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A GELFAND-NAIMARK THEOREM FOR SOME C^*-ALGEBRAS WITH FINITE DIMENSIONAL IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. We present a description of the C^*-algebras with Hausdorff spectrum whose all irreducible representations are finite-dimensional as algebras of continuous cross-sections of certain Banach bundles, thus generalizing a well-known result of [8] and [12] on homogeneous C^*-algebras.

KEYWORDS: *Liminal C^*-algebra, Banach bundle.*

MSC (2000): Primary 46L05; Secondary 55R05.

1. INTRODUCTION

One of the most basic and elegant results of the theory of C^*-algebras is the commutative Gelfand-Naimark theorem. Among the earliest studies that endeavoured to generalize it is the seminal paper of Fell [8] where the C^*-algebras are represented as algebras of operator fields. This method was especially successful in representing the n-homogeneous C^*-algebras as algebras of continuous cross-sections of vector bundles whose fiber is M_n, the algebra of all complex $n \times n$ matrices, and whose group is the group of all the automorphisms of M_n, a result that was obtained independently in [12]. The program was continued in [5], [11] and [7] and many other papers. A different approach was taken in [10].

Here we present a generalization of the result on n-homogeneous C^*-algebras mentioned above to C^*-algebras that have a Hausdorff spectrum and only finite dimensional irreducible representations. To this end we define and investigate in Section 2 a kind of Banach bundle whose fiber is a variable normed linear space of finite dimension. Section 3 contains the main representation result. The fibers are specialized to full algebras of complex matrices and the base spaces are required to be locally compact Hausdorff spaces. It is shown that for the kind of Banach bundles we discuss in this section, the algebras of all continuous cross-sections that vanish at infinity have only finite dimensional irreducible representations that are given by evaluations at the points of the base space. Moreover, every C^*-algebra that has only finite dimensional irreducible representations and a Hausdorff spectrum is isomorphic to such an algebra of continuous cross-sections. The section ends with a condition for the isomorphism of the algebras of continuous cross-sections considered in this paper. In Section 4 we produce a formula for the upper and lower multiplicities of an irreducible representation that were defined in [2] and [4]. One of the reasons that makes these invariants interesting is that they measure the extent to which Fell's condition can fail, see Theorem 4.6 of [2]. As shown in Proposition 10.5.8 of [6], for C^*-algebras with Hausdorff spectrum the presence of Fell's condition is necessary and sufficient for having continuous trace. Another result in the same section is a representation for the multiplier algebra of an algebra of continuous cross-sections of the kind studied here.

2. TOPOLOGICAL PRELIMINARIES

As mentioned in the introduction, in this section we are going to discuss a certain class of Banach bundles. By a bundle we mean a triple $\xi := (E, p, B)$ where E and B are Hausdorff topological spaces and p is a continuous open map of E onto B. Following J.M.G. Fell [9] (see also [7]), a Banach bundle is a bundle $\xi := (E, p, B)$ such that each fiber $p^{-1}(t)$ $(t \in B)$ has a Banach space structure over the field $\mathbb{K}$ of real ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers, compatible with the relative topology, and satisfying the following conditions:

(1) the norm is continuous as a map from E to $\mathbb{R}$;

(2) the addition is continuous as a map from $\{(x_1, x_2) \in E \times E : p(x_1) = p(x_2)\}$ to E;

(3) the scalar multiplication is continuous as a map from $\mathbb{K} \times E$ to E;

(4) if a net $\{x_i\}$ in E satisfies $\lim_i \|x_i\| = 0$ and the net $\{p(x_i)\}$ converges to some $t \in B$ then $\lim_i x_i = 0_t$, where 0_t is the null element of the Banach space $p^{-1}(t)$.

Let now B be a Hausdorff topological space with $\{B_k : 0 \leqslant k\}$ a finite or infinite non-decreasing sequence of closed subsets of B such that $B = \bigcup_{k=1} B_k$. Put $B_0 := \emptyset$, and let $\{X_k : 1 \leqslant k\}$ be a sequence of normed linear spaces of increasing finite dimensions. Denote ${}_kB := B_k \setminus B_{k-1}$, $1 \leqslant k$. We proceed now to construct a bundle whose base space is B. We begin with a set E and a surjective map $p : E \to B$. It is assumed that B has an open cover $\mathcal{V} := \{V_\iota\}$ such that each nonvoid subset ${}_kB$ is covered by a subfamily $\mathcal{V}_k$ of $\mathcal{V}$ consisting of subsets of $B \setminus B_{k-1}$ and $\mathcal{V} = \bigcup_k \mathcal{V}_k$. Moreover, we assume that whenever $V_\iota \in \mathcal{V}_k$ then

${}_kB \cap V_\iota \neq \emptyset$ and there is an injective mapping $\phi_\iota : V_\iota \times X_k \to E$ with the following properties:

(i) each fiber $\{t\} \times X_k$, $t \in V_\iota$, is mapped by ϕ_ι into $p^{-1}(t)$ and $\phi_\iota(\{t\} \times X_k) = p^{-1}(t)$ if $t \in {}_kB \cap V_\iota$;

(ii) for $V_\iota \in \mathcal{V}_k$ and $t \in V_\iota \cap {}_lB \cap V_{\iota'}$ with $V_{\iota'} \in \mathcal{V}_l$, the map

$$\phi_{\iota'}^{-1}(\phi_\iota(t, \cdot)) : X_k \to \{t\} \times X_l$$

is a linear isometry, $\{t\} \times X_l$ being endowed with the linear and norm structure given to it by X_l; this allows us to define unambigously in each fiber $p^{-1}(t)$, $t \in B$, a structure of normed linear space such that for every $V_\iota \in \mathcal{V}_k$ and every $t \in V_\iota$, $\phi_\iota(t, \cdot)$ is a linear isometry of X_k into $p^{-1}(t)$;

(iii) for $V_\iota \in \mathcal{V}_k$, $t \in V_\iota \cap {}_lB \cap V_{\iota'}$ with $V_{\iota'} \in \mathcal{V}_l$, $x_0 \in X_l$ and $x \in X_k$, the function $\psi_{x_0 x}$ defined on $V_\iota \cap V_{\iota'}$ by

$$\psi_{x_0 x}(s) := \|\phi_{\iota'}(s, x_0) - \phi_\iota(s, x)\|,$$

is continuous at t.

