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# HILBERT $C^*$-MODULES AND $*$-ISOMORPHISMS

MOHAMMAD B. ASADI

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ABSTRACT. In this study, it is shown that if $E_1$ and $E_2$ are Hilbert $C^*$-modules over a $C^*$-algebra of (not necessarily all) compact operators and $\Phi$ is a $*$-isomorphism between $C^*$-algebras $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$, then $\Phi$ is in the form $\mathrm{Ad}U$, for some unitary operator $U : E_1 \to E_2$, and so $E_1$ and $E_2$ are isomorphic as Hilbert $C^*$-modules. This implies that if $C^*$-algebras $\mathcal{A}$ and $K(H)$ are strongly Morita equivalent then the Picard group of $\mathcal{A}$ is trivial.

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Let $H$ and $H'$ be Hilbert spaces and $u : H \to H'$ a unitary operator. Then the map

$$\mathrm{Ad}u : B(H) \to B(H'), \quad v \mapsto uvu^*$$

is a $*$-isomorphism. In fact, all $*$-isomorphisms between $B(H)$ and $B(H')$ are obtained in this way. Therefore, whenever $B(H)$ and $B(H')$ are $*$-isomorphic, then the Hilbert spaces $H$ and $H'$ are isomorphic.

Generally, this is not valid for Hilbert $C^*$-modules. For instance, if $\mathcal{A}$ is the hyperfinite type $\mathrm{II}_1$ $\mathrm{W}^*$-factor, then Hilbert $C^*$-modules $E_1 = \mathcal{A}$ and $E_2 = \mathcal{A}^2$, with the usual $\mathcal{A}$-valued inner products, are not isomorphic as Hilbert $C^*$-modules, but the $C^*$-algebras $\mathcal{K}(E_1)$ and $\mathcal{K}(E_2)$ (and so $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$) are $*$-isomorphic to $\mathcal{A}$ [5].

Now we are going to show that this is true for Hilbert $C^*$-modules over $C^*$-algebras of compact operators on some Hilbert space.

Let us denote by $\mathcal{L}(E)$ the set of adjointable operators on Hilbert $C^*$-module $E$, and denote by $\mathcal{K}(E)$ the set of compact operators in $\mathcal{L}(E)$.

As a result derived from the main theorem, if $E, F$ are $K(H)$-imprimitivity bimodules, then we have $\mathcal{K}(E) \cong K(H) \cong \mathcal{K}(F)$, and so $E \cong F$ by Corollary 1. Therefore, one can conclude the previously known fact that the Picard group of any $C^*$-algebra of compact operators has to be trivial [4].

Moreover, one can conclude that if $C^*$-algebras $\mathcal{A}$ and $K(H)$ are strongly Morita equivalent, then all $\mathcal{A}$-$K(H)$-imprimitivity bimodules are isomorphic to each other. Also, every automorphism of $\mathcal{A}$ is a generalized inner automorphism, i.e., for every $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ there exists a unitary $u \in M(\mathcal{A})$ such that $\varphi = \mathrm{Ad}u$, since if $E$ is an $\mathcal{A}$-$K(H)$-imprimitivity bimodule then $\mathcal{A} \cong \mathcal{K}_{K(H)}(E)$ [3]. Consequently, it follows from Theorem 3.9 in [4] that the Picard group of $\mathcal{A}$ is trivial.

Bakić and Guljaš in [2] discussed a concept of an orthonormal basis for Hilbert $C^*$-modules and proved that each Hilbert $C^*$-module $(E, \langle\cdot,\cdot\rangle)$ over a $C^*$-algebra $\mathcal{A}$ of (not necessarily all) compact operators on a Hilbert space $H$ possesses an orthonormal basis, and consequently the $C^*$-algebra $\mathcal{L}(E)$ is naturally represented on a Hilbert space contained in $E$. In fact, let $e_0 \in K(H)$ be a minimal projection and let $E_{e_0} = Ee_0 = \{xe_0 : x \in E\}$.

