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RELATIVE K-HOMOLOGY AND NORMAL OPERATORS

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ABSTRACT. Let A be a C^*-algebra, $J \subset A$ a C^*-subalgebra, and let B be a stable C^*-algebra. Under modest assumptions we organize invertible C^*-extensions of A by B that are trivial when restricted onto J to become a group $\mathrm{Ext}_J^{-1}(A, B)$, which can be computed by a six-term exact sequence which generalizes the excision six-term exact sequence in the first variable of KK-theory. Subsequently we investigate the relative K-homology which arises from the group of relative extensions by specializing to abelian C^*-algebras. It turns out that this relative K-homology carries substantial information also in the operator theoretic setting from which the BDF theory was developed and we conclude the paper by extracting some of this information on approximation of normal operators.

KEYWORDS: *C^*-algebra relative extension, K-homology, normal operator.*

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1. INTRODUCTION

Let X be a compact metric space. By results of Brown, Douglas and Fillmore, [2], the K-homology of X is realized by $\mathrm{Ext}(X)$, the equivalence classes of unital and essential extensions of $C(X)$ by the compact operators $\mathbb{K}$ on a separable infinite dimensional Hilbert space H, or equivalently, the equivalence classes of unital and injective $*$-homomorphisms $C(X) \to Q$, where $Q = \mathbb{L}(H)/\mathbb{K}$ is the Calkin algebra. This discovery came out of questions and problems related to essential normal operators, and it led quickly to the development of a vast new area of mathematics which combines operator theory with algebraic topology. In particular, the BDF-theory was generalized by Kasparov in form of KK-theory, which has proven to be a powerful tool in the theory of operator algebras as well as in algebraic topology.

It is the purpose of the present paper to develop a relative theory in this context. The point of departure here is a generalization of the six-term exact sequence of extension theory which relates the group of extensions of a unital C^*-algebra to the group of unital extensions. This sequence was discovered by Skandalis, cf.

[14], and a construction of it was presented in [10]. It is the latter construction which we here generalize to get a relative extension theory. Subsequently we investigate the relative K-homology which arises from it by specializing to abelian C^*-algebras. It turns out that relative K-homology carries substantial information also in the operator theoretic setting from which the BDF-theory departed, cf. [1], and we conclude the paper by extracting some of this information.

In the remaining part of this introduction we give a more detailed account of the content of the paper. Let A be a C^*-algebra, $J \subseteq A$ a C^*-subalgebra, and let B be a stable C^*-algebra. Under modest assumptions we organize the C^*-extensions of A by B that are trivial when restricted onto J to become a semi-group $\mathrm{Ext}_J(A,B)$ which is the semi-group $\mathrm{Ext}(A,B)$ of Kasparov, [8], when $J = \{0\}$. The group $\mathrm{Ext}_J^{-1}(A,B)$ of invertible elements in $\mathrm{Ext}_J(A,B)$ can be effectively computed by a six-term exact sequence which generalizes the excision six-term exact sequence in the first variable of KK-theory, and it turns out that there is a natural identification $\mathrm{Ext}_J^{-1}(A,B) = KK(C_i,B)$, where C_i is the mapping cone of the inclusion $i : J \to A$. Thus, as an abstract group, the relative extension group is a familiar object, and the six-term exact sequence which calculates it is a version of the Puppe exact sequence of Cuntz and Skandalis; [5]. But the realization of $KK(C_i,B)$ as a relative extension group has non-trivial consequences already in the set-up from which KK-theory developed, namely the setting of (essential) normal operators, and the second half of the paper is devoted to the extraction of the information which the relative extension group contains about normal operators when specialized to the case where $B = \mathbb{K}$ and X and Y are compact metric spaces, and $f : X \to Y$ is a continuous surjection giving rise to an embedding of $J = C(Y)$ into $A = C(X)$. In this setting $\mathrm{Ext}_J(A,\mathbb{K})$ is a group, and we denote it by $\mathrm{Ext}_{Y,f}(X)$. As an abstract group this is the even K-homology of the mapping cone of f, and the above mentioned six-term exact sequence takes the form

(1.1)
$$\begin{tikzcd}
\mathrm{Ext}_{Y,f}(X) \arrow[r] & K_1(X) \arrow[r, "f_*"] & K_1(Y) \arrow[d] \\
K_0(Y) \arrow[u] & K_0(X) \arrow[l, "f_*"'] & \mathrm{Ext}_{SY,Sf}(SX) \arrow[l]
\end{tikzcd}$$

where S is the reduced suspension. An element of $\mathrm{Ext}_{Y,f}(X)$ consists of a commuting diagram

(1.2)
$$\begin{tikzcd}
C(X) \arrow[r, "\varphi"] & Q \\
C(Y) \arrow[u, "f^*"] \arrow[r, "\varphi_0"] & \mathbb{L}(H) \arrow[u]
\end{tikzcd}$$

where φ and φ_0 are unital and injective $*$-homomorphisms. Thus φ is an extension of $C(X)$ by $\mathbb{K}$, in the sense of Brown, Douglas and Fillmore, which is trivial (or split) when restricted to $C(Y)$, and φ_0 is a specified splitting of the restriction. $\mathrm{Ext}_{Y,f}(X)$ can be defined as the homotopy classes of such diagrams, or pairs (φ, φ_0), but as one would expect from experience with BDF-theory and KK-theory, the group admits several other descriptions where the equivalence relation is seemingly stronger and/or the diagrams are required to have special properties. In particular, triviality of the diagram (1.2) is equivalent to the existence of $*$-homomorphisms $\psi_n : C(X) \to \mathbb{L}(H)$ such that the upper triangle in the diagram

$$\begin{tikzcd} C(X) \arrow[r, "\varphi"] \arrow[dr, "\psi_n"] & Q \\ C(Y) \arrow[u, "f^*"] \arrow[r, "\varphi_0"] & \mathbb{L}(H) \arrow[u] \end{tikzcd} \tag{1.3}$$

commutes for each n, and the lower triangle asymptotically commutes in the sense that $\lim_{n\to\infty} \psi_n \circ f^*(g) = \varphi_0(g)$ for all $g \in C(Y)$. Thus the relative extension group $\mathrm{Ext}_{Y,f}(X)$ presents the obstructions for the existence of a splitting of the whole extension φ which respects the given splitting over $C(Y)$ up to any given tolerance. These obstructions are naturally divided in two classes, where the first is the rather obvious obstruction that the diagram (1.3) can only exist when the extension φ is split. This obstruction is described by the presence of an obvious map $\mathrm{Ext}_{Y,f}(X) \to \mathrm{Ext}(X)$. In many cases this map is injective, and then the obvious obstruction is the only obstruction. But generally the map to $\mathrm{Ext}(X)$ is not injective, and the kernel of it consists of the non-trivial obstructions — those that arise because we insist that the given splitting over $C(Y)$ should be respected, at least asymptotically. The six-term exact sequence (1.1) shows that the kernel of the forgetful map $\mathrm{Ext}_{Y,f}(X) \to \mathrm{Ext}(X)$ is isomorphic to the co-kernel of the map $f_* : K_0(X) \to K_0(Y)$. This part of the relative K-homology contains the obstructions for finding a $*$-homomorphic lift $C(X) \to \mathbb{L}(H)$ of φ which agrees with φ_0 on $C(Y)$ up to an arbitrarily small compact perturbation. We show that this part of the relative K-homology vanishes in many cases, and in particular when Y is a compact subset of the complex plane $\mathbb{C}$. This then serves as the main ingredient in the proof of the following operator-theoretic fact:

THEOREM 1.1. *Let $M, N_1, N_2, N_3, \dots, N_k$ be bounded normal operators such that $N_iN_j = N_jN_i$ for all i, j, and let F be a continuous function from the joint spectrum of the N_i's onto the spectrum of M such that*

$$F(N_1, N_2, \dots, N_k) - M \in \mathbb{K}.$$

For every $\varepsilon > 0$ there are normal operators $N_1^\varepsilon, N_2^\varepsilon, N_3^\varepsilon, \dots, N_k^\varepsilon$ such that $N_i^\varepsilon N_j^\varepsilon = N_j^\varepsilon N_i^\varepsilon$, $N_i - N_i^\varepsilon \in \mathbb{K}$ for all i, j, and

$$\|F(N_1^\varepsilon, N_2^\varepsilon, \dots, N_k^\varepsilon) - M\| \leqslant \varepsilon.$$

2. THE RELATIVE EXTENSION GROUP

We begin by recalling the definition of the group of C^*-extensions, as it appears in KK-theory. Let A, B be separable C^*-algebras, B stable. As is well-known, the C^*-algebra extensions of A by B can be identified with $\mathrm{Hom}(A, Q(B))$, the set of $*$-homomorphisms $A \to Q(B)$, where $Q(B) = M(B)/B$ is the generalized Calkin algebra. Let $q_B : M(B) \to Q(B)$ be the quotient map. Two extensions $\varphi, \psi : A \to Q(B)$ are *unitarily equivalent* when there is a unitary $u \in M(B)$ such that $\mathrm{Ad}\, q_B(u) \circ \psi = \varphi$. The unitary equivalence classes of extensions of A by B have the structure of an abelian semi-group thanks to the stability of B: Choose isometries $V_1, V_2 \in M(B)$ such that $V_1 V_1^* + V_2 V_2^* = 1$, and define the sum $\varphi \oplus \psi : A \to Q(B)$ of $\varphi, \psi \in \mathrm{Hom}(A, Q(B))$ to be

$$(\psi \oplus \varphi)(a) = \mathrm{Ad}\, q_B(V_1) \circ \psi(a) + \mathrm{Ad}\, q_B(V_2) \circ \varphi(a). \tag{2.1}$$

An extension $\varphi : A \to Q(B)$ is *split* when there is a $*$-homomorphism $\pi : A \to M(B)$ such that $\varphi = q_B \circ \pi$. To trivialize the split extensions and obtain a neutral element for the composition we declare two extensions $\varphi, \psi : A \to Q(B)$ to be *stably equivalent* when there is a split extension π such that $\psi \oplus \pi$ and $\varphi \oplus \pi$ are unitarily equivalent. The semigroup of stable equivalence classes of extensions of A by B is denoted by $\mathrm{Ext}(A, B)$. As is well-documented by now, the semi-group is generally not a group, and we denote by

$$\mathrm{Ext}^{-1}(A, B)$$

the abelian group of invertible elements in $\mathrm{Ext}(A, B)$.

An *absorbing $*$-homomorphism* $\pi : A \to M(B)$ is a $*$-homomorphism with the property that for every completely positive contraction $\varphi : A \to M(B)$ there is a sequence $V_n \in M(B)$ of isometries such that $V_n^* \pi(a) V_n - \varphi(a) \in B$ for all n, and $\lim_{n\to\infty} V_n^* \pi(a) V_n = \varphi(a)$ for all $a \in A$. When A is unital, a *unitally absorbing $*$-homomorphism* $\pi : A \to M(B)$ is a unital $*$-homomorphism with the property that for every completely positive contraction $\varphi : A \to B$ there is a sequence $W_n \in M(B)$ such that $\lim_{n\to\infty} W_n^* \pi(a) W_n = \varphi(a)$ for all $a \in A$, and $\lim_{n\to\infty} W_n^* b = 0$ for all $b \in B$. We refer the reader to [15] for alternative characterizations of absorbing and unitally absorbing $*$-homomorphisms which justify the names, and a proof that they always exist in the separable case. Of particular importance here is the *essential uniqueness* of such $*$-homomorphisms. Specifically, when $\pi, \lambda : A \to M(B)$ are $*$-homomorphisms that are either both absorbing or both unitally absorbing, there is a sequence U_n of unitaries in $M(B)$ such that $U_n \pi(a) U_n^* - \lambda(a) \in B$ for all n, and $\lim_{n\to\infty} U_n^* \pi(a) U_n = \lambda(a)$ for all $a \in A$.

Let now $J \subseteq A$ be a C^*-subalgebra of A, and consider an absorbing $*$-homomorphism $\alpha_0 : A \to M(B)$. Set $\alpha = q_B \circ \alpha_0 : A \to Q(B)$, and let

$$0 \longrightarrow B \longrightarrow E_0 \longrightarrow J \longrightarrow 0$$

be the extension of J by B whose Busby invariant is $\alpha|_J$. Let $i : J \to A$ be the inclusion. We consider extensions E of A by B which fit into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \\ 0 \arrow[r] & B \arrow[r] \arrow[u, equal] & E_0 \arrow[r] \arrow[u] & J \arrow[r] \arrow[u, "i"'] & 0 \end{tikzcd}$$

of C^*-algebras. In terms of the Busby invariant this corresponds to extensions $\varphi : A \to Q(B)$ such that $\varphi|_J = \alpha|_J$. We say that φ *equals* α *on* J. Two such extensions, $\varphi, \psi : A \to Q(B)$, that both equal α on J, are said to be *unitarily equivalent* when there is a unitary v connected to 1 in the unitary group of the relative commutant $\alpha(J)' \cap Q(B)$ such that $\operatorname{Ad} v \circ \varphi = \psi$.

LEMMA 2.1. *For each* $n \in \mathbb{N}$, *there are isometries* $v_1, v_2, \dots, v_n$ *in* $\alpha(A)' \cap Q(B)$ *such that* $v_i^* v_j = 0, i \neq j$, *and* $\sum_{i=1}^{n} v_i v_i^* = 1$.

Proof. Let $S_1, S_2, S_3, \dots$ be a sequence of isometries in $M(B)$ such that $S_i^* S_j = 0, i \neq j$, and $\sum_{i=1}^{\infty} S_i S_i^* = 1$ with convergence in the strict topology. Set $\beta_0(a) = \sum_{i=1}^{\infty} S_i \alpha_0(a) S_i^*$ for all $a \in A$, and note that β_0 is absorbing because α_0 is. Set $W_i = \sum_{j=1}^{\infty} S_{i+jn} S_j^*$. Then $W_1, W_2, \dots, W_n$ are isometries in $\beta_0(A)' \cap M(B)$ such that $W_i^* W_j = 0$ when $i \neq j$, and $\sum_{i=1}^{n} W_i W_i^* = 1$. The essential uniqueness property of absorbing $*$-homomorphisms guarantees the existence of a unitary $U \in M(B)$ such that $\operatorname{Ad} U \circ \beta_0(a) - \alpha_0(a) \in B$ for all $a \in A$. Set $v_i = q_B(U W_i U^*), i = 1, 2, \dots, n$. ∎

Note that any choice of isometries $v_1, \dots, v_n$ as in Lemma 2.1 gives us a $*$-isomorphism, Θ_n, which maps the C^*-algebra $M_n(\alpha(A)' \cap Q(B))$ onto $\alpha(A)' \cap Q(B)$. Θ_n is given by

$$\Theta_n((x_{ij})) = \sum_{i,j=1}^{n} v_i x_{ij} v_j^*.$$

Any other choice of isometries as in Lemma 2.1 will result in an isomorphism which is conjugate to Θ_n by a unitary from $\alpha(A)' \cap Q(B)$. Thanks to Lemma 2.1 we can define a composition $+_\alpha$ among the extensions of A by B which agree with α on J:

$$(\varphi +_\alpha \psi)(a) = w_1 \varphi(a) w_1^* + w_2 \psi(a) w_2^*, \tag{2.2}$$

where $w_i \in \alpha(A)' \cap Q(B)$ are isometries such that $w_1 w_1^* + w_2 w_2^* = 1$. To show that (2.2) gives the unitary equivalence classes of extensions of A by B that agree with α on J the structure on an abelian semi-group, consider the $*$-homomorphism

β_0 introduced in the proof of Lemma 2.1, and set $\beta = q_B \circ \beta_0$. There are then isometries $V_i, i = 1, 2$, in $\beta_0(A)' \cap M(B)$ such that $V_1V_1^* + V_2V_2^* = 1$. With these isometries we define the sum $\varphi +_{\beta_0} \psi$ of two extensions, φ, ψ, of A by B which both equal β on J to be $\varphi +_{\beta_0} \psi = \operatorname{Ad} q_B(V_1) \circ \varphi + \operatorname{Ad} q_B(V_2) \circ \psi$, in analogy with (2.2).

