE(v^*x)) = \sigma(v^*v)\phi(x)$. Hence for any $x \in \mathcal{C}$,

$$\phi(v)[v,\sigma](x) = \frac{\phi(v)\sigma(v^*x)}{\sigma(v^*v)^{1/2}} = \frac{\phi(vE(v^*x))}{\sigma(v^*v)^{1/2}} = \sigma(v^*v)^{1/2}\phi(x) = [v,\sigma](v)\phi(x).$$

This equality, together with the fact that $\|[v,\sigma]\| = 1$, shows that $\lambda \in \mathbb{T}$ when ϕ has unit norm. ∎

Proposition 4.7 leads to a description of $\mathcal{E}(\mathcal{M})$ in terms of $[v,\sigma]$'s, for a norm-closed $\mathcal{D}$-bimodule $\mathcal{M}$ (Theorem 4.13). Before giving this description, we use Proposition 4.7 as a tool in the following result, which gives the precise relationship between the concepts of regular C^*-inclusion and C^*-diagonal.

THEOREM 4.8. *Let $(\mathcal{C},\mathcal{D})$ be a regular C^*-inclusion, and let $\mathfrak{N} := \{x \in \mathcal{C} : E(x^*x) = 0\}$ be the left kernel of E. Then $\mathfrak{N}$ is a closed (two-sided) ideal of $\mathcal{C}$ and*

$$\mathfrak{N} = \{x \in \mathcal{C} : \phi(x) = 0 \text{ for all } \phi \in \mathcal{E}(\mathcal{C})\}.$$

Let $\pi : \mathcal{C} \to \mathcal{C}/\mathfrak{N}$ be the quotient map. Then $(\pi(\mathcal{C}), \pi(\mathcal{D}))$ is a C^*-diagonal, $\pi|_{\mathcal{D}}$ is a $*$-isomorphism of $\mathcal{D}$ onto $\pi(\mathcal{D})$, and the restriction of the adjoint map $\pi^{\#}|_{\mathcal{E}(\pi(C))}$ is an isometric isomorphism of $\mathcal{E}(\pi(\mathcal{C}))$ onto $\mathcal{E}(\mathcal{C})$.

In particular, the first part of the theorem shows that, in a C^*-diagonal, $\mathcal{E}^1(\mathcal{C})$ separates points.

Proof. Since $\mathfrak{N} = \{x \in \mathcal{C} : E(x^*x) = 0\}$ is clearly a closed left ideal, to show that it is also a right ideal, it suffices to show that, for $x \in \mathfrak{N}$ and a normalizer $v \in \mathcal{C}$, $xv \in \mathfrak{N}$. To do this, we shall prove that for every $\sigma \in \widehat{\mathcal{D}}$, $\sigma(v^*x^*xv) = 0$. When $\sigma(v^*v) = 0$, this holds since $\sigma(v^*x^*xv) \leqslant \|x\|^2\sigma(v^*v)$. When $\sigma(v^*v) \neq 0$, let $\psi \in \widehat{\mathcal{D}}$ be given by

$$\psi(z) = \frac{\sigma(v^*zv)}{\sigma(v^*v)}$$

and observe that, as $x \in \mathfrak{N}$, $\psi(x^*x) = 0$ and hence $\sigma(v^*x^*xv) = 0$ in this case as well. Thus $\mathfrak{N}$ is a closed two-sided ideal.

We now show that $\mathfrak{N} = \{x \in \mathcal{C} : \phi(x) = 0 \text{ for all } \phi \in \mathcal{E}(\mathcal{C})\}$. Suppose that $x \in \mathfrak{N}$. Given $\phi \in \mathcal{E}(\mathcal{C})$, by Proposition 4.7 we may assume $\phi = [v,\sigma]$ where $\sigma \in \widehat{\mathcal{D}}$ and v is a normalizer with $\sigma(v^*v) > 0$. By the Cauchy-Schwarz inequality we have $|\sigma(v^*x)|^2 \leqslant \sigma(v^*v)\sigma(x^*x)$. But $\sigma(x^*x) = \sigma(E(x^*x)) = 0$, so $[v,\sigma](x) = 0$.

Conversely, suppose $x \neq 0$ and $\phi(x) = 0$ for every $\phi \in \mathcal{E}^1(\mathcal{C})$. We shall show that $\sigma(x^*x) = 0$ for every $\sigma \in \widehat{\mathcal{D}}$. So fix $\sigma \in \widehat{\mathcal{D}}$. Notice that for every normalizer $v \in \mathcal{C}$, we have $\sigma(v^*x) = 0$: this follows from the Cauchy-Schwartz inequality when $\sigma(v^*v) = 0$ and from the hypothesis and Proposition 4.7 when $\sigma(v^*v) \neq 0$. Let $\varepsilon > 0$. Since the span of the normalizers is dense in $\mathcal{C}$, we may find normalizers $v_1, \ldots, v_n$ so that $\left\| x - \sum_{i=1}^{n} v_i \right\| < \varepsilon / \|x\|$. Thus,

$$|\sigma(x^*x)| = \left|\sigma\Big(x^*x - \sum_{j=1}^{n} v_j^* x\Big)\right| \leqslant \left\| x^* - \sum_{j=1}^{n} v_j^* \right\| \|x\| < \varepsilon,$$

and we conclude that $\sigma(x^*x) = 0$. Since this holds for every $\sigma \in \widehat{D}$, we have $E(x^*x) = 0$.

Clearly, $\mathfrak{N} \cap \mathcal{D} = 0$, so $\pi|_{\mathcal{D}}$ is an isomorphism of $\mathcal{D}$ onto $\pi(\mathcal{D})$. To see that $(\pi(\mathcal{C}), \pi(\mathcal{D}))$ is a C^*-diagonal, observe first that E gives rise to a faithful conditional expectation on $\mathcal{C}/\mathfrak{N}$, by the definition of $\mathfrak{N}$. Given a pure state σ of $\pi(\mathcal{D})$, let τ_1 and τ_2 be pure states of $\pi(\mathcal{C})$ which extend σ. Now $\tau_i \circ \pi$ are extensions of the pure state $\sigma \circ \pi$ of $\mathcal{D}$ and hence coincide because $\mathcal{D}$ has the extension property in $\mathcal{C}$. Therefore, as π is onto, $\tau_1 = \tau_2$, and $\pi(\mathcal{D})$ has the extension property in $\pi(\mathcal{C})$.

To show that $\pi(\mathcal{C})$ is regular (relative to $\pi(\mathcal{D})$), let $x + \mathfrak{N} \in \pi(\mathcal{C})$, and $\varepsilon > 0$. Since $\mathcal{C}$ is regular, we may find normalizers $v_i \in \mathcal{C}$ such that $y := \sum_{i=1}^{n} v_i$ satisfies $\|x - y\| < \varepsilon$. Then $\pi(v_i)$ is a normalizer, and since π is contractive, $\|(x + \mathfrak{N}) - (y + \mathfrak{N})\| < \varepsilon$. Hence $\pi(\mathcal{C})$ is regular. It is also clear that $\pi^\#|_{\mathcal{E}(\pi(\mathcal{C}))}$ is a homeomorphism of $\mathcal{E}(\pi(\mathcal{C}))$ onto $\mathcal{E}(\mathcal{C})$ which preserves the partially defined product structure. Finally, since the adjoint of a quotient map is always isometric, the proof is complete. ∎

We turn now to additional consequences of Proposition 4.7. Recall that for any $f \in \mathcal{C}^\#$, f^* is the bounded linear functional given by $f^*(x) = \overline{f(x^*)}$. It is easy to see that if ϕ is an eigenfunctional on $\mathcal{C}$, then so is ϕ^* and also that $s(\phi^*) = r(\phi)$ and $r(\phi^*) = s(\phi)$. Thus, $f \mapsto f^*$ provides an involution on $\mathcal{E}^1(\mathcal{C})$. The inverse of $[v, \sigma]$ is $[v^*, \beta_v(\sigma)]$. Thus, we can summarize Proposition 4.7 and the discussion preceding it.

THEOREM 4.9. *For a regular C^*-inclusion $(\mathcal{C}, \mathcal{D})$, $\mathcal{E}^1(\mathcal{C}) = \Gamma$ and the range and source maps, the involution, and topology are all the same.*

COROLLARY 4.10. *If $\phi, \psi \in \mathcal{E}(\mathcal{M})$ satisfy $r(\phi) = r(\psi)$ and $s(\phi) = s(\psi)$, then there exists $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and $\phi = \lambda\psi$.*

Proof. Without loss of generality, assume $\|\phi\| = \|\psi\| = 1$. Let $\sigma = s(\phi) = s(\psi)$. Theorem 2.6 shows ϕ and ψ extend to norm-one eigenfunctionals on $\mathcal{C}$, which we denote by the same symbols. By Proposition 4.7, we have $\phi = [v, \sigma]$ and $\psi = [w, \sigma]$ for some $v, w \in \mathcal{N}(\mathcal{C})$. It follows from the hypothesis that $\beta_v(\sigma) = \beta_w(\sigma)$. Thus $\beta_{v^*w}(\sigma) = \beta_{v^*}(\beta_w(\sigma)) = \sigma$, so we can find $\lambda \in \mathbb{T}$ with $\sigma(\lambda v^* w) > 0$. Hence by Corollary 4.4, $[\overline{\lambda} v, \sigma] = [w, \sigma]$ and therefore $\lambda\phi = \psi$. ∎

REMARK 4.11. Using the expression for $\phi = [v, s(\phi)]$ from Proposition 4.7 and a short calculation with $r(\phi)(x) = \sigma(v^*xv)/\sigma(v^*v)$, where $\sigma = s(\phi)$, we obtain for all $x \in \mathcal{C}$,

$$\phi(x) = \frac{s(\phi)(v^*x)}{[s(\phi)(v^*v)]^{1/2}} = \frac{r(\phi)(xv^*)}{[r(\phi)(vv^*)]^{1/2}}.$$

Also, we can sharpen the inequality of Proposition 2.5 (ii) to an equality.

COROLLARY 4.12. *If $\phi \in \mathcal{E}(\mathcal{M})$, then $|\phi(m)| = \|\phi\| B_{r(\phi),s(\phi)}(m)$ for all $m \in \mathcal{M}$.*

Proof. By Proposition 2.5, we have $|\phi(m)| \leqslant \|\phi\| B_{r(\phi),s(\phi)}(m)$ for every $m \in \mathcal{M}$.

To obtain the reverse inequality, fix $m \in \mathcal{M}$ such that $B_{r(\phi),s(\phi)}(m) \neq 0$. Given $\varepsilon \in (0,1)$, we may find elements $a, b \in \mathcal{D}$ such that $\rho(a) = \sigma(b) = 1$ and $B_{r(\phi),s(\phi)}(m) > (1-\varepsilon)\|amb\|$. Let $m_0 := amb$ and define a linear functional f on $\mathbb{C}m_0$ by

$$f(tm_0) = t\|\phi\| B_{r(\phi),s(\phi)}(m_0) = t\|\phi\| B_{r(\phi),s(\phi)}(m).$$

By the Hahn-Banach Theorem and Proposition 2.5, f extends to an eigenfunctional F on $\mathcal{M}$ such that for every $x \in \mathcal{M}$,

$$|F(x)| \leqslant \|\phi\| B_{r(\phi),s(\phi)}(x).$$

Thus $\|F\| \leqslant \|\phi\|$, $s(F) = s(\phi)$ and $r(F) = r(\phi)$. By Corollary 4.10, there exists a nonzero scalar λ with $|\lambda| \leqslant 1$ and $F = \lambda\phi$. We obtain,

$$\begin{aligned}|\phi(m)| \geqslant |F(m)| = |F(m_0)| = |f(m_0)| = \|\phi\| \frac{B_{r(\phi),s(\phi)}(m)}{\|amb\|}\|amb\| \\ > (1-\varepsilon)\|\phi\|\|amb\| \geqslant (1-\varepsilon)\|\phi\| B_{r(\phi),s(\phi)}(m).\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the result. ∎

We now extend Proposition 4.7 to bimodules and show that the set of eigenfunctionals on a bimodule can be written in the form $[v, \sigma]$ with $v \in \mathcal{M}$.

THEOREM 4.13. *If $\phi \in \mathcal{E}^1(\mathcal{M})$, then there is a intertwiner $v \in \mathcal{M}$ and $\sigma \in \widehat{\mathcal{D}}$ so that $\phi = [v, \sigma]|_{\mathcal{M}}$.*

Conversely, if $v \in \mathcal{N}(\mathcal{M})$, and $\sigma \in \widehat{\mathcal{D}}$ satisfies $\sigma(v^*v) > 0$, then $[v, \sigma]|_{\mathcal{M}}$ belongs to $\mathcal{E}^1(\mathcal{M})$.

Proof. We prove the second assertion first. Given $v \in \mathcal{N}(\mathcal{M})$ and $\sigma \in \widehat{\mathcal{D}}$ with $\sigma(v^*v) \neq 0$, Proposition 4.7 shows that $\phi := [v, \sigma]|_{\mathcal{M}}$ is an eigenfunctional on $\mathcal{M}$. Clearly, $\|\phi\| \leqslant \|[v, \sigma]\| = 1$. To show that $\|\phi\| = 1$, fix $m \in \mathcal{M}$ with $\phi(m) \neq 0$ and set $w = E_v(m)/\sigma(v^*v)$. Proposition 3.10 shows that $w \in \mathcal{N}(\mathcal{M})$. Calculation then yields

$$\sigma(w^*w) = |[v, \sigma](m)|^2 \neq 0, \quad [v, \sigma](w) = [v, \sigma](m). \tag{4.1}$$

Let $\varepsilon > 0$. Pick a norm-one positive element $d \in \mathcal{D}$ with $\sigma(d) = 1$ so that $\|d^2w^*w\| \leqslant (1+\varepsilon)\sigma(w^*w)$. If we let $s = wd/\sigma(w^*w)^{1/2}$, then $\|s\| \leqslant (1+\varepsilon)^{1/2}$ and the equations (4.1) give

$$\Big|\phi\Big(\frac{wd}{\sigma(w^*w)^{1/2}}\Big)\Big| = \Big|[v, \sigma]\Big(\frac{wd}{\sigma(w^*w)^{1/2}}\Big)\Big| = \Big|\frac{[v, \sigma](w)}{\sigma(w^*w)^{1/2}}\Big| = 1.$$

To prove the first assertion, let $\phi \in \mathcal{E}^1(\mathcal{M})$ and fix $m \in \mathcal{M}$ with $\phi(m) \neq 0$. By Theorem 2.6, ϕ extends to a norm-one eigenfunctional, also called ϕ, on $\mathcal{C}$. By Proposition 4.7, there is an intertwiner v and $\sigma \in \widehat{\mathcal{D}}$ with $\sigma(v^*v) > 0$ so that $\phi = [v, \sigma]$. By Proposition 3.10,

$$w := \frac{vE(v^*m)}{\phi(m)\sigma(v^*v)^{1/2}} \in \mathcal{M}.$$

and w is an intertwiner. Thus,

$$\phi(w) = \phi(v)\frac{\sigma(v^*m)}{\phi(m)\sigma(v^*v)^{1/2}} = \phi(v)\frac{[v,\sigma](m)}{\phi(m)} = \phi(v),$$

so that $[w, \sigma] = [v, \sigma] = \phi$. ∎

REMARK 4.14. Let $\mathcal{A} \subseteq \mathcal{C}$ be a norm-closed algebra which is also a $\mathcal{D}$-bimodule. We can use the multiplication defined for elements of Γ to give a (also partially defined) multiplication on $\mathcal{E}^1(\mathcal{A})$ and hence on $\mathcal{E}(\mathcal{A})$. Indeed, call $\phi_1, \phi_2 \in \mathcal{E}^1(\mathcal{A})$ composable if $s(\phi_1) = r(\phi_2)$. By Theorem 4.13, we can write $\phi_i = [v_i, \sigma_i]$, where v_1 and v_2 are in $\mathcal{N}(\mathcal{A})$. Define $\phi_1\phi_2$ to be the product of the $[v_i, \sigma_i]$ restricted to $\mathcal{A}$, namely $[v_1v_2, \sigma_2]|_{\mathcal{A}}$.

We can also improve several of the results of the previous section. First, we immediately have a unique extension in Theorem 2.6.

COROLLARY 4.15. *Suppose $\mathcal{M}_1, \mathcal{M}_2$ are norm-closed $\mathcal{D}$-bimodules with $\mathcal{M}_1 \subseteq \mathcal{M}_2$. There is a unique isometric map $\iota : \mathcal{E}(\mathcal{M}_1) \to \mathcal{E}(\mathcal{M}_2)$ so that, for every $\phi \in \mathcal{E}(\mathcal{M}_1)$, $\iota(\phi)|_{\mathcal{M}_1} = \phi$. The image $\iota(\mathcal{E}(\mathcal{M}_1))$ is an open subset of $\mathcal{E}(\mathcal{M}_2)$.*

If in addition, $\mathcal{M}_2$ is regular, then ι is $\sigma(\mathcal{M}_1^\#, \mathcal{M}_1)$–$\sigma(\mathcal{M}_2^\#, \mathcal{M}_2)$ continuous on bounded subsets of $\mathcal{E}(\mathcal{M}_1)$.

Proof. Fix $\phi \in \mathcal{E}(\mathcal{M}_1)$. Theorem 4.13 shows ϕ extends uniquely to an eigenfunctional $\iota(\phi)$ on $\mathcal{M}_2$ of the same norm. Writing $\phi = [v, \sigma]$ for some normalizer $v \in \mathcal{M}_1$ and $\sigma \in \widehat{\mathcal{D}}$, then $\{\psi \in \mathcal{E}(\mathcal{M}_2) : \psi(v) \neq 0\}$ is a $\sigma(\mathcal{M}_2^\#, \mathcal{M}_2)$-open set containing $\iota(\phi)$, so $\iota(\mathcal{E}(\mathcal{M}_1))$ is an open set in $\mathcal{E}(\mathcal{M}_2)$.

It remains to prove that ι is continuous on bounded subsets of $\mathcal{E}(\mathcal{M}_1)$ when $\mathcal{M}_2$ is regular. Suppose ϕ_λ is a bounded net in $\mathcal{E}(\mathcal{M}_1)$ converging $\sigma(\mathcal{M}_1^\#, \mathcal{M}_1)$ to $\phi \in \mathcal{E}(\mathcal{M}_1)$. Let $\sigma = s(\phi)$ and $\sigma_\lambda = s(\phi_\lambda)$. Then σ_λ converges in the $\sigma(\mathcal{C}^\#, \mathcal{C})$-topology to σ.

By Theorem 4.13, there exists a normalizer $v \in \mathcal{M}_1$ such that $\phi = \|\phi\|[v, \sigma]$. For large enough λ, $\phi_\lambda(v) \neq 0$, so there exist scalars $t_\lambda \in \mathbb{C}$ with $|t_\lambda| = \|\phi_\lambda\|$ and $\phi_\lambda = t_\lambda[v, \sigma_\lambda]$. Since $\phi_\lambda(v) \to \phi(v)$ we have $t_\lambda \to \|\phi\|$.

For any normalizer $w \in \mathcal{M}_2$, we have

$$\iota(\phi_\lambda)(w) = t_\lambda[v, \sigma_\lambda](w) = t_\lambda \frac{\sigma_\lambda(v^*w)}{\sigma_\lambda(v^*v)^{1/2}} \to \|\phi\|[v, \sigma](w) = \iota(\phi)(w).$$

As $\mathcal{M}_2$ is the span of the normalizers it contains and ϕ_λ is a bounded net, we conclude that for any $x \in \mathcal{M}_2$, $\iota(\phi_\lambda)(x) \to \iota(\phi)(x)$. ∎

REMARK 4.16. Given a bimodule $\mathcal{M} \subseteq \mathcal{C}$, the set $\Gamma_{\mathcal{M}} := \{\phi \in \Gamma : \phi|_{\mathcal{M}} \neq 0\}$ is an open subset of Γ which plays a crucial role in the study of $\mathcal{M}$ (see, for example, [24], [25]). Theorem 4.13, together with Corollary 4.15, shows that the restriction map $[v, \sigma] \in \Gamma_{\mathcal{M}} \mapsto [v, \sigma]|_{\mathcal{M}}$ is a homeomorphism of $\Gamma_{\mathcal{M}}$ onto $\mathcal{E}^1(\mathcal{M})$. Thus, $\Gamma_{\mathcal{M}}$ can be defined directly in terms of the bimodule structure of $\mathcal{M}$, without explicit reference to $\mathcal{C}$.

Since the norm is only lower semi-continuous for weak* convergence, it is not possible to show that $\mathcal{E}^1(\mathcal{M})$ is locally compact for general modules. However, for regular C^*-inclusions, we can show this.

PROPOSITION 4.17. *With the relative weak*-topology, $\mathcal{E}^1(\mathcal{M}) \cup \{0\}$ is compact. Thus, $\mathcal{E}^1(\mathcal{M})$ is a locally compact Hausdorff space.*

Proof. Suppose that ϕ_λ is a net in $\mathcal{E}^1(\mathcal{M}) \cup \{0\}$ which converges to $\phi \in (\mathcal{M}^\#)_1$. If $\phi = 0$, there is nothing to do, so we assume that $\phi \neq 0$ and show that $\|\phi\| = 1$.

Fix a normalizer $v \in \mathcal{M}$ with $\phi(v) > 0$. From Theorem 4.13, $\phi = \|\phi\|[v, \sigma]$. Choose a positive element $d \in \mathcal{D}$ so that $0 \leqslant dv^*vd \leqslant I$ and $\widehat{dv^*vd} = 1$ in a neighborhood of σ. For large enough λ, $\phi_\lambda(vd) \neq 0$, so there exists $t_\lambda \in \mathbb{T}$ such that $\phi_\lambda = t_\lambda[vd, s(\phi_\lambda)]$. Thus, $1 = |\phi_\lambda(vd)|$. As ϕ_λ converges to ϕ, we obtain $|\phi(vd)| = 1 = \|vd\|$, so $\|\phi\| = 1$. ∎

As usual, we may regard an element $m \in \mathcal{M}$ as a function on $\mathcal{E}^1(\mathcal{M})$ via $\widehat{m}(\phi) = \phi(m)$, and $\mathcal{E}^1(\mathcal{M})$ can be regarded as a set of coordinates for $\mathcal{M}$. Thus we make the following definition.

DEFINITION 4.18. For a norm-closed $\mathcal{D}$-bimodule $\mathcal{M}$, we call the set $\mathcal{E}^1_{\mathcal{D}}(\mathcal{M})$, equipped with the relative weak*-topology, the $\mathbb{T}$-action, and the range and source mappings, a *coordinate system* for $\mathcal{M}$.

When $\mathcal{A}$ is both a norm-closed algebra and a $\mathcal{D}$-bimodule, the coordinate system $\mathcal{E}^1(\mathcal{A})$ also has the additional structure of a continuous partially defined product as described in Remark 4.14. In this case we will sometimes refer to the coordinate system as a *semitwist*.

DEFINITION 4.19. If $\mathcal{M} \subseteq \mathcal{C}$ is a $\mathcal{D}$-bimodule, let $R(\mathcal{M}) := \{|\phi| : \phi \in \mathcal{E}^1(\mathcal{M})\}$. Then $R(\mathcal{M})$ may be identified with the quotient $\mathcal{E}^1(\mathcal{M})\backslash\mathbb{T}$ of $\mathcal{E}^1(\mathcal{M})$ by the natural action of $\mathbb{T}$. Obviously, $\phi \mapsto |\phi|$ is the quotient map, and the topology on $R(\mathcal{M})$ is the quotient topology. Corollary 4.15 shows that we may regard $R(\mathcal{M})$ as a subset of $R(\mathcal{C})$, and, as $\mathcal{C}$ is regular, Corollary 4.15 also implies that if $v \in \mathcal{M}$ is an intertwiner, then $G_v := \{\phi \in R(\mathcal{M}) : |\phi(v)| > 0\}$ is an open set and $\{G_v : v \in \mathcal{M} \text{ is an intertwiner}\}$ is a base for the topology of $R(\mathcal{M})$. Thus $R(\mathcal{M})$ is a locally compact Hausdorff space.