Observe that $E_{e_0}$ is an invariant subspace for all $K(H)$-linear operators on $E$, and $E_{e_0}$ is a Hilbert space with the inner product $(xe_0, ye_0) = \mathrm{tr}(\langle xe_0, ye_0\rangle)$ for all $x, y \in E$. Also, there exists an orthonormal basis $(v_\lambda)$ for $E$ such that $\langle v_\lambda, v_\lambda\rangle = e_0$, for all $\lambda$, and therefore $E_{e_0}$ contains an orthonormal basis for $E$. This implies that $E_{e_0}$ generates a dense submodule in $E$. The main results of [2] are as follows (Theorems 5 and 6 in [2]):

Let $E$ be a Hilbert $K(H)$-module and $e_0$ be a minimal projection in $K(H)$. Then the map $\Psi : \mathcal{L}(E) \to B(E_{e_0})$, $\Psi(A) = A|_{E_{e_0}}$ is a $*$-isomorphism of $C^*$-algebras. Also, $A \in \mathcal{K}(E)$ if and only if $\Psi(A) = A|_{E_{e_0}}$ is a compact operator on Hilbert space $E_{e_0}$. Therefore $\Psi : \mathcal{K}(E) \to K(E_{e_0})$, $\Psi(A) = A|_{E_{e_0}}$ is a $*$-isomorphism, too.

MAIN THEOREM. *Let $(E_1, \langle\cdot,\cdot\rangle_1)$ and $(E_2, \langle\cdot,\cdot\rangle_2)$ be Hilbert $K(H)$-modules and $\Phi : \mathcal{L}(E_1) \to \mathcal{L}(E_2)$ be a $*$-isomorphism. Then there is a unitary $U : E_1 \to E_2$ such that $\Phi = \mathrm{Ad}U$, and so $E_1$ and $E_2$ are isomorphic as Hilbert $C^*$-modules.*

*Proof.* Let $e_0$ be a nonzero minimal projection in $K(H)$. As mentioned in the above statements there exist Hilbert spaces $(E_{1e_0}, (\cdot,\cdot)_1)$ and $(E_{2e_0}, (\cdot,\cdot)_2)$ and $*$-isomorphisms

$$\Psi_i : \mathcal{L}(E_i) \to B(E_{ie_0}),\ \Psi_i(A) = A|_{E_{ie_0}}, \quad \text{for } i = 1, 2.$$

We consider the linear operator $\Phi' : B(E_{1e_0}) \to B(E_{2e_0})$ given by $\Phi' = \Psi_2\Phi\Psi_1^{-1}$. Clearly, $\Phi'$ is a $*$-isomorphism. Therefore, there exists a unitary $u : E_{1e_0} \to E_{2e_0}$ such that $\Phi' = \mathrm{Ad}u$. Since each Hilbert $K(H)$-module possesses an orthonormal basis, we can choose an orthonormal basis $(v_\lambda)_{\lambda\in I}$ for $E_1$ such that $\langle v_\lambda, v_\lambda\rangle_1 = e_0$. Then $v_\lambda \in E_{1e_0}$, because $v_\lambda = v_\lambda\langle v_\lambda, v_\lambda\rangle_1 = v_\lambda e_0$. Then $(v_\lambda)_{\lambda\in I}$ is an orthonormal basis for Hilbert space $E_{1e_0}$. Let $w_\lambda = u(v_\lambda)$, for all $\lambda \in I$, then $(w_\lambda)_{\lambda\in I}$ is an orthonormal basis for Hilbert space $E_{2e_0}$ such that $\langle w_\lambda, w_\lambda\rangle_2 = e_0$, since $u$ is unitary. Now we can define a linear map $U : E_1 \to E_2$ by letting

$$U(x) = \sum_\lambda w_\lambda\langle v_\lambda, x\rangle_1 \quad \text{for all } x \in E_1.$$

Obviously $U$ is a $K(H)$-linear map. Also, we have

$$\langle U(x),z\rangle_2 = \Big\langle \sum_\lambda w_\lambda\langle v_\lambda,x\rangle_1, z\Big\rangle_2 = \sum_\lambda \langle x,v_\lambda\rangle_1\langle w_\lambda,z\rangle_2 = \Big\langle x,\sum_\lambda v_\lambda\langle w_\lambda,z\rangle_2\Big\rangle_1$$

for all $x\in E_1$ and $z\in E_2$. Therefore $U$ is adjointable and $U^*: E_2\to E_1$ will be given by