LEMMA 2.2. *Let $P \in \beta_0(A)' \cap M(B)$ be a projection such that both P and $1 - P$ are Murray-von Neumann equivalent to 1 in $\beta_0(A)' \cap M(B)$. Let U be a unitary in $\beta_0(A)' \cap M(B)$ such that $UPU^* = P$. It follows that U is connected to 1 in the unitary group of $\beta_0(A)' \cap M(B)$.*

Proof. Note first that $K_1(\beta_0(A)' \cap M(B)) = 0$ by Lemma 3.1 of [15]. Since $U = [PUP + (1-P)][P + (1-P)U(1-P)]$, the lemma follows from this. ∎

LEMMA 2.3. *Let $\varphi, \psi, \lambda : A \to Q(B)$ be extensions of A by B that both equal β on J. There are then unitaries $S, T \in \beta_0(A)' \cap M(B)$, connected to 1 in the unitary group of $\beta_0(A)' \cap M(B)$, such that $\operatorname{Ad} q_B(S) \circ (\varphi +_{\beta_0} \psi) = \psi +_{\beta_0} \varphi$ and $\operatorname{Ad} q_B(T) \circ ((\varphi +_{\beta_0} \psi) +_{\beta_0} \lambda) = \varphi +_{\beta_0} (\psi +_{\beta_0} \lambda)$.*

Proof. Let $\Theta : M_2(B) \to B$ be the $*$-isomorphism given by

$$\Theta \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \sum_{i,j=1}^{2} V_i b_{ij} V_j^*,$$

and let $\overline{\Theta} : M(M_2(B)) = M_2(M(B)) \to M(B)$ be the $*$-isomorphism extending Θ. Then $\overline{\Theta} \circ \begin{pmatrix} \beta_0 & \\ & \beta_0 \end{pmatrix} = \beta_0$, so we see that

$$S = V_1V_2^* + V_2V_1^* = \overline{\Theta}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \tag{2.3}$$

is connected to 1 in the unitary group of $\beta_0(A)' \cap M(B)$. Since $\operatorname{Ad} q_B(S) \circ (\varphi +_{\beta_0} \psi) = \psi +_{\beta_0} \varphi$, this proves the first statement. To prove the second statement we identify $M_3(M(B))$ with $\mathbb{L}_B(B \oplus B \oplus B)$ — the C^*-algebra of adjointable operators on the Hilbert B-module B^3. Define unitaries $W : B^3 \to B$ and $Z : B^3 \to B$ of Hilbert B-modules such that $W(b_1, b_2, b_3) = V_1^2 b_1 + V_1V_2b_2 + V_2b_3$ and $Z(b_1, b_2, b_3) = V_1b_1 + V_2V_1b_2 + V_2^2b_3$. Then $ZW^* \in M(B)$ and $\operatorname{Ad} q_B(ZW^*) \circ ((\varphi +_{\beta_0} \psi) +_{\beta_0} \chi) = \varphi +_{\beta_0} (\psi +_{\beta_0} \lambda)$. It remains to show that ZW^* is connected to 1 in the unitary group of $\beta_0(A)' \cap M(B)$. First observe that ZW^* is connected to

$$Z \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} W^*$$

in the unitary group of $\beta_0(A)' \cap M(B)$. Since the unitary S from (2.3) is connected to 1 in the unitary group of $\beta_0(A)' \cap M(B)$ we see that ZW^* is connected to

$$T = SZ \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} W^*$$

in the unitary group of $\beta_0(A)' \cap M(B)$. Note that

$$W^*V_2V_2^*W = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \quad \text{and} \quad Z^*V_1V_1^*Z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix},$$

which implies that

$$Z \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} W^* V_2 V_2^* W \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} Z^* = V_1 V_1^*,$$

and hence that $TV_2V_2^*T^* = V_2V_2^*$. It follows then from Lemma 2.2 that T, and hence also ZW^* is connected to 1 in the unitary group of $\beta_0(A)' \cap M(B)$. ∎

LEMMA 2.4. *Let $\varphi, \psi, \lambda : A \to Q(B)$ be extensions of A by B that equal α on D. There are then unitaries $v, w \in \alpha(A)' \cap Q(B)$, connected to* 1 *in the unitary group of $\alpha(A)' \cap Q(B)$, such that* $\operatorname{Ad} v \circ (\varphi +_\alpha \psi) = \psi +_\alpha \varphi$ *and* $\operatorname{Ad} w \circ ((\varphi +_\alpha \psi) +_\alpha \lambda) = \varphi +_\alpha (\psi +_\alpha \lambda)$.

Proof. By the essential uniqueness of absorbing $*$-homomorphisms there is a unitary $U \in M(B)$ such that $\operatorname{Ad} U \circ \beta_0(a) - \alpha_0(a) \in B$. Set $t = w_1 q_B(UV_1U^*)^* + w_2 q_B(UV_2U^*)^*$, and let S, T be the unitaries from Lemma 2.3. Then, by Lemma 2.3, $v = tq_B(USU^*)t^*$ and $w = tq_B(UTU^*)t^*$ have the stated properties. ∎

It follows from Lemma 2.4 that (2.2) gives the set of unitary equivalence classes of extensions of A by B which agree with α on J the structure of an abelian semi-group. Furthermore, it follows from the proof of Lemma 2.4 that any other choice of absorbing $*$-homomorphism instead of α would result in an isomorphic semi-group. To obtain a neutral element we declare that two extensions $\varphi, \psi : A \to Q(B)$ that agree with α on J are *stably equivalent* when $\varphi +_\alpha \alpha$ is unitarily equivalent to $\psi +_\alpha \alpha$. That stable equivalence *is* an equivalence relation follows from Lemma 2.4 and the observation that $\alpha +_\alpha \alpha = \alpha$. The formula (2.2) gives a well-defined composition in the set of stable equivalence classes of extensions that agree with α on J, giving us an abelian semi-group with a neutral element (or 0) represented by α, and we denote this semi-group by $\operatorname{Ext}_{J,\alpha}(A, B)$. The group of invertible elements in $\operatorname{Ext}_{J,\alpha}(A, B)$ will be denoted by

$$\operatorname{Ext}^{-1}_{J,\alpha}(A, B).$$

It is clear from the construction, and can be seen from the essential uniqueness of an absorbing $*$-homomorphism, that any other choice of an absorbing $*$-homomorphism $A \to M(B)$ will give rise to an isomorphic group. However, at this point it would seem as if the stable equivalence of two given extensions of A by B, which both agree with α on J, depends on the particular choice of isometries from $\alpha(A)' \cap Q(B)$ used to define the addition $+_\alpha$. The next lemma shows that this is not the case because the addition (2.2) is independent of the w_i's up to conjugation by a unitary from $\alpha(A)' \cap Q(B)$.

Let $n \in \mathbb{N}$. To simplify the notation, we denote by $1_n \otimes \alpha_0 : A \to M_n(M(B))$ the $*$-homomorphism given by

$$(1_n \otimes \alpha_0)(a) = \begin{pmatrix} \alpha_0(a) & & & \\ & \alpha_0(a) & & \\ & & \ddots & \\ & & & \alpha_0(a) \end{pmatrix}.$$

Set $1_n \otimes \alpha = (\mathrm{Id}_{M_n(\mathbb{C})} \otimes q_B) \circ (1_n \otimes \alpha_0)$.

LEMMA 2.5. *Let $\varphi : A \to Q(B)$ be an extension of A by B which is equal to α on J, and assume that $v \in \alpha(A)' \cap Q(B)$ is an isometry such that $vv^*\alpha(a) = \alpha(a)$ for all $a \in A$. It follows that $\operatorname{Ad} v \circ \varphi$ is stably equivalent to φ.*

Proof. Note that

$$\begin{pmatrix} v & 1 - vv^* \\ 0 & v^* \end{pmatrix} \begin{pmatrix} \varphi(a) & 0 \\ 0 & \alpha(a) \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 1 - vv^* & v \end{pmatrix} = \begin{pmatrix} v\varphi(a)v^* & 0 \\ 0 & \alpha(a) \end{pmatrix}$$

and that $\left(\begin{smallmatrix} v & 1-vv^* \\ 0 & v^* \end{smallmatrix}\right)$ is a unitary in $M_2(Q(B)) \cap (1_2 \otimes \alpha)(A)'$. It follows that there is a unitary $u \in Q(B) \cap \alpha(A)'$ such that $\operatorname{Ad} w \circ (\varphi +_\alpha \alpha) = (\operatorname{Ad} v \circ \varphi) +_\alpha \alpha$. Since $\left(\begin{smallmatrix} u & \\ & u^* \end{smallmatrix}\right)$ is connected to 1 in the unitary group of $M_2(Q(B)) \cap (1_2 \otimes \alpha)(A)'$, we deduce that $\operatorname{Ad} v \circ \varphi +_\alpha \alpha +_\alpha \alpha$ is unitarily equivalent to $\varphi +_\alpha \alpha +_\alpha \alpha$. ∎

3. A SIX-TERM EXACT SEQUENCE

We will now assume that there is an absorbing $*$-homomorphism $\alpha_0 : A \to M(B)$ such that $\alpha_0|_J : J \to M(B)$ is also absorbing. This condition is known to be automatically fullfilled in the following cases:

(i) B is nuclear, or
(ii) J is nuclear, or
(iii) J is a hereditary C^*-subalgebra of A; in particular, when J is an ideal, or
(iv) there is a surjective conditional expectation $P : A \to J$.

(i) follows from Kasparov's work, [9], and (ii)–(iv) all follow from Lemma 2.1 and Lemma 2.2 of [16]. In general the existence of α_0 fails, cf. the last section in [16].

Fix an absorbing $*$-homomorphism $\alpha_0 : A \to M(B)$ such that $\alpha_0|_J : J \to M(B)$ is also absorbing, and set $\alpha = q_B \circ \alpha_0$ as before. Set

$$\mathcal{D}_\alpha(J) = \{m \in M(B) : \ m\alpha_0(j) - \alpha_0(j)m \in B \ \forall j \in J\},$$
$$\mathcal{X}_\alpha(J) = \{m \in D_\alpha(J) : \ m\alpha_0(j) \in B \ \forall j \in J\}.$$

It was shown in [15] that there is a natural isomorphism $K_1(\mathcal{D}_\alpha(J)/\mathcal{X}_\alpha(J)) \simeq KK(J, B)$, and then in Lemma 3.1 of [17] that $K_*(\mathcal{X}_\alpha(J)) = 0$, so that we have a natural isomorphism

$$K_1(\mathcal{D}_\alpha(J)) \simeq KK(J, B). \tag{3.1}$$

Similarly, we set $\mathcal{D}_\alpha(A) = \{m \in M(B) : \ m\alpha_0(a) - \alpha_0(a)m \in B \ \forall a \in A\}$, and get a natural isomorphism

$$K_1(\mathcal{D}_\alpha(A)) \simeq KK(A, B). \tag{3.2}$$

As above $i : J \to A$ will denote the inclusion, and we denote also by i the induced inclusion $i : \mathcal{D}_\alpha(A) \to \mathcal{D}_\alpha(J)$. Note that the diagram

$$\begin{tikzcd} K_1(\mathcal{D}_\alpha(A)) \arrow[r, "i_*"] \arrow[d] & K_1(\mathcal{D}_\alpha(J)) \arrow[d] \\ KK(A,B) \arrow[r, "i^*"] & KK(J,B) \end{tikzcd} \tag{3.3}$$

is commutative when the vertical arrows are isomorphisms (3.1) and (3.2).

Let v be a unitary in $M_n(\mathcal{D}_\alpha(J))$. Let $\Theta_n : M_n(Q(B)) \to Q(B)$ be the isomorphism from Lemma 2.1. Then $\Theta_n \circ (\mathrm{Id}_{M_n(\mathbb{C})} \otimes q_B) \circ \mathrm{Ad}\, v \circ (1_n \otimes \alpha_0) : A \to M_n(Q(B))$ is an extension $e(v) : A \to Q(B)$ of A by B which is equal to α on J. Since

$$\Theta_{2n} \circ (\mathrm{Id}_{M_{2n}(\mathbb{C})} \otimes q_B) \begin{pmatrix} v & \\ & v^* \end{pmatrix}$$

is connected to 1 in the unitary group of $\alpha(J)' \cap Q(B)$, we see that $e(v) +_\alpha e(v^*)$ is stably equivalent to α, as an extension of A by B which is equal to α on J, proving that $e(v)$ is invertible, hence it represents an element in $\mathrm{Ext}^{-1}_{J,\alpha}(A,B)$. When v_t, $t \in [0,1]$, is a norm-continuous path of unitaries in $M_n(\mathcal{D}_\alpha(J))$ there is a partition $0 = t_0 < t_1 < t_2 < \cdots < t_N = 1$ of $[0,1]$ such that $(\mathrm{Id}_{M_n(\mathbb{C})} \otimes q_B)(v_{t_i} v^*_{t_{i+1}})$ is in the connected component of 1 in the unitary group of $M_n(\alpha(J)' \cap Q(B))$. Hence $e(v_0) = e(v_1)$ in $\mathrm{Ext}^{-1}_{J,\alpha}(A,B)$. It is then straightforward to check that the construction gives us a group homomorphism

$$\partial : K_1(\mathcal{D}_\alpha(J)) \to \mathrm{Ext}^{-1}_{J,\alpha}(A,B). \tag{3.4}$$

LEMMA 3.1. *The following sequence is exact:*

$$\begin{tikzcd} K_1(\mathcal{D}_\alpha(J)) \arrow[r, "\partial"] & \mathrm{Ext}^{-1}_{J,\alpha}(A,B) \arrow[r] & \mathrm{Ext}^{-1}(A,B) \arrow[d, "i^*"] \\ K_1(\mathcal{D}_\alpha(A)) \arrow[u, "i_*"] & & \mathrm{Ext}^{-1}(J,B). \end{tikzcd}$$