We shall sometimes find it useful to view $R(\mathcal{M})$ as a topological relation on $\widehat{\mathcal{D}}$. The map $|\phi| \mapsto (r(\phi), s(\phi))$ is a bijection between $R(\mathcal{M})$ and $\{(r(\phi), s(\phi)) \in$

$\widehat{\mathcal{D}} \times \widehat{\mathcal{D}} : \phi \in \mathcal{E}^1(\mathcal{M})\}$, and we will sometimes identify these two sets under this bijection. With this identification, $(\rho, \sigma) \in R(\mathcal{M})$ if and only if there is an intertwiner $v \in \mathcal{M}$ with $\sigma(v^*v) \neq 0$ and $\rho = \beta_v(\sigma)$; moreover, the set G_v is the graph of β_v and the collection of such sets gives a base for the topology. We call $R(\mathcal{M})$ the *spectral relation* of $\mathcal{M}$. Also, $R(\mathcal{M})$ is reflexive if $\mathcal{D} \subseteq \mathcal{M}$, is symmetric if $\mathcal{M} = \mathcal{M}^*$ and is transitive if $\mathcal{M}$ is a subalgebra.

A *topological equivalence relation* is a principal topological groupoid. If v, w normalize $\mathcal{D}$, then $G_{vw} = \{|\phi\psi| : s(\phi) = r(\psi), \phi \in G_v, \psi \in G_w\}$ and $G_{v^*} = \{|\phi^*| : \phi \in G_v\}$. It follows that the topology on $R(\mathcal{C})$ is compatible with the groupoid operations, so $R(\mathcal{C})$ is a topological equivalence relation. We will sometimes write $\rho \sim_{\mathcal{C}} \sigma$, or simply $\rho \sim \sigma$, when $(\rho, \sigma) \in R(\mathcal{C})$.

We now show that the regularity of $(\mathcal{C}, \mathcal{D})$ and the faithfulness of E imply that the span of eigenfunctionals is weak* dense in $\mathcal{M}^\#$, for any norm closed $\mathcal{D}$-bimodule $\mathcal{M}$.

PROPOSITION 4.20. *Suppose $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal, and let $\mathcal{M} \subseteq \mathcal{C}$ be a norm-closed $\mathcal{D}$-bimodule. Then* span $\mathcal{E}^1(\mathcal{M})$ *is $\sigma(\mathcal{M}^\#, \mathcal{M})$-dense in $\mathcal{M}^\#$.*

Proof. Let W be the $\sigma(\mathcal{M}^\#, \mathcal{M})$-closure of span $\mathcal{E}^1(\mathcal{M})$. If $W \neq \mathcal{M}^\#$, then there exists a nonzero $\sigma(\mathcal{M}^\#, \mathcal{M})$-continuous linear functional ψ on $\mathcal{M}^\#$ which annihilates W. Since ψ is $\sigma(\mathcal{M}^\#, \mathcal{M})$-continuous, there exists $m \in \mathcal{M}$ such that $\psi(f) = f(m)$ for all $f \in \mathcal{M}^\#$. But then for every $\phi \in \mathcal{E}^1(\mathcal{M})$, we have $\psi(\phi) = \phi(m) = 0$. Since $\mathcal{E}^1(\mathcal{C})$ separates points (Theorem 4.8), there exists an eigenfunctional $\overline{\phi} \in \mathcal{E}^1(\mathcal{C})$ so that $\overline{\phi}(m) \neq 0$. The restriction $\phi := \overline{\phi}|_{\mathcal{M}}$ is an eigenfunctional on $\mathcal{M}$, so $0 = \psi(\phi) = \phi(m) = \overline{\phi}(m) \neq 0$, a contradiction. Therefore, $W = \mathcal{M}^\#$. ∎

We conclude this section with two applications of the results in this section. For our first application, we give a description of the C^*-envelope of an algebra satisfying $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$.

THEOREM 4.21. *Let $(\mathcal{C}, \mathcal{D})$ be a C^*-diagonal and suppose $\mathcal{A}$ is a norm closed algebra satisfying $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$. If $\mathcal{B}$ is the C^*-subalgebra of $\mathcal{C}$ generated by $\mathcal{A}$, then $\mathcal{B}$ is the C^*-envelope of $\mathcal{A}$.*

If in addition, $\mathcal{B} = \mathcal{C}$, then $R(\mathcal{C})$ is the topological equivalence relation generated by $R(\mathcal{A})$.

Proof. Let $\mathcal{B}_e$ be the C^*-envelope of $\mathcal{A}$ and let $j : \mathcal{A} \to \mathcal{B}_e$ be the canonical embedding (see, for example, Section 4.3 of [4]). Then there exists a unique $*$-epimorphism $\pi : \mathcal{B} \to \mathcal{B}_e$ such that $\pi(a) = j(a)$ for every $a \in \mathcal{A}$. In particular, π is faithful on $\mathcal{D}$. Assume, to get a contradiction, that π is not injective. Then $\ker \pi$ is a $\mathcal{D}$-bimodule and let x be a non-zero element of $\ker \pi$. By Theorem 4.8, eigenfunctionals separate points, so there exists an element $\phi \in \mathcal{E}^1(\mathcal{C})$ with $\phi(x) \neq 0$. Writing $\phi = [v, \sigma]$, Proposition 3.10 shows that $u := E_v(x)$ is a nonzero normalizer belonging to $\ker \pi$. But then u^*u is a nonzero element of $\mathcal{D} \cap \ker \pi$, a contradiction. Thus, π is faithful on $\mathcal{B}$, and hence $\mathcal{B}$ is the C^*-envelope of $\mathcal{A}$.

Suppose now that $\mathcal{B} = \mathcal{C}$. By Corollary 4.15, there is an inclusion $\mathcal{E}^1(\mathcal{A}) \subseteq \mathcal{E}^1(\mathcal{C})$. So $R(\mathcal{A}) \subseteq R(\mathcal{C})$, and hence $R(\mathcal{C})$ contains the equivalence relation generated by $R(\mathcal{A})$.

For the other direction, assume that $(\rho, \sigma) \in R(\mathcal{C})$. Then there is $\phi \in \mathcal{E}^1(\mathcal{C})$ with source σ and range ρ. Let $\mathcal{W}$ be the set of all finite products of intertwiners belonging to $\mathcal{A}$ or to $\mathcal{A}^*$. Then $\mathcal{W} \subseteq \mathcal{N}_{\mathcal{D}}(\mathcal{C})$ and the set of finite sums from $\mathcal{W}$ is a $*$-algebra which, because $\mathcal{A}$ generates $\mathcal{C}$, is dense in $\mathcal{C}$. Hence there is some $w \in \mathcal{W}$ so that $\phi(w) \neq 0$. By Proposition 4.7, $\phi = [w, \sigma]$.

Suppose that w factors as $v_{2n}^* v_{2n-1} \cdots v_2^* v_1$, where each v_i is an intertwiner in $\mathcal{A}$. Let $\sigma_1 = \sigma$ and for $i = 2, \dots, 2n$, let σ_i be image of σ under conjugation by the rightmost $i-1$ factors in the factorization of w. It follows that ϕ is the product

$$[v_{2n}, \sigma_{2n}]^* [v_{2n-1}, \sigma_{2n-1}] \cdots [v_2, \sigma_2]^* [v_1, \sigma_1]$$

and each $[v_i, \sigma_i]$ is in $\mathcal{E}^1(\mathcal{A})$. Thus, the equivalence relation generated by $R(\mathcal{A})$ contains (ρ, σ). Similar arguments apply for the other possible factorizations of w, so the equivalence relation generated by $R(\mathcal{A})$ contains $R(\mathcal{C})$.

It remains to show that the usual topology on $R(\mathcal{C})$ equals that generated by $R(\mathcal{A})$, i.e., the smallest topology containing the topology of $R(\mathcal{A})$ which makes $R(\mathcal{C})$ into a topological equivalence relation.

As we noted in Definition 4.19, $R(\mathcal{C})$ is already a topological equivalence relation, and so its topology contains the topology generated by $R(\mathcal{A})$. Since the norm-closed span of $\mathcal{W}$ is $\mathcal{C}$, it follows that $\{G_w : w \in \mathcal{W}\}$ is a base for the topology of $R(\mathcal{C})$, where, as before, $G_w = \{\phi \in R(\mathcal{C}) : |\phi(w)| > 0\}$. For a topological equivalence relation, the inverse map is a homeomorphism. Further, given two precompact open G-sets (i.e., a subset of $R(\mathcal{C})$ on which the two natural projection maps into $\widehat{\mathcal{D}}$ are injective), one can show that their product is again a precompact open G-set (for example, adapt the proof of Proposition I.2.8 in [38]). Since, for v an intertwiner in $\mathcal{A}$, each G_v is a precompact open G-set in $R(\mathcal{A})$, and each $w \in W$ is a finite product of such v's and their inverses, it follows that each G_w, $w \in W$, is open in the topology generated by $R(\mathcal{A})$. Thus, the topology generated by $R(\mathcal{A})$ contains $R(\mathcal{C})$. ∎

Our second application is an application of Theorem 4.8. We show that inductive limits of C^*-diagonals are again C^*-diagonals, when the connecting maps satisfy a certain condition, which we now define. The difficulty in showing that these inductive limits are again C^*-diagonals is in showing that the expectation is faithful, and this is where Theorem 4.8 provides a key tool.

DEFINITION 4.22. Given regular C^*-inclusions $(\mathcal{C}_i, \mathcal{D}_i)$, $i = 1, 2$, and a $*$-homomorphism $\pi : \mathcal{C}_1 \to \mathcal{C}_2$, we say π is *regular* if $\pi(\mathcal{N}(\mathcal{C}_1)) \subseteq \mathcal{N}(\mathcal{C}_2)$.

Of course, if π is regular, then $\pi(\mathcal{D}_1) \subseteq \mathcal{D}_2$. Indeed, for $D \in \mathcal{D}_1$ with $D \geqslant 0$, $D^{1/2} \in \mathcal{N}(\mathcal{C}_1)$ and so $\pi(D) = \pi(D^{1/2}) 1 \pi(D^{1/2}) \in \mathcal{D}_2$.

THEOREM 4.23. *Let $(\mathcal{C}_\lambda, \mathcal{D}_\lambda)$, $\lambda \in \Lambda$, be a directed net of regular C^*-inclusions with regular $*$-monomorphisms $\pi_{\lambda,\mu} : \mathcal{C}_\mu \to C_\lambda$. Then the pair of inductive limit C^*-algebras, $(\varinjlim(\mathcal{C}_\lambda, \pi_{\lambda,\mu}), \varinjlim(\mathcal{D}_\lambda, \pi_{\lambda,\mu}))$, is a regular C^*-inclusion. Moreover, if each $(\mathcal{C}_\lambda, \mathcal{D}_\lambda)$ is a C^*-diagonal, then so is $(\varinjlim(\mathcal{C}_\lambda, \pi_{\lambda,\mu}), \varinjlim(\mathcal{D}_\lambda, \pi_{\lambda,\mu}))$.*

Proof. The first part of the proof is routine. Regard $\mathcal{C}_\lambda$ as a $*$-subalgebras of $\mathcal{C} := \varinjlim(\mathcal{C}_\lambda, \pi_{\lambda,\mu})$ and identify $\pi_{\lambda,\mu}$ with the inclusion map from $\mathcal{C}_\lambda$ to $\mathcal{C}_\mu$. Then $\mathcal{D} = \varinjlim(\mathcal{D}_\lambda, \pi_{\lambda,\mu})$ is a subalgebra of $\mathcal{C}$.

Given a normalizer $v \in \mathcal{C}_\lambda$, by the regularity of the inclusion maps, v normalizes $\mathcal{D}_\mu$ for all $\mu \geqslant \lambda$. Thus, v normalizes $\mathcal{D}$ and so $\mathcal{N}_{\mathcal{D}_\lambda}(\mathcal{C}_\lambda) \subseteq \mathcal{N}_{\mathcal{D}}(\mathcal{C})$. Since $\mathcal{C}$ is the closed union of the $\mathcal{C}_\lambda$, and each $\mathcal{C}_\lambda$ is the span of $\mathcal{N}_{\mathcal{D}_\lambda}(\mathcal{C}_\lambda)$, $\mathcal{C}$ is regular in $\mathcal{D}$.

Given $\sigma \in \widehat{\mathcal{D}}$, suppose ϕ and ψ are extensions of σ to states of $\mathcal{C}$. Then, for each $\lambda \in \Lambda$, $\mathcal{D}_\lambda \subseteq \mathcal{D}$ and so $\phi|_{\mathcal{C}_\lambda}$ and $\psi|_{\mathcal{C}_\lambda}$ are extensions of the pure state $\sigma|_{\mathcal{D}_\lambda} \in \widehat{\mathcal{D}}_\lambda$ and so agree on $\mathcal{C}_\lambda$. Since $\mathcal{C}$ is the closed union of the $\mathcal{C}_\lambda$, $\phi = \psi$. Thus, $(\mathcal{C}, \mathcal{D})$ is a regular C^*-inclusion.

Let $E : \mathcal{C} \to \mathcal{D}$ be the expectation. By Theorem 4.8, $\mathfrak{N} := \{x \in \mathcal{C} : E(x^*x) = 0\}$ is an ideal of $\mathcal{C}$, and, if $q : \mathcal{C} \to \mathcal{C}/\mathfrak{N}$ is the quotient map, then $(q(\mathcal{C}), q(\mathcal{D}))$ is a C^*-diagonal. If $x \in \mathcal{C}_\lambda$ and $E(x^*x) = 0$, then $\sigma(x^*x) = 0$ for all $\sigma \in \widehat{\mathcal{D}}$. Since every $\rho \in \widehat{\mathcal{D}}_\lambda$ has at least one extension to an element of $\widehat{\mathcal{D}}$, we have $E_\lambda(x^*x) = 0$, where E_λ is the expectation for $(\mathcal{C}_\lambda, \mathcal{D}_\lambda)$. Thus, if $(\mathcal{C}_\lambda, \mathcal{D}_\lambda)$ is a C^*-diagonal, then we have $x = 0$, that is, $\mathfrak{N} \cap \mathcal{C}_\lambda = (0)$, so q is faithful on $\mathcal{C}_\lambda$.

Therefore, when each $(\mathcal{C}_\lambda, \mathcal{D}_\lambda)$ is a C^*-diagonal, $q(\mathcal{C})$ contains isomorphic copies of each $\mathcal{C}_\lambda$, and when $\lambda \leqslant \mu$, $q(\mathcal{C}_\lambda) \subseteq q(\mathcal{C}_\mu)$. By the minimality of the inductive limit, q is an isomorphism of $\mathcal{C}$ onto $q(\mathcal{C})$, i.e. $\mathfrak{N} = 0$. Thus, $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal. ∎

5. COMPATIBLE REPRESENTATIONS OF C^*-DIAGONALS

Our goal in this section is to produce a faithful representation π of a C^*-diagonal $(\mathcal{C}, \mathcal{D})$. Because we require the faithfulness of the expectation, we work with C^*-diagonals instead of regular C^*-inclusions.

5.1. STANDING ASSUMPTIONS FOR SECTION 5. We assume that $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal. For $(\mathcal{C}, \mathcal{D})$ a C^*-diagonal, we write $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ if $\mathcal{A} \subseteq \mathcal{C}$ is a norm-closed subalgebra with $\mathcal{D} \subseteq \mathcal{A}$. For $\sigma \in \widehat{\mathcal{D}}$, we use $(\mathcal{H}_\sigma, \pi_\sigma)$ for the GNS representation of $\mathcal{C}$ associated to the (unique) extension of σ.

Eigenfunctionals can be viewed as normal linear functionals on $\mathcal{C}^{\#\#}$ and we start by using the polar decomposition for such functionals to obtain a "minimal" partial isometry for each eigenfunctional. Although these results are implicit in the development of dual groupoids (see p. 435 of [39]), we give a (mostly) self-contained treatment.

Fix a norm-one eigenfunctional ϕ on $\mathcal{C}$. By the polar decomposition for linear functionals (see Theorem III.4.2, Definition III.4.3 of [41]), there is a partial isometry $u^* \in \mathcal{C}^{\#\#}$ and positive linear functionals $|\phi|, |\phi^*| \in \mathcal{C}^{\#}$ so that $\phi = u^* \cdot |\phi| = |\phi^*| \cdot u^*$. Applying the characterization given in Proposition III.4.6 of [41], we find that

$$r(\phi) = |\phi| \quad \text{and} \quad s(\phi) = |\phi^*|. \tag{5.1}$$

Moreover, uu^* and u^*u are the smallest projections in $\mathcal{C}^{\#\#}$ which satisfy,

$$u^*u \cdot s(\phi) = s(\phi) \cdot u^*u = s(\phi) \quad \text{and} \quad uu^* \cdot r(\phi) = r(\phi) \cdot uu^* = r(\phi).$$

DEFINITION 5.1. For $\phi \in \mathcal{E}^1(\mathcal{C})$, we call the partial isometry u above the *partial isometry associated to* ϕ and denote it by v_ϕ. If $\phi \in \widehat{\mathcal{D}}$, then u is a projection and we denote it by p_ϕ.

REMARK 5.2. The above equations show that $v_\phi^* v_\phi = p_{s(\phi)}$ and $v_\phi v_\phi^* = p_{r(\phi)}$. Moreover, given $\phi \in \mathcal{E}^1(\mathcal{C})$, v_ϕ, Proposition 5.3 below implies that may be characterized as the unique minimal partial isometry $w \in \mathcal{C}^{\#\#}$ such that $\phi(w) > 0$.

Our first goal is to show that the initial and final projections of this partial isometry are minimal projections in $\mathcal{C}^{\#\#}$ and compressing by them gives ϕ, in the following sense.

PROPOSITION 5.3. *For $\sigma \in \widehat{\mathcal{D}}$, $p_\sigma = p_{\sigma\circ E}$ is a minimal projection in $\mathcal{C}^{\#\#}$. For all $\phi \in \mathcal{E}^1(\mathcal{C})$ and $x \in \mathcal{C}^{\#\#}$,*

$$p_{r(\phi)} x p_{s(\phi)} = \phi(x) v_\phi.$$

Proof. First, we show that p_σ is a minimal projection in $\mathcal{D}^{\#\#}$. We know that p_σ is the smallest projection in $\mathcal{D}^{\#\#}$ such that $p_\sigma \cdot \sigma = \sigma \cdot p_\sigma = \sigma$. Suppose, to get a contradiction, that p_1, p_2 are nonzero projections in $\mathcal{D}^{\#\#}$ with $0 \leqslant p_1, p_2 \leqslant p$ and $p_\sigma = p_1 + p_2$.

If $\sigma(p_1) = 0$, then for $d \geqslant 0$, $\sigma(p_1 d) = \sigma(p_1 d p_1) \leqslant \sigma(p_1)\|d\| = 0$ and so, for all $d \in \mathcal{D}$, $\sigma(p_1 d) = 0$. But then $p_2 \cdot \sigma = p \cdot \sigma = \sigma = \sigma \cdot p_2$, which yields $p = p_2$, contrary to hypothesis. Hence $\sigma(p_1) \neq 0$ and, similarly, $\sigma(p_2) \neq 0$.

This implies that σ can be written as a nontrivial convex combination of states on $\mathcal{D}$, for

$$\sigma = \sigma(p_1)\frac{p_1 \cdot \sigma}{\sigma(p_1)} + \sigma(p_2)\frac{p_2 \cdot \sigma}{\sigma(p_2)}.$$

But this is a contradiction, since elements of $\widehat{\mathcal{D}}$ are pure states.

Since p_σ is minimal in $\mathcal{D}^{\#\#}$, for $d \in \mathcal{D}$, $p_\sigma d p_\sigma = \sigma(d) p_\sigma$.

Now suppose that $q \in \mathcal{C}^{\#\#}$ is a projection with $0 < q \leqslant p_\sigma$. Since $q \neq 0$, there exists a state $g \in \mathcal{C}^{\#}$ such that $g(q) > 0$. Define

$$f := \frac{q \cdot g \cdot q}{g(q)}.$$

Then f is a state on $\mathcal{C}$ with $f(q) = 1$. As $p_\sigma q = q$, we have for $d \in \mathcal{D}$,

$$f(d) = \frac{g(q(p_\sigma d p_\sigma)q)}{g(q)} = \sigma(d).$$

Since pure states on $\mathcal{D}$ extend uniquely to pure states on $\mathcal{C}$, we conclude that $f = \sigma \circ E$.

If p_σ is not minimal, write $p_\sigma = q_1 + q_2$ where $q_i \in \mathcal{C}^{\#\#}$ are projections with $0 < q_1, q_2 \leqslant p_\sigma$. Apply the argument of the previous paragraph to find states h_1 and h_2 on $\mathcal{C}$ such that $h_i(q_i) = 1$ and $q_i \cdot h_i \cdot q_i = h_i$. Since $q_1 q_2 = 0$, $h_1(q_2) = h_2(q_1) = 0$. But the previous paragraph shows that $h_1 = \sigma \circ E = h_2$, contradicting the extension property. So p_σ is minimal in $\mathcal{C}^{\#\#}$.

The uniqueness of polar decompositions implies that $p_\sigma = p_{\sigma\circ E}$.

To prove the second statement, first note that $p_{r(\phi)} \mathcal{C}^{\#\#} p_{s(\phi)}$ has dimension one, since $p_{r(\phi)}, p_{s(\phi)}$ are minimal projections in $\mathcal{C}^{\#\#}$ and $p_{r(\phi)} v_\phi p_{s(\phi)} = v_\phi \neq 0$. Hence there is a linear functional g on $\mathcal{C}$ such that for every $x \in \mathcal{C}^{\#\#}$, $p_{r(\phi)} x p_{s(\phi)} = g(x) v_\phi$. Then $g|_{\mathcal{C}}$ is an eigenfunctional with the same source and range as ϕ. Since $g(v_\phi) = \phi(v_\phi)$, $g = \phi$. ∎

Recall ([41], Lemma III.2.2) that any $*$-representation π of a C^*-algebra $\mathcal{C}$ has a unique extension to a $*$-representation $\widetilde{\pi} : \mathcal{C}^{\#\#} \to \pi(\mathcal{C})''$, continuous from the $\sigma(\mathcal{C}^{\#\#}, \mathcal{C}^{\#})$-topology to the ρ-weak topology on $\pi(\mathcal{C})''$, i.e., the $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})_*)$-topology. Let $\mathcal{J} = \ker \widetilde{\pi} \subseteq \mathcal{C}^{\#\#}$. By the continuity of $\widetilde{\pi}$, $\mathcal{J}$ is $\sigma(\mathcal{C}^{\#\#}, \mathcal{C}^{\#})$ closed, and hence (see Proposition 1.10.5 of [40]) there exists a unique central projection $P \in \mathcal{C}^{\#\#}$ such that $\mathcal{J} = \mathcal{C}^{\#\#}(I - P)$. Further, $\widetilde{\pi}|_{\mathcal{C}^{\#\#}P}$ is one-to-one (see Definition 1.21.14 of [40]) and is onto $\mathcal{B}(\mathcal{H})$. The projection P is called the *support projection* for π.