LEMMA 2.1. *With the above notations, if C is a bounded subset of X_k then the family of functions $\{\psi_{x_0 x} : x \in C\}$ is equicontinuous at t.*

Proof. Let $\varepsilon > 0$ and choose an ε-net $\{x_j : 1 \leqslant j \leqslant m\}$ in C. There is a neighbourhood $U \subset V_\iota \cap V_{\iota'}$ of t such that for $s \in U$ we have $|\psi_{x_0 x_j}(t) - \psi_{x_0 x_j}(s)| < \varepsilon$, $1 \leqslant j \leqslant m$. Let now $x \in C$ and choose x_j such that $\|x - x_j\| < \varepsilon$. Then, for $s \in U$, we get

$$\begin{aligned}
&|[\psi_{x_0 x}(t) - \psi_{x_0 x}(s)] - [\psi_{x_0 x_j}(t) - \psi_{x_0 x_j}(s)]| \\
&\qquad \leqslant |\|\phi_{\iota'}(t, x_0) - \phi_\iota(t, x)\| - \|\phi_{\iota'}(t, x_0) - \phi_\iota(t, x_j)\|| \\
&\qquad\qquad + |\|\phi_{\iota'}(s, x_0) - \phi_\iota(s, x_j)\| - \|\phi_{\iota'}(s, x_0) - \phi_\iota(s, x)\|| \\
&\qquad \leqslant \|\phi_\iota(t, x_j) - \phi_\iota(t, x)\| + \|\phi_\iota(s, x) - \phi_\iota(s, x_j)\| = 2\|x_j - x\| < 2\varepsilon.
\end{aligned}$$

Thus, if $s \in U$, we have $|\psi_{x_0 x}(t) - \psi_{x_0 x}(s)| < 3\varepsilon$. ∎

REMARK 2.2. One can similarly prove that the family $\{\psi_{x_0 x}\}$ obtained by letting both x_0 and x run through bounded subsets is equicontinuous at t but we will not need this stronger statement here.

We are going to define a topology on E such that $\xi := (E, p, B)$ will be a Banach bundle. We specify a local base of open neighbourhoods of $y \in p^{-1}(t)$ with $t \in {}_kB$. Let ε be a positive number and U an open neighbourhood of t contained in some $V_\iota \in \mathcal{V}_k$. Putting $(t, x) = \phi_\iota^{-1}(y)$, this local base will consist of all the sets of the form

$$W(y, U, \iota, \varepsilon) := \{y' \in E : p(y') \in U,\ \|\phi_\iota(p(y'), x) - y'\| < \varepsilon\}.$$

THEOREM 2.3. *The family of sets $W(y, U, \iota, \varepsilon)$ is the basis of a Hausdorff topology on E such that for this topology ξ is a Banach bundle, each fiber being endowed with the* *normed structure mentioned above. If $V_\iota \in \mathcal{V}_k$ then ϕ_ι is a homeomorphism of $V_\iota \times X_k$ onto its image. The restriction of ξ to ${}_kB$ is a vector bundle.*

Proof. Let t_j belong to ${}_{k_j}B$, let U_j be an open neighbourhood of t_j contained in $V_{\iota_j} \in \mathcal{V}_{k_j}$, $y_j \in p^{-1}(t_j)$, and $\varepsilon_j > 0$, $j = 1,2$. Suppose $y \in W(y_1, U_1, \iota_1, \varepsilon_1) \cap W(y_2, U_2, \iota_2, \varepsilon_2)$ and $t := p(y) \in {}_kB$, say. Let $V_\iota \in \mathcal{V}_k$ be such that $t \in V_\iota$. We have to show that there exist an open neighbourhood U of t contained in V_ι and $\varepsilon > 0$ such that $W(y, U, \iota, \varepsilon) \subset W(y_j, U_j, \iota_j, \varepsilon_j)$, $j = 1,2$. By condition (i) above, we know that there are $x_j \in X_{k_j}$ and $x \in X_k$ such that $y_j = \phi_{\iota_j}(t_j, x_j)$, $y = \phi_\iota(t, x)$. Then y satisfies $\|y - \phi_{\iota_j}(t, x_j)\| < \varepsilon_j$, j = 1,2. Choose δ so that

$$0 < \delta < \min\{\varepsilon_j - \|y - \phi_{\iota_j}(t, x_j)\| : j = 1,2\}. \tag{2.1}$$

From condition (iii) we infer that there is an open neighbourhood U of t such that $U \subset U_1 \cap U_2 \cap V_\iota$ and

$$|\|\phi_\iota(t,x) - \phi_{\iota_j}(t,x_j)\| - \|\phi_\iota(s,x) - \phi_{\iota_j}(s,x_j)\|| < \frac{\delta}{2}, \quad j = 1,2, \tag{2.2}$$

for every $s \in U$. By (2.1) and (2.2) we get for $z \in W(y, U, \iota, \delta/2)$:

$$\begin{aligned}\|z - \phi_{\iota_j}(p(z), x_j)\| &\leqslant \|z - \phi_\iota(p(z), x)\| + \|\phi_\iota(p(z), x) - \phi_{\iota_j}(p(z), x_j)\| \\ &\leqslant \frac{\delta}{2} + \|\phi_\iota(t,x) - \phi_{\iota_j}(t,x_j)\| + \frac{\delta}{2} < \varepsilon_j, \quad j = 1,2,\end{aligned}$$

hence $W(y, U, \iota, \delta/2) \subset W(y_1, U_1, \iota_1, \varepsilon_1) \cap W(y_2, U_2, \iota_2, \varepsilon_2)$.

Thus we have shown that the family of sets described above is the basis of a topology on E. Clearly this topology is Hausdorff and the projection p is continuous and open. It is also easily seen that the relative topology of each fiber $p^{-1}(t)$ coincides with the topology given to it by the Banach space structure.

Now we want to show that ξ is a Banach bundle. If $y' \in W(y, U, \iota, \varepsilon)$ then $|\|y\| - \|y'\|| < \varepsilon$ and we infer that the norm is continuous on E. Suppose $y_1, y_2 \in p^{-1}(t)$ with $t \in {}_kB$, U is an open neighbourhood of t contained in some $V_\iota \in \mathcal{V}_k$ and $\varepsilon > 0$. If $z_j \in W(y_j, U, \iota, \varepsilon)$, $j = 1,2$, then $z_1 + z_2 \in W(y_1 + y_2, U, \iota, 2\varepsilon)$ and from this follows the continuity of the addition on the subset of E on which it is defined. One checks in a similar manner that the scalar multiplication is continuous. Suppose now that the net $\{x_i\}$ in E satisfies $\lim_i \|x_i\| = 0$ and the net $\{p(x_i)\}$ converges to $t \in {}_kB$. Given $V_\iota \in \mathcal{V}_k$ to which t belongs, U an open neighbourhood of t contained in V_ι and $\varepsilon > 0$, we have eventually $\|x_i\| < \varepsilon$ and $p(x_i) \in U$. But then $x_i \in W(0_t, U, \iota, \varepsilon)$ and the last condition in the definition of a Banach bundle is verified.