Proof. Exactness at $K_1(\mathcal{D}_\alpha(J))$: If v is a unitary in $M_n(\mathcal{D}_\alpha(A))$, we find that $(\mathrm{Id}_{M_n(\mathbb{C})} \otimes q_B) \circ \mathrm{Ad}\, v \circ (1_n \otimes \alpha_0) = 1_n \otimes \alpha$ so that $e(v) = [\Theta_n \circ (1_n \otimes \alpha)] = [\alpha] = 0$ in $\mathrm{Ext}^{-1}_{J,\alpha}(A,B)$. To show that $\ker \partial \subseteq \mathrm{Im}\, i^*$, let $v \in \mathcal{D}_\alpha(J)$ be a unitary such that $\partial[v] = 0$. Then $\mathrm{Ad}\, q_B(v) \circ \alpha +_\alpha \alpha$ is unitarily equivalent to $\alpha +_\alpha \alpha$, which means that there is a unitary S connected to 1 in the unitary group of $M_2(\alpha(J)' \cap Q(B))$ such that

$$\mathrm{Ad}\left[S \begin{pmatrix} q_B(v) & \\ & 1 \end{pmatrix} \right] \begin{pmatrix} \alpha(a) & \\ & \alpha(a) \end{pmatrix} = \begin{pmatrix} \alpha(a) & \\ & \alpha(a) \end{pmatrix}. \tag{3.5}$$

Since $q_B : \mathcal{D}_\alpha(J) \to \alpha(J)' \cap Q(B)$ is surjective, there is a unitary S_0 connected to 1 in the unitary group of $M_2(\mathcal{D}_\alpha(J))$ such that $\mathrm{Id}_{M_2(\mathbb{C})} \otimes q_B(S_0) = S$. Then $[v] = [S_0 \begin{pmatrix} v & \\ & 1 \end{pmatrix}]$ in $K_1(\mathcal{D}_\alpha(J))$. It follows from (3.5) that $S_0 \begin{pmatrix} v & \\ & 1 \end{pmatrix}$ belongs to the commutant of $(1_2 \otimes \alpha)(A)$, hence $[S_0 \left(\begin{smallmatrix} v & \\ & 1 \end{smallmatrix}\right)] \in i^*(K_1(\mathcal{D}_\alpha(A)))$. The same argument works when v is a unitary in $M_n(\mathcal{D}_\alpha(J))$ for some $n \geqslant 2$.

Exactness at $\mathrm{Ext}^{-1}_{J,\alpha}(A,B)$: Let v be a unitary in $\mathcal{D}_\alpha(J)$. Then $(\mathrm{Ad}\, q_B(v) \circ \alpha)$ is a split extension of A by B, proving that $[v] \in K_1(\mathcal{D}_\alpha(J))$ goes to zero under the composition $K_1(\mathcal{D}_\alpha(J)) \to \mathrm{Ext}^{-1}_{J,\alpha}(A,B) \to \mathrm{Ext}^{-1}(A,B)$. The same argument works when $v \in M_n(\mathcal{D}_\alpha(J))$, so we see that the composition is zero. Let $\varphi : A \to Q(B)$ be an extension of A by B which is equal to α over J, and assume that $[\varphi] = 0$ in $\mathrm{Ext}^{-1}(A,B)$. This means that there is a unitary $T \in M(M_2(B))$ such that

$$\mathrm{Ad}(\mathrm{Id}_{M_2(\mathbb{C})} \otimes q_B)(T) \circ \left(\begin{smallmatrix} \varphi & \\ & \alpha \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & \\ & \alpha \end{smallmatrix}\right). \tag{3.6}$$

Since φ is equal to α over J this implies that $T \in M_2(\mathcal{D}_\alpha(J))$ and we see from (3.6) that $[\varphi] = \partial[T^*]$. Note that we did not assume that φ represented an *invertible* element in $\mathrm{Ext}_{J,\alpha}(A,B)$, so besides establishing the exactness at $\mathrm{Ext}^{-1}_{J,\alpha}(A,B)$ the argument also shows that every element of $\mathrm{Ext}_{J,\alpha}(A,B)$ which goes to 0 in $\mathrm{Ext}^{-1}(A,B)$ comes from $K_1(\mathcal{D}_\alpha(J))$, and hence *is* invertible in $\mathrm{Ext}_{J,\alpha}(A,B)$. This point will be used shortly.

Exactness at $\mathrm{Ext}^{-1}(A,B)$: It is obvious that i^* kills the image of $\mathrm{Ext}^{-1}_J(A,B)$ in $\mathrm{Ext}^{-1}(A,B)$, so consider an invertible extension $\varphi : A \to Q(B)$ such that $[\varphi \circ i] = 0$ in $\mathrm{Ext}^{-1}(J,B)$. This means that there is a unitary $T \in M(B)$ such that

$$\mathrm{Ad}\, q_B(T)(\varphi \oplus \alpha)(j) = \alpha(j)$$

for any $j \in J$, i.e. $\mathrm{Ad}\, q_B(T)(\varphi \oplus \alpha)$ equals α on J. Since $[\varphi] = [\mathrm{Ad}\, q_B(T)(\varphi \oplus \alpha)]$ in $\mathrm{Ext}^{-1}(A,B)$, this completes the proof, provided we can show that $\mathrm{Ad}\, q_B(T)(\varphi \oplus \alpha)$ represents an invertible element of $\mathrm{Ext}_{J,\alpha}(A,B)$. To this end, let $\psi : A \to Q(B)$ be an extension of A by B which represents the inverse of φ in $\mathrm{Ext}^{-1}(A,B)$. Then $i^*[\psi] = 0$ in $\mathrm{Ext}^{-1}(J,B)$ so we deduce as in case of φ that there is a unitary $T' \in M(B)$ such that $\mathrm{Ad}\, q_B(T')(\psi \oplus \alpha)(j) = \alpha(j)$ for all $j \in J$. Thus $\mathrm{Ad}\, q_B(T')(\psi \oplus \alpha)$ and $\mathrm{Ad}\, q_B(T)(\varphi \oplus \alpha)$ both represent elements of $\mathrm{Ext}_{J,\alpha}(A,B)$. Since the sum

$$\mathrm{Ad}\, q_B(T')(\psi \oplus \alpha) +_\alpha \mathrm{Ad}\, q_B(T)(\varphi \oplus \alpha)$$

represents 0 in $\mathrm{Ext}^{-1}(A,B)$, it follows from the argument that proved exactness at $\mathrm{Ext}^{-1}_{J,\alpha}(A,B)$ that $[\mathrm{Ad}\, q_B(T)(\varphi \oplus \alpha)]$ is invertible in $\mathrm{Ext}_{J,\alpha}(A,B)$. ∎

The proof of Lemma 3.1 has the following corollary:

LEMMA 3.2. *Let φ be an extension of A by B which equals α on J. Assume that φ is invertible in* $\mathrm{Ext}(A,B)$. *It follows that φ is invertible in* $\mathrm{Ext}_{J,\alpha}(A,B)$.

Proof. The image of $[\varphi]$ in $\mathrm{Ext}^{-1}(A,B)$ is killed by

$$i^* : \mathrm{Ext}^{-1}(A,B) \to \mathrm{Ext}^{-1}(J,B)$$

and the proof of Lemma 3.1, more precisely the proof of exactness at $\mathrm{Ext}^{-1}(A,B)$, shows that $[\varphi]$ is invertible in $\mathrm{Ext}_{J,\alpha}(A,B)$. ∎

In particular, $\mathrm{Ext}_{J,\alpha}(A,B)$ is a group when $\mathrm{Ext}(A,B)$ is.

Consider now the suspension $SB = C_0(0,1) \otimes B$ of B. If we combine (3.3) with Lemma 3.1 and use the natural identification $KK(-,B) = \mathrm{Ext}^{-1}(-,SB)$ we get the exact sequence

(3.7)
$$\begin{tikzcd} \mathrm{Ext}^{-1}(J,SB) \arrow[r, "\partial_0"] & \mathrm{Ext}^{-1}_{J,\alpha}(A,B) \arrow[r] & \mathrm{Ext}^{-1}(A,B) \arrow[d, "i^*"] \\ \mathrm{Ext}^{-1}(A,SB) \arrow[u, "i^*"] & & \mathrm{Ext}^{-1}(J,B) \end{tikzcd}$$

where ∂_0 is the composition of $\partial : K_1(D_\alpha(J)) \to \mathrm{Ext}^{-1}_J(A,B)$ with the isomorphism $\mathrm{Ext}^{-1}(J,SB) \to K_1(D_\alpha(J))$ coming from (3.1).

Let now $\beta_0 : A \to M(SB)$ be an absorbing $*$-homomorphism such that also $\beta_0|_J : J \to M(SB)$ is absorbing. The existence of such a β_0 does not require any additional asumptions because β_0 can be constructed from α_0 by use of Lemma 3.2 of [17]. Hence there is also a map $\partial' : K_1(\mathcal{D}_\beta(J)) \to \mathrm{Ext}^{-1}_{J,\beta}(A,SB)$, defined in the same way as ∂, but with SB in place of B. This leads to the following version of (3.7):

(3.8)
$$\begin{tikzcd} \mathrm{Ext}^{-1}(J,SB) & & \mathrm{Ext}^{-1}(A,B) \arrow[d, "i^*"] \\ \mathrm{Ext}^{-1}(A,SB) \arrow[u, "i^*"] & \mathrm{Ext}^{-1}_{J,\beta}(A,SB) \arrow[l] & \mathrm{Ext}^{-1}(J,B), \arrow[l, "\partial_1"'] \end{tikzcd}$$

where ∂_1 is the composition of $\partial' : K_1(\mathcal{D}_\beta(J)) \to \mathrm{Ext}^{-1}_{J,\beta}(A,SB)$ with the isomorphism $\mathrm{Ext}^{-1}(J,B) {=} KK(J,SB) {\to} K_1(\mathcal{D}_\beta(J))$. By combining (3.7) and (3.8) we get

THEOREM 3.3. *The following sequence is exact:*

$$\begin{tikzcd} \mathrm{Ext}^{-1}_{J,\alpha}(A,B) \arrow[r] & \mathrm{Ext}^{-1}(A,B) \arrow[r, "i^*"] & \mathrm{Ext}^{-1}(J,B) \arrow[d, "\partial_1"] \\ \mathrm{Ext}^{-1}(J,SB) \arrow[u, "\partial_0"] & \mathrm{Ext}^{-1}(A,SB) \arrow[l, "i^*"'] & \mathrm{Ext}^{-1}_{J,\beta}(A,SB). \arrow[l] \end{tikzcd}$$

4. OTHER REALIZATIONS OF THE RELATIVE EXTENSION GROUP

As above we assume that there is an absorbing $*$-homomorphism $\alpha_0 : A \to M(B)$ such that $\alpha_0|_J : J \to M(B)$ is also absorbing. By an *extension of A by B which splits over J* we mean a pair (φ, φ_0) where $\varphi : A \to Q(B)$ is an extension of A by B, and $\varphi_0 : J \to M(B)$ is a $*$-homomorphism such that $q_B \circ \varphi_0 = \varphi|_J$. We say that (φ, φ_0) is *invertible* when φ is an invertible extension of A by B, i.e.

when there is another extension ψ of A by B with the property that $\varphi \oplus \psi$ is a split extension (of A by B). Two invertible extensions, (φ, φ_0) and (ψ, ψ_0), of A by B which split over J are *homotopic in norm* when there is a path $(\Phi^t, \Phi_0^t), t \in [0,1]$, of invertible extensions of A by B which split over J such that $[0,1] \ni t \mapsto \Phi^t(a)$ is norm-continuous for all $a \in A$ and $[0,1] \ni t \mapsto \Phi_0^t(j)$ is norm-continuous for all $j \in J$, $(\Phi^0, \Phi_0^0) = (\varphi, \varphi_0)$ and $(\Phi^1, \Phi_0^1) = (\psi, \psi_0)$. We say that (φ, φ_0) and (ψ, ψ_0) are *stably homotopic in norm* when there is a $*$-homomorphism $\pi : A \to M(B)$ such that $(\varphi \oplus q_B \circ \pi, \varphi_0 \oplus \pi|_J)$ and $(\psi \oplus q_B \circ \pi, \psi_0 \oplus \pi|_J)$ are homotopic in norm. We denote by $\mathrm{Ext}_J^{-1}(A, B)$ the abelian semi-group of stable homotopy classes of invertible extensions of A by B which split over J. As we shall see shortly, $\mathrm{Ext}_J^{-1}(A, B)$ is actually a group.

Choose a sequence $W_1, W_2, W_3, \dots$ of isometries in $M(B)$ such that $W_i^* W_j = 0$ when $i \neq j$, and $\sum_{i=1}^{\infty} W_i W_i^* = 1$, with convergence in the strict topology. Set $\beta_0(a) = \sum_{i=2}^{\infty} W_i \alpha_0(a) W_i^*$, and note that $\beta_0 : A \to M(B)$ and $\beta_0|_J : J \to M(B)$ are both absorbing. We shall work with β_0 instead of α_0. The point is that unlike α_0, the absorbing $*$-homomorphisms β_0 and $\beta_0|_J$ are both guaranteed to be saturated in the sense of [17]. Recall that a $*$-homomorphism $\varphi_0 : A \to M(B)$ is *saturated* if it is unitarily equivalent to $(\varphi_0)_\infty \oplus (0)_\infty$, where $(\varphi_0)_\infty = \varphi_0 \oplus \varphi_0 \oplus \cdots$.

Let $\varphi : A \to Q(B)$ be an extension of A by B which equals β on J. Then (φ, β_0) is an extension of A by B which splits over J, and it is straightforward to see that the recipe $[\varphi] \to [\varphi, \beta_0]$ is a group homomorphism

$$\mathrm{Ext}_{J,\beta}^{-1}(A, B) \to \mathrm{Ext}_J^{-1}(A, B). \tag{4.1}$$

The aim is to show that (4.1) is an isomorphism.

Set $IB = C[0,1] \otimes B$, and let $\mathrm{ev}_t : IB \to B$ be the $*$-homomorphism given by evaluation at $t \in [0,1]$. Then e_t extends to a $*$-homomorphism $\overline{\mathrm{ev}_t} : M(IB) \to M(B)$ and induces in turn a $*$-homomorphism $\widehat{\mathrm{ev}_t} : Q(IB) \to Q(B)$. Let $\gamma_0 : A \to M(IB)$ be the $*$-homomorphism such that $(\gamma_0(a) f)(t) = \beta_0(a) f(t), t \in [0,1], f \in IB$. Since β_0 is saturated it follows from Lemma 2.3 of [17] that γ_0 is absorbing. Set $\gamma = q_{IB} \circ \gamma_0$, and note that we have, for any $t \in [0,1]$, a homomorphism

$$e_{t*} : \mathrm{Ext}_{J,\gamma}^{-1}(A, IB) \to \mathrm{Ext}_{J,\beta}^{-1}(A, B)$$

defined such that $e_{t*}[\varphi] = [\widehat{e_t} \circ \varphi]$ when $\varphi : A \to Q(IB)$ is an extension of A by IB which equals γ on J.