$$U^*(z)=\sum_\lambda v_\lambda\langle w_\lambda,z\rangle_1$$

for all $z\in E_2$. Also we have

$$\begin{aligned}\langle U(x),U(y)\rangle_2 &= \Big\langle \sum_\lambda w_\lambda\langle v_\lambda,x\rangle_1,\sum_\lambda w_\lambda\langle v_\lambda,y\rangle_1\Big\rangle_2 = \sum_\lambda\langle x,v_\lambda\rangle_1\langle w_\lambda,w_\lambda\rangle_2\langle v_\lambda,y\rangle_1\\ &= \sum_\lambda\langle x,v_\lambda\rangle_1\langle v_\lambda,y\rangle_1 = \langle x,y\rangle_1\end{aligned}$$

where in the third equality, the fact was used that $v_\lambda = v_\lambda\langle v_\lambda,v_\lambda\rangle = v_\lambda\langle w_\lambda,w_\lambda\rangle$ for all $\lambda\in I$ and the last equality holds by Theorem 1 of [2]. Then $U$ is an isometry. In a similar way, it can be shown that $U^*$ is an isometry, too. Therefore $U$ is a unitary operator between $E_1$ and $E_2$. Also, we can show that $U(xe_0)=u(xe_0)$ for all $x\in E_1$. It can be shown that $U^*(ze_0)=u^*(ze_0)$, for all $z\in E_2$, too. Therefore $(UAU^*)|_{E_{2e_0}}(ze_0) = uA|_{E_{1e_0}}u^*(ze_0)$, for all $A\in\mathcal{L}(E_1)$ and $z\in E_2$.

Finally, assume that $A\in\mathcal{L}(E_1)$ and $z\in E_2$. Then

$$\begin{aligned}\Psi_2\Phi(A)(ze_0) &= \Phi'\Psi_1(A)(ze_0) = \Phi'(A|_{E_{1e_0}})(ze_0)\\ &= u(A|_{E_{1e_0}})u^*(ze_0) = UAU^*|_{E_{2e_0}}(ze_0) = \Psi_2(UAU^*)(ze_0).\end{aligned}$$

Hence $\Phi(A)(ze_0)=(UAU^*)(ze_0)$ and this implies that $\Phi(A)=UAU^*$, since $E_{2e_0}$ generates a dense submodule in $E_2$, and $\Phi(A)$ and $UAU^*$ are bounded module maps. ∎

Assume that $\mathcal{A}$ is a $C^*$-algebra of (not necessarily all) compact operators. It is well known that $\mathcal{A}$ must be in the form $\mathcal{A}=\bigoplus_{j\in J}K(H_j)$ for a family $\{H_j\}_{j\in J}$ of Hilbert spaces.

Also, it is well known that every $*$-isomorphism between $K(H_1)$ and $K(H_2)$ is in the form of Ad$u$, for some unitary operator $u: H_1\to H_2$.

Now, by Theorems 8 and 9 in [2], the following result can be obtained:

COROLLARY 1. *Let $(E_1,\langle\cdot,\cdot\rangle_1)$ and $(E_2,\langle\cdot,\cdot\rangle_2)$ be Hilbert $C^*$-modules over a $C^*$-algebra $\mathcal{A}$ of (not necessarily all) compact operators. Then for every $*$-isomorphism $\Phi:\mathcal{L}(E_1)\to\mathcal{L}(E_2)$ there is a unitary $U: E_1\to E_2$ such that $\Phi=$ Ad$U$. Also, if $\Psi:\mathcal{K}(E_1)\to\mathcal{K}(E_2)$ is a $*$-isomorphism, then $\Psi$ is in the form Ad$U$, for some unitary operator $U: E_1\to E_2$, and so $E_1$ and $E_2$ are isomorphic as Hilbert $C^*$-modules.*

The following interesting result can be concluded from Theorem 3.9 in [4] and Lemma 8.1.15 in [3].

COROLLARY 2. *If $C^*$-algebra $\mathcal{A}$ is strongly Morita equivalent to $C^*$-algebra of compact operators, then every automorphism of $\mathcal{A}$ is a generalized inner automorphism. Consequently, the Picard group of $\mathcal{A}$ is trivial.*

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MOHAMMAD B. ASADI, DEPARTMENT OF MATHEMATICAL SCIENCES, SHAHED UNIVERSITY,TEHRAN, IRAN

*E-mail address*: mb.asadi@gmail.com

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