Let now $V_\iota \in \mathcal{V}_k$ and $(t_0, x_0) \in (V_\iota \cap {}_lB) \times X_k$. We are going to show that ϕ_ι is continuous at (t_0, x_0). Denote $y_0 := \phi_\iota(t_0, x_0)$ and let $V_{\iota'} \in \mathcal{V}_l$ contain t_0. Put $(t_0, x_1) := \phi_{\iota'}^{-1}(y_0)$. Let U be a neighbourhood of t_0 contained in $V_{\iota'}$ and $\varepsilon > 0$. For x in the open ball C of X_k around x_0 with radius $\varepsilon/2$ we have

$$\|\phi_{\iota'}(t_0, x_1) - \phi_\iota(t_0, x)\| = \|\phi_\iota(t_0, x_0) - \phi_\iota(t_0, x)\| = \|x_0 - x\| < \varepsilon/2.$$

By Lemma 2.1 there exists a neighbourhood U_0 of t_0 included in $U \cap V_\iota$ such that $|\|\phi_{\iota'}(s,x_1) - \phi_\iota(s,x)\| - \|\phi_{\iota'}(t_0,x_1) - \phi_\iota(t_0,x)\|| < \frac{\varepsilon}{2}$ for every $s \in U_0$ and $x \in C$. Thus, if $(s,x) \in U_0 \times C$ then $\|\phi_{\iota'}(s,x_1) - \phi_\iota(s,x)\| < \varepsilon$. From this it follows that ϕ_ι maps the neighbourhood $U_0 \times C$ of (t_0, x_0) into

$$W(y_o, U, \iota', \varepsilon) = \{y \in E : p(y) \in U,\ \|\phi_{\iota'}(p(y), x_1) - y\| < \varepsilon\}$$

and the proof of the continuity of ϕ_ι at (t_0, x_0) is complete.

Let now $V_\iota \in \mathcal{V}_k$. Our next step is to show that $\phi_\iota : V_\iota \times X_k \to \phi_\iota(V_\iota \times X_k)$ is open. Suppose $U \subset V_\iota$ is open and let $D \subset X_k$ be the open ball of radius $r > 0$ and center x. Pick $y \in \phi_\iota(U \times D)$, $y = \phi_\iota(t_1, x_1)$ where $t_1 \in U$, $x_1 \in D$. Then

$$\|y - \phi_\iota(t_1, x)\| = \|\phi_\iota(t_1, x_1) - \phi_\iota(t_1, x)\| = \|x_1 - x\| < r.$$

Put $\varepsilon := (1/2)(r - \|x_1 - x\|) > 0$. We have $t_1 \in {}_lB \cap V_{\iota_1}$ for some l and some $V_{\iota_1} \in \mathcal{V}_l$. There is $x_1' \in X_l$ such that $y = \phi_{\iota_1}(t_1, x_1')$. Let $U_1 \subset V_{\iota_1} \cap U$ be an open neighbourhood of t_1 with

$$\begin{aligned} \|\phi_{\iota_1}(s, x_1') - \phi_\iota(s, x)\| &< \|\phi_{\iota_1}(t_1, x_1') - \phi_\iota(t_1, x)\| + \varepsilon \\ &= \|y - \phi_\iota(t_1, x)\| + \varepsilon = \|x_1 - x\| + \varepsilon \end{aligned} \tag{2.3}$$

for every $s \in U_1$. We assert that

$$\begin{aligned} W(y, U_1, \iota_1, \varepsilon) \cap \phi_\iota(V_\iota \times X_k) \subset \phi_\iota(U \times D) = \\ \{z \in E : p(z) \in U,\ \|z - \phi_\iota(p(z), x)\| < r\} \cap \phi_\iota(V_\iota \times X_k) \end{aligned}$$

and this will substantiate our claim about ϕ_ι being open on its image. Let $z \in W(y, U_1, \iota_1, \varepsilon) \cap \phi_\iota(V_\iota \times X_k)$. Then $p(z) \in U_1$ and $\|z - \phi_{\iota_1}(p(z), x_1')\| < \varepsilon$ which with (2.3) yields

$$\|z - \phi_\iota(p(z), x)\| \leqslant \|z - \phi_{\iota_1}(p(z), x_1')\| + \|\phi_{\iota_1}(p(z), x_1') - \phi_\iota(p(z), x)\| < \varepsilon + \|x_1 - x\| + \varepsilon = r$$

and this means that $z \in \phi_\iota(U \times D)$.

From the above we gather that whenever $V_\iota \in \mathcal{V}_k$, ϕ_ι maps $({}_kB \cap V_\iota) \times X_k$ homeomorphically onto $p^{-1}({}_kB \cap V_\iota)$. Thus the restriction of ξ to ${}_kB$ is a vector bundle whose fiber is X_k. ∎

REMARK 2.4. Clearly, a point $t \in {}_kB$ can belong to more than one $V_\iota \in \mathcal{V}_k$. However, it is a consequence of the first part of the preceeding proof that in order to get a local base of neighbourhoods for an element $y \in p^{-1}(t)$ it is enough to restrict oneself to sets $W(y, U, \iota, \varepsilon)$ defined by using a fixed $V_\iota \in \mathcal{V}_k$ that contains t.

A Banach bundle $\xi = (E, p, B)$ constructed as above will be called a scaled Banach bundle. We shall call the sequence $\{B_k\}$ of closed subsets of B its gradation and the sequence $\{X_k\}$ of finite dimensional Banach spaces will be its scale. The elements of $\mathcal{V}_k$ are the coordinate neighbourhoods for the points of ${}_kB$ and the maps ϕ_ι will be called coordinate functions of the bundle. Of course, for a triple (E, p, B), with a given gradation and a given scale one may consider various families of coordinate neighbourhoods and their associate coordinate functions. One may define a natural equivalence relation between such structures. We shall discuss at the end of this section a somewhat more general equivalence relation.

The last statement of Theorem 2.3 has a converse. Namely, in a suitable environment, every vector bundle is a scaled Banach bundle.

PROPOSITION 2.5. *Let $\xi := (E, p, B)$ be a vector bundle whose fiber X is a normed finite dimensional space and whose coordinate transformations are isometries. Then ξ is a scaled Banach bundle.*

Proof. Let $\{V_\iota\}$ be a family of coordinate neighbourhoods for ξ that covers B and let $\phi_\iota : V_\iota \times X :\rightarrow p^{-1}(V_\iota)$ be the coordinate functions. Then one can infer immediately from the definition of a vector bundle that for every two indices ι and ι' and every $x_0, x \in X$ the map $s \rightarrow \phi_{\iota'}(s, x_0) - \phi_\iota(s, x)$ from $V_\iota \cap V_{\iota'}$ to $p^{-1}(V_\iota \cap V_{\iota'})$ is continuous. ∎

Let now $\xi = (E, p, B)$, $\xi' = (E', p', B')$ be scaled Banach bundles with the same scale $\{X_k\}$, gradations $\{B_k\}$, $\{B'_k\}$, coordinate neighbourhoods $\{V_\iota\}$, $\{V'_\varkappa\}$ and coordinate functions $\{\phi_\iota\}$, $\{\phi'_\varkappa\}$, respectively. If θ is a fiberwise map from E to E' then $\widehat{\theta}$ will denote the unique map from B to B' such that $p'\theta = \widehat{\theta}p$. We shall say that ξ and ξ' are isomorphic if there is a fiberwise homeomorphism $\theta : E \rightarrow E'$ such that the restriction of θ to $p^{-1}(t)$ is a linear isometry onto $p'^{-1}(\widehat{\theta}(t))$ for each $t \in B$. In this case $\widehat{\theta}$ is a homeomorphism onto B' since p and p' are continuous and open and it respects the gradations. By abuse of language, we shall sometimes say that θ is an isomorphism from ξ to ξ'. Obviously, this is an equivalence relation between scaled Banach bundles with a given scale.