LEMMA 4.1. *The homomorphisms $e_{t*}, t \in [0,1]$, are all the same group isomorphism.*

Proof. Define $c : B \to IB$ such that $c(b)(t) = b$, and note that c induces $*$-homomorphisms $\overline{c} : M(B) \to M(IB)$ and $\widehat{c} : Q(B) \to Q(IB)$. Since $\widehat{c} \circ \beta = \gamma$ there is a homomorphism $c_* : \mathrm{Ext}_{J,\beta}^{-1}(A, B) \to \mathrm{Ext}_{J,\gamma}^{-1}(A, IB)$ such that $c_*[\psi] =$

$[\widehat{c} \circ \psi]$. Since $e_{t*} \circ c_*$ is the identity on $\mathrm{Ext}^{-1}_{J,\beta}(A,B)$ for all $t \in [0,1]$, it suffices to show that c_* is an isomorphism. This follows from Theorem 3.3 by an obvious application of the five-lemma. ∎

LEMMA 4.2. *Let A_1 and B_1 be separable $*$-algebras, B_1 stable. Let $\varphi, \psi : A_1 \to M(B_1)$ be $*$-homomorphisms and $W_t, t \in [1,\infty)$, a continuous path of unitaries in $M(B_1)$ such that:*
(i) $W_t\varphi(a)W_t^* - \psi(a) \in B_1,\ t \in [1,\infty), a \in A_1$;
(ii) $\lim_{t\to\infty} W_t\varphi(a)W_t^* = \psi(a),\ a \in A_1$.
Then $[\mathrm{Ad}\, W_1 \circ \varphi, \psi] = 0$ *in* $KK(A_1, B_1)$.

Proof. The lemma and its proof are essentially identical to Lemma 3.1 of [6]. Note, however, that one of the crucial assumptions has mysteriously disappeared in the lemma in [6]. ∎

PROPOSITION 4.3. *The map* (4.1) *is an isomorphism. In particular,* $\mathrm{Ext}^{-1}_J(A,B)$ *is a group.*

Proof. Surjectivity: Let (φ, φ_0) be an invertible extension of A by B which splits over J. Then $\varphi_0 \oplus \beta_0$ is approximately unitarily equivalent to β_0 because β_0 is absorbing, i.e. there exists a sequence of unitaries $U_n \in M(B)$, $n \in \mathbb{N}$, such that $\mathrm{Ad}\, U_n \circ (\varphi_0 \oplus \beta_0)(j) - \beta_0(j) \in B$ for all n, and $\lim_{n\to\infty} \mathrm{Ad}\, U_n \circ (\varphi_0 \oplus \beta_0)(j) - \beta_0(j) = 0$ for any $j \in J$. It follows then from Lemma 2.4 of [6] that $\varphi_0 \oplus (\beta_0)_\infty$ is asymptotically unitarily equivalent to $(\beta_0)_\infty$, i.e. there exists a norm-continuous path $V_t, t \in [1,\infty)$, of unitaries in $M(B)$ such that $\mathrm{Ad}\, V_t \circ (\varphi_0 \oplus \beta_0)(j) - \beta_0(j) \in B$ for all t, and $\lim_{t\to\infty} \mathrm{Ad}\, V_t \circ (\varphi_0 \oplus \beta_0)(j) - \beta_0(j) = 0$ for any $j \in J$. Since β_0 is unitarily equivalent to $(\beta_0)_\infty$ because β_0 is saturated, we conclude that there is a norm-continuous path $W_t, t \in [1,\infty)$, of unitaries in $M(B)$ such that $W_t(\varphi_0 \oplus \beta_0)(j)W_t^* - \beta_0(j) \in B$ for all t, and $\lim_{t\to\infty} W_t(\varphi_0 \oplus \beta_0)(j)W_t^* - \beta_0(j) = 0$ for any $j \in J$. Since the unitary group of $M(B)$ is connected in norm, it holds that $[\varphi, \varphi_0] = [\mathrm{Ad}\, q_B(W_1) \circ (\varphi \oplus \beta), \mathrm{Ad}\, W_1 \circ (\varphi_0 \oplus \beta_0)]$ in $\mathrm{Ext}_{J,s}(A,B)$. Set

$$\Psi^t = \begin{cases} \mathrm{Ad}\, W_{1/t} \circ (\varphi_0 \oplus \beta_0) & t \in]0,1], \\ \beta_0 & t = 0. \end{cases}$$

Then $(\mathrm{Ad}\, q_B(W_1) \circ (\varphi \oplus \beta), \Psi^t), t \in [0,1]$, is a homotopy in norm showing that

$$[\mathrm{Ad}\, q_B(W_1) \circ (\varphi \oplus \beta), \mathrm{Ad}\, W_1 \circ (\varphi_0 \oplus \beta_0)] = [\mathrm{Ad}\, q_B(W_1) \circ (\varphi \oplus \beta), \beta_0].$$

Note that $\mathrm{Ad}\, q_B(W_1) \circ (\varphi \oplus \beta)$ is equal to β on J, and is invertible in $\mathrm{Ext}_{J,\beta}(A,B)$ by the proof of Lemma 3.1 since φ is an invertible extension. Thus $\mathrm{Ad}\, q_B(W_1) \circ (\varphi \oplus \beta)$ represents an element of $\mathrm{Ext}^{-1}_{J,\beta}(A,B)$, and we conclude that (4.1) is surjective. In particular $\mathrm{Ext}^{-1}_J(A,B)$ is a group.

Injectivity: Let φ, ψ be extensions of A by B which both equal β on J. Assume that $[\varphi, \beta_0] = [\psi, \beta_0]$ in $\mathrm{Ext}^{-1}_J(A,B)$. There is then a $*$-homomorphism

$\pi : A \to M(B)$ such that $(\varphi \oplus q_B \circ \pi, \beta_0 \oplus \pi)$ is homotopic in norm to $(\psi \oplus q_B \circ \pi, \beta_0 \oplus \pi)$. As in the proof of surjectivity we can find a norm-continuous path $W_t, t \in [1, \infty)$, of unitaries in $M(B)$ such that $W_t(\pi \oplus \beta_0)(a)W_t^* - \beta_0(a) \in B$ for all t, and $\lim_{t\to\infty} W_t(\pi \oplus \beta_0)(a)W_t^* - \beta_0(a) = 0$ for all $a \in A$. Set $S_t = 1 \oplus W_t$ and note that this gives a homotopy in norm between $(\varphi \oplus q_B \circ \pi \oplus \beta, \beta_0 \oplus \pi \oplus \beta)$ and $(\varphi \oplus \beta, \beta_0 \oplus \beta_0)$. Similarly, there is homotopy in norm between $(\psi \oplus q_B \circ \pi \oplus \beta, \beta_0 \oplus \pi \oplus \beta)$ and $(\psi \oplus \beta, \beta_0 \oplus \beta_0)$. It follows that there is a homotopy in norm, (Ψ^t, Ψ_0^t), between $(\varphi +_\beta \beta, \beta_0 +_\beta \beta_0)$ and $(\psi +_\beta \beta, \beta_0 +_\beta \beta_0)$. This homotopy defines in an obvious way an extension $\Phi : A \to Q(IB)$ and a $*$-homomorphism $\Phi_0 : J \to M(IB)$ such that $q_{IB} \circ \Phi_0 = \Phi|_J$, $(\widehat{e_0} \circ \Phi, \overline{e_0} \circ \Phi_0) = (\varphi +_\beta \beta, \beta_0 +_\beta \beta_0)$ and $(\widehat{e_1} \circ \Phi, \overline{e_1} \circ \Phi_0) = (\psi +_\beta \beta, \beta_0 +_\beta \beta_0)$. Note that Φ is invertible since each Ψ^t is. Let $\gamma_0 : A \to M(IB)$ be the $*$-homomorphism such that $(\gamma_0(a)f)(t) = \beta_0(a)f(t), t \in [0,1], f \in IB$. As in the proof of surjectivity, it follows from [6] that there is a norm-continuous path $W_t, t \in [1, \infty)$, of unitaries in $M(IB)$ such that $W_t(\Phi_0 \oplus \gamma_0)(j)W_t^* - \gamma_0(j) \in IB$ for all t, and $\lim_{t\to\infty} W_t(\Phi_0 \oplus \gamma_0)(j)W_t^* - \gamma_0(j) = 0$ for any $j \in J$. There is therefore also a norm-continuous path V_t, $t \in [1, \infty)$, of unitaries in $M(IB)$ such that

$$V_t(\Phi_0 +_\gamma \gamma_0)(j)V_t^* - ((\gamma_0 +_\gamma \gamma_0) +_\gamma \gamma_0)(j) \in IB$$

for all t, and $\lim_{t\to\infty} V_t(\Phi_0 +_\gamma \gamma_0)(j)V_t^* - ((\gamma_0 +_\gamma \gamma_0) +_\gamma \gamma_0)(j) = 0$ for any $j \in J$. It follows from Lemma 4.2 that

$$[\mathrm{Ad}\, V_1 \circ (\Phi_0 +_\gamma \gamma_0), (\gamma_0 +_\gamma \gamma_0) +_\gamma \gamma_0] = 0 \tag{4.2}$$

in $KK(J, IB)$. Set $S_i = \overline{e_i}(V_1), i = 0, 1$, and note that it follows from (4.2) that

$$[\mathrm{Ad}\, S_0 \circ ((\beta_0 +_\beta \beta_0) +_\beta \beta_0), \beta_0 +_\beta \beta_0) +_\beta \beta_0] = 0, \tag{4.3}$$

$$[\mathrm{Ad}\, S_1 \circ ((\beta_0 +_\beta \beta_0) +_\beta \beta_0), (\beta_0 +_\beta \beta_0) +_\beta \beta_0] = 0, \tag{4.4}$$

in $KK(J, B)$. Let $W \in M(B)$ be a unitary such that $\mathrm{Ad}\, W \circ \beta_0 = (\beta_0 +_\beta \beta_0)$. It follows then from (4.3) and (4.4) that

$$[\mathrm{Ad}\, W^*S_0W \circ \beta_0, \beta_0] = [\mathrm{Ad}\, W^*S_1W \circ \beta_0, \beta_0] = 0 \tag{4.5}$$

in $KK(J, B)$. Note that $[\mathrm{Ad}\, W^*S_0W \circ \beta_0, \beta_0] \in KK(J, B)$ is the image of the class of the unitary W^*S_0W under the isomorphism (3.1). Thus (4.5) implies that

$$[q_B(W^*S_0W)] = 0$$

in $K_1(\beta(J)' \cap Q(B))$. Similarly, it also implies that

$$[q_B(W^*S_1W)] = 0$$

in $K_1(\beta(J)' \cap Q(B))$. It follows therefore that

$$[\varphi] = [\mathrm{Ad}\, q_B(W^*S_0W) \circ (\widehat{e_0} \circ \Phi +_\beta \beta)] \quad \text{and} \quad [\psi] = [\mathrm{Ad}\, q_B(W^*S_1W) \circ (\widehat{e_1} \circ \Phi +_\beta \beta)]$$

in $\mathrm{Ext}^{-1}_{J,\beta}(A,B)$. Consider W as a unitary in $M(IB)$ via the map $\overline{c} : M(B) \to M(IB)$ from the proof of Lemma 4.3, and let $\Lambda : A \to Q(IB)$ be the extension given by

$$\Lambda = \mathrm{Ad}\, q_{IB}(W^* V_1 W) \circ (\Phi +_\gamma \gamma).$$

Then Λ is equal to γ on J. By assumption Φ is invertible which means that it represents an invertible element of $\mathrm{Ext}(A, IB)$. Hence Λ represents also an invertible element of $\mathrm{Ext}(A, IB)$. As we saw in the proof of Lemma 3.1 this implies that Λ represents an invertible element of $\mathrm{Ext}_{J,\gamma}(A, IB)$. It follows therefore from Lemma 4.1 that

$$[\mathrm{Ad}\, q_B(W^* S_0 W) \circ (\widehat{e_0} \circ \Phi +_\beta \beta)] = e_{0*}[\Lambda] = e_{1*}[\Lambda] = [\mathrm{Ad}\, q_B(W^* S_1 W) \circ (\widehat{e_1} \circ \Phi +_\beta \beta)]$$

in $\mathrm{Ext}^{-1}_J(A,B)$. Hence $[\psi] = [\varphi]$ in $\mathrm{Ext}^{-1}_J(A,B)$. ∎

The main virtue of Proposition 4.3, which we shall exploit below, is that it gives a description of the relative extension group without any reference to absorbing $*$-homomorphism. Furthermore, the description makes it easy to make the relative extension group functorial, covariantly in the “coefficient algebra” B, and contravariantly in the pair $J \subseteq A$.

5. MAPPING CONES AND THE RELATIVE EXTENSION GROUP

Let C_i be the mapping cone of the inclusion $i : J \to A$ which we realize as

$$C_i = \{f \in IA : f(0) = 0,\ f(s) \in J,\ s \in [\tfrac{1}{2}, 1]\}.$$

Let $\varphi : A \to Q(B)$ be an invertible extension of A by B which equals α on J. We can then choose a completely positive contraction $\xi : A \to M(B)$ such that $q_B \circ \xi = \varphi$. Note that $\xi(j) - \alpha_0(j) \in B$ for any $j \in J$ since $\varphi|_J = q_B \circ \alpha_0$. We define $\Phi : C_i \to IM(B)$ such that

$$\Phi(f)(s) = \begin{cases} \xi(f(s)) & s \in [0, \frac{1}{2}], \\ (2-2s)\xi(f(s)) + (2s-1)\alpha_0(f(s)) & s \in [\frac{1}{2}, 1]. \end{cases}$$

Then Φ is a completely positive contraction and $\Phi(fg) - \Phi(f)\Phi(g) \in SB$ for all $f, g \in C_i$. Thus $\mu(\varphi) = q_{SB} \circ \Phi : C_i \to Q(SB)$ is an invertible extension of C_i by SB. It is easy to see that we get a group homomorphism $\mu : \mathrm{Ext}^{-1}_{J,\alpha}(A,B) \to \mathrm{Ext}^{-1}(C_i, SB)$ such that $\mu[\varphi] = [\mu(\varphi)]$.