Recall that $\rho, \sigma \in \widehat{\mathcal{D}}$ have $(\sigma, \rho) \in R(\mathcal{C})$ if and only if there is $\phi \in \mathcal{E}^1(\mathcal{C})$ with $r(\phi) = \sigma$ and $s(\phi) = \rho$. For brevity, we write $\rho \sim \sigma$ in this case.

PROPOSITION 5.4. *If $\sigma \in \widehat{\mathcal{D}}$, then $\pi_\sigma(\mathcal{D})''$ is an atomic* MASA *in $\mathcal{B}(\mathcal{H}_\sigma)$ and the support projection for π_σ is $\sum_{\rho\sim\sigma} p_\rho$. Moreover, the map from $\pi_\sigma(\mathcal{C})$ to $\pi_\sigma(\mathcal{D})$ given by $\pi_\sigma(c) \mapsto \pi_\sigma(E(c))$ is well-defined and extends to a faithful normal expectation $\widetilde{E} : \pi_\sigma(\mathcal{C})'' \to \pi_\sigma(\mathcal{D})''$.*

Proof. Let $\pi = \pi_\sigma$ and $\mathcal{H} = \mathcal{H}_\sigma$. Since the extension of σ to $\mathcal{C}$ is pure, the representation π is irreducible, so $\pi(\mathcal{C})'' = \mathcal{B}(\mathcal{H})$.

Recall from Corollary 1 of [19] that if $\mathcal{M} := \{x \in \mathcal{C} : \sigma(x^*x) = 0\}$, then $\mathcal{C}/\mathcal{M}$ is complete relative to the norm induced by the inner product $\langle x + \mathcal{M}, y + \mathcal{M}\rangle = \sigma(y^*x)$, and thus $\mathcal{H} = \mathcal{C}/\mathcal{M}$. Our first task is to obtain a convenient orthonormal basis for $\mathcal{H}$ and, towards this end, we require the following observation.

If $v, w \in \mathcal{N}(\mathcal{C})$ with $\sigma(v^*v) = 1$ and $\sigma(v^*w) \neq 0$, then

$$w - [v, \sigma](w)v \in \mathcal{M}. \tag{5.2}$$

To see this, let $w_1 = \sigma(w^*v)w$. Then $\sigma(v^*w_1) > 0$, so that by Corollary 4.4, $[v, \sigma](w_1) = [w_1, \sigma](w_1)$, and thus $\sigma(v^*w_1) = \sigma(w_1^*w_1)^{1/2}$. Therefore $|\sigma(v^*w)| =$

$\sigma(w^*w)^{1/2}$. A calculation then shows that $w - [v,\sigma](w)v = w - \sigma(v^*w)v \in \mathcal{M}$, as required.

Choose a set $\mathcal{Z} \subseteq \mathcal{C}$ of normalizers such that for each $z \in \mathcal{Z}$, $\sigma(z^*z) = 1$ and the map $z \mapsto r([z,\sigma])$ is a bijection of $\mathcal{Z}$ onto $\mathcal{O} := \{\rho \in \widehat{\mathcal{D}} : \rho \sim \sigma\}$.

We claim that if $X = \sum_{j=1}^{n} w_j \in \mathcal{C}$ with each $w_j \in \mathcal{N}_{\mathcal{D}}(\mathcal{C})$, then

$$X + \mathcal{M} = \sum_{z\in\mathcal{Z}} [z,\sigma](X)(z+\mathcal{M}). \tag{5.3}$$

To see that the summation is well-defined, first observe that if, for some $z \in \mathcal{Z}$, $\sigma(z^*w_j) \neq 0$, then $r([w_j,\sigma]) = r([z,\sigma]) \in \mathcal{O}$. Thus, for each j, $\{z \in \mathcal{Z} : \sigma(z^*w_j) \neq 0\}$ is a singleton, and so $\{z \in \mathcal{Z} : \sigma(z^*X) \neq 0\}$ is finite. To prove the equality, interchange the order of summation and use (5.2) as follows:

$$\begin{aligned}
\sum_{z\in\mathcal{Z}} [z,\sigma](X)(z+\mathcal{M}) &= \sum_{z\in\mathcal{Z}}\sum_{j=1}^{n} [z,\sigma](w_j)(z+\mathcal{M}) = \sum_{j=1}^{n}\sum_{z\in\mathcal{Z}} [z,\sigma](w_j)(z+\mathcal{M}) \\
&= \sum_{j=1}^{n} \sum_{z\in\mathcal{Z}, \sigma(z^*w_j)\neq 0} [z,\sigma](w_j)(z+\mathcal{M}) \\
&= \sum_{j=1}^{n} w_j + \mathcal{M} \quad \text{by (5.2)} \\
&= X + \mathcal{M},
\end{aligned}$$

Next, we show that $\{z + \mathcal{M} : z \in \mathcal{Z}\}$ is an orthonormal basis for $\mathcal{H}$. By Corollary 4.4, it is an orthonormal set. Given any $Y \in \mathcal{C}$ and $\varepsilon > 0$, we may find a finite sum of normalizers X so that $\|X - Y\|_{\mathcal{C}} < \varepsilon$. Then $\|X - Y + \mathcal{M}\|^2_{\mathcal{H}_\sigma} = \sigma((X-Y)^*(X-Y)) \leqslant \|X-Y\|^2_{\mathcal{C}} < \varepsilon^2$. As $X + \mathcal{M}$ is in $\operatorname{span}\mathcal{Z}$ and ε is arbitrary, $Y + \mathcal{M} \in \overline{\operatorname{span}\mathcal{Z}}$, so that $\{z+\mathcal{M} : z \in \mathcal{Z}\}$ is an orthonormal basis.

For $d \in \mathcal{D}$ and $z \in \mathcal{Z}$, using (5.2)

$$\pi(d)(z+\mathcal{M}) = dz + \mathcal{M} = [z,\sigma](dz)z + \mathcal{M} = \sigma(z^*dz)(z+\mathcal{M}). \tag{5.4}$$

Thus, $\pi(d)$ is diagonal with respect to the basis $\{z+\mathcal{M}\}_{z\in\mathcal{Z}}$. Fixing $z \in \mathcal{Z}$, let $\Lambda = \{d \in \mathcal{D} : d \geqslant 0, \text{ and } r([z,\sigma])(d) = 1\}$. Then Λ is a directed set under the ordering $d_1 \preceq d_2$ if and only if $d_1 \geqslant d_2$. It is easy to see that the net $\{\pi(d)\}_{d\in\Lambda}$ decreases to the rank-one projection onto $z + \mathcal{M}$. Thus $\pi(\mathcal{D})''$ is an atomic MASA.

Fix $z \in \mathcal{Z}$ and let P_z be the orthogonal projection of $\mathcal{H}$ onto $z + \mathcal{M}$. Then a calculation shows that for $x \in \mathcal{C}$,

$$P_z\pi(x)P_z = \sigma(z^*xz)P_z. \tag{5.5}$$

Since $r([z,\sigma])$ is the vector state corresponding to $z + \mathcal{M}$ and $\widetilde{\pi}$ is normal, (5.5) holds when π is replaced by $\widetilde{\pi}$ and $x \in \mathcal{C}^{\#\#}$. As $p_{r([z,\sigma])}$ is a minimal projection in $\mathcal{C}^{\#\#}$, $\widetilde{\pi}(p_{r([z,\sigma])}) = P_z$. If $Q = \sum_{\rho\sim\sigma} p_\rho$, then $\widetilde{\pi}(Q) = I$. Letting P be the support projection of π, this implies that $P \leqslant Q$. If $\rho \sim \sigma$, then p_ρ is a a minimal projection satisfying $\widetilde{\pi}(p_\rho) \neq 0$. Thus, $p_\rho \leqslant P$ and so $Q \leqslant P$. This shows that Q is the support projection for π and that $Q \in \mathcal{D}^{\#\#}$.

Finally, (5.4) implies that for $x \in \mathcal{C}$,

$$\begin{aligned}\pi(E(x))(z+\mathcal{M}) &= \sigma(z^*E(x)z)(z+M) = r([z,\sigma])(E(x))(z+\mathcal{M})\\ &= r([z,\sigma])(x)(z+\mathcal{M}) = \sigma(z^*xv)(z+\mathcal{M}).\end{aligned}$$

Hence, for $x \in \mathcal{C}$, we obtain

$$\pi(E(x)) = \sum_{z\in\mathcal{Z}} P_z\pi(x)P_z.$$

If $\widetilde{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is defined by $\widetilde{E}(T) = \sum_{z\in\mathcal{Z}} P_zTP_z$, this shows that $\widetilde{E}$ is a faithful normal conditional expectation of $\mathcal{B}(\mathcal{H}) = \pi(\mathcal{C})''$ onto $\pi(\mathcal{D})''$ satisfying $\widetilde{E}(\pi(x)) = \pi(E(x))$ for $x \in \mathcal{C}$. ∎

We next record a simple consequence of the proof of Proposition 5.4.

COROLLARY 5.5. *Suppose $\sigma \in \widehat{\mathcal{D}}$ and $\phi \in \mathcal{E}^1(\mathcal{C})$ satisfies $\sigma \sim s(\phi)$ and $\sigma \neq \phi$. Then there exist orthogonal unit vectors $\xi, \eta \in \mathcal{H}_\sigma$ such that for every $x \in \mathcal{C}$, $\phi(x) = \langle \pi_\sigma(x)\xi, \eta\rangle$.*

Proof. With the same notation as in the proof of Proposition 5.4, there exists $z, w \in \mathcal{Z}$ such that $r([z,\sigma]) = s(\phi)$ and $r([w,\sigma]) = r(\phi)$. Let $\xi_0 = z + \mathcal{M}$ and $\eta = w + \mathcal{M}$. For $x \in \mathcal{C}$, let $\psi(x) = \langle \pi(x)\xi_0, \eta\rangle$. Observe that $\psi(wz^*) = \sigma(w^*wz^*z) = 1$, so ψ is nonzero. Also, for $x \in \mathcal{C}$ and $d \in \mathcal{D}$, we have

$$\psi(xd) = \sigma(w^*xdz) = \frac{\sigma(w^*xdzz^*z)}{\sigma(z^*z)} = \sigma(w^*xz)\frac{\sigma(z^*dz)}{\sigma(z^*z)} = \psi(x)s(\phi)(d).$$

Similarly, $\psi(dx) = r(\phi)(d)\psi(x)$. Thus ψ is an eigenfunctional with the same range and source as ϕ. Hence, there exists $t \in \mathbb{T}$ so that $\phi = t\psi$. Take $\xi = t\xi_0$. ∎

DEFINITION 5.6. Given a C^*-diagonal $(\mathcal{C}, \mathcal{D})$, a representation π of $\mathcal{C}$ is $\mathcal{D}$-*compatible*, (or simply *compatible*) if $\pi(\mathcal{D})''$ is a MASA in $\pi(\mathcal{C})''$ and there exists a faithful conditional expectation $\widetilde{E} : \pi(\mathcal{C})'' \to \pi(\mathcal{D})''$ such that for every $x \in \mathcal{C}$, $\widetilde{E}(\pi(x)) = \pi(E(x))$.

Proposition 5.4 shows that the GNS representation of $\mathcal{C}$ associated to an element of $\widehat{\mathcal{D}}$ is compatible. For an example of a compatible representation π where $\pi(\mathcal{D})$ has no minimal projections, let $\mathcal{C}_n = M_{2^n}(\mathbb{C})$ and $\mathcal{D}_n \subseteq \mathcal{C}_n$ be the diagonal matrices. Put $(\mathcal{C}, \mathcal{D}) = \varinjlim(\mathcal{C}_n, \mathcal{D}_n)$ with connecting maps $A \mapsto A \oplus A$. The GNS construction using the usual trace on $\mathcal{C}$ produces a compatible representation π such that $\pi(\mathcal{D})''$ contains no minimal projections.

REMARK 5.7. Let X be an r-discrete locally compact principal groupoid with unit space X^0 and suppose ρ is a 2-cocycle. Then Drinen [13] shows that $(C_r(X,\rho), C_0(X^0))$ is a C^*-diagonal. We expect that when λ^u is a Haar system on X, and μ is a measure on X^0, the induced representation $\mathrm{Ind}(^-)$ of $C_r(X,\rho)$ (see for example, page 44 of [26]) is a compatible representation of $(C_r(X,\rho), C_0(X^0))$, and it would not be surprising if every compatible representation for $(C_r(X,\rho), C_0(X^0))$ arises in this way. We do not pursue this issue here, however.

LEMMA 5.8. *Let $\sigma, \rho \in \widehat{\mathcal{D}}$. Then $\sigma \sim \rho$ if and only if the* GNS *representations π_ρ and π_σ are unitarily equivalent.*

Proof. If $U \in \mathcal{B}(\mathcal{H}_\sigma, \mathcal{H}_\rho)$ is a unitary such that $U\pi_\sigma U^* = \pi_\rho$, the fact that $\mathcal{C}/\mathcal{M}_\sigma = \mathcal{H}_\sigma$ allows us to find $X \in \mathcal{C}$ such that $U^*(I + \mathcal{M}_\rho) = X + \mathcal{M}_\sigma$. Then for all $x \in \mathcal{C}$, we have $\rho(x) = \langle \pi_\rho(x)(I + \mathcal{M}_\rho), (I + \mathcal{M}_\rho)\rangle = \sigma(X^* x X)$. As ρ and σ are normal states on $\mathcal{C}^{\#\#}$, this equality is valid for $x \in \mathcal{C}^{\#\#}$ as well; in particular, $1 = \sigma(X^* p_\rho X) = \rho(p_\rho)$. For $x \in \mathcal{C}$, define $\phi(x) = \sigma(X^* p_\rho x)$. Then $\phi \neq 0$ because $\phi(X) = 1$ and ϕ is an eigenfunctional with $s(\phi) = \sigma$ and $r(\phi) = \rho$. Hence $\sigma \sim \rho$. Conversely, if $\sigma \sim \rho$, then find a normalizer v with $\sigma(v^* v) = 1$ and $\rho(x) = \sigma(v^* x v)$ for every $x \in \mathcal{C}$. Then $\rho(x) = \langle \pi_\sigma(x) v + \mathcal{M}_\sigma, v + \mathcal{M}_\sigma\rangle$. By Kadison's Transitivity Theorem, there exists a unitary $V \in \mathcal{C}$ such that $V^*(1 + N_\sigma) = v + \mathcal{M}_\sigma$. Then $\pi_\rho = V\pi_\sigma V^*$. ∎

THEOREM 5.9. *If $\mathcal{X} \subseteq \widehat{\mathcal{D}}$ contains exactly one element from each equivalence class in $R(\mathcal{C})$, then $\pi = \bigoplus_{\sigma\in\mathcal{X}} \pi_\sigma$ on $\mathcal{H} = \bigoplus_{\sigma\in\mathcal{X}} \mathcal{H}_\sigma$ is a faithful compatible representation of $\mathcal{C}$ on $\mathcal{B}(\mathcal{H})$ and $\pi(\mathcal{D})''$ is an atomic* MASA *in $\mathcal{B}(\mathcal{H})$.*

Moreover, if $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ and P is the support projection for σ, then $\ker \widetilde{\pi}|_{\mathcal{A}^{\#\#}} = P^\perp \mathcal{A}^{\#\#}$ *and $\widetilde{\pi}(\mathcal{A}^{\#\#})$ is a* CSL *algebra.*

Proof. For $\sigma \in \mathcal{X}$, let $P_\sigma \in \mathcal{C}^{\#\#}$ be the support projection of π_σ. Then for $\rho \in \widehat{\mathcal{D}}$,

$$P_\sigma p_\rho = \begin{cases} p_\rho & \text{if } \rho \sim \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\widetilde{\pi}(p_\rho)$ is a minimal projection for every $\rho \in \widehat{\mathcal{D}}$. Since

$$I_{\mathcal{H}} = \sum_{\sigma\in\mathcal{X}} \widetilde{\pi}(P_\sigma) = \sum_{\sigma\in\mathcal{X}} \sum_{\rho\sim\sigma} \widetilde{\pi}(p_\rho),$$

which is a sum of minimal projections, $\pi(\mathcal{D})''$ is an atomic MASA in $\mathcal{B}(\mathcal{H})$. If, for each $\sigma \in \mathcal{X}$, $E'_\sigma : \mathcal{B}(\mathcal{H}_\sigma) \to \pi_\sigma(\mathcal{D})''$ is the expectation on $\mathcal{B}(\mathcal{H}_\sigma)$ induced by E, the map $E' = \bigoplus_{\sigma\in\mathcal{X}} E'_\sigma$ is faithful and satisfies $E' \circ \pi = \pi \circ E$ so π is compatible. For any $x \in \mathcal{C}$ such that $\pi(x^* x) = 0$,

$$0 = E'(\pi(x^* x)) = \pi(E(x^* x)),$$

so $E(x^* x) = 0$ since π is faithful on $\mathcal{D}$. Thus, as E is faithful, we conclude that $x^* x = 0$, hence π is faithful on $\mathcal{C}$.

Suppose $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$. As $P = \sum_{\sigma\in\mathcal{X}} P_\sigma$, where P_σ is the support projection for π_σ, Proposition 5.4 implies that $P \in \mathcal{D}^{\#\#} \subseteq \mathcal{A}^{\#\#}$. Thus $\ker \widetilde{\pi}|_{\mathcal{A}^{\#\#}} = P^\perp \mathcal{A}^{\#\#}$.

We claim that $\tilde{\pi}(\mathcal{A}^{\#\#})$ is weak* closed in $\mathcal{B}(\mathcal{H})$. By the Krein-Smullian Theorem, it suffices to prove the (norm-closed) unit ball of $\tilde{\pi}(\mathcal{A}^{\#\#})$ is weak* closed. Suppose (y_λ) is a net in $\tilde{\pi}(\mathcal{A}^{\#\#})$ converging weak* to $y \in \mathcal{B}(\mathcal{H})$ with $\|y_\lambda\| \leqslant 1$ for all λ. For each λ, choose $x_\lambda \in \mathcal{A}^{\#\#}$ with $\tilde{\pi}(x_\lambda) = y_\lambda$. Since $\tilde{\pi}$ is isometric on $P\mathcal{C}^{\#\#}$, the net (Px_λ) in $\mathcal{A}^{\#\#}$ satisfies $\|Px_\lambda\| \leqslant 1$ for every λ. Hence a subnet Px_{λ_μ} converges weak* to some $x \in \mathcal{A}^{\#\#}$. Then $\tilde{\pi}(x) = y$, as required.

Since $\tilde{\pi}(\mathcal{A}^{\#\#})$ contains the atomic MASA $\pi(\mathcal{D})''$, $\mathcal{L} := \mathrm{Lat}(\tilde{\pi}(\mathcal{A}^{\#\#}))$ is an atomic CSL. We claim that $\tilde{\pi}(\mathcal{A}^{\#\#}) = \mathrm{Alg}\mathcal{L}$. As $\mathrm{Alg}\mathcal{L}$ is the largest algebra whose lattice of invariant subspaces is $\mathcal{L}$, $\tilde{\pi}(\mathcal{A}^{\#\#}) \subseteq \mathrm{Alg}\mathcal{L}$.

To obtain the reverse inclusion, first observe that for minimal projections $p, q \in \pi(\mathcal{D})''$, we have $q\mathrm{Alg}\mathcal{L}p = q\tilde{\pi}(\mathcal{A}^{\#\#})p$. Indeed, the subspaces $\overline{\mathrm{Alg}\mathcal{L}p\mathcal{H}}$ and $\overline{\tilde{\pi}(\mathcal{A}^{\#\#})p\mathcal{H}}$ are each the smallest element of $\mathcal{L}$ containing the range of p, so the two subspaces coincide. Thus $q\overline{\mathrm{Alg}\mathcal{L}p\mathcal{H}} = q\overline{\tilde{\pi}(\mathcal{A}^{\#\#})p\mathcal{H}}$, which yields the observation.

Let $\mathbb{A}$ be the set of minimal projections of $\pi(\mathcal{D})''$. Given $Y \in \mathrm{Alg}\mathcal{L}$, we may write Y as the weak* convergent sum, $Y = \sum_{q,p\in\mathbb{A}} qYp$. As each $qYp \in \tilde{\pi}(\mathcal{A}^{\#\#})$, and $\tilde{\pi}(\mathcal{A}^{\#\#})$ is weak* closed, $Y \in \tilde{\pi}(\mathcal{A}^{\#\#})$, as desired. ∎

Muhly, Qiu and Solel ([24], Theorem 4.7) prove that if $\mathcal{C}$ is nuclear and $\mathcal{A} \subset (\mathcal{C}, \mathcal{D})$ with $\mathcal{A}$ triangular, then the expectation $E|_{\mathcal{A}}$ is a homomorphism. The connection with CSL algebras provided by Theorem 5.9 allows us to remove the hypothesis of nuclearity.

THEOREM 5.10. *If $\mathcal{A}$ is a triangular subalgebra of the C^*-diagonal $(\mathcal{C}, \mathcal{D})$, then $E|_{\mathcal{A}}$ is a homomorphism of $\mathcal{A}$ onto $\mathcal{D}$.*

Proof. Let π be the faithful compatible representation of $\mathcal{C}$ provided by Theorem 5.9, and again write $\mathrm{Alg}\mathcal{L}$ for $\tilde{\pi}(\mathcal{A}^{\#\#})$. Theorem 5.9 also shows that $\tilde{\pi}(\mathcal{D}^{\#\#})$ is a MASA in $\mathcal{B}(\mathcal{H})$. We claim that $\mathcal{L}$ is multiplicity free. To show this, we prove that $\mathrm{Alg}\mathcal{L} \cap (\mathrm{Alg}\mathcal{L})^* = \tilde{\pi}(\mathcal{D}^{\#\#})$. Clearly, $\tilde{\pi}(\mathcal{D}^{\#\#}) \subseteq \mathrm{Alg}\mathcal{L} \cap (\mathrm{Alg}\mathcal{L})^*$. To show the reverse implication, suppose $q_1, q_2 \in \tilde{\pi}(\mathcal{D}^{\#\#})$ are distinct nonzero minimal projections and $q_2(\mathrm{Alg}\mathcal{L})q_1 \neq (0)$. It suffices to show that

$$q_1(\mathrm{Alg}\mathcal{L})q_2 = (0). \tag{5.6}$$

We may find $\sigma, \rho \in \widehat{\mathcal{D}}$ so that $q_1 = \tilde{\pi}(p_\sigma)$ and $q_2 = \tilde{\pi}(p_\rho)$, where p_σ and p_ρ are as in Definition 5.1. Since $\pi(\mathcal{A})$ is weak* dense in $\mathrm{Alg}\mathcal{L}$, we see that $p_\rho \mathcal{A} p_\sigma \neq (0)$. Therefore, $B_{\rho,\sigma}$ is nonzero on $\mathcal{A}$, so that there exists an eigenfunctional $\phi \in \mathcal{E}^1(\mathcal{A})$ with $s(\phi) = \sigma$ and $r(\phi) = \rho$.

We claim that ϕ^* vanishes on $\mathcal{A}$. Indeed, as $q_1 \neq q_2$, $\sigma \neq \rho$, so that we may find a normalizer $v \in \mathcal{A}$ so that $(v^*v)(vv^*) = 0$ and $\phi = [v, \sigma]$. Suppose to obtain a contradiction, that $\phi^* = [v^*, \rho]$ does not vanish on $\mathcal{A}$, and let $y \in \mathcal{A}$ satisfy $\phi^*(y) \neq 0$. Proposition 3.10 shows that $w := v^*E(vy)$ is a nonzero element of $\mathcal{A}$.