PROPOSITION 2.6. *Let $\theta : E \rightarrow E'$ be a fiberwise bijective map that is a linear isometry on each fiber. Moreover, suppose that $\widehat{\theta}$ is a homeomorphism. Then θ is an isomorphism from ξ to ξ' if and only if for every $t \in {}_kB$, $V_\iota \in \mathcal{V}_k$, $V'_\varkappa \in \mathcal{V}'_k$ with $t \in U_{\iota\varkappa} := V_\iota \cap \widehat{\theta}^{-1}(V'_\varkappa)$ and $x, x' \in X_k$, the map $s \rightarrow \|\phi_\iota(s, x) - \theta^{-1}(\phi'_\varkappa(\widehat{\theta}(s), x'))\|$ defined on $U_{\iota\varkappa}$ is continuous at t.*

Proof. If θ is an isomorphism then, by Theorem 2.3, the maps $s \rightarrow \phi_\iota(s, x)$ and $s \rightarrow \theta^{-1}(\phi'_\varkappa(\widehat{\theta}(s), x'))$ from $U_{\iota\varkappa}$ to E are continuous and the conclusion follows.

Conversely, let θ and $\widehat{\theta}$ be as in the statement of the proposition. Then $\widehat{\theta}({}_kB) = {}_kB'$ for every k. Besides, suppose that whenever $t \in {}_kB \cap U_{\iota\varkappa}$ where $V_\iota \in \mathcal{V}_k$, $V'_\varkappa \in \mathcal{V}'_k$, and $x, x' \in X_k$ we have

$$\lim_{s \rightarrow t} \|\phi_\iota(s, x) - \theta^{-1}(\phi'_\varkappa(\widehat{\theta}(s), x'))\| = \|\phi_\iota(t, x) - \theta^{-1}(\phi'_\varkappa(\widehat{\theta}(t), x'))\|.$$

Let $\varepsilon > 0$, $y' := \phi'_\varkappa(\widehat{\theta}(t), x') \in E'$ and

$$W' = W'(y', U', \varkappa, \varepsilon) := \{z' \in E' : p'(z') \in U',\ \|\phi'_\varkappa(p'(z'), x') - z'\| < \varepsilon\},$$

U' being a neighbourhood of $\widehat{\theta}(t)$ contained in $V'_\varkappa$. We shall show that $\theta^{-1}(W')$ contains a neighbourhood of $y := \theta^{-1}(y')$ and this will establish the continuity of

θ. Let $x \in X_k$ be such that $y = \phi_\iota(t, x)$. There is an open neighbourhood U of t contained in $V_\iota \cap \widehat{\theta}^{-1}(U')$ such that for $s \in U$ we have

$$\begin{aligned}\|\phi'_\varkappa(\widehat{\theta}(s), x') - \theta(\phi_\iota(s, x))\| &= \|\theta^{-1}(\phi'_\varkappa(\widehat{\theta}(s), x')) - \phi_\iota(s, x)\| \\ &< \|\theta^{-1}(\phi'_\varkappa(\widehat{\theta}(t), x')) - \phi_\iota(t, x)\| + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.\end{aligned}$$

Then, if $z_0 \in W(y, U, \iota, \varepsilon/2) := \{z \in E : p(z) \in U, \ \|\phi_\iota(p(z), x) - z\| < \varepsilon/2\}$, we have $\widehat{\theta}(p(z_0)) \in U'$ and

$$\begin{aligned}&\|\phi'_\varkappa(\widehat{\theta}(p(z_0)), x') - \theta(z_0)\| \\ &\quad \leqslant \|\phi'_\varkappa(\widehat{\theta}(p(z_0)), x') - \theta(\phi_\iota(p(z_0), x))\| + \|\theta(\phi_\iota(p(z_0), x)) - \theta(z_0)\| < \varepsilon.\end{aligned}$$

Thus $\theta(z_0) \in W'$ and we have proved that $W(y, U, \iota, \varepsilon/2) \subset \theta^{-1}(W')$. Clearly, the continuity of θ^{-1} can be established by a similar argument and the proof is complete. ∎

3. LIMINAL C^*-ALGEBRAS

Let now $\xi = (E, p, B)$ be a scaled Banach bundle whose base space B is locally compact Hausdorff and whose scale is $\{\mathbf{M}_k\}$, where $\mathbf{M}_k$ is the C^*-algebra of all $k \times k$ complex matrices (identified with the algebra of linear operators on $\mathbb{C}^k$), and for which the condition (ii) for coordinate functions is strengthened to: if $V_\iota \in \mathcal{V}_k$ and $t \in V_\iota \cap {}_lB \cap V_{\iota'}$ with $V_{\iota'} \in \mathcal{V}_l$ then $x \to \phi_{\iota'}^{-1}(\phi_\iota(t, x))$ is an injective homomorphism of $\mathbf{M}_k$ into $\{t\} \times \mathbf{M}_l$, the algebra structure of $\{t\} \times \mathbf{M}_l$ being that given to it by $\mathbf{M}_l$. Each fiber $p^{-1}(t)$ of such a scaled Banach bundle becomes in a natural manner a C^*-algebra. Such a bundle will be called a scaled C^*-bundle, cf. p. 9 of [7]. Of course, these are the bundles relevant for the description of the C^*-algebras we intend to discuss.

If $\xi = (E, p, B)$ is a scaled C^*-bundle then a continuous section of ξ is a continuous function $f : B \to E$ such that $p(f(t)) = t$ for each $t \in B$. The set $C_0(\xi)$ of all the continuous cross-sections of ξ that vanish at infinity on B is a C^*-algebra with pointwisely defined operations and the supremum norm. Moreover, $t \to p^{-1}(t)$ paired with $C(\xi)$, the $*$-algebra of all the continuous cross-sections of ξ, is a continuous field of elementary C^*-algebras. Indeed, if $y \in p^{-1}(t)$ where $t \in {}_kB$, we can find $V_\iota \in \mathcal{V}_k$ that contains t; choose an open neighbourhood U of t with compact closure $\overline{U} \subset V_\iota$ and a continuous function $f : B \to [0, 1]$ that vanishes off U and assumes the value 1 at t. With $\phi_\iota(t, x) = y$, $x \in \mathbf{M}_k$, we define $\varphi : B \to E$ by $\varphi(s) = f(s)\phi_\iota(s, x)$ for $s \in U$ and $\varphi(s) = 0_s$ if $s \notin U$, to get a continuous cross-section of ξ such that $\varphi(t) = y$. It is easily seen that $C_0(\xi)$ is the C^*-algebra defined by this continuous field of C^*-algebras as in 10.4.1 of [6].