THEOREM 5.1. *$\mu : \mathrm{Ext}^{-1}_{J,\alpha}(A,B) \to \mathrm{Ext}^{-1}(C_i, SB)$ is an isomorphism.*

Proof. Let $\iota : SA \to C_i$ be the natural embedding, i.e.

$$\iota(g)(s) = \begin{cases} g(2s) & s \in [0, \frac{1}{2}], \\ 0 & s \in [\frac{1}{2}, 1], \end{cases}$$

and $p : C_i \to J$ the $*$-homomorphism $p(f) = f(1)$. By comparing the six-term exact sequence of Theorem 3.3 with the Puppe sequence of [5] we see that the five-lemma will give the theorem if we show that the diagram

$$
\begin{tikzcd}
\operatorname{Ext}^{-1}(J, SB) \arrow[r, "\partial_0"] \arrow[d, equal] & \operatorname{Ext}^{-1}_{J,\alpha}(A, B) \arrow[r] \arrow[d, "\mu"] & \operatorname{Ext}^{-1}(A, B) \arrow[d, equal] \\
\operatorname{Ext}^{-1}(J, SB) \arrow[r, "p^*"] & \operatorname{Ext}^{-1}(C_i, SB) \arrow[r, "S^{-1}\circ\iota^*"] & \operatorname{Ext}^{-1}(A, B)
\end{tikzcd}
\tag{5.1}
$$

commutes, where S^{-1} is the inverse of the suspension isomorphism

$$S : \operatorname{Ext}^{-1}(A, B) \to \operatorname{Ext}^{-1}(SA, SB).$$

To this end, only the left square requires some care. To prove commutativity here we consider a unitary v in $M(B)$ such that $v\alpha_0(j) - \alpha_0(j)v \in B$ for any $j \in J$. Under the isomorphism $K_1(D_\alpha(J)) \simeq \operatorname{Ext}^{-1}(J, SB)$, v becomes the extension $\psi = q_{SB} \circ \psi_0$, where $\psi_0 : J \to IM(B)$ is given by $\psi_0(j)(s) = (1 - s)v\alpha_0(j)v^* + s\alpha_0(j)$. Hence $p^*[v]$ is represented by the extension $q_{SB} \circ \Phi : C_i \to Q(SB)$, where $\Phi : C_i \to IM(B)$ is given by

$$\Phi(f)(s) = (1 - s)v\alpha_0(f(1))v^* + s\alpha_0(f(1)).$$

For comparison $\mu \circ \partial_0[v]$ is represented by the extension $q_{SB} \circ \Psi : C_i \to Q(SB)$, where $\Psi : C_i \to IM(B)$ is given by

$$\Psi(f)(s) = \begin{cases} v\alpha_0(f(s))v^* & s \in [0, \frac{1}{2}], \\ (2 - 2s)v\alpha_0(f(s))v^* + (2s - 1)\alpha_0(f(s)) & s \in [\frac{1}{2}, 1]. \end{cases}$$

Set $h_\lambda(s) = \max\{\lambda, s\}$, and define $\Lambda : C_i \to I^2M(B)$ such that

$$\Lambda(f)(\lambda, s) = \begin{cases} v\alpha_0(f(h_\lambda(s)))v^* & s \in [0, \frac{1}{2}], \\ (2 - 2s)v\alpha_0(f(h_\lambda(s)))v^* + (2s - 1)\alpha_0((h_\lambda(s))) & s \in [\frac{1}{2}, 1]. \end{cases}$$

Then $q_{ISB} \circ \Lambda$ is an extension of C_i by ISB which gives us a homotopy between $q_{SB} \circ \Psi$ and $q_{SB} \circ \Psi'$, where

$$\Psi'(f)(s) = \begin{cases} v\alpha_0(f(1))v^* & s \in [0, \frac{1}{2}], \\ (2 - 2s)v\alpha_0(f(1))v^* + (2s - 1)\alpha_0(f(1)) & s \in [\frac{1}{2}, 1]. \end{cases}$$

It is easy to construct a homotopy of invertible extensions connecting $q_{SB} \circ \Psi'$ to $q_{SB} \circ \Phi$, and we therefore conclude that the diagram (5.1) commutes. ∎

Theorem 5.1 has many consequences for the relative extension group. One is that $\operatorname{Ext}^{-1}_{J,\alpha}(A, B) = \operatorname{Ext}^{-1}(A/J, B)$ when J is a semi-split ideal. Another virtue of Theorem 5.1 is that it makes it easy to give the following description of the map ∂_0 from Theorem 3.3 — a description which we shall need in Section 7.

LEMMA 5.2. *When the group* $\mathrm{Ext}^{-1}(J, SB)$ *is identified with* $KK(J, B)$ *in the Cuntz picture, and* $\mathrm{Ext}^{-1}_{J,\alpha}(A, B)$ *is identified with* $\mathrm{Ext}^{-1}_{J}(A, B)$ *via the isomorphism* (4.1), *we have that* $\partial_0[\varphi_+|_J, \varphi_-] = [q_B \circ \varphi_+, \varphi_-]$, *where* $\varphi_+ : A \to M(B)$ *and* $\varphi_- : J \to M(B)$ *are* $*$*-homomorphisms such that* $\varphi_+(j) - \varphi_-(j) \in B$ *for all* $j \in J$.

Proof. Note that the map (4.1) was defined for a particular absorbing $*$-homomorphism β_0. Let $u \in M(B)$ be a unitary such that $\mathrm{Ad}\, u \circ \alpha_0(a) - \beta_0(a) \in B$ for all $a \in A$. There is then an isomorphism $\mathrm{Ext}^{-1}_{J,\alpha}(A, B) \to \mathrm{Ext}^{-1}_{J,\beta}(A, B)$ defined by $[\varphi] \mapsto [\mathrm{Ad}\, q_B(u) \circ \varphi]$. By composing with the isomorphism (4.1) we obtain an isomorphism $\nu : \mathrm{Ext}^{-1}_{J,\alpha}(A, B) \to \mathrm{Ext}^{-1}_{J}(A, B)$. When (ψ, ψ_0) is an invertible extension of A by B which splits over J we can define $\Psi : C_i \to IM(B)$ such that

$$\Psi(f)(s) = \begin{cases} \xi(f(s)) & s \in [0, \frac{1}{2}], \\ (2-2s)\xi(f(s)) + (2s-1)\varphi_0(f(s)) & s \in [\frac{1}{2}, 1], \end{cases}$$

where $\xi : A \to M(B)$ is a completely positive contractive lift of φ. Then $q_{SB} \circ \Psi$ is an invertible extension of C_i by SB and we can define a homomorphism $\mu' : \mathrm{Ext}^{-1}(J, SB) \to \mathrm{Ext}^{-1}(C_i, SB)$ such that $\mu'[\psi, \psi_0] = [q_{SB} \circ \Psi]$. Then the diagram

(5.2)
$$\begin{tikzcd} \mathrm{Ext}^{-1}(J, SB) \arrow[r, "\partial_0"] \arrow[d, equal] & \mathrm{Ext}^{-1}_{J,\alpha}(A, B) \arrow[r, "\nu"] \arrow[d, "\mu"] & \mathrm{Ext}^{-1}_{J}(A, B) \arrow[dl, "\mu'"] \\ \mathrm{Ext}^{-1}(J, SB) \arrow[r, "p^*"] & \mathrm{Ext}^{-1}(C_i, SB) & \end{tikzcd}$$

commutes. The commutativity of the square was established in the proof of Theorem 5.1 and it is easy to see that the triangle commutes. The diagram (5.2) gives us the lemma in the following way: The element of $\mathrm{Ext}^{-1}(J, SB)$ corresponding to the Cuntz pair $(\varphi_+|_J, \varphi_-)$ is represented by $q_{SB} \circ \Phi$, where $\Phi : J \to IM(B)$ is given by $\Phi(j)(s) = (1-s)\varphi_+(j) + s\varphi_-(j)$. Thus $p^*[\varphi_+|_J, \varphi_-]$ is represented by $q_{SB} \circ \Lambda$, where $\Lambda(f)(s) = (1-s)\varphi_+(f(1)) + s\varphi_-(f(1))$. For comparison $\mu'[q_B \circ \varphi_+, \varphi_-]$ is represented by the extension $q_{SB} \circ \Psi'$ where

$$\Psi'(f)(s) = \begin{cases} \varphi_+(f(s)) & s \in [0, \frac{1}{2}], \\ (2-2s)\varphi_+(f(s)) + (2s-1)\varphi_-(f(s)) & s \in [\frac{1}{2}, 1]. \end{cases}$$

Homotopies almost identical to the ones used in the proof of Theorem 5.1 now prove that $p^*[\varphi_+|_J, \varphi_-] = \mu'[q_B \circ \varphi_+, \varphi_-]$. The conclusion of the lemma follows then from the commutativity in (5.2) because μ' is injective. ∎

Under our standing assumption that there is an absorbing $*$-homorphism $A \to M(B)$ whose restriction to J is also absorbing, every element of $KK(J, B)$ can be represented by a Cuntz pair of the form considered in Lemma 5.2.

6. WHEN A AND J HAVE THE SAME UNIT

Assume now that the pair $J \subseteq A$ share the same unit $1 \in J$. It is then often natural and convenient to consider extensions that are unital. This is certainly the case for the relative K-homology of compact spaces which we investigate in the following section. In the present section we describe how to adjust the definitions and results of the previous sections in order to "fix the unit". Most of the considerations are standard, so we will be brief (but, hopefully, precise).

First of all the role of the absorbing $*$-homomorphisms must now be taken by the *unitally* absorbing $*$-homomorphisms. The first lemma shows that this does not effect the fundamental condition of Section 3.

LEMMA 6.1. *There is an absorbing $*$-homomorphism $\alpha_0 : A \to M(B)$ such that $\alpha_0|_J : J \to M(B)$ is also absorbing if and only if there is a unitally absorbing $*$-homomorphism $\beta_0 : A \to M(B)$ such that $\beta_0|_J : J \to M(B)$ is also unitally absorbing.*

Proof. Assume first that α_0 exists. Since there exists a unitally absorbing $*$-homomorphism $A \to M(B)$, [15], it follows from Lemma 1.1 in [10] that there is an absorbing $*$-homomorphism $A \to M(B)$ such that the image of 1 is the range projection of an isometry in $M(B)$, and then it follows from the essential uniqueness of absorbing $*$-homomorphisms that this is the case for all of them. In particular, there is an isometry $W \in M(B)$ such that $WW^* = \alpha_0(1)$. Then $W^*\alpha_0(\cdot)W : A \to M(B)$ is a unital $*$-homomorphism and we claim that it is unitally absorbing. To show this we check that condition 1) of Theorem 2.1 of [15] is fullfilled. Let $\varphi : A \to B$ be a completely positive contraction. Extend φ to $A^+ = A \oplus \mathbb{C}$ by annihilating the direct summand $\mathbb{C}$. Since $\alpha_0^+ : A^+ \to M(B)$ is unitally absorbing by Theorem 2.7 of [15], there is a sequence $\{W_n\} \subseteq M(B)$ such that $\lim_{n\to\infty} W_n^* b = 0$ for all $b \in B$ and $\lim_{n\to\infty} W_n^* \alpha_0(a) W_n = \varphi(a)$ for all $a \in A$. It follows that $\lim_{n\to\infty} W_n^* W b = 0$ for all $b \in B$ and $\lim_{n\to\infty} W_n^* WW^* \alpha_0(a) WW^* W_n = \varphi(a)$ for all a, verifying that $W^*\alpha_0(\cdot)W$ is unitally absorbing. The same argument applies to its restriction to J, and hence $\beta_0 = W^*\alpha_0(\cdot)W$ is unitally absorbing on both A and J.

Conversely, when $\beta_0 : A \to M(B)$ is unitally absorbing on both A and J, Lemma 1.1 of [10] shows that there is an isometry $V \in M(B)$ such that $\alpha_0 = \operatorname{Ad} V \circ \beta_0$ is absorbing on both A and J. ∎

Assume now that $\beta_0 : A \to M(B)$ is a unitally absorbing $*$-homomorphism such that $\beta_0|_J : J \to M(B)$ is also unitally absorbing. Set $\beta = q_B \circ \beta_0$. We say that two unital extensions, $\varphi, \psi : A \to Q(B)$, that are equal to β on J, are unitarily equivalent when there is a unitary connected to 1 in the unitary group of $\beta(J)' \cap Q(B)$ such that $\operatorname{Ad} v \circ \varphi = \psi$. By repeating the arguments that proved Lemma 2.4 we find that the addition $+_\beta$, defined using isometries from $\beta(A)' \cap Q(B)$, gives the unitary equivalence classes of unital extensions of A by B which equal β on J the structure of an abelian semi-group. We define stable equivalence in this setting in the natural way: φ and ψ are stably equivalent, as unital extensions which equal β on J, when $\varphi +_\beta \alpha$ is unitarily equivalent to $\psi +_\beta \alpha$. The stable equivalence classes of unital extensions of A by B which equal β on J is then an abelian semi-group with 0, and the invertible elements of this semi-group form an abelian group which we denote by $\mathrm{Ext}^{-1}_{J,\beta}(A,B)$. Let V,W be isometries in $M(B)$ such that $VV^* + WW^* = 1$. By Lemma 1.1 of [10], $\alpha_0 = \mathrm{Ad}\, V \circ \beta_0$ will be absorbing on both A and J and we can define a group homomorphism $\chi_0 : \mathrm{Ext}^{-1}_{J,\beta}(A,B) \to \mathrm{Ext}^{-1}_{J,\alpha}(A,B)$ such that $\chi_0[\varphi] = [\mathrm{Ad}\, q_B(V) \circ \varphi]$. In the other direction, if $\psi : A \to Q(B)$ is an extension which equals α on J, note that $\mathrm{Ad}\, q_B(V)^* \circ \psi$ is a unital extension of A by B which equals β on J. We define a homomorphism $\chi_1 : \mathrm{Ext}^{-1}_{J,\alpha}(A,B) \to \mathrm{Ext}^{-1}_{J,\beta}(A,B)$ such that $\chi_1[\psi] = [\mathrm{Ad}\, q_B(V)^* \circ \psi]$. It is easy to see that χ_1 is the inverse of χ_0. Hence we see that the unital version of the relative extension group agrees with non-unital version.