But since $v \in \mathcal{A}$ and $E(vy) \in \mathcal{D}$, we also have $w^* \in \mathcal{A}$. However,

$$ww^* = v^*E(vy)E(vy)^*v \leqslant \|E(vy)\|^2 v^*v, \quad w^*w = E(vy)^*vv^*E(vy) \leqslant \|E(vy)\|^2 vv^*.$$

Thus w is a non-normal element of $\mathcal{A} \cap \mathcal{A}^*$, violating triangularity of $\mathcal{A}$.

Therefore, ϕ^* vanishes on $\mathcal{A}$, so that $p_\sigma \mathcal{A} p_\rho = (0)$. Applying $\widetilde{\pi}$, we obtain $q_1(\mathrm{Alg}\mathcal{L})q_2 = (0)$. Therefore, $\mathrm{Alg}\mathcal{L}$ is multiplicity free as desired.

Each minimal projection e in $\widetilde{\pi}(\mathcal{D}^{\#\#})$ is the difference of elements of $\mathcal{L}$, and hence the compression $x \mapsto exe$ is a homomorphism on $\mathrm{Alg}\mathcal{L}$. The extension of $\widetilde{E}$ of E to all of $\mathcal{B}(\mathcal{H})$is faithful and normal, and is the sum of such compressions. Thus, $\widetilde{E}$ is a homomorphism on $\mathrm{Alg}\mathcal{L}$ and, by restriction, on $\mathcal{A}$. ∎

We describe the maximal ideals of $\mathcal{A}$ and identify $\ker E|_{\mathcal{A}}$ in algebraic terms.

PROPOSITION 5.11. *Let $(\mathcal{C}, \mathcal{D})$ be a C^*-diagonal and $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ be triangular. The map $J \mapsto J \cap \mathcal{D}$ is a bijection between the proper maximal ideals of $\mathcal{A}$ and the proper maximal ideals of $\mathcal{D}$. Further,*

$$\ker E|_{\mathcal{A}} = \bigcap \{J \subseteq \mathcal{A} : J \textit{ is a maximal ideal of } \mathcal{A}\}.$$

Proof. Let J be a proper maximal ideal of $\mathcal{A}$. For any $x \in \mathcal{C}$, $E(x)$ belongs to the norm-closed convex hull of $\{gxg^{-1} : g \in \mathcal{D}, g \text{ is unitary}\}$ (Theorem 3.7). Since J is a closed ideal, we see that $E(J) \subseteq J \cap \mathcal{D} \subseteq E(J)$, so $E(J) = J \cap \mathcal{D}$. Since J is proper, $J \cap \mathcal{D} \neq \mathcal{D}$. Hence there exists $\sigma \in \widehat{\mathcal{D}}$ such that $\ker \sigma \supseteq J \cap \mathcal{D}$. The unique extension of σ to $\mathcal{A}$ is $\sigma \circ E|_{\mathcal{A}}$, so we have $J \subseteq \ker(\sigma \circ E|_{\mathcal{A}})$. Since J is maximal, $J = \ker \sigma \circ E|_{\mathcal{A}}$. Therefore, $J \cap \mathcal{D} = \ker \sigma$, which is a proper maximal ideal of $\mathcal{D}$.

To show that the map $J \mapsto J \cap \mathcal{D}$ is a bijection, we need only consider proper maximal ideals. The previous paragraph shows that if J_1 is another proper maximal ideal of $\mathcal{A}$ and $J_1 \cap \mathcal{D} = J \cap D$, then $J = J_1$. Also, if $K \subseteq \mathcal{D}$ is a proper maximal ideal, then $K = \ker \sigma$ for some $\sigma \in \widehat{\mathcal{D}}$, and then $J := \ker \sigma \circ E_{\mathcal{A}}$ is a proper maximal ideal of $\mathcal{A}$ with $J \cap \mathcal{D} = K$. Thus the map is onto.

For $x \in \mathcal{A}$, we have $x \in \ker E|_{\mathcal{A}}$ if and only if $x \in \ker \sigma \circ E|_{\mathcal{A}}$ for every $\sigma \in \widehat{\mathcal{D}}$. Using the map above, we find that this is equivalent to x belonging to $\bigcap \{J \subseteq \mathcal{A} : J \text{ is a maximal ideal of } \mathcal{A}\}$. ∎

6. INVARIANCE UNDER DIAGONAL-PRESERVING ISOMORPHISMS

As an application of the results obtained so far, we will show that coordinate systems are preserved under isomorphisms of algebras which preserve the diagonal. These results extend those for isometric isomorphisms and we compare our results with them.

DEFINITION 6.1. For $i = 1, 2$, suppose $(\mathcal{C}_i, \mathcal{D}_i)$ are regular C^*-inclusions and that $\mathcal{A}_i$ are subalgebras with $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$, i.e., $\mathcal{A}_i \subseteq \mathcal{C}_i$ is a norm-closed subalgebra with $\mathcal{D}_i \subset \mathcal{A}_i$. We say that a (bounded) isomorphism $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is *diagonal preserving* if $\theta(\mathcal{D}_1) = \mathcal{D}_2$.

REMARK 6.2. (i) If θ is diagonal preserving, then $\theta|_{\mathcal{D}_1}$ is a $*$-isomorphism of $\mathcal{D}_1$ onto $\mathcal{D}_2$.

(ii) While we are interested in coordinates for modules, when studying invariance properties, we note that it suffices to consider subalgebras of regular C^*-inclusions. This is because of the well-known "2×2 matrix trick". If $(\mathcal{C}, \mathcal{D})$ is a regular C^*-inclusion, so is $(M_2(\mathcal{C}), \mathcal{D} \oplus \mathcal{D})$. For a $\mathcal{D}$-bimodule $\mathcal{M} \subseteq \mathcal{C}$, let $\mathfrak{T}(\mathcal{M})$ be the subalgebra of $M_2(\mathcal{C})$,

$$\mathfrak{T}(\mathcal{M}) := \left\{ \begin{bmatrix} d_1 & m \\ 0 & d_2 \end{bmatrix} : d_1, d_2 \in \mathcal{D}, m \in \mathcal{M} \right\}$$

contained in $(M_2(\mathcal{C}), \mathcal{D} \oplus \mathcal{D})$. An isomorphism of $\mathcal{D}_i$-bimodules $\mathcal{M}_i$ ($i = 1, 2$), that is, a bounded map $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ together with an isomorphism $\alpha : \mathcal{D}_1 \to \mathcal{D}_2$ satisfying $\theta(dme) = \alpha(d)\theta(m)\alpha(e)$ ($d, e \in \mathcal{D}_1, m \in \mathcal{M}$) can be equivalently described as a diagonal preserving isomorphism of $\mathfrak{T}(\mathcal{M}_1)$ onto $\mathfrak{T}(\mathcal{M}_2)$.

We have noted that $\theta^\#$ is a bicontinuous isomorphism from $\mathcal{E}(\mathcal{A}_2)$ onto $\mathcal{E}(\mathcal{A}_1)$ (Proposition 2.9). The next result shows that normalizing $\theta^\#$ pointwise gives an isomorphism of the norm-one eigenfunctionals.

THEOREM 6.3. *For $i = 1, 2$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be regular C^*-inclusions, let $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$, and suppose $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is a bounded diagonal-preserving isomorphism. There exists a bicontinuous isomorphism of coordinate systems $\gamma : \mathcal{E}^1(\mathcal{A}_1) \to \mathcal{E}^1(\mathcal{A}_2)$ given by*

$$\gamma(\phi) = \frac{\phi \circ \theta^{-1}}{\|\phi \circ \theta^{-1}\|}.$$

Moreover, if $\phi \in \mathcal{E}^1(\mathcal{A}_1)$ is written as $[v, \sigma]$, then $(\sigma \circ \theta^{-1})(\theta(v)^*\theta(v))$ is nonzero, and

$$\gamma(\phi) = [\theta(v), \sigma \circ \theta^{-1}].$$

When necessary for clarity, we use γ_θ to denote the dependence of γ on θ.

REMARK 6.4. As a special case, if $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is a contractive isomorphism, then it is diagonal preserving, as its restriction to $\mathcal{D}_1$ is then a $*$-homomorphism. Thus, Theorem 6.3 extends previous work for isometric isomorphisms of TAF algebras and of subalgebras of (nuclear) groupoid C^*-algebras; see Theorem 3 of [34] and Theorem 2.1 of [25].

Proof of Theorem 6.3. Given $\phi \in \mathcal{E}^1(\mathcal{A}_1)$, write $\phi = [v, \sigma]$ where $v \in \mathcal{A}_1$ is an intertwiner and $\sigma(v^*v) \neq 0$. That $r(\gamma(\phi)) = r(\phi) \circ \theta^{-1}$ and $s(\gamma(\phi)) = s(\phi) \circ \theta^{-1}$ follows from the definition of γ.

Clearly $\theta(v)$ is a $\mathcal{D}_2$-intertwiner, so $\theta(v)^*\theta(v) \in \mathcal{D}_2$. For $\sigma \in \widehat{\mathcal{D}}_1$ with $\sigma(v^*v) \neq 0$,

$$\begin{aligned}(\sigma \circ \theta^{-1})(\theta(v)^*\theta(v)) &= \inf\{\|\theta(d)^*\theta(v)^*\theta(v)\theta(d)\| : d \in \mathcal{D}_1, \sigma(d) = 1\} \\ &= \inf\{\|\theta(vd)\|^2 : d \in \mathcal{D}_1, \sigma(d) = 1\} \\ &\geqslant \|\theta^{-1}\|^{-2}\inf\{\|vd\|^2 : d \in \mathcal{D}_1, \sigma(d) = 1\} = \|\theta^{-1}\|^{-2}\sigma(v^*v) \neq 0.\end{aligned}$$

Simple calculations show that $[\theta(v), \sigma \circ \theta^{-1}]$ and $\phi \circ \theta^{-1}/\|\phi \circ \theta^{-1}\|$ are elements of $\mathcal{E}^1_{\mathcal{D}_2}(\mathcal{A}_2)$ with the same range. Since both are positive on $\theta(v)$,

$$\gamma(\phi) = [\theta(v), \sigma \circ \theta^{-1}].$$

This formula implies that $\gamma(\phi_1\phi_2) = \gamma(\phi_1)\gamma(\phi_2)$ whenever $\phi_1\phi_2$ is defined, so that γ is an algebraic isomorphism of coordinate systems.

To show continuity, let (ϕ_λ) be a net in $\mathcal{E}^1(\mathcal{A}_1)$ converging weak* to $\phi \in \mathcal{E}^1(\mathcal{A}_1)$. Write $\phi = [v, \sigma]$ for some intertwiner $v \in \mathcal{A}_1$ and $\sigma \in \widehat{\mathcal{D}}_1$. Since $\sigma(v^*v) \neq 0$, there exists $d \in \mathcal{D}_1$ and a neighborhood of σ, $G \subseteq \widehat{\mathcal{D}}_1$, with $d \geqslant 0$ and $\rho(d^*v^*vd) = 1$ for all $\rho \in G$. Since $\sigma(d) \neq 0$, $\phi = [vd, \sigma]$. Replacing v with vd, we may assume that $\rho(v^*v) = 1$ for every ρ in $G \subseteq \widehat{\mathcal{D}}_1$, a neighborhood of σ.

Since $\sigma_\lambda := s(\phi_\lambda)$ converges weak* to $s(\phi) = \sigma$, by deleting the first part of the net, we may assume that $\sigma_\lambda \in G$ and $\phi_\lambda(v) \neq 0$ for all λ. By Proposition 4.7, there exist scalars $t_\lambda \in \mathbb{T}$ such that $\phi_\lambda = [t_\lambda v, \sigma_\lambda]$. Since

$$\bar{t}_\lambda = [t_\lambda v, \sigma_\lambda](v) = \phi_\lambda(v) \to \phi(v) = 1,$$

and $[\theta(v), \sigma_\lambda \circ \theta^{-1}]$ converges weak* to $[\theta(v), \sigma \circ \theta^{-1}]$, we conclude

$$\gamma(\phi_\lambda) = [t_\lambda\theta(v), \sigma_\lambda \circ \theta] = \bar{t}_\lambda[\theta(v), \sigma_\lambda \circ \theta^{-1}] \to [\theta(v), \sigma \circ \theta^{-1}] = \gamma(\phi).$$

Thus, γ is continuous. Similarly, γ^{-1} is continuous. ∎

We give several applications of Theorem 6.3 to isomorphisms of subalgebras. The following corollary is immediate.

COROLLARY 6.5. *For $i = 1, 2, 3$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be regular C^*-inclusions, and let $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ be norm-closed algebras. For $j = 1, 2$, let $\theta_j : \mathcal{D}_j \to \mathcal{D}_{j+1}$ be bounded diagonal preserving isomorphisms. Then $\gamma_{\theta_2 \circ \theta_1} = \gamma_{\theta_2} \circ \gamma_{\theta_1}$.*

DEFINITION 6.6. Suppose $(\mathcal{C}, \mathcal{D})$ is a regular C^*-inclusion and $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$. By a *(1-)cocycle* on $\mathcal{E}^1(\mathcal{A})$, we mean a map $c : \mathcal{E}^1(\mathcal{A}) \to \mathbb{C}_*$, the group of nonzero complex numbers under multiplication, satisfying, for all composable elements $\phi, \psi \in \mathcal{E}^1(\mathcal{A})$,

$$c(\phi\psi) = c(\phi) \cdot c(\psi).$$

COROLLARY 6.7. *Suppose $(\mathcal{C}, \mathcal{D})$ is a regular C^*-inclusion, $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ is a norm-closed algebra, and $\theta : \mathcal{A} \to \mathcal{A}$ is a bounded automorphism fixing $\mathcal{D}$ elementwise. Assume further that γ_θ is the identity map on $\mathcal{E}^1(\mathcal{A})$. Then, for all $\phi \in \mathcal{E}^1(\mathcal{A})$,*

$$\phi(\theta(x)) = \phi(x)\|\phi \circ \theta\|. \tag{6.1}$$

If $c : \mathcal{E}^1(\mathcal{A}) \to \mathbb{R}$ is defined by $c(\phi) = \|\phi \circ \theta\|$, then c is a positive cocycle on $\mathcal{E}^1(\mathcal{A})$.

If, in addition, θ is isometric, then $\theta = \mathrm{id}_{\mathcal{A}}$.

REMARK 6.8. In essence, (6.1) shows that θ is given by multiplication by a cocycle on $\mathcal{E}^1(\mathcal{A})$. Thus, we will call θ a *cocycle automorphism*.

Proof of Corollary 6.7. Formula (6.1) follows immediately from Theorem 6.3 applied to θ^{-1}. If θ is isometric, then (6.1) shows that $\phi(x) = \phi(\theta(x))$ for all $\phi \in \mathcal{E}^1(\mathcal{A})$. By Theorem 4.8, $\mathcal{E}^1(\mathcal{A})$ separates points and so $x = \theta(x)$ for all $x \in \mathcal{A}$.

It remains to show that c is a cocycle. Observe that if $v \in \mathcal{C}$ is a $\mathcal{D}$-intertwiner, then $v^*\theta(v) \in \mathcal{D}$. Indeed, as in the proof of Proposition 3.3, given a self-adjoint $d \in \mathcal{D}$, we may find a self-adjoint $d' \in \mathcal{D}$ so that $vd = d'v$. Then $v^*\theta(v)d = v^*d'\theta(v) = dv^*\theta(v)$, so that $v^*\theta(v)$ commutes with the self-adjoint elements of $\mathcal{D}$ and hence belongs to $\mathcal{D}$.

Finally, suppose for $i = 1, 2$, that $\phi_i = [v_i, \sigma_i] \in \mathcal{E}^1(\mathcal{A})$ such that the product $\phi_1\phi_2$ is defined. Then $\phi_1\phi_2 = [v_1v_2, \sigma_2]$, and $\sigma_1 = r([v_2, \sigma_2])$, i.e., $\sigma_1(x) = \sigma_2(v_2^*xv_2)/\sigma_2(v_2^*v_2)$. Using (6.1) and these facts, we have

$$
\begin{aligned}
c(\phi_1\phi_2) &= \frac{[v_1v_2, \sigma_2](\theta(v_1v_2))}{[v_1v_2, \sigma_2](v_1v_2)} = \frac{\sigma_2(v_2^*v_1^*\theta(v_1)\theta(v_2))}{\sigma_2(v_2^*v_1^*v_1v_2)} \\
&= \frac{\sigma_2(v_2^*v_2)\sigma_2(v_2^*v_1^*\theta(v_1)\theta(v_2))}{\sigma_2(v_2^*v_2)\sigma_2(v_2^*v_1^*v_1v_2)} \\
&= \frac{\sigma_1(v_1^*\theta(v_1))\sigma_2(v_2^*\theta(v_2))}{\sigma_2(v_2^*v_1^*v_1v_2)} \\
&= \frac{\sigma_1(v_1^*\theta(v_1))\sigma_2(v_2^*\theta(v_2))}{\sigma_1(v_1^*v_1)\sigma_2(v_2^*v_2)} = c(\phi_1)c(\phi_2). \quad \blacksquare
\end{aligned}
$$

For our next application, we need two technical lemmas.

LEMMA 6.9. *For $i = 1, 2$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be regular C^*-inclusions and let $\mathcal{B}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ be selfadjoint. If $\gamma : \mathcal{E}^1(\mathcal{B}_1) \to \mathcal{E}^1(\mathcal{B}_2)$ is an isomorphism of coordinate systems, then, for all $\phi \in \mathcal{E}^1(\mathcal{B}_1)$, $\gamma(\phi^*) = \gamma(\phi)^*$.*

Proof. Since each $\mathcal{B}_i$ is selfadjoint, $\tau^* \in \mathcal{E}^1(\mathcal{B}_i)$ if $\tau \in \mathcal{E}^1(\mathcal{B}_i)$, and so $\gamma(\phi^*)$ and $\gamma(\phi)^*$ are defined. As $\gamma(\phi^*)$ and $\gamma(\phi)^*$ have the same range and domain, by Corollary 4.10, there is $\lambda \in \mathbb{T}$ with $\gamma(\phi^*) = \lambda\gamma(\phi)^*$. As $\phi \cdot \phi^* = r(\phi)$, we have

$$\gamma(r(\phi)) = \gamma(\phi \cdot \phi^*) = \gamma(\phi) \cdot \gamma(\phi^*) = \gamma(\phi) \cdot (\lambda\gamma(\phi)^*) = \lambda r(\gamma(\phi)).$$

As $\gamma(r(\phi)) = r(\gamma(\phi))$, the result follows. $\blacksquare$

LEMMA 6.10. *For $i = 1, 2$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be regular C^*-inclusions, let $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ be regular, and let $\pi : \mathcal{C}_1 \to C_2$ be a $*$-isomorphism with $\pi(\mathcal{D}_1) = \mathcal{D}_2$. If γ_π maps $\mathcal{E}^1(\mathcal{A}_1)$ into $\mathcal{E}^1(\mathcal{A}_2)$, then $\pi(\mathcal{A}_1) \subseteq \mathcal{A}_2$.*

Proof. Since each $\mathcal{A}_i$ is regular, it is enough to show that for an intertwiner $v \in \mathcal{A}_1$, we have $\pi(v) \in \mathcal{A}_2$. Clearly, $\pi(v)$ is an intertwiner in $\mathcal{C}_2$, and it is easy to see that the closed $\mathcal{D}_2$-bimodule generated by $\pi(v)$ is isometrically isomorphic to $\overline{|\pi(v)|\mathcal{D}_2}$ via the map $\pi(v)d \mapsto |\pi(v)|d$. Given $\sigma \in \widehat{\mathcal{D}}_2$ with $\sigma(\pi(v^*v)) \neq 0$, we have $\gamma([v, \sigma \circ \pi^{-1}]) = [\pi(v), \sigma] \in \mathcal{E}^1(\mathcal{A}_2)$. Applying Proposition 3.10, we conclude that there exists $d \in \mathcal{D}_2$ with $d \geqslant 0$, $\sigma(d) = 1$, and $\pi(v)d \in \mathcal{A}_2$. Given $\varepsilon > 0$, let $f_\varepsilon \in \mathcal{D}$ be chosen so that $0 \leqslant f_\varepsilon \leqslant I$, $\widehat{f_\varepsilon}$ is compactly supported in $G := \{\sigma \in \widehat{\mathcal{D}}_2 : \sigma(|\pi(v)|) > 0\}$ and $\|\pi(v) - \pi(v)f_\varepsilon\| < \varepsilon$. Compactness of $F_\varepsilon := \overline{\operatorname{supp} f_\varepsilon}$ ensures that there exist $d_1, \ldots, d_n \in \mathcal{D}_2$ such that $\pi(v)d_i \in \mathcal{A}_2$ and $\sigma(\sum_{i=1}^n d_i) > 0$ for every $\sigma \in F_\varepsilon$. Hence there exists an element $g \in \mathcal{D}_2$ such that for every $\sigma \in F_\varepsilon$, $\sigma(g \sum_{i=1}^n d_i) = 1$. Then

$$\pi(v)f_\varepsilon = \sum_{i=1}^n \pi(v)d_i g f_\varepsilon \in \mathcal{A}_2.$$

Letting $\varepsilon \to 0$, we conclude that $\pi(v) \in \mathcal{A}_2$ as well. ∎

Recall that a subalgebra $\mathcal{A} \subseteq \mathcal{C}$ is said to be *Dirichlet* if $\mathcal{A} + \mathcal{A}^*$ is norm dense in $\mathcal{C}$. This implies that $\mathcal{E}^1(\mathcal{C}) = \mathcal{E}^1(\mathcal{A}) \cup \mathcal{E}^1(\mathcal{A})^*$. To see this, let $\phi \in \mathcal{E}^1(\mathcal{C})$. By density, ϕ does not vanish on one of $\mathcal{A}$ or $\mathcal{A}^*$ and hence is in either $\mathcal{E}^1(\mathcal{A})$ or $\mathcal{E}^1(\mathcal{A})^*$.

THEOREM 6.11. *For $i = 1, 2$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be regular C^*-inclusions and let $\mathcal{A}_i$ be Dirichlet subalgebras with $\mathcal{D}_i \subseteq \mathcal{A}_i \subseteq \mathcal{C}_i$. Consider the following statements:*

(i) *$\mathcal{A}_1$ and $\mathcal{A}_2$ are isometrically isomorphic.*

(ii) *There exists a bounded isomorphism $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\theta(\mathcal{D}_1) = \mathcal{D}_2$.*

(iii) *$\mathcal{E}^1(\mathcal{A}_1)$ and $\mathcal{E}^1(\mathcal{A}_2)$ are isomorphic coordinate systems.*

(iv) *$\mathcal{E}^1(\mathcal{C}_1)$ and $\mathcal{E}^1(\mathcal{C}_2)$ are isomorphic twists and the isomorphism maps $\mathcal{E}^1(\mathcal{A}_1)$ onto $\mathcal{E}^1(\mathcal{A}_2)$.*

Then statement (n) *implies statement* (n + 1), *$n = 1, 2, 3$.*

If, in addition, $(\mathcal{C}_i, \mathcal{D}_i)$ are C^*-diagonals and $\mathcal{A}_i$ are regular, then the four statements are equivalent.

Proof. That (i) implies (ii) is obvious. To show that (ii) implies (iii), apply Theorem 6.3.

To show (iii) implies (iv) suppose $\gamma : \mathcal{E}^1(\mathcal{A}_1) \to \mathcal{E}^1(\mathcal{A}_2)$ is a continuous isomorphism. We extend γ to a map, call it δ, on all of $\mathcal{E}^1(\mathcal{C}_1)$ by mapping $\phi \in \mathcal{E}^1(\mathcal{A}_1)^* = \mathcal{E}^1(\mathcal{A}_1^*)$ to $\gamma(\phi^*)^*$. By Lemma 6.9, δ is well-defined on $\mathcal{E}^1(\mathcal{A}_1 \cap \mathcal{A}_1^*)$, and hence well-defined on all of $\mathcal{E}^1(\mathcal{C}_1)$.