We now proceed to construct scaled C^*-bundles related to a class of liminal C^*-algebras. Let A be a liminal C^*-algebra whose spectrum is Hausdorff and which has only finite dimensional irreducible representations. For $1 \leqslant k$ we denote

$$\widehat{A}_k := \{\pi \in \widehat{A} : \dim(\pi) \leqslant k\}, \quad \widehat{A}_0 := \emptyset; \quad {}_k\widehat{A} := \widehat{A}_k \setminus \widehat{A}_{k-1}.$$

The closed two-sided ideal of A whose spectrum is (canonically homeomorphic to) $\widehat{A} \setminus \widehat{A}_k$ is denoted by I_k. As it is well known, $\widehat{A}_k$, ${}_k\widehat{A}$ can be canonically identified with the spectra of A/I_k, I_{k-1}/I_k, respectively. It follows from Theorem 3.1 of [8] that for each $\pi \in {}_k\widehat{A}$ there exist an open neighbourhood V_π of π in $\widehat{A}$, $V_\pi \subset \widehat{A} \setminus \widehat{A}_{k-1}$, and elements $\{a^\pi_{ij} : 1 \leqslant i,j \leqslant k\}$ in A such that for each $\rho \in V_\pi$, $\{\rho(a^\pi_{ij}) : 1 \leqslant i,j \leqslant k\}$ are matrix units in $\rho(A)$ and the linear subspace of $\rho(A)$ spanned by them is a subalgebra isomorphic to $\mathbf{M}_k$.

Let now $E := \bigcup\{(\pi, \pi(A)) : \pi \in \widehat{A}\}$ and let $p : E \to \widehat{A}$ be the natural projection. Given $\pi \in {}_k\widehat{A}$, V_π, and $\{a^\pi_{ij}\}$ as above, one can define the map

$$\phi_\pi : V_\pi \times \mathbf{M}_k \to E \quad \text{by } \phi_\pi\Big(\rho, \sum \alpha_{ij} e^k_{ij}\Big) := \Big(\rho, \sum \alpha_{ij} \rho(a^\pi_{ij})\Big), \quad \{e^k_{ij} : 1 \leqslant i,j \leqslant k\}$$

being the standard matrix units of $\mathbf{M}_k$. The restriction of ϕ_π to any fiber $\{\rho\} \times \mathbf{M}_k$ is an injective homomorphism into $p^{-1}(\rho) = \{\rho\} \times \rho(A)$; in particular, the image of $\{\rho\} \times \mathbf{M}_k$ is $p^{-1}(\rho)$ whenever $\rho \in {}_k\widehat{A}$.

THEOREM 3.1. *Preserving the above notations, $\xi := (E, p, \widehat{A})$ is a scaled C^*-bundle with the gradation $\{\widehat{A}_k\}$, the coordinate neighbourhoods $\{V_\pi : \pi \in \widehat{A}\}$ and the coordinate functions $\{\phi_\pi : \pi \in \widehat{A}\}$. A is naturally isomorphic to $C_0(\xi)$.*

Conversely, if $\xi = (E, p, B)$ is a scaled C^*-bundle then the spectrum of $C_0(\xi)$ is homeomorphic to B and all its irreducible representations are given by point evaluations at the points of B.

Proof. For the first assertion of the theorem we need only to check condition (iii) of the previous section. Thus let $\pi \in {}_k\widehat{A}$, $\pi' \in V_\pi \cap {}_l\widehat{A}$, $\alpha = \sum \alpha_{ij} e^l_{ij} \in \mathbf{M}_l$ and $\beta = \sum \beta_{mq} e^k_{mq} \in \mathbf{M}_k$. It is well known that the function

$$\rho \to \psi_{\alpha\beta}(\rho) := \|\phi_{\pi'}(\rho, \alpha) - \phi_\pi(\rho, \beta)\| = \Big\| \sum \alpha_{ij} \rho(a^{\pi'}_{ij}) - \sum \beta_{mq} \rho(a^\pi_{mq}) \Big\|$$

is continuous on the Hausdorff spectrum $\widehat{A}$, cf. Corollary 3.3.9 of [6].

Now, for each $a \in A$, define $\widetilde{a} : \widehat{A} \to E$ by

$$\widetilde{a}(\pi) := (\pi, \pi(a)), \quad \pi \in \widehat{A}.$$

Then, for $a \in A$ and $\pi \in {}_k\widehat{A}$ we have $\pi(a) = \sum \alpha_{ij} \pi(a^\pi_{ij})$ for some scalars $\{\alpha_{ij}\}$. It follows from Corollary 3.3.9 of [6] that given $\varepsilon > 0$ there is an open neighbourhood $U \subset V_\pi$ of π such that, for $\rho \in U$,

$$\|\phi_\pi(\rho, \sum \alpha_{ij} e^k_{ij}) - \widetilde{a}(\rho)\| = \Big\| \sum \alpha_{ij} \rho(a^\pi_{ij}) - \rho(a) \Big\| < \varepsilon$$

and this together with Proposition 3.3.7 of [6] implies that $\widetilde{a} \in C_0(\xi)$. Thus $a \to \widetilde{a}$ is an injective homomorphism of A into $C_0(\xi)$. If $\pi_1, \pi_2 \in \widehat{A}$, $b_1 \in \pi_1(A)$,

$b_2 \in \pi_2(A)$ (with $b_1 = b_2$ if $\pi_1 = \pi_2$), then by Proposition 4.2.5 of [6] there is $a \in A$ such that $\pi_1(a) = b_1$, $\pi_2(a) = b_2$. We infer from Lemma 10.5.3 of [6] that $\{\widetilde{a} : a \in A\} = C_0(\xi)$.

The last assertion in the statement of the theorem is an immediate consequence of the discussion at the beginning of this section and Corollary 10.4.4 of [6]. ∎

REMARK 3.2. Suppose that A is a C^*-algebra that has only finite dimensional irreducible representations and whose spectrum is Hausdorff.

We use the notations introduced in the paragraph preceding Theorem 3.1. Let $(\pi, b) = (\pi, \sum \alpha_{rs}\pi(a_{rs}^{\pi})) \in p^{-1}({}_k\widehat{A})$ and suppose $\{(\pi_\iota, b_\iota)\}$ is a net in $p^{-1}(V_\pi \cap {}_k\widehat{A})$ with $b_\iota = \sum \alpha_{rs}^{\iota}\pi_\iota(a_{rs}^{\pi})$. By the definition of the topology of E we have $(\pi, b) = \lim\limits_{\iota}(\pi_\iota, b_\iota)$ if and only if $\pi = \lim\limits_{\iota}\pi_\iota$ in $\widehat{A}$ and the net of matrices $\{(\alpha_{rs}^{\iota})_{1\leqslant r,s\leqslant k}\}_\iota$ converges to the matrix $(\alpha_{rs})_{1\leqslant r,s\leqslant k}$ in $\mathbf{M}_k$. Thus the relative topology of $p^{-1}({}_k\widehat{A})$ coincides with that defined in the proof of Theorem 3.2 of [8] when the algebra under the consideration is the homogeneous C^*-algebra I_{k-1}/I_k.