Note that in the particular case where A is unital and $J = \mathbb{C}1 \subseteq A$, the group $\mathrm{Ext}^{-1}_{J,\beta}(A,B)$ is the group $\mathrm{Ext}^{-1}_{\mathrm{unital}}(A,B)$ considered in [10], and the six-term exact sequence of Theorem 3.3 is then the six-term exact sequence of Skandalis, [14], a construction of which was exhibited in [10]. In fact, in the present setting where J and A have a common unit the six-term exact sequence of Theorem 3.3 can be modified so that the involved extension groups are "unital" in the sense that they are defined using unital extensions only. The key point for this is that since β_0 is unitally absorbing, the isomorphism (3.2) can be substituted by an isomorphism $K_1(\beta(A)' \cap Q(B)) \simeq KK(A,B)$ because $\mathcal{D}_\alpha(A)/\mathcal{X}_\alpha(A) \simeq \beta(A)' \cap Q(B)$, cf. [10]. By using this isomorphism in place of (3.2) and the isomorphism $K_1(\beta(J)' \cap Q(B)) \simeq KK(J,B)$ in place of (3.1), the proof of Theorem 3.3 can easily be adopted to yield the following six-term exact sequence in the present case:

$$\begin{tikzcd}
\mathrm{Ext}^{-1}_J(A,B) \arrow[r] & \mathrm{Ext}^{-1}_{\mathrm{unital}}(A,B) \arrow[r, "i^*"] & \mathrm{Ext}^{-1}_{\mathrm{unital}}(J,B) \arrow[d, "\partial_1"] \\
\mathrm{Ext}^{-1}_{\mathrm{unital}}(J,SB) \arrow[u, "\partial_0"] & \mathrm{Ext}^{-1}_{\mathrm{unital}}(A,SB) \arrow[l, "i^*"'] & \mathrm{Ext}^{-1}_J(A,SB). \arrow[l]
\end{tikzcd}$$

As one would expect by now, the alternative picture of the relative extension group given in Section 4 is also not changed by restricting attention entirely to unital extensions. This will be very useful in the following, so let us make it precise: By a unital extension of A by B which splits over J we mean a pair (φ, φ_0) where $\varphi : A \to Q(B)$ is a unital extension of A by B, and $\varphi_0 : J \to M(B)$ is a unital $*$-homomorphism such that $q_B \circ \varphi_0 = \varphi|_J$. Recall that if φ is invertible, it is actually unitally invertible, i.e. there is another unital extension ψ of A by B with the property that $\varphi \oplus \psi$ is a split extension (of A by B). Two unital invertible extensions, (φ, φ_0) and (ψ, ψ_0), of A by B which split over J are now homotopic in norm when there is a path $(\Phi^t, \Phi_0^t), t \in [0,1]$, of unital invertible extensions of A by B which split over J such that $[0,1] \ni t \mapsto \Phi^t(a)$ is norm-continuous for all $a \in A$ and $[0,1] \ni t \mapsto \Phi_0^t(j)$ is norm-continuous for all $j \in J$, $(\Phi^0, \Phi_0^0) = (\varphi, \varphi_0)$ and $(\Phi^1, \Phi_0^1) = (\psi, \psi_0)$. We say that (φ, φ_0) and (ψ, ψ_0) are stably homotopic in norm when there is a unital $*$-homomorphism $\pi : A \to M(B)$ such that $(\varphi \oplus q_B \circ \pi, \varphi_0 \oplus \pi|_J)$ and $(\psi \oplus q_B \circ \pi, \psi_0 \oplus \pi|_J)$ are homotopic in norm. The stable homotopy classes of unital invertible extensions of A by B which split over J form a group which we temporarily denote by $\mathrm{Ext}_J^{-1,+}(A,B)$.

LEMMA 6.2. *The forgetful map* $\mathrm{Ext}_J^{-1,+}(A,B) \to \mathrm{Ext}_J^{-1}(A,B)$ *is an isomorphism.*

Proof. Surjectivity: Let (φ, φ_0) be an invertible extension of A by B which splits over J. By adding $(0,0)$ we do not change the class of (φ, φ_0), but reach a situation where there is a $*$-homomorphism $\pi : A \to M(B)$ such that $\pi(1) = \varphi_0(1)^\perp$. Then, with $\overline{\Theta}$ the $*$-isomorphism from the proof of Lemma 2.3, $w = \overline{\Theta}\begin{pmatrix} \varphi_0(1) & \pi(1) \\ 0 & 0 \end{pmatrix}$ is a partial isometry such that $\mathrm{Ad}\, q_B(w) \circ (\varphi \oplus q_B \circ \pi)$ and $\mathrm{Ad}\, w \circ (\varphi_0 \oplus \pi|_J)$ are both unital. By choosing a unitary dilation of w and using that the unitary group of $M(B)$ is connected in the norm-topology, [11], [4], we see that the class of (φ, φ_0) in $\mathrm{Ext}_J^{-1}(A,B)$ is also represented by the unital pair

$$(\mathrm{Ad}\, q_B(w) \circ (\varphi \oplus q_B \circ \pi), \mathrm{Ad}\, w \circ (\varphi_0 \oplus \pi|_J)).$$

Injectivity: Let (φ, φ_0) and (ψ, ψ_0) be unital and invertible extensions of A by B which split over J and define the same element of $\mathrm{Ext}_J^{-1}(A,B)$. After the addition of a pair $(q_B \circ \pi, \pi|_J)$ we have a homotopy in norm, $(\Phi_t, \Phi_t^0), t \in [0,1]$, connecting $(\varphi \oplus q_B \circ \pi, \varphi_0 \oplus \pi|_J)$ to $(\psi \oplus q_B \circ \pi, \psi_0 \oplus \pi|_J)$. Standard arguments show that there is a norm-continuous path $U_t, t \in [0,1]$, of unitaries in $M(B)$ such that $\Phi_t^0(1) = U_t p U_t^*$ for all $t \in [0,1]$, where $p = 1 \oplus \pi(1)$. Note that $U_0 p U_0^* = U_1 p U_1^* = p$. By the same trick of adding $(0,0)$ as above, we can arrange that there is a $*$-homomorphism $\chi : A \to M(B)$ such $\chi(1) = p^\perp$. Define $\Psi_t^0(j) = \Phi_t^0(j) + \mathrm{Ad}\, U_t \circ \chi(j)$ and $\Psi_t(a) = \Phi_t(a) + \mathrm{Ad}\, q_B(U_t) \circ q_B \circ \chi(a)$. Then (Ψ_t, Ψ_t^0) is a homotopy in norm connecting

$$((\varphi \oplus q_B \circ \pi) + q \circ \mathrm{Ad}\, U_0 \circ \chi, (\varphi_0 \oplus \pi|_J) + \mathrm{Ad}\, U_0 \circ \chi|_J)$$

to

$$((\psi \oplus q_B \circ \pi) + q_B \circ \mathrm{Ad}\, U_1 \circ \chi, (\psi_0 \oplus \pi|_J) + \mathrm{Ad}\, U_1 \circ \chi|_J).$$

Since there are unital $*$-homomorphisms $\pi_\pm : A \to M(B)$ such that

$$((\varphi \oplus q_B \circ \pi) + q_B \circ \mathrm{Ad}\, U_0 \circ \chi, (\varphi_0 \oplus \pi) + \mathrm{Ad}\, U_0 \circ \chi|_J) = (\varphi \oplus q_B \circ \pi_+, \varphi_0 \oplus \pi_+|_J),$$
$$((\psi \oplus q_B \circ \pi) + q_B \circ \mathrm{Ad}\, U_1 \circ \chi, (\psi_0 \oplus \pi) + \mathrm{Ad}\, U_1 \circ \chi|_J) = (\psi \oplus q_B \circ \pi_-, \psi_0 \oplus \pi_-|_J),$$

we conclude that (φ, φ_0) and (ψ, ψ_0) define the same element of $\mathrm{Ext}_J^{-1,+}(A,B)$. ∎

Lemma 6.2 is our excuse for not distinguishing between $\mathrm{Ext}_J^{-1,+}(A,B)$ and $\mathrm{Ext}_J^{-1}(A,B)$ in the following.

7. RELATIVE K-HOMOLOGY FOR SPACES

Fix a separable infinite-dimensional Hilbert space H and denote by $\mathbb{L}(H)$ the algebra of bounded operators on H, and by $\mathbb{K}$ the ideal in $\mathbb{L}(H)$ consisting of the compact operators. In this section we will study the relative extension group $\mathrm{Ext}_J^{-1}(A,B)$ in the case when the "coefficient algebra" is $B=\mathbb{K}$ and $J\subseteq A$ is a unital inclusion of one abelian C^*-algebra into another. Since the coefficients are now fixed we drop them in the notation. In the same vein we write Q for the Calkin algebra and $q:\mathbb{L}(H)\to Q$ for the quotient map. In order to draw directly on the work of Brown, Douglas and Fillmore, [2], we will use the "unital version" of the relative extension group, as explained in Section 6.

LEMMA 7.1. *Let $A\subseteq\mathbb{L}(H)$ be a separable C^*-subalgebra such that $q(A)\subseteq Q$ is abelian. Let $\omega_i: q(A)\to\mathbb{C}, i=1,2,\dots,$ be a sequence of characters of $q(A)$, and let $\{a_i\}$ be a dense sequence in A.*

There is then a family of continuous maps $\chi_i:[1,\infty)\to H, i{=}1,2,3,\dots,$ such that:

(i) $(\chi_1(t),\chi_2(t),\chi_3(t),\dots)$ *is an orthonormal set for all* $t\in[1,\infty)$;

(ii) $\chi_i(t)=\chi_i(s), i\geqslant n,$ *when* $s,t\in[1,n]$;

(iii) $\sum_{i=1}^{\infty}\|a_i\chi_i(t)-\omega_i(q(a_i))\chi_i(t)\|^2<\infty$ *for all* $t\in[1,\infty)$;

(iv) $\lim_{t\to\infty}\sup_i\|a\chi_i(t)-\omega_i(q(a))\chi_i(t)\|=0$ *for all* $a\in A$; *and*

(v) $\lim_{t\to\infty}\sup_i\|k\chi_i(t)\|=0$ *for all* $k\in\mathbb{K}$.

Proof. Let $\{\mu_i\}$ be a sequence of characters on $q(A)$ with the property that each ω_j occurs infinitely many times in $\{\mu_i\}$, and let $b_1,b_2,b_3,\dots,$ be a dense sequence in $\mathbb{K}$. By Lemma 1.3 of [2] there is an orthonormal sequence $\psi_i, i=1,2,\dots,$ in H such that

$$\|a_k\psi_i-\mu_i(q(a_k))\psi_i\|\leqslant 2^{-i}$$

when $k\leqslant i$. Since each ω_j occurs infinitely often in $\{\mu_i\}$ we can select subsequences $\{\varphi_i^k\}_{i=1}^{\infty}, k=1,2,3,\dots,$ from $\{\psi_i\}$ such that:

(a) $\{\varphi_i^k\}_{i=1}^{\infty}\cap\{\varphi_i^{k'}\}_{i=1}^{\infty}=\emptyset$ when $k\neq k'$; and

(b) $\sup_i\|a_j\varphi_i^k-\omega_i(q(a_j))\varphi_i^k\|\leqslant\frac{1}{k}$;

(c) $\sup_i\|b_j\varphi_i^k\|\leqslant\frac{1}{k}$;

for all $j = 1, 2, \ldots, k$. Set

$$\gamma_i^k = \begin{cases} \varphi_i^k & i \leqslant k, \\ \varphi_i^1 & i > k, \end{cases} \quad \text{and} \quad \chi_i(t) = \sqrt{t-n}\,\gamma_i^{n+1} + \sqrt{n+1-t}\,\gamma_i^n$$

when $t \in [n, n+1]$. It is straigthforward to show that $\{\chi_i\}$ has the properties (i)–(v). ∎

Let X and Y be compact Hausdorff spaces and $f : X \to Y$ a continuous surjection. There is then a unital embedding $i : C(Y) \to C(X)$ such that $i(g) = g \circ f$, and we will sometimes identify $C(Y)$ with $i(C(Y)) \subseteq C(X)$. It follows from [9] that any unital $*$-homomorphism $\alpha_0 : C(X) \to \mathbb{L}(H)$ such that $\alpha = q \circ \alpha_0$ is injective, is also unitally absorbing. In particular, it follows that $\alpha_0|_{C(Y)}$ is unitally absorbing whenever α_0 is. We fix a unitally absorbing $*$-homomorphism $\alpha_0 : C(X) \to \mathbb{L}(H)$, and denote $\mathrm{Ext}_{C(Y),\alpha}(C(X), \mathbb{K})$ by

$$\mathrm{Ext}_{Y,f}(X).$$

It is this group we shall investigate in this section.

LEMMA 7.2. *Let $\varphi : C(X) \to Q$ be an injective and unital extension of $C(X)$ by $\mathbb{K}$. Let $\varphi' : C(X) \to \mathbb{L}(H)$ be a continuous map such that $q_{\mathbb{K}} \circ \varphi' = \varphi$. There is then a continuous path $V_t, t \in [1, \infty)$, of unitaries in $\mathbb{L}(H)$ such that:*

(i) $\mathrm{Ad}\, q(V_t) \circ (\varphi \oplus \alpha) = \varphi$ *for all* $t \in [1, \infty)$;

(ii) $\lim\limits_{t\to\infty} V_t(\varphi' \oplus \alpha_0)(g)V_t^* = \varphi'(g)$ *for all* $g \in C(X)$;

(iii) $V_1 - V_t \in \mathbb{K}$ *for all* $t \in [1, \infty)$.

Proof. We pick a dense sequence $\{x_i\}_{i=1}^\infty$ in X and an orthonormal basis $\{e_i\}_{i=1}^\infty$ in H. We arrange that each point x_j is repeated infinitely many times in $\{x_i\}_{i=1}^\infty$. Define $D : C(X) \to \mathbb{L}(H)$ such that $D(g)\psi = \sum\limits_{i=1}^\infty g(x_i)\langle \psi, e_i\rangle e_i$. Apply then Lemma 7.1 with A the C^*-algebra generated by $\varphi'(C(X))$ and $\omega_i(a) = \varphi^{-1}(a)(x_i)$ to get the continuous functions $\chi_i : [1, \infty) \to H$ with the properties listed there. Let $\sigma_1, \sigma_2 : \mathbb{N} \to \mathbb{N}$ be injective maps such that $\sigma_1(\mathbb{N}) \cap \sigma_2(\mathbb{N}) = \emptyset$, $\mathbb{N} = \sigma_1(\mathbb{N}) \cup \sigma_2(\mathbb{N})$ and $x_i = x_{\sigma_1(i)} = x_{\sigma_2(i)}$ for all $i \in \mathbb{N}$ (such maps exist because each x_j is repeated infinitely many times).

Set $P_t\psi = \sum\limits_{i=1}^\infty \langle \psi, \chi_i(t)\rangle \chi_i(t)$. Then P_t is a projection and we define $V_t : P_tH \to H$ such that $V_t\chi_i(t) = \chi_{\sigma_1(i)}(t)$. Define isometries $S_t, T_t : H \to H$ such that $S_t = P_t^\perp + V_tP_t$ and $T_te_i = \chi_{\sigma_2(i)}(t)$. It follows from the properties (i)–(v) of $\{\chi_i\}$ that both $t \mapsto S_t$ and $t \mapsto T_t$ are continuous in norm, that $S_t\varphi'(g)S_t^* + T_tD(g)T_t^* - \varphi'(g) \in \mathbb{K}$ and that $\lim\limits_{t\to\infty} S_t\varphi'(g)S_t^* + T_tD(g)T_t^* = \varphi'(g)$. Let V_1, V_2 be the isometries used to define the addition $\oplus$, and set $U_t = S_tV_1^* + T_tV_2^*$. Then

(a) $U_t(\varphi' \oplus D)(g)U_t^* - \varphi'(g) \in \mathbb{K}$ for all $t \in [1, \infty)$ and all $g \in C(X)$;

(b) $\lim\limits_{t\to\infty} U_t(\varphi' \oplus D)(g)U_t^* = \varphi'(g)$ for all $g \in C(X)$; and

(c) $U_1 - U_t \in \mathbb{K}$ for all $t \in [1, \infty)$.