Since the adjoint map is continuous, δ is a continuous homomorphism restricted to $\mathcal{E}^1(\mathcal{A}_1)$ or to $\mathcal{E}^1(\mathcal{A}_1^*)$. By Corollary 4.15, these are open sets in $\mathcal{E}^1(\mathcal{C}_1)$, and so δ is continuous on their union, $\mathcal{E}^1(\mathcal{C}_1)$. Applying the same argument to δ^{-1}, we see that δ^{-1} is continuous as well.

The restriction of δ to $\mathcal{E}^1(\mathcal{A}_1^*)$ or to $\mathcal{E}^1(\mathcal{A}_1)$ is a homomorphism. To show that δ is a homomorphism, fix $\phi, \psi \in \mathcal{E}^1(\mathcal{A}_1)$. We claim that if $\phi^*\psi$ is defined, then $\delta(\phi^*\psi) = \delta(\phi^*)\delta(\psi)$ and if $\phi\psi^*$ is defined, then $\delta(\phi\psi^*) = \delta(\phi)\delta(\psi)^*$. We only show the first equality; the proof of the second is similar.

As δ maps $\widehat{\mathcal{D}}_1$ to $\widehat{\mathcal{D}}_2$, we have, for $\phi \in \mathcal{E}^1(\mathcal{A}_1)$,

$$\delta(\phi\phi^*) = \gamma(r(\phi)) = r(\gamma(\phi)) = \gamma(\phi)\gamma(\phi^*). \tag{6.2}$$

Let $\eta = \phi^*\psi$. If $\eta \in \mathcal{E}^1(\mathcal{A}_1)$, we have $\phi\eta = \phi\phi^*\psi = \psi$, and hence $\gamma(\phi)\gamma(\eta) = \gamma(\psi)$. Multiplying each side of this equality by $\gamma(\phi^*) = \delta(\phi)^*$ and using (6.2) yields $\delta(\phi^*\psi) = \delta(\phi)^*\delta(\psi)$. If $\eta \in \mathcal{E}^1(\mathcal{A}_1^*)$, then apply the previous argument to $\eta^* = \psi^*\phi$ and take adjoints to obtain the equation. Thus δ is a homomorphism and part (iv) holds.

Finally, if $(\mathcal{C}_i, \mathcal{D}_i)$ are C^*-diagonals and (iv) holds, then Kumjian's theorem implies that there is an isomorphism $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ with $\Phi(\mathcal{D}_1) = \mathcal{D}_2$. Since Φ is induced by an isomorphism of twists mapping $\mathcal{E}^1(\mathcal{A}_1)$ onto $\mathcal{E}^1(\mathcal{A}_2)$, Lemma 6.10 shows that $\Phi(\mathcal{A}_1) = \mathcal{A}_2$. Thus, we obtain (i) with $\Phi|_{\mathcal{A}_1}$ the isometric isomorphism. ∎

REMARK 6.12. Theorem 6.11 is related to a result of Muhly, Qiu and Solel, ([25], Theorem 2.1). Their result, expressed in terms of eigenfunctionals, takes the following form. When the $\mathcal{C}_i$ are nuclear and $(\mathcal{C}_i, \mathcal{D}_i)$ are C^*-diagonals, the following are equivalent, for subalgebras $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ so that $\mathcal{A}_i$ generates $\mathcal{C}_i$ as C^*-algebras (with *no* Dirichlet hypothesis):

(i) there is an isometric isomorphism $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ (necessarily, $\theta(\mathcal{D}_1) = \mathcal{D}_2$);

(ii) there is a coordinate system isomorphism, $\gamma : \mathcal{E}^1(\mathcal{A}_1) \to \mathcal{E}^1(\mathcal{A}_2)$ which extends to a coordinate system isomorphism $\gamma' : \mathcal{E}^1(\mathcal{C}_1) \to \mathcal{E}^1(\mathcal{C}_2)$;

(iii) there is a $*$-isomorphism $\tau : \mathcal{C}_1 \to \mathcal{C}_2$ such that $\tau|_{\mathcal{A}_1}$ is an isomorphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$.

Theorem 6.11 extends the Muhly-Qiu-Solel result to not-necessarily-isometric diagonal preserving isomorphisms, assuming the Dirichlet condition instead of the hypothesis in (ii) that γ extends to an isomorphism of $\mathcal{E}^1(\mathcal{C}_1)$ onto $\mathcal{E}^1(\mathcal{C}_2)$. Example 6.13 shows that in the absence of the Dirichlet condition, isomorphisms of coordinate systems need not extend to isomorphisms of the enveloping twists. Thus, the hypothesis that γ extends in (ii) is essential.

Also, since Theorem 6.11 did not use the Spectral Theorem for Bimodules ([24], Theorem 4.1), we do not need the $\mathcal{C}_i$ to be nuclear. In Theorem 8.9, we prove the full Muhly-Qiu-Solel result, without requiring nuclearity.

EXAMPLE 6.13. Without the Dirichlet hypothesis, an isomorphism of coordinate systems need not induce an isometric isomorphism of the algebras. Let $(\mathcal{C}, \mathcal{D}) = (M_4(\mathbb{C}), \mathcal{D}_4)$, where $\mathcal{D}_4$ is the algebra of diagonal matrices.

Let $\mathcal{A}_1 = \mathcal{A}_2$ be the algebra

$$\mathcal{A} := \left\{ \begin{pmatrix} t_{11} & 0 & t_{13} & t_{14} \\ 0 & t_{22} & t_{23} & t_{24} \\ 0 & 0 & t_{33} & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix} : t_{ij} \in \mathbb{C} \right\}.$$

The automorphism

$$\begin{pmatrix} t_{11} & 0 & t_{13} & t_{14} \\ 0 & t_{22} & t_{23} & t_{24} \\ 0 & 0 & t_{33} & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix} \mapsto \begin{pmatrix} t_{11} & 0 & t_{13} & t_{14} \\ 0 & t_{22} & t_{23} & -t_{24} \\ 0 & 0 & t_{33} & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix}$$

is not isometric, and induces an automorphism of $\mathcal{E}^1(\mathcal{A})$ which does not extend to an automorphism of $\mathcal{E}^1(\mathcal{C})$.

The Dirichlet condition can be removed if one assumes a continuous section from $R(\mathcal{C}_i)$ into $\mathcal{E}^1(\mathcal{C}_i)$. Since TAF algebras always admit such sections, the following result generalizes Theorem 7.5 of [37].

THEOREM 6.14. *For $i = 1, 2$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be C^*-diagonals and suppose that $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ are norm closed subalgebras such that $\mathcal{A}_i$ generates $\mathcal{C}_i$ as a C^*-algebra. Consider the following statements:*

(i) *$\mathcal{A}_1$ and $\mathcal{A}_2$ are isometrically isomorphic.*

(ii) *There exists a bounded isomorphism $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\theta(\mathcal{D}_1) = \mathcal{D}_2$.*

(iii) *$R(\mathcal{A}_1)$ and $R(\mathcal{A}_2)$ are isomorphic topological binary relations.*

(iv) *$R(\mathcal{C}_1)$ and $R(\mathcal{C}_2)$ are isomorphic topological equivalence relations, and the isomorphism maps $R(\mathcal{A}_1)$ onto $R(\mathcal{A}_2)$.*

Then for $n = 1, 2, 3$, statement (n) implies statement (n + 1).

If, in addition, $\mathcal{A}_i$ are regular and there exist continuous sections $h_i : R(\mathcal{C}_i) \to \mathcal{E}^1(\mathcal{C}_i)$, then the statements are equivalent.

Proof. That (i)⇒(ii) is obvious and (ii)⇒(iii) follows as in the proof of Theorem 6.11.

Suppose (iii) holds. By Proposition 4.21, the isomorphism of $R(\mathcal{A}_1)$ onto $R(\mathcal{A}_2)$ extends uniquely to an isomorphism of $R(\mathcal{C}_1)$ onto $R(\mathcal{C}_2)$, so (iv) holds.

To complete the proof, we show that, when the $\mathcal{A}_i$ are regular and there exist continuous sections $h_i : R(\mathcal{C}_i) \to \mathcal{E}^1(\mathcal{C}_i)$, then (iv) implies (i). The existence of the sections and (iv) gives a coordinate system isomorphism γ of $\mathcal{E}^1(\mathcal{C}_1)$ and $\mathcal{E}^1(\mathcal{C}_2)$ such that $\gamma|_{\mathcal{E}^1(\mathcal{A}_1)}$ is an isomorphism of $\mathcal{E}^1(\mathcal{A}_1)$ onto $\mathcal{E}^1(\mathcal{A}_2)$. By Kumjian's theorem, there is a (regular) $*$-isomorphism π of $(\mathcal{C}_1, \mathcal{D}_1)$ onto $(\mathcal{C}_2, \mathcal{D}_2)$.

Finally, Lemma 6.10 implies $\pi(\mathcal{A}_1) = \mathcal{A}_2$. ∎

REMARK 6.15. A modification of Example 6.13 shows that in general, there may exist a section for $R(\mathcal{A})$ which cannot be extended to a section of $R(\mathcal{C})$. Thus, one cannot replace the hypothesis of a section for $R(\mathcal{C}_i)$ with a hypothesis of a section for $R(\mathcal{A}_i)$ in Theorem 6.14.

To drop the Dirichlet condition in Theorem 6.11 without assuming the existence of a continuous section, we would need to replace γ with a new isomorphism from $\mathcal{E}^1(\mathcal{A}_1)$ to $\mathcal{E}^1(\mathcal{A}_2)$ that could extend to the twist of $\mathcal{C}_1$, which would be larger than $\mathcal{E}^1(\mathcal{A}_1) \cup \mathcal{E}^1(\mathcal{A}_1)^*$. In particular, while isometric bimodule maps on finite relations are always $*$-extendible (a key ingredient in [25]), this is not true for general bimodule maps ([8], Theorem 1.2, Proposition 1.4). Indeed, based on [8], there are homological obstructions to be considered.

7. INVARIANCE UNDER GENERAL ISOMORPHISMS

In this section and the next, we come to the core of the paper, the study of bounded isomorphisms of triangular algebras which need not map the diagonal to the diagonal. The principal result of this section is Theorem 7.7, which shows that such isomorphisms induce *algebraic* isomorphisms of the corresponding coordinate systems. We would particularly like to know if this algebraic isomorphism is always continuous. We can prove that it is continuous in certain cases, Theorem 7.11, extending results of Donsig-Hudson-Katsoulis.

7.1. STANDING ASSUMPTIONS FOR SECTIONS 7 AND 8. For $i = 1, 2$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be C^*-diagonals, and let $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ be (norm-closed) triangular subalgebras. Let $E_i : \mathcal{C}_i \to \mathcal{D}_i$ be the unique conditional expectations. Suppose $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is a bounded isomorphism.

Our first task is to show that θ induces an *algebraic* isomorphism of coordinate systems, $\gamma : \mathcal{E}^1(\mathcal{A}_1) \to \mathcal{E}^1(\mathcal{A}_2)$. We have been unable to show that γ is continuous in general. Since we do not assume $\theta(\mathcal{D}_1) = \mathcal{D}_2$, it is not possible to use Theorem 6.3, so we proceed along different lines.

DEFINITION 7.1. Define $\alpha : \mathcal{D}_1 \to \mathcal{D}_2$ by $\alpha(d) = E_2(\theta(d))$.

PROPOSITION 7.2. *The map α is a $*$-isomorphism of $\mathcal{D}_1$ onto $\mathcal{D}_2$.*

Proof. Theorem 5.10 shows $E_2|_{\mathcal{A}_2}$ is a homomorphism; hence α is a homomorphism. Since any algebraic isomorphism of commutative C^*-algebras is a $*$-isomorphism, it suffices to show that α is bijective.

Since E_1 is idempotent, $\mathcal{D}_1 \cap \ker E_1 = \{0\}$, so $\mathcal{A}_1 = \mathcal{D}_1 + \ker E_1|_{\mathcal{A}_1}$ is a direct sum decomposition. By Proposition 5.11, $\theta(\ker E_1|_{\mathcal{A}_1}) = \ker E_2|_{\mathcal{A}_2}$, so that we have two direct sum decompositions of $\mathcal{A}_2$:

$$\mathcal{A}_2 = \mathcal{D}_2 + \ker E_2|_{\mathcal{A}_2} = \theta(\mathcal{D}_1) + \ker E_2|_{\mathcal{A}_2}.$$

Therefore, $\ker E_2|_{\theta(\mathcal{D}_1)} = \ker \alpha$ is trivial. If $d \in \mathcal{D}_2$ we may write $d = x + y$ where $x \in \theta(\mathcal{D}_1)$ and $y \in \ker E_2$. Then $E_2(x) = d$, so α is onto. ∎

Given Banach spaces X and Y, and a bounded linear map $R : X \to Y$, the double transpose map $R^{\#\#} : X^{\#\#} \to Y^{\#\#}$ is a norm-continuous extension of R which is also $\sigma(X^{\#\#}, X^{\#})$–$\sigma(Y^{\#\#}, Y^{\#})$ continuous. In light of our standing assumptions, $\theta^{\#\#} : \mathcal{A}_1^{\#\#} \to \mathcal{A}_2^{\#\#}$ and $\alpha^{\#\#} : \mathcal{D}_1^{\#\#} \to \mathcal{D}_2^{\#\#}$ are also isomorphisms. Similarly, the $E_i^{\#\#}$ are homomorphisms of $\mathcal{A}_i^{\#\#}$ onto $\mathcal{D}_i^{\#\#}$. For notational ease, we will sometimes identify the double transpose map with the original map. Thus we often write θ or α instead of $\theta^{\#\#}$ or $\alpha^{\#\#}$.

REMARK 7.3. Notice that $\mathcal{A}_2$ may be regarded as a $\mathcal{D}_1$-bimodule in two natural ways: for $d, e \in \mathcal{D}_1$ and $x \in \mathcal{A}_2$, define $d \cdot_\alpha x \cdot_\alpha e := \alpha(d)x\alpha(e)$ and $d \cdot_\theta x \cdot_\theta e := \theta(d)x\theta(e)$. When these two modules are (boundedly) isomorphic, methods similar to those used in the proof of Theorem 6.3 show there exists an isomorphism of the coordinate systems $\mathcal{E}^1(\mathcal{A}_1)$ and $\mathcal{E}^1(\mathcal{A}_2)$. Unfortunately, we do not know in general whether the α and θ module actions of $\mathcal{D}_1$ on $\mathcal{A}_2$ are isomorphic. However, there is enough structure present to show that these modules are "virtually" isomorphic.

DEFINITION 7.4. Let $G = \mathcal{U}(\mathcal{D}_1)$ be the group of unitary elements of $\mathcal{D}_1$, regarded as a discrete abelian group and fix, once and for all, an invariant mean Λ on G.

Define elements $S, T \in \mathcal{A}_2^{\#\#}$ by requiring that for every $f \in \mathcal{A}_2^{\#}$,

$$f(S) = \underset{g\in G}{\Lambda} f(\theta(g)\alpha(g^{-1})) \quad \text{and} \quad f(T) = \underset{g\in G}{\Lambda} f(\alpha(g)\theta(g^{-1})).$$

Notice that $S \in \overline{\text{co}}^\sigma\{\theta(g)\alpha(g^{-1}) : g \in G\}$ and $T \in \overline{\text{co}}^\sigma\{\alpha(g)\theta(g^{-1}) : g \in G\}$, where $\overline{\text{co}}^\sigma Z$ is the $\sigma(\mathcal{A}_2^{\#\#}, \mathcal{A}_2^{\#})$-closed convex hull of the set Z. (We implicitly embed $\mathcal{A}_2$ into $\mathcal{A}_2^{\#\#}$ using the canonical inclusion.)

We next collect some properties of S and T. Of particular interest to us is the fact that they intertwine $\alpha(\mathcal{D}_1)$ and $\theta(\mathcal{D}_1)$.

PROPOSITION 7.5. *For S and T as above, we have:*

(i) *For every $d \in \mathcal{D}_1$, $T\theta(d) = \alpha(d)T$ and $S\alpha(d) = \theta(d)S$.*

(ii) *$E_2^{\#\#}(S) = I = E_2^{\#\#}(T)$ and $E_1^{\#\#}(\theta^{-1}(S)) = I = E_1^{\#\#}(\theta^{-1}(T))$.*

(iii) *Given $\sigma \in \widehat{\mathcal{D}}_1$, let $p = p_\sigma$ (see Definition 5.1). Then*

$$\alpha(p)T = \alpha(p)T\theta(p) = T\theta(p) = \alpha(p)\theta(p) \ \text{ and } \ \theta(p)S = \theta(p)S\alpha(p) = S\alpha(p) = \theta(p)\alpha(p).$$

(iv) *For all $x \in \mathcal{A}_2$ and for all $\phi \in \mathcal{E}^1(\mathcal{A}_1)$, $\phi(\theta^{-1}(STxST)) = \phi(\theta^{-1}(x))$.*

Proof. If $h \in G = \mathcal{U}(\mathcal{D}_1)$, then $T\theta(h) = \alpha(h)T$ follows from the invariance of Λ. Indeed, for every $f \in \mathcal{A}_2^{\#}$ we have,

$$\begin{aligned} f(\alpha(h)T) &= \Lambda_g(f \cdot \alpha(h))(\alpha(g)\theta(g)^{-1}) = \Lambda_g f(\alpha(hg)\theta(hg)^{-1}\theta(h)) \\ &= \Lambda_g f(\alpha(g)\theta(g^{-1})\theta(h)) = \Lambda_g(\theta(h) \cdot f)(\alpha(g)\theta(g^{-1})) = f(T\theta(h)). \end{aligned}$$

Since $\mathcal{A}_2^{\#}$ separates points of $\mathcal{A}_2^{\#\#}$, we see that $T\theta(h) = \alpha(h)T$ for every $h \in G$. But the span of G is norm dense in $\mathcal{D}_1$, which yields $T\theta(d) = \alpha(d)T$ for $d \in \mathcal{D}_1$. The proof that $\theta(d)S = S\alpha(d)$ is similar.

To prove $E_1^{\#\#}(\theta^{-1}(S)) = I$, first observe that weak* continuity of $E_1^{\#\#}$ (and ${\theta^{-1}}^{\#\#}$) implies

$$E_1^{\#\#}(\theta^{-1}(S)) \in E_1^{\#\#}\overline{\text{co}}^\sigma\{g\theta^{-1}(\alpha(g^{-1}))\} = \overline{\text{co}}^\sigma\{E_1(g\theta^{-1}(\alpha(g^{-1})))\}.$$

Modifying the proof of Proposition 7.2 yields $\alpha^{-1} = E_1 \circ \theta^{-1}$ and, for each $g \in G$, $E_1(g\theta^{-1}(\alpha(g^{-1}))) = I$; the equality follows. The remaining equalities in part (ii) have similar proofs.

For part (iii), the first two equalities follow from statement (i), as p is a $\sigma(\mathcal{A}_1^{\#\#}, \mathcal{A}_1^{\#})$-limit of elements of $\mathcal{D}_1$. For the third equality, first observe that for $g \in G$, Theorem 5.3 gives $\alpha(p)\alpha(g) = \sigma(g)\alpha(p)$; similarly, $\theta(g)^{-1}\theta(p) = \sigma(g^{-1})\theta(p)$. Hence, $\alpha(p)\alpha(g)\theta(g)^{-1}\theta(p) = \alpha(p)\theta(p)$. Since $T \in \overline{\text{co}}^{\sigma}\{\alpha(g)\theta(g)^{-1} : g \in G\}$, we find that

$$\alpha(p)T\theta(p) \in \overline{\text{co}}^{\sigma}\{\alpha(p)\alpha(g)\theta(g)^{-1}\theta(p) : g \in G\}.$$

This set is a singleton, so $\alpha(p)T\theta(p) = \alpha(p)\theta(p)$. The proofs of the equalities involving S are similar.

For part (iv), fix $\phi \in \mathcal{E}^1(\mathcal{A}_1)$, and let q and p be the minimal projections in $\mathcal{D}_1^{\#\#}$ corresponding to $r(\phi)$ and $s(\phi)$, respectively. Part (i) implies that p and q commute with $\theta^{-1}(ST)$ and by part (ii), we have

$$r(\phi)(\theta^{-1}(ST)) = r(\phi)(E_1(\theta^{-1}(ST))) = r(\phi)(I) = 1.$$

Hence by Proposition 5.3, $q\theta^{-1}(ST) = q$. Likewise $p\theta^{-1}(ST) = p$. As, again by Proposition 5.3, $\phi(a) = \phi(qap)$, we have, as desired:

$$\phi(\theta^{-1}(STxST)) = \phi(q\theta^{-1}(ST)\theta^{-1}(x)\theta^{-1}(ST))p?) = \phi(q\theta^{-1}(x)p) = \phi(\theta^{-1}(x)). \qquad \blacksquare$$

We now obtain a bijective mapping between the eigenfunctionals of $\mathcal{A}_1$ and those of $\mathcal{A}_2$.

PROPOSITION 7.6. *For $\phi \in \mathcal{E}(\mathcal{A}_1)$, let*

$$f = T \cdot (\phi \circ \theta^{-1}) \cdot S.$$

Then f is an eigenfunctional for $\mathcal{A}_2$ with $r(f) = r(\phi) \circ \alpha^{-1}$ and $s(f) = s(\phi) \circ \alpha^{-1}$. Moreover, $\phi \circ \theta^{-1} = S \cdot f \cdot T$.

Proof. For clarity, let $\psi = \phi \circ \theta^{-1}$. For all $d, e \in \mathcal{D}_1$ and $x \in \mathcal{A}_2$,

$$\begin{aligned} f(\alpha(d)x\alpha(e)) &= \psi(S\alpha(d)x\alpha(e)T) = \psi(\theta(d)SxT\theta(e)) = r(\phi)(d)\psi(SxT)s(\phi)(e) \\ &= (r(\phi) \circ \alpha^{-1})(\alpha(d))f(x)(s(\phi) \circ \alpha^{-1})(\alpha(e)), \end{aligned}$$

showing f is an eigenfunctional with the claimed range and source.

The last equality follows from part (iv) of Proposition 7.5. $\blacksquare$

We now show the existence of an algebraic isomorphism between the coordinate systems which is the non-diagonal preserving analog of Theorem 6.3.

THEOREM 7.7. *The map $\gamma : \mathcal{E}^1(\mathcal{A}_1) \to \mathcal{E}^1(\mathcal{A}_2)$ given by*

$$\gamma(\phi) = \frac{T \cdot (\phi \circ \theta^{-1}) \cdot S}{\|T \cdot (\phi \circ \theta^{-1}) \cdot S\|}$$

is an algebraic isomorphism of coordinate systems such that for every $\sigma \in \widehat{\mathcal{D}}_1$, $\gamma(\sigma) = \sigma \circ \alpha^{-1}$.

Proof. The fact that γ is a bijection between $\mathcal{E}^1(\mathcal{A}_1)$ and $\mathcal{E}^1(\mathcal{A}_2)$ such that for every $\phi \in \mathcal{E}^1(\mathcal{A}_1)$, $r(\gamma(\phi)) = r(\phi) \circ \alpha^{-1}$ and $s(\gamma(\phi)) = s(\phi) \circ \alpha^{-1}$ follows immediately from Proposition 7.6, so we need only show γ is multiplicative on composable elements.