Moreover, the restriction of ξ to ${}_k\widehat{A}$ is a vector bundle with fiber $\mathbf{M}_k$ whose group is the group of all the automorphisms of $\mathbf{M}_k$.

In particular, Theorem 3.1 generalizes the well known characterization of homogeneous C^*-algebras from [8] and [12].

COROLLARY 3.3. *With the same notations as in Theorem* 3.1*, if A is a C^*-algebra with continuous trace that has only finite dimensional irreducible representations then the coordinate neighbourhoods V_π and the coordinate functions can be chosen such that for each k and each $\pi \in {}_k\widehat{A}$, the restriction of ϕ_π to each fiber $\{\rho\} \times \mathbf{M}_k$, $\rho \in V_\pi$ is trace preserving.*

Conversely, if the scaled C^*-bundle ξ has the property that for each k, each $t \in {}_kB$ and each $V_\iota \in \mathcal{V}_k$ to which t belongs, the restriction of ϕ_ι to each fiber $\{s\} \times \mathbf{M}_k$ is trace preserving then $C_0(\xi)$ is a C^*-algebra with continuous trace.

Proof. Both assertions result immediately from the well known connection between Fell's condition and C^*-algebras with continuous trace, see Proposition 10.5.8 of [6]. ∎

We shall say that two scaled C^*-bundles $\xi = (E, p, B)$ and $\xi' = (E', p', B')$ are isomorphic if there is a homeomorphism θ of E onto E' that respects the fibers and whose restriction to each fiber of ξ is an isomorphism of C^*-algebras onto a fiber of ξ'.

PROPOSITION 3.4. *The scaled C^*-bundles $\xi = (E, p, B)$, $\xi' = (E', p', B')$ are isomorphic if and only if the C^*-algebras $C_0(\xi)$ and $C_0(\xi')$ are isomorphic.*

Proof. It is obvious that if two scaled C^*-bundles are isomorphic then the C^*-algebras defined by them are isomorphic so we turn now to the converse.

We shall let $\{B_k\}$, $\{B'_k\}$ be the gradations of ξ and ξ', $\{V_\iota\}$, $\{V'_\varkappa\}$ be their coordinate neighbourhoods and $\{\phi_\iota\}$, $\{\phi'_\varkappa\}$ be their coordinate functions, respectively. Let φ be an isomorphism from $C_0(\xi)$ onto $C_0(\xi')$. It induces a homeomorphism $\widetilde{\varphi}$ from B' which is the spectrum of $C_0(\xi')$, onto B, the spectrum of $C_0(\xi)$: $f(\widetilde{\varphi}(t')) = \varphi(f)(t')$ for $f \in C_0(\xi)$ and $t' \in B'$. We shall define now a map $\theta : E \to E'$ as follows: let $b \in p^{-1}(t)$ with $t = \widetilde{\varphi}(t')$ and choose $f \in C_0(\xi)$ such that $f(t) = b$; we define $\theta(b) := \varphi(f)(t')$. Since the primitive ideal of $C_0(\xi)$ determined by t is mapped injectively by φ onto the primitive ideal of $C_0(\xi')$ determined by t', θ is a well defined fiberwise injective map of E onto E'. Also, it is easily seen that its restriction to each fiber is an isomorphism of C^*-algebras.

We want to show now that θ is continuous. Let b, t, t' and f be as in the previous paragraph, that is $b = f(t)$, $t = \widetilde{\varphi}(t')$, and $\varepsilon > 0$. Thus $b' := \theta(b) = \varphi(f)(t')$. We shall suppose that $t \in {}_kB \cap V_\iota$ for some $V_\iota \in \mathcal{V}_k$ hence $t' \in {}_kB' \cap V'_\varkappa$ for some $V'_\varkappa \in \mathcal{V}'_k$. Then $b = \phi_\iota(t, x)$ and $b' = \phi'_\varkappa(t', x')$ for some $x, x' \in \mathbf{M}_k$. Let $U' \subset V'_\varkappa$ be a neighbourhood of t'; the continuity of $\varphi(f)$ allows us to suppose, without any loss of generality, that $\|\varphi(f)(s') - \phi'_\varkappa(s', x')\| < \varepsilon/3$ if $s' \in U'$. Let

$$W' := W'(b', U', \varkappa, \varepsilon) = \{c' \in E' : p'(c') \in U',\ \|\phi'_\varkappa(p'(c'), x') - c'\| < \varepsilon\}.$$

By the continuity of f there is an open neighbourhood U of t contained in $V_\iota \cap \widetilde{\varphi}(U')$ such that $\|f(s) - \phi_\iota(s, x)\| < \varepsilon/3$ if $s \in U$. Then, if

$$c_0 \in W(b, U, \iota, \varepsilon/3) = \left\{c \in E : p(c) \in U,\ \|\phi_\iota(p(c), x) - c\| < \frac{\varepsilon}{3}\right\}$$

and $c'_0 := \theta(c_0)$, we have $p'(c'_0) \in U'$ and $\|\varphi(f)(p'(c'_0)) - c'_0\| = \|\theta(f(p(c_0))) - \theta(c_0)\| = \|f(p(c_0)) - c_0\|$ since θ is an isomorphism of C^*-algebras on each fiber. Thus $\|\varphi(f)(p'(c'_0)) - c'_0\| \leqslant \|f(p(c_0)) - \phi_\iota(p(c_0), x)\| + \|\phi_\iota(p(c_0), x) - c_0\| < 2\varepsilon/3$ hence

$$\|\phi'_\varkappa(p'(c'_0), x') - c'_0\| \leqslant \|\phi'_\varkappa(p'(c'_0), x') - \varphi(f)(p'(c'_0))\| + \|\varphi(f)(p'(c'_0)) - c'_0\| < \varepsilon$$

which shows that $\theta(c_0) \in W'$. The continuity of θ^{-1} can be established in the same way and the proof is complete. ∎

4. APPLICATIONS

Throughout this section $\xi = (E, p, B)$ will be a scaled C^*-bundle, $\{B_k\}$ its gradation, $\{V_\iota\}$ its coordinate neighbourhoods and $\{\phi_\iota\}$ its coordinate functions. Each fiber of E will be identified with the algebra of all the operators on a suitable finite dimensional Hilbert space.

As a first application of the discussion in the previous section we shall establish some formulas for the multiplicities of irreducible representations of $C_0(\xi)$. Given an irreducible representation π of a C^*-algebra A, the upper and lower multiplicities $\mathrm{M_U}(\pi, \Omega)$ and $\mathrm{M_L}(\pi, \Omega)$ relative to a net Ω in $\widehat{A}$ were defined in [4]. The definitions of these quantities are unfortunately rather complicated and we shall not detail them here. Instead, we shall reproduce below a lemma from [3] that allows us to compute them rather easily. Let us mention only that both are non-negative integers or ∞ and always $\mathrm{M_L}(\pi,\Omega) \leqslant \mathrm{M_U}(\pi,\Omega)$.