By Theorem 3.11 of [6] there is is a norm-continuous path $W_t, t \in [1, \infty)$, of unitaries in $\mathbb{L}(H)$ such that $\operatorname{Ad} W_t \circ D(g) - \alpha_0(g) \in \mathbb{K}$ for all $t \in [1, \infty), g \in C(X)$, and $\lim_{t\to\infty} \operatorname{Ad} W_t \circ D(g) = \alpha_0(g)$ for all $g \in C(X)$. It follows then from Lemma 4.2 that the Cuntz-pair $(\operatorname{Ad} W_1 \circ D, \alpha_0)$ represents zero in $KK(C(X), \mathbb{K})$, and then an application of Theorem 3.12 of [6] shows that we can assume that $W_t - W_1 \in \mathbb{K}$ for all t. Set $V_t = U_t(1 \oplus W_t^*)$. ∎

THEOREM 7.3. *Let $\varphi : C(X) \to Q(\mathbb{K})$ be an injective and unital extension of $C(X)$ by $\mathbb{K}$. Assume that there is a unital $*$-homomorphism $\varphi_0 : C(Y) \to \mathbb{L}(H)$ such that $\varphi \circ i = q \circ \varphi_0$. Then the following are equivalent:*

(i) *(φ, φ_0) represents zero in $\operatorname{Ext}_{Y,f}(X)$.*

(ii) *There is a path $\psi_t, t \in [1, \infty)$, of $*$-homomorphisms $\psi_t : C(X) \to \mathbb{L}(H)$ such that $t \mapsto \psi_t(g)$ is continuous for all $g \in C(X)$, $\varphi = q \circ \psi_t$ for all $t \in [1, \infty)$, and $\lim_{t\to\infty} \psi_t \circ i(h) = \varphi_0(h)$ for all $h \in C(Y)$.*

(iii) *There is a sequence of $*$-homomorphisms $\psi_n : C(X) \to \mathbb{L}(H)$ such that $q \circ \psi_n = \varphi$ for all n, and $\lim_{n\to\infty} \psi_n \circ i(h) = \varphi_0(h)$ for all $h \in C(Y)$.*

Proof. The implication (ii) ⇒ (iii) is trivial so it suffices to prove that (i) ⇒ (ii) and that (iii) ⇒ (i). First (i) ⇒ (ii): It follows from Section 6 that there is a unital $*$-homomorphism $\pi : C(X) \to \mathbb{L}(H)$ such that $(\varphi \oplus q_B \circ \pi, \varphi_0 \oplus \pi|_{C(Y)})$ is homotopic in norm, as a unital extension of $C(X)$ by $\mathbb{K}$ which splits over $C(Y)$, to the pair $(q_B \circ \pi, \pi|_{C(Y)})$. It follows then from [2] that φ represents zero in $\operatorname{Ext}(X)$. There is therefore a unital $*$-homomorphism $\varphi' : C(X) \to \mathbb{L}(H)$ such that $q \circ \varphi' = \varphi$. Then $(\varphi' \circ i, \varphi_0)$ is a Cuntz pair and from the description of ∂_0 given in Lemma 5.2, we find that $\partial_0[\varphi' \circ i, \varphi_0] = [\varphi, \varphi_0]$. Since $[\varphi, \varphi_0] = 0$, the six-term exact sequence of Theorem 3.3 shows that $[\varphi' \circ i, \varphi_0] = i^*[\psi_+, \psi_-]$, where $\psi_\pm : C(X) \to \mathbb{L}(H)$ are (not necessarily unital) $*$-homomorphisms such that $\psi_+(g) - \psi_-(g) \in \mathbb{K}$ for all $g \in C(X)$. By adding the same $*$-homomorphism to ψ_+ and ψ_- we may assume that $q \circ \psi_\pm$ are both injective. Since $[\varphi' \circ i \oplus \psi_- \circ i, \varphi_0 \oplus \psi_+ \circ i] = 0$ in $KK(C(Y), \mathbb{K})$ and φ' and φ_0 are both unital it follows that $[\psi_+(1), \psi_-(1)] = 0$ in $KK(\mathbb{C}, \mathbb{K})$. Thus also $[1 - \psi_+(1), 1 - \psi_-(1)] = 0$ in $KK(\mathbb{C}, \mathbb{K})$. Let $\chi : C(X) \to \mathbb{C}$ be any character. Then $[\chi(1 - \psi_+(1)), \chi(1 - \psi_-(1))] = 0$ in $KK(C(X), \mathbb{K})$. It follows that $[\psi_+, \psi_-] = [\psi_+, \psi_-] + [\chi(1 - \psi_+(1)), \chi(1 - \psi_-(1))]$ can be represented by a Cuntz pair $\gamma_\pm$ of unital $*$-homomorphisms $\gamma_\pm : C(X) \to \mathbb{L}(H)$ such that $q \circ \gamma_+$ and $q \circ \gamma_-$ are both injective. Since $[\varphi' \circ i \oplus \gamma_- \circ i, \varphi_0 \oplus \gamma_+ \circ i] = 0$ in $KK(C(Y), \mathbb{K})$, it follows from Theorem 3.12 of [6] there is a continuous path $V_t, t \in [1, \infty)$, of unitaries in $1 + \mathbb{K}$ such that

$$\lim_{t\to\infty} \operatorname{Ad} V_t \circ (\varphi' \circ i \oplus \gamma_- \circ i) = \varphi_0 \oplus \gamma_+ \circ i.$$

Set $\gamma = q \circ \gamma_+ = q \circ \gamma_-$. It follows from Lemma 7.2 that there are paths of unitaries U_t, W_t, $t \in [1, \infty)$, such that:

$$\begin{aligned} &\operatorname{Ad} q(W_t)(\varphi \oplus \gamma) = \varphi = \operatorname{Ad} q(U_t)(\varphi \oplus \gamma) \quad \text{for all } t, \\ &\lim_{t\to\infty} \operatorname{Ad} W_t(\varphi' \oplus \gamma_-)(g) = \varphi'(g) \quad \text{for all } g \in C(X), \text{ and} \\ &\lim_{t\to\infty} \operatorname{Ad} U_t(\varphi_0 \oplus \gamma_+ \circ i)(h) = \varphi_0(h) \quad \text{for all } h \in C(Y). \end{aligned}$$

Set $T_t = U_t V_t W_t^*$ and $\psi_t = \operatorname{Ad} T_t \circ \varphi'$.

(iii) $\Rightarrow$ (i): Let $\Psi : C(X) \to \mathbb{L}(H)$ be the $*$-homomorphism obtained from the representation $\operatorname{diag}(\psi_1, \psi_2, \psi_3, \dots)$. Then $(q \circ \Psi, \Psi)$ represents zero in $\operatorname{Ext}_{Y,f}(X)$. Let $\varphi' : C(X) \to \mathbb{L}(H)$ be a continuous function such that $q \circ \varphi' = \varphi$ and $\varphi' \circ i = \varphi_0$. Let χ_t, $t \in [1, \infty)$, be the path of maps such that $\chi_t, t \in [n, n+1]$, connects

$$\operatorname{diag}(\psi_1, \psi_2, \dots, \psi_{n-1}, \varphi', \psi_n, \psi_{n+1}, \dots\dots)$$

to

$$\operatorname{diag}(\psi_1, \psi_2, \dots, \psi_n, \varphi', \psi_{n+1}, \psi_{n+2}, \dots\dots)$$

by rotation. Considered as maps $\chi_t : C(X) \to \mathbb{L}(H)$ they give us a path of maps such that $q \circ \chi_t = \varphi \oplus q \circ \Psi$ while $\chi_t \circ i$ is a $*$-homomophism, and $\lim_{t\to\infty} \chi_t \circ i(h) = \Psi(h)$ for all $h \in C(Y)$. It follows that $(\varphi, \varphi_0) \oplus (q \circ \Psi, \Psi)$ is homotopic in norm to $(q \circ \Psi, \Psi)$. ∎

Two injective unital extensions $\varphi, \psi : C(X) \to Q$ that equal α on $C(Y)$ are said to be *equivalent* when there is a norm-continuous path $V_t, t \in [1, \infty)$, of unitaries in $\mathbb{L}(H)$ such that:

(i) $\operatorname{Ad} q(V_t) \circ \varphi = \psi$ for all $t \in [1, \infty)$;
(ii) $\lim_{t\to\infty} V_t \alpha_0 \circ i(h) V_t^* = \alpha_0 \circ i(h)$ for all $h \in C(Y)$; and
(iii) $V_1 - V_t \in \mathbb{K}$ for all $t \in [1, \infty)$.

We write $\varphi \simeq \psi$ in this case.

THEOREM 7.4. *Let $\varphi, \psi : C(X) \to Q$ be injective unital extensions that equal α on $C(Y)$. Then the following are equivalent:*
(i) *φ and ψ define the same element of* $\operatorname{Ext}_{Y,f}(X)$.
(ii) *$\varphi \simeq \psi$.*
(iii) *There is a unitary V connected to* 1 *in the unitary group of a*

$$D = \{m \in \mathbb{L}(H) : \ m\alpha_0 \circ i(h) - \alpha_0 \circ i(h)m \in \mathbb{K},\ h \in C(Y)\}$$

such that $\operatorname{Ad} q(V) \circ \varphi = \psi$.

Proof. It is trivial that (iii) implies (i).

(i) $\Rightarrow$ (ii): Assume that $[\varphi] = [\psi]$ in $\operatorname{Ext}_{Y,f}(X)$. There is then a unitary V connected to 1 in the unitary group of D such that

$$\operatorname{Ad} q(V) \circ (\varphi +_\alpha \alpha) = \psi +_\alpha \alpha.$$

In particular, it follows that $(\beta_1, \beta_2) = (\operatorname{Ad} V \circ \alpha_0 \circ i, \alpha_0 \circ i)$ is a Cuntz pair, and since V is connected to 1 in the unitary group of D, the pair represents zero in $KK(C(Y), \mathbb{K})$. It follows therefore from Theorem 3.12 of [6] that there is a norm-continuous path $S_t, t \in [1, \infty)$, of unitaries in $1 + \mathbb{K}$ such that $\lim_{t\to\infty} S_t \beta_1(h) S_t^* - \beta_2(h) = 0$ for all $h \in C(Y)$. It follows that $U_t = S_t V, t \in [1, \infty)$, is a norm-continuous path of unitaries in $\mathbb{L}(H)$ giving rise to an equivalence $\varphi +_\alpha \alpha \simeq \psi +_\alpha \alpha$. By Lemma 7.2 $\varphi +_\alpha \alpha$ and $\psi +_\alpha \alpha$ are equivalent to φ and ψ, respectively. Thus φ and ψ are equivalent.

(ii) $\Rightarrow$ (iii): Let $V_t, t \in [1, \infty)$, be a continuous path of unitaries in $\mathbb{L}(H)$ giving rise to the equivalence $\varphi \simeq \psi$. It suffices to show that V_1 is connected to 1 in the unitary group of D. Note that the Cuntz pair $(\operatorname{Ad} V_1 \circ \alpha_0, \alpha_0)$ represents 0 in $KK(C(Y), \mathbb{K})$ by Lemma 4.2. It follows therefore from Paschke's duality result, [12], that $q(V_1)$ is connected to 1 in the unitary group of the relative commutant $(q \circ \alpha_0 \circ i(C(Y)))' \cap Q$. It follows that there is a unitary W connected to 1 in the unitary group of D such that $q(W) = q(V_1)$. Then $V_1 W^* \in 1 + \mathbb{K}$ and since the unitary group of $1 + \mathbb{K}$ is connected, we see that also V_1 is connected to 1 in the unitary group of D. ∎

It follows from Theorem 7.4 that $\operatorname{Ext}_{Y,f}(X)$ is the group of equivalence classes of injective (or essential) unital extensions of $C(X)$ by $\mathbb{K}$ that equal α on $C(Y)$ with the addition defined by (2.1).

8. NORMAL OPERATORS

In this section we use the results of the previous sections to prove Theorem 1.1 from the introduction. The key point is the following

THEOREM 8.1. *Let X be a compact metric space and Y a compact subset of the complex plane $\mathbb{C}$. Let $f : X \to Y$ be a continuous surjection. Then the map f_* : $K_0(X) \to K_0(Y)$ is surjective.*

To prove this recall that for every compact metric space Y there is a natural decomposition $K_0(C(Y)) = \widetilde{K}^0(Y) \oplus C(Y, \mathbb{Z})$, where the summand $\widetilde{K}^0(Y)$ is called the reduced K-theory of Y, at least when Y is connected. We say that Y *has trivial reduced K-theory* when $\widetilde{K}^0(Y) = 0$. It is well-known and easy to see that a compact subset of the complex plane has trivial reduced K-theory and trivial K^1-group. Therefore Theorem 8.1 is a consequence of the following more general result.

THEOREM 8.2. *Let X and Y be compact metric spaces and $f : X \to Y$ a continuous surjection. Assume the Y has trivial reduced K-theory and that* $\operatorname{Ext}(K^1(Y), \mathbb{Z}) = 0$. *Then the map $f_* : K_0(X) \to K_0(Y)$ is surjective.*

Proof. Let
$$X = X_1^n \sqcup X_2^n \sqcup \cdots \sqcup X_{m_n}^n,$$
$n \in \mathbb{N}$, be a sequence of partitions of X into non-empty closed and open subsets such that:

(i) the $n+1$'st partition is a refinement of the n'th partition,

(ii) $m_{n+1} \leqslant m_n + 1$,

(iii) $C(X,\mathbb{Z}) = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, where $\mathcal{A}_n$ is the subgroup consisting of the continuous $\mathbb{Z}$-valued functions on X that are constant on each X_i^n.