Given $\phi \in \mathcal{E}^1(\mathcal{A}_1)$ with minimal partial isometry v_ϕ (see Definition 5.1), we first identify the minimal partial isometry for $\gamma(\phi)$. In fact, we claim that

$$v_{\gamma(\phi)} = \frac{T\theta(v_\phi)S}{\|T\theta(v_\phi)S\|}. \tag{7.1}$$

To see this, first observe that part (iv) of Proposition 7.5 also holds for $x \in \mathcal{A}_2^{\#\#}$. Thus, $(T \cdot (\phi \circ \theta^{-1}) \cdot S)(T\theta(v_\phi)S) = \phi(\theta^{-1}(ST\theta(v_\phi)ST)) = \phi(v_\phi) = 1$. Therefore,

$$\gamma(\phi)(T\theta(v_\phi)S) > 0.$$

Moreover, if $q = p_{r(\phi)}$ and $p = p_{s(\phi)}$, then by part (iii) of Proposition 7.5,

$$T\theta(v_\phi)S = T\theta(q)\theta(v_\phi)\theta(p)S = \alpha(p)\theta(v_\phi)\alpha(q).$$

Hence $\dfrac{T\theta(v_\phi)S}{\|T\theta(v_\phi)S\|}$ is a minimal partial isometry in $\mathcal{A}_2^{\#\#}$ on which $\gamma(\phi)$ takes a positive value. Thus, equation (7.1) holds by Remark 5.2.

Now suppose $\phi_1, \phi_2 \in \mathcal{E}^1(\mathcal{A}_1)$ are such that $\phi_1\phi_2$ is defined. Notice that the minimal partial isometry for the product $\phi_1\phi_2$ is the product of the minimal partial isometries for ϕ_1 and ϕ_2, that is, $v_{\phi_1\phi_2} = v_{\phi_1}v_{\phi_2}$. To show that $\gamma(\phi_1\phi_2) = \gamma(\phi_1)\gamma(\phi_2)$, it suffices to show that

$$v_{\gamma(\phi_1\phi_2)} = v_{\gamma(\phi_1)}v_{\gamma(\phi_2)}. \tag{7.2}$$

To do this, we first show that for all $\sigma \in \widehat{\mathcal{D}}_1$, we have

$$\theta(p_\sigma)\alpha(p_\sigma)\theta(p_\sigma) = \theta(p_\sigma). \tag{7.3}$$

Indeed, by Proposition 5.3,

$$\begin{aligned} p_\sigma\theta^{-1}(\alpha(p_\sigma))p_\sigma &= \sigma(\theta^{-1}(\alpha(p_\sigma)))p_\sigma = \sigma((E_1 \circ \theta^{-1})(\alpha(p_\sigma)))p_\sigma \\ &= \sigma(\alpha^{-1}(\alpha(p_\sigma)))p_\sigma = \sigma(p_\sigma)p_\sigma = p_\sigma. \end{aligned}$$

Applying θ to the ends of this equality yields (7.3).

Noting that $p_{s(\phi_1)} = p_{r(\phi_2)}$, we have

$$\begin{aligned} &(T\theta(v_{\phi_1})S)(T\theta(v_{\phi_2})S) \\ &= [\alpha(p_{r(\phi_1)})\theta(v_{\phi_1})\alpha(p_{s(\phi_1)})][\alpha(p_{r(\phi_2)})\theta(v_{\phi_2})\alpha(p_{s(\phi_2)})] \\ &= [\alpha(p_{r(\phi_1)})\theta(v_{\phi_1})\theta(p_{s(\phi_1)})\alpha(p_{s(\phi_1)})][\alpha(p_{r(\phi_2)})\theta(p_{r(\phi_2)})\theta(v_{\phi_2})\alpha(p_{s(\phi_2)})] \\ &= \alpha(p_{r(\phi_1)})\theta(v_{\phi_1})[\theta(p_{s(\phi_1)})\alpha(p_{s(\phi_1)})\alpha(p_{r(\phi_2)})\theta(p_{r(\phi_2)})]\theta(v_{\phi_2})\alpha(p_{s(\phi_2)}) \\ &= \alpha(p_{r(\phi_1)})\theta(v_{\phi_1})\theta(v_{\phi_2})\alpha(p_{s(\phi_2)}) = \alpha(p_{r(\phi_1)})\theta(v_{\phi_1\phi_2})\alpha(p_{s(\phi_2)}) = T\theta(v_{\phi_1\phi_2})S. \end{aligned}$$

This relation, together with equation (7.1), shows that equation (7.2) holds, and the proof is complete. ∎

The continuity of the map γ appearing in Theorem 7.7 is a particularly vexing issue; in general, we do not know whether it is continuous. Theorem 7.7 does imply the restriction of γ to $\widehat{\mathcal{D}}$ is continuous, a fact we will use in Example 7.13.

In the following corollary, we show that in some circumstances, γ is "nearly continuous", in the sense that it is possible to alter γ by multiplying by an appropriate $\mathbb{T}$-valued cocycle to obtain a continuous isomorphism of coordinate systems. The key hypothesis, that α from Definition 7.1 extends to a $*$-isomorphism of C^*-envelopes, is in part motivated by Theorem 8.9, which shows that an *isometric* isomorphism θ between triangular algebras is $*$-extendible to their C^*-envelopes, and in particular, $\alpha = \theta|_{\mathcal{D}_1}$ is $*$-extendible to the envelopes.

COROLLARY 7.8. *If there is a $*$-isomorphism $\pi : \mathcal{C}_1 \to \mathcal{C}_2$ so that $\pi|_{\mathcal{D}_1} = \alpha$, where $\alpha = E_2 \circ \theta|_{\mathcal{D}_1}$, then the map $\delta : \mathcal{E}^1(\mathcal{A}_1) \to \mathcal{E}^1(\mathcal{A}_2)$, defined by $\phi \mapsto \phi \circ \pi^{-1}$, is an isomorphism of the coordinate systems. Moreover, there exists a cocycle $c : \mathcal{E}^1(\mathcal{A}_1) \to \mathbb{T}$ such that for every $\phi \in \mathcal{E}^1(\mathcal{A}_1)$,*

$$\delta(\phi) = c(\phi)\gamma(\phi).$$

Proof. Clearly, $\phi \mapsto \phi \circ \pi^{-1}$ is a bicontinuous isomorphism of $\mathcal{E}^1(\mathcal{C}_1)$ onto $\mathcal{E}^1(\mathcal{C}_2)$. We must show that this map sends $\mathcal{E}^1(\mathcal{A}_1)$ into $\mathcal{E}^1(\mathcal{A}_2)$. Fix $\phi \in \mathcal{E}^1(\mathcal{A}_1)$ and let γ be the map from Theorem 7.7. By Proposition 7.6,

$$r(\phi \circ \pi^{-1}) = r(\phi) \circ \pi^{-1} = r(\phi) \circ \alpha^{-1} = r(\gamma(\phi)),$$

and similarly, $s(\phi \circ \pi^{-1}) = s(\gamma(\phi))$. By Corollary 4.10, there is $c(\phi) \in \mathbb{T}$ so that $\phi \circ \pi^{-1} = c(\phi)\gamma(\phi)$, and so $\phi \circ \pi^{-1} \in \mathcal{E}^1(\mathcal{A}_2)$. Since both γ and δ are multiplicative on composable elements of $\mathcal{E}^1(\mathcal{A}_1)$, so is c, whence c is a cocycle. ∎

We now introduce a new class of algebras for which γ is continuous for bounded isomorphisms between triangular algebras in the class. This class includes those algebras $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ admits a cover by monotone G-sets with respect to $\mathcal{A}$ (see p. 57 of [26]). In the context of limit algebras, this class includes limit algebras generated by their order-preserving normalizers (see [10], [9]).

As the definition of the class does not require our standing assumptions for the section, we relax them momentarily.

DEFINITION 7.9. Let $(\mathcal{C}, \mathcal{D})$ be a C^*-diagonal and $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ be a subalgebra (not necessarily triangular). We say a normalizer $v \in \mathcal{A}$ is *algebra-preserving* if either $v\mathcal{A}v^* \subseteq \mathcal{A}$ or $v^*\mathcal{A}v \subseteq \mathcal{A}$.

This is related to the notion of order-preserving normalizers, which can be described as those $v \in \mathcal{N}_{\mathcal{D}}(\mathcal{A})$ satisfying both $v\mathcal{A}v^* \subseteq \mathcal{A}$ and $v^*\mathcal{A}v \subseteq \mathcal{A}$.

As a trivial example of an algebra-preserving normalizer that is not order preserving, let $\mathcal{C}$ be $M_4(\mathbb{C})$, $\mathcal{D}$ the diagonal matrices, and $\mathcal{A}$ the span of all upper-triangular matrix units except $e_{1,2}$. Then $v = e_{1,3} + e_{2,4}$ normalizes $\mathcal{D}$ but is not order preserving, since $ve_{3,4}v^* = e_{1,2} \notin \mathcal{A}$. However, v is algebra preserving, since $v^*\mathcal{A}v = v^*\mathcal{D}v \subset \mathcal{A}$.

LEMMA 7.10. *Let $(\mathcal{C}, \mathcal{D})$ be a C^*-diagonaland let $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ be triangular. If $\phi \in \mathcal{E}^1(\mathcal{A})$ and $v \in \mathcal{A}$ is an algebra-preserving normalizer with $\phi(v) \neq 0$, then for $x, y \in \mathcal{A}^{\#\#} \cap \ker E^{\#\#}$,*

$$\phi(xv) = \phi(vy) = \phi(xvy) = 0.$$

Proof. By Remark 4.11 and the fact that $\ker E^{\#\#} \cap \mathcal{A}^{\#\#}$ is a $\mathcal{D}$-bimodule,

$$\phi(xv) = \frac{r(\phi)(xvv^*)}{[r(\phi)(vv^*)]^{1/2}} = 0, \quad \phi(vy) = \frac{s(\phi)(v^*vy)}{[s(\phi)(v^*v)]^{1/2}} = 0,$$

as $xvv^*, v^*vy \in \ker E^{\#\#}$.

For the last equality, assume first that $v\mathcal{A}v^* \subseteq \mathcal{A}$. Then $v\mathcal{A}^{\#\#}v^* \subseteq \mathcal{A}^{\#\#}$ and, as $\ker E^{\#\#} \cap \mathcal{A}^{\#\#}$ is an ideal in $\mathcal{A}^{\#\#}$ (Theorem 5.10),

$$\phi(xvy) = \frac{r(\phi)(x(vyv^*))}{[r(\phi)(vv^*)]^{1/2}} = \frac{r(\phi)(E^{\#\#}(x(vyv^*)))}{[r(\phi)(vv^*)]^{1/2}} = 0.$$

If $v^*\mathcal{A}v \subseteq \mathcal{A}$, then, similarly, $\phi(xvy) = s(\phi)((v^*xv)y)/[s(\phi)(v^*v)]^{1/2}$ shows that $\phi(xvy) = 0$. ∎

We now reimpose the Standing Assumptions for Section 7; they remain in force through the remainder of the section.

THEOREM 7.11. *If $\phi \in \mathcal{E}^1(\mathcal{A}_1)$ and $v \in \mathcal{A}_1$ is an algebra-preserving normalizer such that $\phi(v) \neq 0$, then γ^{-1} is continuous at $\gamma(\phi)$.*

In particular, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are the closed span of their algebra-preserving normalizers, then γ is a homeomorphism.

Proof. Letting $\sigma = s(\phi)$, $\phi = \lambda[v, \sigma]$ for some $\lambda \in \mathbb{T}$, by Theorem 4.7. Without loss of generality, we may replace v by λv.

Fix $\psi \in \mathcal{E}^1(\mathcal{A}_1)$. By Proposition 7.5, $\theta^{-1}(S) = I + X$, $\theta^{-1}(T) = I + Y$ where $X, Y \in \ker E_1^{\#\#}|_{\mathcal{A}_1^{\#\#}}$. Thus,

$$\psi(\theta^{-1}(S\theta(v)T)) = \psi(v + Xv + vY + XvY) = \psi(v),$$

by Lemma 7.10. Putting $n(\psi) = \|T \cdot (\psi \circ \theta^{-1}) \cdot S\|^{-1}$, we can conclude that, for all $\psi \in \mathcal{E}^1(\mathcal{A}_1)$,

$$\gamma(\psi)(\theta(v)) = n(\psi)\psi(v). \tag{7.4}$$

Let (ϕ_λ) be a net in $\mathcal{E}^1(\mathcal{A}_1)$ such that $\gamma(\phi_\lambda) \to \gamma(\phi)$ and let $\sigma_\lambda = s(\phi_\lambda)$. Then $s(\gamma(\phi_\lambda)) \to s(\gamma(\phi))$, and so, by Proposition 7.6, $\sigma_\lambda \to \sigma$. Since $\gamma(\phi_\lambda)(\theta(v)) \to \gamma(\phi)(\theta(v)) = n(\phi)\phi(v) > 0$, by (7.4), we may assume that $\phi_\lambda(v) \neq 0$ for all λ. Thus, there exist $t_\lambda \in \mathbb{T}$ such that $\phi_\lambda = t_\lambda[v, \sigma_\lambda]$. Using (7.4) and the convergence of $\gamma(\phi_\lambda)$,

$$n(\phi_\lambda)t_\lambda[v, \sigma_\lambda](v) = \gamma(\phi_\lambda)(\theta(v)) \to \gamma(\phi)(\theta(v)) = n(\phi)\phi(v) = n(\phi)[v, \sigma](v).$$

Since $\sigma_\lambda \to \sigma$, $[v,\sigma_\lambda](v) \to [v,\sigma](v) \neq 0$, and hence $t_\lambda n(\phi_\lambda) \to n(\phi)$. Taking absolute values shows that $n(\phi_\lambda) \to n(\phi)$, and hence $t_\lambda \to 1$. Therefore, we have

$$\phi_\lambda = t_\lambda[v,\sigma_\lambda] \to [v,\sigma] = \phi. \quad ∎$$

COROLLARY 7.12. *Suppose, in addition, that each $\mathcal{A}_i$ is Dirichlet and is the norm closure of the span of its algebra-preserving normalizers. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are boundedly isomorphic if and only if they are isometrically isomorphic.*

Proof. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are boundedly isomorphic, Theorem 7.11 shows $\mathcal{E}^1(\mathcal{A}_1)$ and $\mathcal{E}^1(\mathcal{A}_2)$ are isomorphic coordinate systems. As the $\mathcal{A}_i$ are Dirichlet and regular by hypothesis, the result follows from an application of Theorem 6.11. ∎

This corollary extends Theorem 2.3 of [11], which proves the corresponding result for strongly maximal TAF algebras generated by their order-preserving normalizers. The cited theorem does somewhat more, as it shows that algebraic isomorphism implies isometric isomorphism.

In light of Theorem 7.7, it is natural to ask what implications can be drawn from the existence of the algebraic isomorphism of coordinate systems. It has been known for more than a decade that algebraic isomorphism of coordinate systems does not imply isometric isomorphism ([18], Remark on page 120). It is easily shown that the algebras in their example fail to be boundedly isomorphic and the algebraic isomorphism of coordinates they exhibit is continuous on the diagonals.

We give a somewhat different example, one in which the algebras are anti-isomorphic, have no minimal projections, and, most importantly, continuity on the diagonal can be exploited to show that the algebras are not boundedly isomorphic.

EXAMPLE 7.13. Let M_k, D_k, T_k be the algebra of $2^k \times 2^k$ matrices and the subalgebras in M_k of diagonal and upper-triangular matrices, respectively. Where necessary, we will equip an M_k with a matrix unit system $\{e_{i,j}\}$. Let A_k (respectively, B_k) be the permutation unitary in M_k that interchanges the first (respectively, last) two entries of a vector in $\mathbb{C}^{2^k}$. We consider three embeddings from M_k to M_{k+1}. First, we have π_k that sends a matrix $[a_{ij}]$ to the block matrix $[a_{ij}I_2]$ where I_2 is the 2×2 identity matrix. Let $\alpha_k = \operatorname{Ad} A_{k+1} \circ \pi_k$ and $\beta_k = \operatorname{Ad} B_{k+1} \circ \pi_k$. Then $\mathcal{M}_a = \varinjlim(M_k, \alpha_k)$ and $\mathcal{M}_b = \varinjlim(M_k, \beta_k)$ are both the 2^∞ UHF C^*-algebra. Since $\alpha_k|_{D_k} = \beta_k|_{D_k} = \pi_k|_{D_k}$, we have $\varinjlim(D_k, \alpha_k|_{D_k}) = \varinjlim(D_k, \alpha_k|_{D_k})$, which we denote D.

The operator algebras $T_a = \varinjlim(T_k, \alpha_k|_{T_k})$ and $T_b = \varinjlim(T_k, \beta_k|_{T_k})$ are anti-isomorphic. Indeed, if $\phi_k : T_k \to T_k$ sends $e_{i,j}$ to $e_{2^k+1-j,2^k+1-i}$, then $\beta_{k+1} \circ \phi_k = \phi_{k+1} \circ \alpha_k$, so the limit of the ϕ_k defines an anti-isomorphism between T_a and T_b.

Suppose that T_a and T_b were boundedly isomorphic. By Proposition 7.2, we would have a $*$-isomorphism $\alpha : D \to D$ and by Theorem 7.7, there would be an algebraic isomorphism $\gamma : \mathcal{E}^1(T_a) \to \mathcal{E}^1(T_b)$. This induces $\delta : R(T_a) \to R(T_b)$, an (algebraic) isomorphism of spectral relations, namely $\delta(|\phi|) = |\gamma(\phi)|$. Although δ need not be continuous, we know that on the diagonals of $R(T_a)$ and $R(T_b)$, δ can be identified with $\widehat{\alpha} : \widehat{D} \to \widehat{D}$, the map induced by α, and so is continuous. Moreover, by the range and source condition in Theorem 7.7, $(\sigma, \rho) \in R(T_a)$ if and only if $(\widehat{\alpha}(\sigma), \widehat{\alpha}(\rho)) \in R(T_b)$.

Let f (respectively, l) be the element of $\widehat{D}$ that equals 1 on the $e_{1,1}$ matrix unit (respectively, $e_{2^k,2^k}$ matrix unit) in each D_k. Now (f, l) is in both $R(T_a)$ and $R(T_b)$. Let $\mathcal{O}_a = \{\sigma \in \widehat{\mathcal{D}} : (f, \sigma) \in R(T_a)\}$ and define $\mathcal{O}_b$ similarly. The existence of δ implies that $\widehat{\alpha}$ maps $\mathcal{O}_a$ onto $\mathcal{O}_b$. We claim that this is impossible for a continuous α. The essence of the following argument is that every element of $\mathcal{O}_a \backslash \{f\}$ has a neighborhood N where it is maximal in $N \cap \mathcal{O}_a$, while no element of $\mathcal{O}_b \backslash \{l\}$ has such a neighborhood.

The basic neighborhoods for f are given by elements of $\widehat{D}$ that are nonzero on a $e_{1,1}$ matrix unit in some D_k. If we consider some $\sigma \in \mathcal{O}_a \backslash \{f\}$, then there is some k and some $j \in \{2, \dots, 2^k\}$ so that $\sigma = e_{j,1} \cdot f \cdot e_{1,j}$. where $e_{1,j}$ is a matrix unit in M_k. In particular, $N := \{\rho \in \widehat{D} : \rho(e_{j,j}) \neq 0\}$ is a basic neighborhood of σ. Every element of $N \cap \mathcal{O}_a$ is smaller than σ in the ordering induced by $R(T_a)$, since N is given by conjugating a basic neighborhood of f by $e_{1,j}$ and conjugation by $e_{1,j}$ reverses the diagonal ordering for pairs (f, ψ), $\psi \in \mathcal{O}_a \backslash \{f\}$. This last fact follows from considering the image of $e_{1,j}$ in M_l, $l > k$.

On the other hand, if $\rho \in \mathcal{O}_b \backslash \{l\}$, then every neighborhood of ρ contains elements $\tau \in \mathcal{O}_b$ with $(\rho, \tau) \in R(T_b)$. This follows from repeating the argument of the previous paragraph, observing that conjugation by $e_{1,j}$ preserves the $R(T_b)$-ordering for pairs (f, ψ).

Pick some $\sigma \in \mathcal{O}_a \backslash \{f, l\}$ and a neighborhood, N, of σ with all elements of $N \cap \mathcal{O}_a$ less than σ in the diagonal order. Now $\widehat{\alpha}$ maps σ to an element of $\mathcal{O}_b \backslash \{f, l\}$, call it ρ. By the previous paragraph, every neighborhood of ρ, including $\widehat{\alpha}(N)$, contains points of $\mathcal{O}_b$ greater than ρ in the $R(T_b)$-ordering. This contradicts $\widehat{\alpha}$ mapping $R(T_a)$ onto $R(T_b)$ and so T_a and T_b are not boundedly isomorphic. In fact, by the automatic continuity result of [11], there is not even an algebraic isomorphism between T_a and T_b.

We show there is an algebraic isomorphism from $R(T_a)$ to $R(T_b)$, and hence, using the continuous section, between $\mathcal{E}^1(T_a)$ and $\mathcal{E}^1(T_b)$. It suffices to construct a map $h : \widehat{D} \to \widehat{D}$ so that $(\rho, \tau) \in R(T_a)$ if and only if $(h(\rho), h(\tau)) \in R(T_b)$.

For $u \in \{a, b\}$, define

$$X_u := \{(\rho, \tau) \in R(T_u) : \rho = f\} \cup \{(\rho, \tau) \in R(T_u) : \tau = l\}.$$

Before defining h, we observe that $R(T_a) \backslash X_a = R(T_b) \backslash X_b$. To see this, first let $\widetilde{T}_k := (e_1 e_1^*)^\perp T_k (e_{2^k} e_{2^k}^*)^\perp$. If $(\rho, \tau) \in R(T_a) \backslash X_a$, there is some $p \in \mathbb{N}$ and some matrix unit $e \in \widetilde{T}_p$ such that $\rho = e \cdot \tau \cdot e^*$. For $k \geqslant p$, α_k and β_k agree on the image of e in $\widetilde{T}_k$. Thus, the image of e in T_a and the image of e in T_b induce the same partial homeomorphism of $\widehat{D}$. In particular, $(\rho,\tau) \in R(T_b)\backslash X_b$. The reverse inclusion is similar.

Thus, each of $\mathcal{O}_a\backslash\{f,l\}$ and $\mathcal{O}_b\backslash\{f,l\}$ is ordered the same way by both $R(T_a)$ and $R(T_b)$. In fact, we can show that each set is ordered like $\mathbb{Q}$ — for example, given $(\rho,\tau) \in \mathcal{O}_a\backslash\{f,l\}$ with $\rho \neq \tau$, find, as above, an off-diagonal matrix unit e and use the images of ee^* and e^*e in a later matrix algebra to show there is $\eta \in \widehat{D}$ with $(\rho,\eta),(\eta,\tau) \in R(T_a)$ and η different from ρ and τ.

Define h to be the identity map everywhere except $\mathcal{O}_a\backslash\{f,l\}$ and $\mathcal{O}_b\backslash\{f,l\}$. Since these two sets have the same order type, there is a bijection $g : \mathcal{O}_a\backslash\{f,l\} \to \mathcal{O}_b\backslash\{f,l\}$ so that $(\rho,\tau) \in \mathcal{O}_a\backslash\{f,l\}$ if and only if $(g(\rho),g(\tau)) \in \mathcal{O}_b\backslash\{f,l\}$. As $\mathcal{O}_a \setminus \{f,l\}$ and $\mathcal{O}_b \setminus \{f,l\}$ are disjoint sets, we define h on $\mathcal{O}_a\backslash\{f,l\}$ to be g and on $\mathcal{O}_b\backslash\{f,l\}$ to be g^{-1}. Using the observation above and the definitions of $\mathcal{O}_a$ and $\mathcal{O}_b$, it is straightforward to show that h has the required property.