LEMMA 4.1 (Lemma 5.2 of [3]). *Let A be a C^*-algebra and Ω a net in $\widehat{A}$. Let ω be a pure state associated with $\pi \in \widehat{A}$. For a natural number n the following hold:*

(i) *$n \leqslant \mathrm{M_U}(\pi,\Omega)$ if and only if there is a subnet $\{\pi_\alpha\}$ of Ω and an orthonormal set $\{\eta_1^\alpha,\dots,\eta_n^\alpha\}$ in the space of π_α, for each α, such that*

$$\omega(a) = \lim\langle \pi_\alpha(a)\eta_j^\alpha,\eta_j^\alpha\rangle,\quad 1\leqslant j\leqslant n,\ a\in A;$$

(ii) *$n \leqslant \mathrm{M_L}(\pi,\Omega)$ if and only if each subnet Ω' of Ω has a subnet $\{\pi_\alpha\}$ and there is an orthonormal set $\{\eta_1^\alpha,\dots,\eta_n^\alpha\}$ in the space of π_α, for each α, such that*

$$\omega(a) = \lim\langle \pi_\alpha(a)\eta_j^\alpha,\eta_j^\alpha\rangle,\quad 1\leqslant j\leqslant n,\ a\in A.$$

It follows from this lemma and also directly from the definitions that $\mathrm{M_U}(\pi,\Omega) > 0$ if and only if π is a cluster point of the net Ω and $\mathrm{M_L}(\pi,\Omega) > 0$ if and only if the net Ω converges to π. Moreover, if π is a cluster point of the net Ω then there is a subnet Ω' of Ω that converges to π and which satisfies $\mathrm{M_U}(\pi,\Omega) = \mathrm{M_U}(\pi,\Omega')$. Indeed, by Proposition 2.3 of [4], the net Ω has a subnet Ω_1 such that

$$\mathrm{M_L}(\pi,\Omega_1) = \mathrm{M_U}(\pi,\Omega_1) = \mathrm{M_U}(\pi,\Omega) > 0.$$

Since π is a cluster point of Ω_1 too, there is a subnet Ω' of Ω_1 that converges to π. By p. 206 of [4] we have

$$\mathrm{M_L}(\pi,\Omega_1) = \mathrm{M_L}(\pi,\Omega') = \mathrm{M_U}(\pi,\Omega') = \mathrm{M_U}(\pi,\Omega_1) = \mathrm{M_U}(\pi,\Omega).$$

Before we state the result that shows how to compute the multiplicities for $C_0(\xi)$ we shall need a simple lemma.

LEMMA 4.2. *Let $t \in {}_kB\cap V_\iota$ with $V_\iota \in \mathcal{V}_k$ and let e be a one dimensional projection in $\mathbf{M}_k$. Choose a unit vector η_s in the image of the projection $\phi_\iota(s,e)$ for each $s\in V_\iota$. Then $\lim\limits_{s\to t}\langle f(s)\eta_s,\eta_s\rangle = \langle f(t)\eta_t,\eta_t\rangle$ for every $f\in C_0(\xi)$.*

Proof. For $f\in C_0(\xi)$ let $x_f\in\mathbf{M}_k$ be such that $\phi_\iota(t,x_f) = f(t)$. Then $ex_fe = \lambda e$ for some scalar λ and so:

$$\begin{aligned}\langle\phi_\iota(s,ex_fe)\eta_s,\eta_s\rangle &= \lambda = \langle f(t)\eta_t,\eta_t\rangle,\quad s\in V_\iota;\\ |\langle f(s)\eta_s,\eta_s\rangle - \langle f(t)\eta_t,\eta_t\rangle| &= |\langle f(s)\eta_s,\eta_s\rangle - \langle\phi_\iota(s,ex_fe)\eta_s,\eta_s\rangle|\\ &\leqslant \|\phi_\iota(s,e)f(s)\phi_\iota(s,e) - \phi_\iota(s,ex_fe)\| \leqslant \|f(s)-\phi_\iota(s,x_f)\|.\end{aligned}$$

Now $\lim\limits_{s\to t}\|f(s)-\phi_\iota(s,x_f)\| = 0$ by the continuity of f and the conclusion follows. ∎

In the next result the algebra for which we compute the multiplicities is $C_0(\xi)$ for a scaled C^*-bundle as mentioned at the beginning of the present section.

THEOREM 4.3. *Suppose $t \in {}_kB \cap V_\iota$ with $V_\iota \in \mathcal{V}_k$ and e is a one dimensional projection in $\mathbf{M}_k$. Let $\Omega = \{t_\alpha\}$ be a net in B that converges to t. Then* $\mathrm{M_U}(t,\Omega) = \limsup_\alpha \mathrm{Tr}(\phi_\iota(t_\alpha, e))$ *and* $\mathrm{M_L}(t,\Omega) = \liminf_\alpha \mathrm{Tr}(\phi_\iota(t_\alpha, e))$.

Proof. There are a neighbourhood U of t whose closure is compact and included in V_ι and a section $f \in C_0(\xi)$ of ξ such that $f(s) = \phi_\iota(s,e)$ for every $s \in U$. Choose a unit vector η_t in the image of the projection $f(t)$. Of course, there is no harm in assuming that the net Ω is included in the neighbourhood U of t.

Let n be a natural number such that

$$\limsup_\alpha \mathrm{Tr}(f(t_\alpha)) = \limsup_\alpha \mathrm{Tr}(\phi_\iota(t_\alpha, e)) \geqslant n.$$

Then there is a subnet $\{t_{\alpha'}\}$ of Ω such that $\mathrm{Tr}(f(t_{\alpha'})) \geqslant n$ for each α'. Thus we can choose orthonormal vectors $\{\eta_{\alpha',j} : 1 \leqslant j \leqslant n\}$ in the image of $f(t_{\alpha'})$ for each α'. By Lemma 4.2 we have $\lim_{\alpha'} \langle f(t_{\alpha'})\eta_{\alpha',j}, \eta_{\alpha',j}\rangle = \langle f(t)\eta_t, \eta_t\rangle, 1 \leqslant j \leqslant n$. But then Lemma 4.1 implies that $\mathrm{M_U}(t,\Omega) \geqslant n$. Thus we have proved that $\mathrm{M_U}(t,\Omega) \geqslant \limsup_\alpha \mathrm{Tr}(\phi_\iota(t_\alpha, e))$.