Let $\mathcal{B}_n$ denote the subgroup of $C(Y,\mathbb{Z})$ consisting of the continuous $\mathbb{Z}$-valued functions on Y that are constant on each $f(X_i^n)$. Then $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$ and $C(Y,\mathbb{Z}) = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. Note that $\mathcal{A}_n = \mathbb{Z}^{m_n}$ and $\mathcal{B}_n = \mathbb{Z}^{k_n}$, when $k_n \leqslant m_n$ is the number of elements in the partition of Y which consists of unions of the $f(X_i^n)$'s. Hence we also have $\mathrm{Hom}(\mathcal{A}_n,\mathbb{Z}) = \mathbb{Z}^{m_n}$ and $\mathrm{Hom}(\mathcal{B}_n,\mathbb{Z}) = \mathbb{Z}^{k_n}$. Since the map $\mathrm{Hom}(\mathcal{A}_{n+1},\mathbb{Z}) \to \mathrm{Hom}(\mathcal{A}_n,\mathbb{Z})$ is surjective for each n, there is an identification $\mathrm{Hom}(C(X,\mathbb{Z}),\mathbb{Z}) = \varprojlim \mathrm{Hom}(\mathcal{A}_n,\mathbb{Z})$. Similarly, there is also an isomorphism $\mathrm{Hom}(C(Y,\mathbb{Z}),\mathbb{Z}) = \varprojlim \mathrm{Hom}(\mathcal{B}_n,\mathbb{Z})$. Let $\gamma_X : K_0(X) \to \mathrm{Hom}(K_0(C(X)),\mathbb{Z})$ and $\gamma_Y : K_0(Y) \to \mathrm{Hom}(K_0(C(Y)),\mathbb{Z})$ be the canonical maps. Combined with the above inverse limit decompositions we get a commutative diagram

$$\begin{tikzcd} K_0(X) \arrow[r, "\gamma_X"] \arrow[d, "f_*"'] & \mathrm{Hom}(K_0(C(X)),\mathbb{Z}) \arrow[r] \arrow[d, "f_*"'] & \varprojlim \mathrm{Hom}(\mathcal{A}_n,\mathbb{Z}) \arrow[d, "f_*"'] \\ K_0(Y) \arrow[r, "\gamma_Y"] & \mathrm{Hom}(K_0(C(Y)),\mathbb{Z}) \arrow[r] & \varprojlim \mathrm{Hom}(\mathcal{B}_n,\mathbb{Z}). \end{tikzcd} \tag{8.1}$$

It follows from the universal coefficient theorem of Rosenberg and Schochet, [13], that γ_Y is an isomorphism since $\mathrm{Ext}(K^1(Y),\mathbb{Z}) = 0$ by assumption. It follows that the composition of the maps on the lower row is an isomorphism because Y has trivial reduced K-theory by assumption. In the upper row the first map γ_X is surjective by [13] and the second is surjective because $C(X,\mathbb{Z})$ is a direct summand of $K_0(C(X))$. It suffices therefore to show that

$$f_* : \varprojlim \mathrm{Hom}(\mathcal{A}_n,\mathbb{Z}) \to \varprojlim \mathrm{Hom}(\mathcal{B}_n,\mathbb{Z}) \tag{8.2}$$

is surjective. To this end we consider a square

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{A}_n,\mathbb{Z}) \arrow[d, "f_*"'] & \mathrm{Hom}(\mathcal{A}_{n+1},\mathbb{Z}) \arrow[l, "{i_n}^*"'] \arrow[d, "f_*"] \\ \mathrm{Hom}(\mathcal{B}_n,\mathbb{Z}) & \mathrm{Hom}(\mathcal{B}_{n+1},\mathbb{Z}), \arrow[l, "{j_n}^*"] \end{tikzcd} \tag{8.3}$$

where $i_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ and $j_n : \mathcal{B}_n \to \mathcal{B}_{n+1}$ are the inclusions. To prove surjectivity of (8.2) it suffices to show that if we are given $u \in \mathrm{Hom}(\mathcal{A}_n,\mathbb{Z})$ and $v \in$

$\operatorname{Hom}(\mathcal{B}_{n+1}, \mathbb{Z})$ such that ${j_n}^*(v) = f_*(u)$, there is an element $z \in \operatorname{Hom}(\mathcal{A}_{n+1}, \mathbb{Z})$ such that ${i_n}^*(z) = u$ and $f_*(z) = v$. This is trivial when $m_n = m_{n+1}$ because ${i_n}^*$ and ${j_n}^*$ are identity maps in this case. So assume that $m_{n+1} = m_n + 1$. Let $c_1, c_2, \dots, c_{m_n}$ be the elements of the partition $\{X_i^n\}_{i=1}^{m_n}$ and $c_0', c_1', c_2, c_3, \dots, c_{m_n}$ the elements of the partition $\{X_i^{n+1}\}_{i=1}^{m_{n+1}}$, so that c_0' and c_1' are the only new elements, obtained by a splitting of c_1. We divide the considerations into the following cases:

$k_{n+1} = k_n + 1$: In this case the diagram (8.3) takes the form

$$\begin{tikzcd} \mathbb{Z} \oplus \mathbb{Z}^{m_n - 1} \arrow[d, "A"'] & \mathbb{Z}^2 \oplus \mathbb{Z}^{m_n - 1} \arrow[l, "C"'] \arrow[d, "D"] \\ \mathbb{Z} \oplus \mathbb{Z}^{k_n - 1} & \mathbb{Z}^2 \oplus \mathbb{Z}^{k_n - 1} \arrow[l, "B"] \end{tikzcd} \tag{8.4}$$

where A, B, C and D are matrices of zeroes and ones such that every column contains one and only one non-zero entry. In the present case the matrices take the form

$$A = \begin{pmatrix} 1 & \varepsilon_2 & \varepsilon_3 & \dots & \varepsilon_{m_n} \\ 0 & x_{22} & x_{23} & \dots & x_{2m_n} \\ 0 & x_{32} & x_{33} & \dots & x_{3m_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_{k_n,2} & x_{k_n,3} & \dots & x_{k_n,m_n} \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & a_2 & a_3 & \dots & a_{m_n} \\ 0 & 1 & b_2 & b_3 & \dots & b_{m_n} \\ 0 & 0 & x_{22} & x_{23} & \dots & x_{2m_n} \\ 0 & 0 & x_{32} & x_{33} & \dots & x_{3m_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & x_{k_n,2} & x_{k_n,3} & \dots & x_{k_n,m_n} \end{pmatrix},$$

where $a_i = b_i = 0$, unless $\varepsilon_i = 1$ in which case $(a_i, b_i) = (1, 0)$ or $(a_i, b_i) = (0, 1)$. To find the desired $z \in \mathbb{Z}^{m_n+1}$, write $u = (u_1, u_2, \dots, u_{m_n})$ and $v = (v_1, v_2, \dots, v_{k_n+1})$. Then

$$z = \left(v_1 - \sum_{i=2}^{m_n} a_i u_i, v_2 - \sum_{i=2}^{m_n} b_i u_i, u_2, u_3, \dots, u_{m_n} \right)$$

has the right properties. z is unique in this case.

$k_{n+1} = k_n$: In this case diagram (8.3) takes the form

$$\begin{tikzcd} \mathbb{Z} \oplus \mathbb{Z}^{m_n - 1} \arrow[d, "A"'] & \mathbb{Z}^2 \oplus \mathbb{Z}^{m_n - 1} \arrow[l, "C"'] \arrow[d, "D"] \\ \mathbb{Z} \oplus \mathbb{Z}^{k_n - 1} & \mathbb{Z} \oplus \mathbb{Z}^{k_n - 1} \arrow[l, "B"] \end{tikzcd} \tag{8.5}$$

where A and C are as before, but B and D have changed to the identity matrix and

$$D = \begin{pmatrix} 1 & 1 & \varepsilon_2 & \varepsilon_3 & \dots & \varepsilon_{m_n} \\ 0 & 0 & x_{22} & x_{23} & \dots & x_{2m_n} \\ 0 & 0 & x_{32} & x_{33} & \dots & x_{3m_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & x_{k_n,2} & x_{k_n,3} & \dots & x_{k_n,m_n} \end{pmatrix},$$

respectively. In this case the solution is not unique; if $\alpha, \beta \in \mathbb{Z}$ satisfy that $\alpha + \beta = u_1$, we can use $z = (\alpha, \beta, u_2, u_3, \dots, u_{m_n})$. ∎

As a first step in the proof of Theorem 1.1 from the Introduction, we prove the following:

THEOREM 8.3. *Let $M, N_1, N_2, N_3, \dots, N_k$ be bounded normal operators such that $N_iN_j = N_jN_i$ for all i, j, and let F be a continuous function from the joint spectrum of the N_i's onto the spectrum of M such that*

$$F(N_1, N_2, \dots, N_k) - M \in \mathbb{K}. \tag{8.6}$$

Assume that the spectrum of M contains no isolated eigenvalue of finite multiplicity. There are then norm-continuous paths $N_i^t, i = 1, 2, \dots, k,\ t \in [1, \infty)$, of bounded normal operators such that $N_i^t N_j^t = N_j^t N_i^t$, $N_i - N_i^t \in \mathbb{K}$ for all i, j, t, and

$$\lim_{t\to\infty} \|F(N_1^t, N_2^t, \dots, N_k^t) - M\| = 0.$$

Proof. Let X_0 be the joint spectrum of $N_1, N_2, \dots, N_k$, and X the joint spectrum of $q(N_1), q(N_2), \dots, q(N_k)$. Then X is a closed subset of X_0 and X/X_0 is totally disconnected. It follows therefore from Lemma 6.4 of [2] that the extension of $C(X)$ given by the $q(N_i)$'s is split. Consequently there are commuting normal operators $N_1', N_2', \dots, N_k'$ such that $N_i - N_i' \in \mathbb{K}$ for all i, and the joint spectrum of $N_1', N_2', \dots, N_k'$ is X. We may therefore assume from the beginning that the joint spectrum of $N_1, N_2, \dots, N_k$ is equal to the joint spectrum of $q(N_1), q(N_2), \dots, q(N_k)$. Let $\varphi_0 : C(\sigma(M)) \to \mathbb{L}(H)$ and $\varphi : C(X) \to \mathbb{L}(H)$ be the unital $*$-homomorphisms coming from the spectral theory of M and $N_1, N_2, \dots, N_k$, respectively. With $\sigma(M)$ in the role of Y and F in the role of f, we are in the setting of Section 7. By combining the six-term exact sequence of Theorem 3.3 with Theorem 8.1 above we conclude that the pair (φ, φ_0) represents zero in $\mathrm{Ext}_{\sigma(M),F}(X)$. The desired path of normal operators arise then from condition (ii) of Theorem 7.3 in the obvious way. ∎

REMARK 8.4. As observed in the proof of Theorem 8.3, it is straightforward to reduce the theorem, by use of [2], to the case where the joint spectrum of the N_i's is equal to the joint spectrum X of the $q(N_i)$'s. After this reduction the assumption (8.6) is exactly that the $*$-homomorphisms $\varphi_+ : C(\sigma(M)) \to \mathbb{L}(H)$ and $\varphi_- : C(\sigma(M)) \to \mathbb{L}(H)$ arising by spectral theory from $F(N_1, N_2, \dots, N_k)$ and M, respectively, is a Cuntz-pair, i.e. define an element of $KK(C(\sigma(M)), \mathbb{K})$. In the case where this element is trivial in $KK(C(\sigma(M)),\mathbb{K})$ the conclusion of the theorem follows from Theorem 3.12 of [6]. On the other hand, it is clear that the conclusion of Theorem 8.3 implies that the element $[\varphi_+,\varphi_-]\in KK(C(\sigma(M)),\mathbb{K})$ is in the range of $F_*: KK(C(X),\mathbb{K})\to KK(C(\sigma(M)),\mathbb{K})$, cf. Lemma 4.2. What the proof of Theorem 8.3 does, is to show that this condition is also sufficient, and always satisfied.

We can now give the proof of Theorem 1.1:

Proof. By spectral theory there is a finite rank projection P such that $PM = MP$, and a normal operator $M'\in\mathbb{L}(P^{\perp}H)$ such that

$$\|M'-M|_{P^{\perp}H}\|\leqslant\varepsilon,\quad M|_{P^{\perp}H}-M'\in\mathbb{K}(P^{\perp}H)\quad\text{and}\quad\sigma(M')=\sigma_{\mathrm{ess}}(M').$$

Set $N_i' = P^{\perp}N_i|_{P^{\perp}H}\in\mathbb{L}(P^{\perp}H)$. As in the proof of Theorem 8.3 we let X denote the joint spectrum of $q(N_1),(N_2),\dots,q(N_k)$. Both the N_i's and the N_i''s define an extension of $\mathbb{K}$ by $C(X)$ in the sense of [2], and as argued in the proof of Theorem 8.3 the extension arising from the N_i's is split. It follows therefore from Theorem (4.3) of [2] that the same is true of the extension arising from the N_i''s. This means that there are commuting normal operators $D_i, i=1,2,\dots,k$, acting on $P^{\perp}H$ such that $N_i'-D_i\in\mathbb{K}(P^{\perp}H)$ for all i, and such that the joint spectrum of the D_i's is X. It follows from the conditions on F that $F(X)=\sigma_{\mathrm{ess}}(M)$. Since $\sigma(M')=\sigma_{\mathrm{ess}}(M')=\sigma_{\mathrm{ess}}(M)$ and $F(D_1,D_2,\dots,D_k)-M'\in\mathbb{K}(P^{\perp}H)$, it follows from Theorem 8.3 that there are commuting normal operators D_i^{ε} on $P^{\perp}H$ such that $D_i-D_i^{\varepsilon}\in\mathbb{K}(P^{\perp}H)$ for all i and $\|F(D_1^{\varepsilon},D_2^{\varepsilon},\dots,D_k^{\varepsilon})-M'\|\leqslant\varepsilon$. Let $\mu_1,\mu_2,\dots,\mu_L$ be the eigenvalues of MP on PH, each repeated according to its multiplicity so that L is the rank of P. Let $e_1,e_2,\dots,e_L$ be the corresponding one-dimensional eigenprojections. Since F is surjective by assumption, there is an $L\times k$ complex matrix (a_{ij}) such that $F(a_{i1},a_{i2},\dots,a_{ik})=\mu_i$ for all i, and we set

$$N_j^{\varepsilon}=D_j^{\varepsilon}P^{\perp}+\sum_{i=1}^{L}a_{ij}e_i.$$

Then $N_j-N_j^{\varepsilon}\in\mathbb{K}$ and $\|F(N_1^{\varepsilon},N_2^{\varepsilon},\dots,N_k^{\varepsilon})-M\|\leqslant 2\varepsilon$. ∎

We want to point out that the approximation aspect in Theorem 1.1 and Theorem 8.3, and hence also in the theorems of Section 7, is inevitable. Specifically, we want to show that in general it is not possible, in the setting of Theorem 8.3 to find commuting normal operators $N_1^0,N_2^0,\dots,N_k^0$ such that each N_i^0 is a compact perturbation of N_i and $F(N_1^0,N_2^0,\dots,N_k^0)=M$; not even when $k=1$. To this end, let $e_i, i\in\mathbb{Z}$, be an orthonormal basis in H. Let $z_i, i\in\mathbb{Z}$, be a dense sequence in $\mathbb{T}$ such that $\operatorname{Re} z_i\neq\operatorname{Re} z_j$ when $i\neq j$ and $\lim_{i\to\infty} z_i=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$. Define $D\in\mathbb{L}(H)$ such that $De_i=2\operatorname{Re} z_ie_i$ for all i, and $T\in\mathbb{L}(H)$ such that $Te_i=z_ie_i, i\leqslant 0$, while $Te_i=\frac{1}{\sqrt{2}}e_i+\frac{i}{\sqrt{2}}e_{i+1}$ and $Te_{i+1}=\frac{i}{\sqrt{2}}e_i+\frac{1}{\sqrt{2}}e_{i+1}$ when $i\geqslant 1$ is odd. Then T is unitary with $\sigma(T)=\sigma_{\mathrm{ess}}(T)=\mathbb{T}$ and $T^*+T-D\in\mathbb{K}$. Since any normal operator N

with $N^* + N = D$ must be diagonal with respect to the basis $\{e_i\}_{i \in \mathbb{Z}}$, such an N can not be a compact perturbation of T.

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