8. STRUCTURE OF GENERAL ISOMORPHISMS

We now turn to the structure of isomorphisms of triangular algebras. First, we build on the representation results from Section 5, obtaining Theorem 8.2, which extends such an isomorphism to an isomorphism of CSL algebras. After several results about isomorphisms of CSL algebras, we obtain an analogue of a factorization result of Arveson-Josephson, Theorem 8.7. A main result of the paper is Theorem 8.8, which shows that such an isomorphism is completely bounded. Finally, we extend a result of Muhly-Qiu-Solel, Theorem 8.9, showing that an isometric isomorphism extends to $*$-isomorphism of the C^*-diagonals.

Our standing assumptions are the same as those of the previous section.

We start with a technical lemma.

LEMMA 8.1. *Suppose that* $C^*(\mathcal{A}_i)=\mathcal{C}_i$. *Given* $\sigma_2\in\widehat{\mathcal{D}}_2$, *let* $\sigma_1=\sigma_2\circ\alpha\in\widehat{\mathcal{D}}_1$, *and let* $(\pi_i,\mathcal{H}_i)$ *be the (compatible)* GNS *representations of* $(\mathcal{C}_i,\mathcal{D}_i)$ *on* $\mathcal{H}_i$ *corresponding to* σ_i. *If, for* $i=1,2$, P_i *is the support projection for* π_i, *then* $\theta(P_1)=P_2$.

Proof. By Proposition 5.4, $P_i \in \mathcal{D}_i^{\#\#}$ and $P_i = \sum_{q\in\mathcal{O}_i} q$, where

$$\mathcal{O}_i := \{q \in \mathcal{D}_i^{\#\#} : q = p_\rho \text{ for some } \rho \in \widehat{\mathcal{D}}_i \text{ such that } (\sigma_i,\rho) \in R(\mathcal{C}_i)\}.$$

Since $\mathcal{A}_i$ generates $\mathcal{C}_i$, Theorem 4.21 shows that the equivalence relation generated by $R(\mathcal{A}_i)$ is $R(\mathcal{C}_i)$. By Theorem 7.7, we have $(\rho,\sigma_2) \in R(\mathcal{A}_2)$ if and only if $(\rho\circ\alpha,\sigma_1) \in R(\mathcal{A}_1)$. Hence $(\rho,\sigma_2) \in R(\mathcal{C}_2)$ if and only if $(\rho\circ\alpha,\sigma_1) \in R(\mathcal{C}_1)$. Theorem 7.7 also shows that for $\rho_1 \in \widehat{\mathcal{D}}_1$, $\alpha(p_{\rho_1}) = p_{\rho_1\circ\alpha^{-1}}$. Therefore, $\alpha(\mathcal{O}_1)=\mathcal{O}_2$ and we obtain,

$$P_2 = \sum_{p\in\mathcal{O}_1} \alpha(p). \tag{8.1}$$

Fix $p \in \mathcal{O}_1$. Since p is a minimal projection in $\mathcal{C}_1^{\#\#}$, it is a minimal idempotent in $\mathcal{A}_1^{\#\#}$, so $\theta(p)$ is a minimal idempotent in $\mathcal{A}_2^{\#\#}$. As P_2 is a central projection in $\mathcal{C}_2^{\#\#}$, $\theta(p)P_2$ is an idempotent in $\mathcal{A}_2^{\#\#}$ and so $\theta(p)P_2$ is either $\theta(p)$ or 0. As done earlier, we again use $\widetilde{\pi}_i$ for the unique extension of π_i to $\mathcal{C}_i^{\#\#}$. Since

$$\widetilde{\pi}_2(E_2(\theta(p)P_2)) = \widetilde{\pi}_2(\alpha(p)) \neq 0,$$

we must have $\theta(p)P_2 = \theta(p)$.

The $\sigma(\mathcal{A}_1^{\#\#}, \mathcal{A}_1^{\#})$-$\sigma(\mathcal{A}_2^{\#\#}, \mathcal{A}_2^{\#})$ continuity of θ yields

$$\theta(P_1) = \theta\Big(\sum_{p\in\mathcal{O}_1} p\Big) = \sum_{p\in\mathcal{O}_1} \theta(p) = \sum_{p\in\mathcal{O}_1} P_2\theta(p) = \theta(P_1)P_2. \tag{8.2}$$

Similar considerations show that for every $p \in \mathcal{O}_1$, $\theta^{-1}(\alpha(p))P_1 = \theta^{-1}(\alpha(p))$ and

$$\theta^{-1}(P_2) = \sum_{p\in\mathcal{O}_1} P_1\theta^{-1}(\alpha(p)) = P_1\theta^{-1}(P_2). \tag{8.3}$$

Applying θ to (8.3) and using (8.2) yields $\theta(P_1) = P_2$. ∎

The support projection of a direct sum of inequivalent representations of a C^*-algebra is the sum of the support projections of the individual representations. Thus, Lemma 8.1 and Theorem 5.9 combine to produce the following result, which connects our context to the theory of CSL algebras.

When the C^*-envelope of $\mathcal{A}_i$ is $\mathcal{C}_i$, Theorem 4.21 shows that $R(\mathcal{C}_1)$ and $R(\mathcal{C}_2)$ are isomorphic as topological equivalence relations. Thus, the assumption on $\mathcal{X}_2$ below implies that $\mathcal{X}_2$ also has exactly one element from each $R(\mathcal{C}_2)$-equivalence class.

THEOREM 8.2. *Suppose that* $C^*(\mathcal{A}_i) = \mathcal{C}_i$. *Let* $\mathcal{X}_2 \subseteq \widehat{\mathcal{D}}_2$ *contain exactly one element from each* $R(\mathcal{C}_2)$ *equivalence class and let* $\mathcal{X}_1 = \{\sigma \circ \alpha : \sigma \in \mathcal{X}_2\}$. *Let* $\pi_i = \bigoplus_{\sigma\in\mathcal{X}_i} \pi_\sigma$ *be the faithful, compatible representations of* $(\mathcal{C}_i, \mathcal{D}_i)$ *as constructed in Theorem* 5.9.

If $\theta' : \pi_1(\mathcal{A}_1) \to \pi_2(\mathcal{A}_2)$ *is the map given by* $\theta'(\pi_1(a)) = \pi_2(\theta(a))$, *then* θ' *extends uniquely to a (bounded) isomorphism* $\overline{\theta} : \widetilde{\pi}_1(\mathcal{A}_1^{\#\#}) \to \widetilde{\pi}_2(\mathcal{A}_2^{\#\#})$.

Proof. Let P_i be the support projections of π_i. Using Proposition 5.4 and Lemma 8.1, we obtain $P_i \in \mathcal{D}_i^{\#\#}$ and $\theta(P_1) = P_2$. By Theorem 5.9, $\ker \widetilde{\pi}_i|_{\mathcal{A}_i^{\#\#}} = P_i^{\perp}\mathcal{A}_i^{\#\#}$ and so $\widetilde{\pi}_i$ is faithful on $P_i\mathcal{A}_i^{\#\#}$. As $\widetilde{\pi}_i|_{P_i\mathcal{A}_i^{\#\#}}$ has image $\widetilde{\pi}_i(\mathcal{A}_i^{\#\#})$, the map $\overline{\theta} : \widetilde{\pi}_1(\mathcal{A}_1^{\#\#}) \to \widetilde{\pi}_2(\mathcal{A}_2^{\#\#})$ given by $\widetilde{\pi}_1(a) \mapsto \widetilde{\pi}_2(\theta(a))$ is well defined.

Uniqueness follows from the weak* density of $\mathcal{A}_i$ in $\mathcal{A}_i^{\#\#}$. ∎

For a representation, π, of $\mathcal{C}_2$, we suspect that $\widetilde{\pi}(S)$ and $\widetilde{\pi}(T)$ are inverses of each other whenever π is a compatible representation. The next two propositions offer some evidence for this. Indeed, Theorem 8.4 proves it when π_2 is the faithful compatible atomic representations of Theorem 8.2.

PROPOSITION 8.3. *If* π *is a compatible representation of* $\mathcal{C}_2$ *on* $\mathcal{H}$, *then* $\widetilde{\pi}(TS)=I$.

Proof. From Proposition 7.5, we have $\widetilde{E}_2(\widetilde{\pi}(S)) = \widetilde{E}_2(\widetilde{\pi}(T)) = I$. Proposition 7.5 also shows that $\widetilde{\pi}(TS) \in \pi(\alpha(\mathcal{D}_1))'$, so $\widetilde{\pi}(TS) \in \pi(\mathcal{D}_2)''$ since $\pi(\mathcal{D}_2)''$ is a MASA in $\pi(\mathcal{C})''$. Since $\widetilde{E}_2$ is a homomorphism on $\pi(\mathcal{A}_2)$ it is also a homomorphism on $\widetilde{\pi}(\mathcal{A}_2^{\#\#})$. Hence,

$$\widetilde{\pi}(TS) = \widetilde{E}_2(\widetilde{\pi}(TS)) = \widetilde{E}_2(\widetilde{\pi}(T))\widetilde{E}_2(\widetilde{\pi}(S)) = I. \qquad ∎$$

THEOREM 8.4. *For π_2 as in Theorem 8.2, $\widetilde{\pi}_2(S)^{-1} = \widetilde{\pi}_2(T)$.*

Proof. We use the same notation as in Lemma 8.1, Theorem 8.2 and their proofs. Fix $\sigma_2 \in \widehat{\mathcal{D}}_2$. We claim that $\widetilde{\pi}_{\sigma_2}(S)$ is invertible and $\widetilde{\pi}_{\sigma_2}(S)^{-1} = \widetilde{\pi}_{\sigma_2}(T)$.

Applying $\widetilde{\pi}_{\sigma_2}$ to $\sum_{p\in\mathcal{O}_1} \theta(p) = P_2$ (obtained from Lemma 8.1 and the first equality in (8.2)) yields the important equality,

$$\sum_{p\in\mathcal{O}_1} \widetilde{\pi}_{\sigma_2}(\theta(p)) = I_{\mathcal{H}_{\sigma_2}}. \tag{8.4}$$

By Proposition 7.5, $S\alpha(p) = \theta(p)\alpha(p)$ for all p, and using (8.4) gives

$$\widetilde{\pi}_{\sigma_2}(S) = \sum_{p\in\mathcal{O}_1} \widetilde{\pi}_{\sigma_2}(S\alpha(p)) = \sum_{p\in\mathcal{O}_1} \widetilde{\pi}_{\sigma_2}(\theta(p)\alpha(p)).$$

A similar calculation with T gives $\widetilde{\pi}_{\sigma_2}(T) = \sum_{p\in\mathcal{O}_1} \widetilde{\pi}_{\sigma_2}(\alpha(p)\theta(p))$. Finally, we then have

$$\widetilde{\pi}_{\sigma_2}(ST) = \sum_{p\in\mathcal{O}_1} \widetilde{\pi}_{\sigma_2}(\theta(p)) = I_{\mathcal{H}_{\sigma_2}}.$$

Proposition 8.3 established $\widetilde{\pi}_{\sigma_2}(TS) = I_{\mathcal{H}_{\sigma_2}}$, and hence our claim holds.

As $\pi_2 = \bigoplus_{\sigma_2\in\mathcal{X}_2} \pi_{\sigma_2}$, the result follows. ∎

We need two structural results for CSL algebras. The factorization result, Lemma 8.6, is well known, and we only sketch its proof.

THEOREM 8.5 ([17], Theorem 2.1). *Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are* CSL*s on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and that $\pi : \mathrm{Alg}\mathcal{L}_1 \to \mathrm{Alg}\mathcal{L}_2$ is an algebra isomorphism. Then, given a* MASA *$\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H}_1)$ which is also contained in $\mathrm{Alg}\mathcal{L}_1$, there exist an invertible operator $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and an automorphism $\beta : \mathrm{Alg}\mathcal{L}_1 \to \mathrm{Alg}\mathcal{L}_1$ such that, for every $T \in \mathrm{Alg}\mathcal{L}_1$,*

$$\pi(T) = X\beta(T)X^{-1} \quad \textit{and} \quad \beta|_{\mathfrak{M}} = \mathrm{Id}_{\mathfrak{M}}.$$

Gilfeather and Moore attribute this result to Ringrose in the nest algebra case and to Hopenwasser for CSL algebras. However, Gilfeather and Moore show that β is a bounded automorphism.

LEMMA 8.6. *Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are atomic* CSL*s on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and that $X\mathrm{Alg}\mathcal{L}_1X^{-1} = \mathrm{Alg}\mathcal{L}_2$ for an invertible operator $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. There exists a unitary operator $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and an invertible operator $A \in \mathrm{Alg}\mathcal{L}_1$ such that $A^{-1} \in \mathrm{Alg}\mathcal{L}_1$ and $X = UA$.*

Proof. Regard $\mathcal{L}_i$ as a commuting family of projections in $\mathcal{B}(\mathcal{H}_i)$. Let $\mathbb{A}_i$ be the set of minimal projections in $\mathcal{L}_i''$. By hypothesis, $I = \sum_{a \in \mathbb{A}_i} a$. Define $\mu : \mathcal{L}_1 \to \mathcal{L}_2$ by $\mu(P) = [XP]$, where $[XP]$ denotes the projection onto the range of XP. Then μ is a complete lattice isomorphism. As each minimal projection in $\mathbb{A}_1$ has the form $PQ^\perp$ for some $P, Q \in \mathcal{L}_1$, we see that μ induces a map, $\mu' : \mathbb{A}_1 \to \mathbb{A}_2$ given by $\mu'(PQ^\perp) = \mu(P)\mu(Q)^\perp$. This map is well defined and bijective. Also, for each $P \in \mathcal{L}_1$,

$$P = \sum_{\{a \in \mathbb{A}_1 : a \leqslant P\}} a, \quad \mu(P) = \sum_{\{a \in \mathbb{A}_1 : a \leqslant P\}} \mu'(a). \tag{8.5}$$

Since $\mathcal{L}_i$ atomic and the isomorphism between $\mathrm{Alg}\mathcal{L}_1$ and $\mathrm{Alg}\mathcal{L}_2$ is given by an invertible element X, $\dim a\mathcal{H}_1 = \dim \mu'(a)\mathcal{H}_2$ for every $a \in \mathbb{A}_1$. For $a \in \mathbb{A}_1$, let $u_a : \mathcal{H}_1 \to \mathcal{H}_2$ be a partial isometry with $u_a u_a^* = \mu'(a)$ and $u_a^* u_a = a$. Put

$$U = \sum_{a \in \mathbb{A}_1} u_a.$$

Then U is unitary and it follows from (8.5) that $A := U^* X$ satisfies $A, A^{-1} \in \mathrm{Alg}\mathcal{L}_1$. ∎

The next two results are structural results for bounded isomorphisms of triangular algebras. The first is an analog of a result of Arveson and Josephson ([2], Theorem 4.10) appropriate to our setting. Briefly, Arveson and Josephson study a variant of the crossed product algebra associated to a homeomorphism of a locally compact Hausdorff space. If the homeomorphism has no periodic points, then results in [2] show easily that the resulting algebra is a triangular subalgebra of a C^*-diagonal (see also Section 4 of [28]). Arveson and Josephson show that a bounded isomorphism of these algebras factors into three maps, the first an isometric map arising from a homeomorphism of the underlying spaces, the second an isometric map arising from a diagonal unitary, and the third a weakly inner automorphism, i.e., one implemented by an invertible in the ultraweak closure of a suitable representation.

The main difference in the form of the Arveson-Josephson factorization and the factorization in Theorem 8.7 below is that we do not know if the approximately inner part of our factorization carries $\mathcal{A}_1$ to itself, so we need to introduce an algebra $\mathcal{A}_3$. We also remark that isomorphisms which fix the diagonal pointwise are essentially cocycle automorphisms (see Definition 6.6).

THEOREM 8.7. *Assume that $C^*(\mathcal{A}_i) = \mathcal{C}_i$. Let $\pi : \mathcal{C}_1 \to \mathcal{B}(\mathcal{H})$ be the faithful compatible representation of $\mathcal{A}_1$ constructed in Theorem* 5.9, *and let* $\mathrm{Alg}\mathcal{L}$ *be the weak* closure (in $\mathcal{B}(\mathcal{H})$) of $\pi(\mathcal{A}_1)$. Then θ factors as*

$$\theta = \tau \circ \mathrm{Ad}\, A \circ \beta \circ \pi|_{\mathcal{A}_1},$$

where $\beta \in \operatorname{Aut}(\operatorname{Alg}\mathcal{L})$ *with* $\beta(x) = x$ *for* $x \in \pi(\mathcal{D}_1)''$, $A \in \operatorname{Alg}\mathcal{L}$ *with* A *invertible and* $A^{-1} \in \operatorname{Alg}\mathcal{L}$, *and, finally, if* $\mathcal{A}_3 := (\operatorname{Ad} A \circ \beta)(\pi(\mathcal{A}_1))$, *then* $\mathcal{A}_3 \subseteq \operatorname{Alg}\mathcal{L}$ *and* $\tau : \mathcal{A}_3 \to \mathcal{A}_2$ *is an isometric isomorphism.*

Proof. Apply Theorem 8.2 and Theorem 8.5 to obtain an invertible operator $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ which implements a similarity between $\tilde{\pi}_1(\mathcal{A}_1^{\#\#})$ and $\tilde{\pi}_2(\mathcal{A}_2^{\#\#})$. Factor X as UA where U is unitary and $A, A^{-1} \in \operatorname{Alg}\mathcal{L}$, as in Lemma 8.6. Take $\tau = \operatorname{Ad} U|_{\mathcal{A}_3}$. The result follows from Theorem 8.5. ∎

We come now to a main result.

THEOREM 8.8. *Suppose that* $C^*(\mathcal{A}_i) = \mathcal{C}_i$. *If* $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ *is a bounded isomorphism, then* θ *is completely bounded and* $\|\theta\|_{\mathrm{cb}} = \|\theta\|$.

Proof. Using Theorem 8.2 (and its notation), we obtain a map $\overline{\theta} : \tilde{\pi}_1(\mathcal{A}_1^{\#\#}) \to \tilde{\pi}_2(\mathcal{A}_2^{\#\#})$. By Theorem 8.5, $\overline{\theta}$ factors as $\overline{\theta} = \operatorname{Ad} X \circ \beta$, where $X : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ is a bounded invertible operator and β is an automorphism of $\tilde{\pi}_1(A_1^{\#\#})$ fixing $\tilde{\pi}_1(\mathcal{D}_1^{\#\#})$ pointwise. By Lemma 8.6, $X = UA$ where A and A^{-1} both belong to $\tilde{\pi}_1(\mathcal{A}_1^{\#\#})$ and U is a unitary operator. Then $\operatorname{Ad} A \circ \beta$ is an automorphism of $\tilde{\pi}_1(\mathcal{A}_1^{\#\#})$ whose norm is $\|\theta\|$.

By Corollary 2.5 and Theorem 2.6 of [8], $\|\operatorname{Ad} A \circ \beta\|_{\mathrm{cb}} = \|\operatorname{Ad} A \circ \beta\| = \|\theta\|$. Thus, $\|\overline{\theta}\|_{\mathrm{cb}} = \|\theta\|$. Therefore, for θ' as in Theorem 8.2, θ' is completely bounded. Since $\|\theta'\|_{\mathrm{cb}} \leqslant \|\overline{\theta}\|_{\mathrm{cb}}$, we have $\|\theta'\|_{\mathrm{cb}} = \|\theta\|$. Noting that each π_i is a complete isometry of $\mathcal{A}_i$ onto its respective range completes the proof. ∎

Finally, we use the universal property of C^*-envelopes to generalize a result of Muhly, Qiu, and Solel, ([25], Theorem 1.1). Their result includes a corresponding statement for anti-isomorphisms, which can be deduced from the statement below by considering appropriate opposite algebras. Our generalization does not require nuclearity of the $\mathcal{C}_i$ or the second countability of the $\widehat{\mathcal{D}}_i$, as we do not use the spectral theorem for bimodules, ([24], Theorem 4.1).

This result also generalizes Corollary 6.11 for isometric θ from triangular subdiagonal algebras to general triangular subalgebras.

THEOREM 8.9. *For* $i = 1, 2$, *let* $\mathcal{A}_i$ *be a triangular subalgebra of the* C^**-diagonal* $(\mathcal{C}_i, \mathcal{D}_i)$ *and assume that* $\mathcal{A}_i$ *generates* $\mathcal{C}_i$ *as a* C^**-algebra. If* $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ *is an isometric isomorphism, then there is a unique* $*$*-isomorphism* $\pi : \mathcal{C}_1 \to \mathcal{C}_2$ *with* $\pi|_{\mathcal{A}_1} = \theta$.

Proof. By Proposition 4.21, we know that $\mathcal{C}_i$ is the C^*-envelope of $\mathcal{A}_i$. Since θ is completely isometric by Theorem 8.8, the universal property for C^*-envelopes shows that there exist unique $*$-epimorphisms $\pi_{12} : \mathcal{C}_2 \to \mathcal{C}_1$ and $\pi_{21} : \mathcal{C}_1 \to \mathcal{C}_2$ such that

$$\pi_{12} \circ i_2 = i_1 \circ \theta^{-1} \quad \text{and} \quad \pi_{21} \circ i_1 = i_2 \circ \theta,$$

where, for $k = 1, 2$, i_k is the inclusion mapping of $\mathcal{A}_k$ into $\mathcal{C}_k$. Thus, $\pi_{12} \circ \pi_{21} \circ i_1 = \operatorname{Id}|_{i_1(\mathcal{A}_1)}$, and hence $\pi_{12} \circ \pi_{21} = \operatorname{Id}|_{\mathcal{C}_1}$. Thus π_{21} is injective, and is the required $*$-isomorphism of $\mathcal{C}_1$ onto $\mathcal{C}_2$. ∎

REMARK 8.10. In Theorems 8.2, 8.7, 8.8 and 8.9, we require that $\mathcal{C}_i$ is the C^*-envelope of $\mathcal{A}_i$, which is somewhat unsatisfying, as we would prefer conditions in terms of $\mathcal{A}_i$ alone. The hypothesis that $C^*(\mathcal{A}_i) = \mathcal{C}_i$ could be removed if we knew that every C^*-algebra$\mathcal{B} \subseteq (\mathcal{C}, \mathcal{D})$ is regular, for then $(C^*(\mathcal{A}_i), \mathcal{D}_i)$ would again be a C^*-diagonal.

9. BOUNDED ISOMORPHISM TO $*$-EXTENDIBLE ISOMORPHISM

Roughly speaking, Theorem 8.9 states that an isometric isomorphism of triangular algebras is $*$-extendible. Clearly a bounded, non-isometric isomorphism between triangular algebras cannot be extended to a $*$-isomorphism of the C^*-envelopes, but it still may be the case that the C^*-envelopes of the triangular algebras are $*$-isomorphic.

QUESTION 9.1. Suppose $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ are triangular algebras such that $C^*(\mathcal{A}_i) = \mathcal{C}_i$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are boundedly isomorphic, are $\mathcal{C}_1$ and $\mathcal{C}_2$ $*$-isomorphic?

In view of Theorems 6.14 and 7.7, one might expect an affirmative answer when there exists a continuous section of $R(\mathcal{C}_i)$ into $\mathcal{E}^1(\mathcal{C}_i)$, since $R(\mathcal{A}_i)$ generates $R(\mathcal{C}_i)$ by Theorem 4.21. However, Theorem 7.7 only implies an *algebraic* isomorphism of $R(\mathcal{A}_1)$ onto $R(\mathcal{A}_2)$; to apply Theorem 6.14, we need to know that the isomorphism of $R(\mathcal{A}_1)$ onto $R(\mathcal{A}_2)$ is continuous. Establishing the continuity of the map γ from Theorem 7.7 would immediately do this.