Suppose now $\mathrm{M_U}(t,\Omega) \geqslant n$. Then, by Lemma 4.1, Ω has a subnet $\{t_{\alpha'}\}$ such that there is an orthonormal set $\{\eta_{\alpha',j} : 1 \leqslant j \leqslant n\}$ for each α' that satisfies $\langle g(t)\eta_t, \eta_t\rangle = \lim_{\alpha'} \langle g(t_{\alpha'})\eta_{\alpha',j}, \eta_{\alpha',j}\rangle$, $1 \leqslant j \leqslant n$ for every $g \in C_0(\xi)$. Then

$$\begin{aligned}\limsup_{\alpha'} \mathrm{Tr}(f(t_{\alpha'})) &\geqslant \limsup_{\alpha'} \sum_{j=1}^n \langle f(t_{\alpha'})\eta_{\alpha',j}, \eta_{\alpha',j}\rangle = \sum_{j=1}^n \lim_{\alpha'} \langle f(t_{\alpha'})\eta_{\alpha',j}, \eta_{\alpha',j}\rangle \\ &= \sum_{j=1}^n \langle f(t)\eta_t, \eta_t\rangle = n.\end{aligned}$$

So

$$\limsup_\alpha \mathrm{Tr}(\phi_\iota(t_\alpha, e)) = \limsup_\alpha \mathrm{Tr}(f(t_\alpha)) \geqslant n.$$

We have proved $\limsup_\alpha \mathrm{Tr}(\phi_\iota(t_\alpha, e)) \geqslant \mathrm{M_U}(t,\Omega)$ and this concludes the proof of the first assertion from the statement of the theorem.

The proof of the second assertion is quite similar so we shall omit it. ∎

The definitions of the upper and lower multiplicities $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ of an irreducible representation π of a C^*-algebra A were first given by R.J. Archbold in [2]. As stated in [4], they satisfy $\mathrm{M_L}(\pi,\Omega) \leqslant \mathrm{M_U}(\pi,\Omega) \leqslant \mathrm{M_U}(\pi)$ for every net Ω in $\widehat{A}$ and if π is not isolated in $\widehat{A}$ and Ω converges to π then also $\mathrm{M_L}(\pi) \leqslant \mathrm{M_L}(\pi,\Omega)$ excluding some exceptional situations (see Proposition 2.1 of [4]). Moreover, by Proposition 2.2 of [4], if $\{\pi\}$ is not open in $\widehat{A}$ then there is a net Ω in $\widehat{A} \setminus \{\pi\}$ converging to π such that $\mathrm{M_U}(\pi) = \mathrm{M_U}(\pi,\Omega)$ and $\mathrm{M_L}(\pi) = \mathrm{M_L}(\pi,\Omega)$. If $\{\pi\}$ is open in $\widehat{A}$ then $\mathrm{M_L}(\pi)$ is undefined and if in addition A is liminal then Proposition 4.11 of [2] specifies that $\mathrm{M_U}(\pi) = 1$. From these facts and Theorem 4.3 the following corollary follows immediately.

COROLLARY 4.4. *We keep the notations of Theorem* 4.3. *If t is not isolated in B then*

$$\mathrm{M_U}(t) = \limsup_{s\to t, s\neq t} \mathrm{Tr}(\phi_\iota(s,e)), \quad \mathrm{M_L}(t) = \liminf_{s\to t, s\neq t} \mathrm{Tr}(\phi_\iota(s,e)).$$

As in Corollary 3.5 of [1] and Lemma 2 of [11], we use the description of the class of liminal C^*-algebras we investigated to get some information about their multiplier algebras.

THEOREM 4.5. *The multiplier algebra $M(C_0(\xi))$ of $C_0(\xi)$ is isomorphic (by an isomorphism that reduces to the identity on $C_0(\xi)$) with the C^*-algebra of all the bounded sections h of ξ that have the following property: for every $t \in {}_kB$ there is $V_\iota \in \mathcal{V}_k$ that contains t such that $s \to h(s)\phi_\iota(s,\mathbf{1}_k)$ and $s \to \phi_\iota(s,\mathbf{1}_k)h(s)$ are continuous at t.*

Proof. Every (irreducible) representation of $C_0(\xi)$ defined by some $t \in B$ has a unique extension to a representation of $M(C_0(\xi))$ whose image is also the algebra $p^{-1}(t)$. If all these representations vanish on some element of $M(C_0(\xi))$ then that element is orthogonal on $C_0(\xi)$ hence is the null element. Thus we get naturally an isomorphism of $M(C_0(\xi))$ onto some C^*-algebra of bounded sections of ξ and we shall use this isomorphism to identify these two C^*-algebras.

It is easily seen that a multiplier of $C_0(\xi)$ must have the continuity property described in the statement of the theorem. Let us now suppose that h is a bounded section of ξ which satisfies the above condition. Given $f \in C_0(\xi)$ we want to show that hf is a continuous section. Let $t \in {}_kB$ and take V_ι as above. Let $\varepsilon > 0$. We have

$$\phi_\iota(t,x_1) = h(t), \quad \phi_\iota(t,x_2) = f(t)$$

for some $x_1, x_2 \in \mathbf{M}_k$ and $\phi_\iota(t,x_1x_2) = h(t)f(t)$. There is a neighbourhood U of t such that for $s \in U$ we have

$$\|\phi_\iota(s,x_1) - h(s)\phi_\iota(s,\mathbf{1}_k)\| < \varepsilon, \quad \|\phi_\iota(s,x_2) - f(s)\| < \varepsilon.$$

Thus, for $s \in U$ we get

$$\begin{aligned}
&\|\phi_\iota(s,x_1x_2) - h(s)f(s)\| \\
&\leqslant \|\phi_\iota(s,x_1)\phi_\iota(s,x_2) - h(s)\phi_\iota(s,\mathbf{1}_k)\phi_\iota(s,x_2)\| + \|h(s)\phi_\iota(s,\mathbf{1}_k)\phi_\iota(s,x_2) - h(s)f(s)\| \\
&\leqslant \|\phi_\iota(s,x_1)\mathrm{i}{-}h(s)\phi_\iota(s,\mathbf{1}_k)\|(\|f\|{+}\varepsilon){+}\|h\|\|\phi_\iota(s,x_2){-}f(s)\| \leqslant (\|f\|{+}\|h\|{+}\varepsilon)\varepsilon
\end{aligned}$$

and the continuity of hf at t is established. The continuity of fh can be proved in the same way. ∎

COROLLARY 4.6. *Suppose that for each k and each $t \in {}_kB$ there is $V_\iota \in \mathcal{V}_k$ that contains it and a neighbourhood U of t included in V_ι such that $\phi_\iota(s,\cdot)$ is a unital homomorphism for $s \in U$. Then $M(C_0(\xi))$ is isomorphic with the C^*-algebra $C_\mathrm{b}(\xi)$ of all the bounded continuous cross-sections of ξ. Conversely, if ξ does not have this property then $C_\mathrm{b}(\xi)$ is a proper C^*-subalgebra of $M(C_0(\xi)$.*

Proof. Indeed, if there is $t_0 \in B$ that does not satisfies the above condition then the cross-section of ξ whose value at every $t \in B$ is the unit of $p^{-1}(t)$ is a multiplier but does not belong to $C_b(\xi)$. ∎

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