In this section, we provide an affirmative answer to Question 9.1 for the class of triangular limit algebras. Perhaps surprisingly, our proof of this result uses K-theory. We do not know whether the isomorphism obtained satisfies the hypotheses of Corollary 7.8, so we cannot use that corollary to establish the existence of a continuous mapping between the coordinate systems or spectral relations of the triangular limit algebras.

We start with a theorem about Murray-von Neumann equivalence. The proof uses the ideas developed in the previous section. Recall that for any Banach algebra $\mathcal{B}$, two idempotents $e, f \in \mathcal{B}$ are *algebraically equivalent* if there exist $x, y \in \mathcal{B}$ such that $xy = e$ and $yx = f$.

THEOREM 9.2. *For $i = 1, 2$, suppose $(\mathcal{C}_i, \mathcal{D}_i)$ are C^*-diagonals, $\mathcal{A}_i \subseteq \mathcal{C}_i$ are triangular subalgebras, and $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is a bounded isomorphism. If $u \in \mathcal{A}_1$ is a partial isometry intertwiner, then $\theta(uu^*)$ and $\theta(u^*u)$ are algebraically equivalent in $\mathcal{C}_2$.*

To prove the theorem, we need the following well-known result. We give a proof to be self-contained.

LEMMA 9.3. *Let $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ be a concrete unital C^*-algebra. If e is an idempotent in $\mathcal{C}$, then the projection onto the range of e is $P_e := (ee^* + (1-e)^*(1-e))^{-1}ee^*$ and so belongs to $\mathcal{C}$. Moreover, if $z = I - eP_e^{\perp}$, then z is an invertible element of $\mathcal{C}$ and $zP_ez^{-1} = e$.*

Suppose e and f are idempotents in $\mathcal{C}$ and there exists an element $x \in \mathcal{C}$ so that $xe = x = fx$ and, as a map from $e\mathcal{H}$ to $f\mathcal{H}$, x is invertible. If $x = v|x|$ is the polar decomposition for x, then $v \in \mathcal{C}$.

Proof. Let p be the orthogonal projection onto the range of e. Then $pe = e$ and $ep = p$, so that $e = p + pep^{\perp}$. Thus $ee^* + (1-e)^*(1-e) = I + pep^{\perp}e^*p + p^{\perp}e^*pep^{\perp} \geqslant I$, so that $ee^* + (1-e)^*(1-e)$ is an invertible element of $\mathcal{C}$. The product of ee^* with $(1-e)^*(1-e)$ is zero, so a computation shows that $P_e := (ee^* + (1-e)^*(1-e))^{-1}ee^*$ is a self-adjoint idempotent satisfying $P_e e = e$ and $eP_e = P_e$. Hence P_e is a projection with the same range as e, so $P_e = p$.

A calculation shows that $z^{-1} = I + eP_e^{\perp}$ and $zP_e = P_e = ez$.

Turning to x, since $|x|$ is invertible from $e\mathcal{H}$ to $e\mathcal{H}$, we find $|x| + (1-P_e)$ is an invertible operator on $\mathcal{H}$. As v^*v is the projection onto $e\mathcal{H} = |x|\mathcal{H}$, $v(1-P_e) = 0$. Thus, $v = x(|x| + (1-P_e))^{-1} \in \mathcal{C}$. ∎

Proof of Theorem 9.2. By Theorem 8.2, there exists an isomorphism $\overline{\theta}$ between the atomic CSL algebras $\widetilde{\pi}_1(\mathcal{A}_1^{\#\#})$ and $\widetilde{\pi}_2(\mathcal{A}_2^{\#\#})$. Invoking Theorem 8.5, we can factor $\overline{\theta}$ as $\operatorname{Ad} X \circ \beta$ where β is an automorphism of $\widetilde{\pi}_1(\mathcal{A}_1^{\#\#})$ that fixes $\pi(\mathcal{D}_1)''$ pointwise and X is invertible.

For ease of notation, identify $\mathcal{C}_i$ with its image $\pi_i(\mathcal{C}_i)$; in particular, we write u instead of $\pi_1(u)$, etc.

Since β fixes $\mathcal{D}_1''$ and u is a partial isometry intertwiner, for every $d \in \mathcal{D}_1$, we have $\beta(u)d = udu^*\beta(u)$; and hence $u^*\beta(u) \in \mathcal{D}_1' = \mathcal{D}_1''$. Let $r = u^*\beta^{-1}(u)u^*$. We claim that

$$r\beta(u) = u^*u \quad \text{and} \quad \beta(u)r = uu^*. \tag{9.1}$$

Indeed, $r\beta(u) = u^*\beta^{-1}(u)u^*\beta(u) = u^*\beta^{-1}(uu^*\beta(u)) = u^*\beta^{-1}(\beta(u)) = u^*u$. The other equality is similar. We have $Xu^*uX^{-1} = X\beta(u^*u)X^{-1} = \theta(u^*u)$, and similarly, $Xuu^*X^{-1} = \theta(uu^*)$. Thus (9.1) yields,

$$(XrX^{-1})\theta(u) = \theta(u^*u) \quad \text{and} \quad \theta(u)(XrX^{-1}) = \theta(uu^*),$$

so that $\theta(u)$ is invertible as an operator from the range of $\theta(u^*u)$ onto the range of $\theta(uu^*)$.

Invoking the second part of Proposition 9.3, $\theta(u) = v|\theta(u)|$ with $v \in \mathcal{C}_2$. Thus, the range projections of $\theta(u^*u)$ and $\theta(uu^*)$ are algebraically equivalent, and hence $\theta(u^*u)$ and $\theta(uu^*)$ are algebraically equivalent in $\mathcal{C}_2$. ∎

REMARK 9.4. Uniqueness of inverses shows that actually $XrX^{-1} \in \mathcal{C}_2$.

DEFINITION 9.5. For $n \in \mathbb{N}$, let $(\mathcal{C}_n, \mathcal{D}_n)$ be a C^*-diagonal, where $\mathcal{C}_n$ is a unital finite dimensional C^*-algebra, and suppose each $\alpha_n : \mathcal{C}_n \to \mathcal{C}_{n+1}$ is a regular $*$-monomorphism. Theorem 4.23 shows that the inductive limit, $(\varinjlim \mathcal{C}_n, \varinjlim \mathcal{D}_n)$ is a C^*-diagonal, which we call an *AF-C^*-diagonal*. The MASA $\varinjlim \mathcal{D}$ is often called a *canonical MASA*. A *limit algebra* is a norm-closed subalgebra $\mathcal{A} \subseteq (\varinjlim \mathcal{C}_n, \varinjlim \mathcal{D}_n)$.

We reprise some of the results on limit algebras we require. For a more detailed exposition, see [37] or the introduction to [12].

Let $(\mathcal{C}, \mathcal{D}) = (\varinjlim \mathcal{C}_n, \varinjlim \mathcal{D}_n)$ be an AF C^*-diagonal, and let $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ be a limit algebra. For $v \in N_{\mathcal{D}}(\mathcal{C})$, there is some i so that we can write $v = dw$ where $d \in \mathcal{D}$ and $w \in N_{\mathcal{D}_i}(\mathcal{C}_i)$. We should also point out that, if $v \in \bigcup_{k=1}^{\infty} N_{\mathcal{D}_k}(\mathcal{C}_k)$, the sets $S(v)$ and $S(v^*)$ are closed and open, so by Propositions 3.4 and 3.3, v is also an intertwiner. Also, $\mathcal{A}_k := \mathcal{C}_k \cap \mathcal{A}$ is a finite-dimensional CSL algebra in $\mathcal{C}_k$, and $\mathcal{A}$ is the closed union of the $\mathcal{A}_k$.

The C^*-subalgebra $\mathcal{B}$ of $\mathcal{C}$ generated by $\mathcal{A}$ is again an AF-algebra containing $\mathcal{D}$, and $(\mathcal{B}, \mathcal{D})$ is again an AF-C^*-diagonal. By Proposition 4.21, $\mathcal{B}$ is the C^*-envelope of $\mathcal{A}$. Thus by replacing $\mathcal{C}$ with $\mathcal{B}$ if necessary, we may, and shall, always assume that $\mathcal{A}$ generates $\mathcal{C}$ as a C^*-algebra.

The spectrum, or fundamental relation, of $\mathcal{A}$, was first defined in [33], as pairs $(\rho, \sigma) \in \widehat{\mathcal{D}} \times \widehat{\mathcal{D}}$ for which there is a partial isometry normalizer $v \in \mathcal{A}$ with $\rho = v \cdot \sigma \cdot v^*$. In our notation, this is $R(\mathcal{A})$. The spectrum can also be described by picking systems of matrix units for each $\mathcal{C}_n$ so that matrix units in $\mathcal{C}_n$ are sums of matrix units in $\mathcal{C}_{n+1}$ and then considering those elements of $\mathcal{A}^{\#}$ that are either 0 or 1 on all matrix units. These elements of $\mathcal{A}^{\#}$ are eigenfunctionals and this description provides a continuous section from $\mathcal{R}(\mathcal{A})$ to $\mathcal{E}^1(\mathcal{A})$.

We require a technical result on normalizing idempotents in a triangular subalgebra of a *finite dimensional* C^*-diagonal. The method is similar to that of Proposition 7.5, and is in fact what led to the constructions of S and T.

LEMMA 9.6. *Suppose that $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal with $\mathcal{C}$ finite dimensional, and $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ is triangular. Let $\mathfrak{B} \subseteq \mathcal{A}$ be a (necessarily finite) Boolean algebra of commuting idempotents. Then there exists an invertible element $A \in \mathcal{A}$ such that $A\mathfrak{B}A^{-1} \subseteq \mathcal{D}$ is a Boolean algebra of idempotents.*

Proof. Let $G = \{I - 2e : e \in \mathfrak{B}\}$; then G is a finite group whose elements are all square roots of the identity. Clearly G is in bijective correspondence with $\mathfrak{B}$. Define

$$S = \frac{1}{|G|} \sum_{g \in G} E(g) g^{-1}.$$

A calculation shows that for any $g \in G$, $E(g)S = Sg$, and we have $E(S) = I$. Thus, $S = I + Y$ where $Y \in \mathcal{A}$ is nilpotent, and we conclude that S is invertible. Then for every $e \in \mathfrak{B}$, $SeS^{-1} = E(e)$, and we are done. ∎

COROLLARY 9.7. *Suppose $(\mathcal{C}, \mathcal{D})$ is an* AF *C^*-diagonal and $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ is a triangular subalgebra. If $e \in \mathcal{A}$ is an idempotent, then there exist $A \in \mathcal{A}$ such that $AeA^{-1} = E(e)$.*

Proof. Write $(\mathcal{C}, \mathcal{D}) = \varinjlim(\mathcal{C}_n, \mathcal{D}_n)$ where $(\mathcal{C}_n, \mathcal{D}_n)$ are finite dimensional C^*-diagonals, and let $\mathcal{A}_n = \mathcal{C}_n \cap \mathcal{A}$, so that $\mathcal{A} = \varinjlim \mathcal{A}_n$. By Proposition 4.5.1 of

[3], there exists $n \in \mathbb{N}$, an idempotent $f \in \mathcal{A}_n$ and an invertible element $X \in \mathcal{A}$ so that $XeX^{-1} = f$. Lemma 9.6 (applied to $\{0, e, I-e, I\}$) shows that there exists $S \in \mathcal{A}_n$ so that $SfS^{-1} = E(f)$. Thus $(SX)e(SX)^{-1} = E(f)$. Since $E|_{\mathcal{A}}$ is a homomorphism, applying E to the previous equality yields $E(e) = E(f)$, and the proof is complete. ∎

REMARK 9.8. Let $\iota : \mathcal{D} \to \mathcal{A}$ be the inclusion map of $\mathcal{D}$ into the triangular limit algebra $\mathcal{A}$. As in [31], Corollary 9.7 implies $\iota_* : K_0(\mathcal{D}) \to K_0(\mathcal{A})$ is an isomorphism of scaled dimension groups and $\iota_*^{-1} = E_*$.

We now show that an isomorphism of triangular limit algebras implies the existence of a $*$-isomorphism of the C^*-envelopes.

THEOREM 9.9. *Suppose $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is an algebra isomorphism of the triangular limit algebras $\mathcal{A}_i$. For $i = 1, 2$, let $\mathcal{C}_i$ be the C^*-envelope of $\mathcal{A}_i$ and let $h_i : \mathcal{D}_i \to \mathcal{A}_i$ and $k_i : \mathcal{A}_i \to \mathcal{C}_i$ be the inclusion maps. Then there exists a $*$-isomorphism $\tau : \mathcal{C}_1 \to \mathcal{C}_2$ such that the following diagram of scaled dimension groups commutes:*

$$\begin{array}{ccccc} K_0(\mathcal{D}_1) & \xrightarrow{h_{1*}} & K_0(\mathcal{A}_1) & \xrightarrow{k_{1*}} & K_0(\mathcal{C}_1) \\ \Big\downarrow{\scriptstyle \alpha_*} & & \Big\downarrow{\scriptstyle \theta_*} & & \Big\downarrow{\scriptstyle \tau_*} \\ K_0(\mathcal{D}_2) & \xrightarrow{h_{2*}} & K_0(\mathcal{A}_2) & \xrightarrow{k_{2*}} & K_0(\mathcal{C}_2) \end{array} . \tag{9.2}$$

Proof. Recall from [11] that algebraic isomorphisms of limit algebras are necessarily bounded.

That $\theta_* \circ h_{1*} = h_{2*} \circ \alpha_*$ follows from the fact that $\alpha = E_2 \circ \theta|_{\mathcal{D}_1}$ and Remark 9.8. For $j = 1, 2$, let $\iota_j = k_j \circ h_j$ be the inclusion mapping of $\mathcal{D}_j$ into $\mathcal{C}_j$. To complete the proof, we shall show the existence of $\tau_* : K_0(\mathcal{C}_1) \to K_0(\mathcal{C}_2)$ so that $\tau_* \circ \iota_{1*} = \iota_{2*} \circ \alpha_*$.

For $j = 1, 2$, write $\mathcal{C}_j = \varinjlim_k (\mathcal{C}_{jk}, \mathcal{D}_{jk})$ where $(\mathcal{C}_{jk}, \mathcal{D}_{jk})$ are finite dimensional C^*-inclusions. Without loss of generality, we may assume that $\mathcal{A}_{jk} := \mathcal{A}_j \cap \mathcal{C}_{jk}$ satisfies $C^*(\mathcal{A}_{jk}) = \mathcal{C}_{jk}$. Any projection $p \in \mathcal{C}_1$ is algebraically equivalent to a projection in $\mathcal{C}_{1k}$ for some k, so p is algebraically equivalent to a projection $p' \in \mathcal{D}_{1k}$. It follows that the induced mapping of scaled dimension groups, $(\iota_j)_* : K_0(\mathcal{D}_j) \to K_0(\mathcal{C}_j)$ is onto.

We claim that if p and q are projections in $\mathcal{D}_1$ which are algebraically equivalent in $\mathcal{C}_1$, then $\alpha(p)$ and $\alpha(q)$ are algebraically equivalent in $\mathcal{C}_2$, and we modify ideas of Lemma 2.2 of [36] for this. We may assume $p, q \in \mathcal{D}_{1k}$ for some k, and are algebraically equivalent in $\mathcal{C}_{1k}$. In fact, we shall show that they are equivalent via an element of $\mathcal{N}_{\mathcal{D}_{1k}}(\mathcal{C}_{1k})$.

Since p and q are algebraically equivalent, they have the same center-valued trace, and hence there exists a positive integer r and minimal projections p_i, q_i belonging to $\mathcal{D}_{1k}$ so that $p = p_1 + \cdots + p_r$, $q = q_1 + \cdots + q_r$. By relabeling if necessary, we may assume that for each i with $1 \leqslant i \leqslant r$, p_i and q_i are algebraically equivalent. Let $w_i \in \mathcal{C}_{1k}$ be a partial isometry so that $q_i w_i p_i = w_i$, $w_i^* w_i = p_i$ and $w_i w_i^* = q_i$. Since p_i and q_i are minimal projections in $\mathcal{C}_{1k}$, w_i are minimal partial isometries in $\mathcal{C}_{1k}$. Moreover, $w = \sum_{i=1}^{r} w_i \in \mathcal{N}_{\mathcal{D}_{1k}}(\mathcal{C}_{1k})$ satisfies $w^*w = p$ and $ww^* = q$.

Since $C^*(\mathcal{A}_{1k}) = \mathcal{C}_{1k}$, we can write $w_i = v_{i_1} v_{i_2} \cdots v_{i_l}$ as a finite product of partial isometries, with each v_{i_j} a normalizer of $\mathcal{D}_{1k}$ and belonging to either $\mathcal{A}_{1k}$ or $\mathcal{A}_{1k}^*$. Theorem 9.2 shows that $\alpha(v_{i_j}^* v_{i_j})$ and $\alpha(v_{i_j} v_{i_j}^*)$ are algebraically equivalent in $\mathcal{C}_2$. Thus, $\alpha(p_i)$ and $\alpha(q_i)$ are equivalent in $\mathcal{C}_2$ also, whence $\alpha(p)$ and $\alpha(q)$ are algebraically equivalent in $\mathcal{C}_2$ as desired.

Thus, if $p \in \mathcal{C}_1$ is a projection, we may define $\tau_*([p]) = (\iota_{2*} \circ \alpha_*)([p'])$, where $p' \in \mathcal{D}_1$ is any projection in $\mathcal{D}_1$ with $\iota_{1*}([p']) = [p]$. The previous paragraph shows τ_* is well-defined, and it determines an isomorphism of the scaled dimension groups $K_0(\mathcal{C}_1)$ and $K_0(\mathcal{C}_2)$ satisfying (9.9).

An application of Elliott's Theorem now completes the proof. ∎

We would very much like to know whether it is possible to choose τ in the conclusion of Theorem 9.9 so that $\tau|_{\mathcal{D}_1} = \alpha$. When this is the case, Corollary 7.8 implies the existence of a continuous isomorphism of coordinate systems, and hence spectra. The next example shows that more than the K-theoretic data provided by the conclusion of Theorem 9.9 is required to prove the existence of such a $*$-isomorphism.

EXAMPLE 9.10. Suppose, for $j = 1, 2$, that $(\mathcal{C}_j, \mathcal{D}_j)$, are AF C^*-diagonals and $i_j : \mathcal{D}_j \to \mathcal{C}_j$ are the natural inclusions. Given an isomorphism $\alpha : \mathcal{D}_1 \to \mathcal{D}_2$ with an isomorphism of scaled dimension groups, $h : K_0(\mathcal{C}_1) \to K_0(\mathcal{C}_2)$ with $i_{2*} \circ \alpha_* = h \circ i_{1*}$, there need not exist a $*$-isomorphism $\tau : \mathcal{C}_1 \to \mathcal{C}_2$ with $\tau|_{\mathcal{D}_1} = \alpha$.

To see this, we use two well-known direct systems for triangular AF algebras, the refinement system and refinement with twist system. Define $\sigma_k : M_{2^k} \to M_{2^{k+1}}$ by sending a matrix $A = [a_{ij}]$ to $\sigma_k(A) = [a_{ij} \otimes I_2]$, i.e., replacing each entry of A with the corresponding multiple of a 2×2 identity matrix. Define $\phi_k : M_{2^k} \to M_{2^{k+1}}$ to be $\operatorname{Ad} U_k \circ \sigma_k$, where U_k is the $2^{k+1} \times 2^{k+1}$ permutation unitary which is the direct sum of a $2^{k+1} - 2$ identity matrix and $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Let $\mathcal{C}_1 = \varinjlim(M_{2^k}, \sigma_k)$ and $\mathcal{C}_2 = \varinjlim(M_{2^k}, \phi_k)$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the direct limits of the diagonal matrices in each direct system. Since σ_k and ϕ_k agree on D_{2^k}, the direct limit of the identity maps $\mathrm{id} : D_{2^k} \to D_{2^k}$ defines an isomorphism, α, from $\mathcal{D}_1$ to $\mathcal{D}_2$. Now, $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic, as they are UHF C^*-algebras with the same "supernatural" number, 2^∞. Further, $K_0(\mathcal{C}_j)$ can be identified with $G = \{k/2^n : k \in \mathbb{Z}, n \in \mathbb{N}\}$, with the usual order and scale $G \cap [0, 1]$. With this identification, i_{j*} is the usual trace from $K_0(\mathcal{D}_j)$ into $G \subset \mathbb{R}$. Since α is the identity on $K_0(\mathcal{D}_j)$, we have $i_{2*} \circ \alpha_* = i_{1*}$. Thus, we can take h to be the identity map on G.

It remains to show that there is no $*$-isomorphism $\tau : \mathcal{C}_1 \to \mathcal{C}_2$ with $\tau|_{\mathcal{D}_1} = \alpha$. We argue by contradiction, so assume such a τ exists.

We may build an intertwining diagram as follows. For brevity, let C_i be M_{2^i}, $\sigma_{i,j}$ denote $\sigma_i \circ \sigma_{i+1} \circ \cdots \circ \sigma_{j-1}$, and define $\phi_{i,j}$ similarly. By Theorem 2.7 of [5], there are sequences (m_i) and (n_i) and $*$-monomorphisms (ψ_i) and (η_i) so that the following diagram commutes:

$$\begin{tikzcd}
C_1 \arrow[r, "\sigma_{1,m_1}"] \arrow[dr, "\psi_1"] & C_{m_1} \arrow[r, "\sigma_{m_1,m_2}"] \arrow[dr, "\psi_2"] & C_{m_2} \arrow[r, "\sigma_{m_2,m_3}"] \arrow[dr, "\psi_3"] & C_{m_3} \arrow[r] \arrow[dr, "\psi_4"] & \cdots \mathcal{C}_1 \arrow[d, "\tau"] \\
C_1 \arrow[r, "\phi_{1,n_1}"] & C_{n_1} \arrow[u, "\eta_1"] \arrow[r, "\phi_{n_1,n_2}"] & C_{n_2} \arrow[u, "\eta_2"] \arrow[r, "\phi_{n_2,n_3}"] & C_{n_3} \arrow[u, "\eta_3"] \arrow[r] & \cdots \mathcal{C}_2
\end{tikzcd} .$$

Since τ maps $\mathcal{D}_1$ onto $\mathcal{D}_2$, we can use this diagram to show that each ψ_k and η_k are restrictions of α and α^{-1}, respectively, and so are the identity map at the level of matrix algebras.

To obtain the contradiction, first fix $C_k = M_{2^k}$ and observe that if e is the $(1,1)$ matrix unit for C_k and f the $(2^k, 2^k)$ matrix unit in C_k, then for any $l < k$, $e\sigma_{l,k}(C_l)f = 0$ while $e\phi_{l,k}(C_l)f \neq 0$. Now consider the two maps $\lambda = \phi_{n_1,n_3}$ and $\mu = \psi_3 \circ \sigma_{m_1,m_2} \circ \eta_1$ from C_{n_1} into C_{n_3}. Letting e and f be the $(1,1)$ and $(2^{n_3}, 2^{n_3})$ matrix units in C_{n_3}, the observation implies that $e\lambda(C_{n_1})f \neq 0$. To see that $e\mu(C_{n_1})f = 0$, let e' and f' be the $(1,1)$ and $(2^{m_2}, 2^{m_2})$ matrix units in C_{m_2} and observe that $e'\sigma_{m_1,m_2}(\eta_1(C_{n_1}))f' = 0$ by the observation. Applying ϕ_3 and noting that e,f are subprojections of $\phi_3(e'),\phi_3(f')$ respectively completes the argument.

Thus, no such diagram exists, and hence no such τ exists.

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ADDED IN PROOFS. Using norming algebras, the second author has recently given a conceptually simple proof of the generalization of Theorem 8.9 ([32], Theorem 2.16).