J. OPERATOR THEORY
62:2(2009), 297–326

THE *a*-DRAZIN INVERSE AND ERGODIC BEHAVIOUR OF SEMIGROUPS AND COSINE OPERATOR FUNCTIONS

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Communicated by Şerban Strătilă

ABSTRACT. The paper introduces a special type of a Drazin-like inverse for closed linear operators that arises naturally in ergodic theory of operator semigroups and cosine operator functions. The Drazin inverse for closed linear operators defined by Nashed and Zhao [30] and in a more general form by Koliha and Tran [21] is not sufficiently general to be applicable to operator semigroups. The *a*-Drazin inverse is in general a closed, not necessarily bounded, operator. The paper gives applications of the inverse to partial differential equations.

KEYWORDS: *Drazin inverse, closed linear operator, continuous semigroup, cosine operator function.*

MSC (2000): 47A10, 47A60, 34G10.

1. INTRODUCTION

In the investigation of ergodic properties of operator semigroups and cosine operator functions, Butzer and Westphal [7], [8] and later Butzer and Gessinger [3], [4], [5], [6] defined and used a certain type of an “almost inverse” of the infinitesimal generator. It has been then employed by other researchers in the field, notably by Shaw and his collaborators (see [9], [24], [32], [33], [34]). The concept is implicit in Koliha’s paper [19].

Even though the inverse described by Butzer and Westphal proved its usefulness and applicability in the context of operator semigroups and cosine operator functions, it has not been studied or utilized in the setting of operator theory as a generalized operator inverse.

We believe that the concept of what we call the "*a*-Drazin inverse" deserves a detailed study, which is the aim of this paper. The name is chosen to highlight the historical background of the inverse, which was born in the Aachen school of approximation and extensively used by them, thus the "*a*" in the *a*-Drazin inverse refers to Aachen. Our particular concern is to separate the operator-theoretical properties of the new inverse from the properties derived from its linkage with cosine operator functions and operator semigroups.

In 1958 Drazin [10] defined a pseudoinverse of an element a of an associative semigroup as an element b of the semigroup such that

$$b \text{ commutes with } a, \quad bab = b, \quad a^{k+1}b = a^k \text{ for some } k \in \mathbb{N}. \tag{1.1}$$

In a unital ring $\mathcal{R}$, the last condition in (1.1) is equivalent to $a - aba$ being nilpotent: There exists $k \in \mathbb{N}$ such that $(a(1-ab))^k = a^k - a^{k+1}b = 0$. We will refer to this pseudoinverse as the *conventional Drazin inverse*. Harte gave a definition of a *quasinilpotent* element a of a unital ring by requiring that $1 - xa$ is invertible for every $x \in \mathcal{R}$ commuting with a, and paved the way for a further generalization of the Drazin inverse in unital rings. A *generalized Drazin inverse* (see [18], [20]) of an element $a \in \mathcal{R}$ is $b \in \mathcal{R}$ such that

$$b \text{ double commutes with } a, \quad bab = b, \quad a - aba \text{ is quasinilpotent.} \tag{1.2}$$

The *Drazin index* of $a \in \mathcal{R}$ is 0 if a is invertible in $\mathcal{R}$, $k \in \mathbb{N}$ if a is Drazin invertible and $a - aba$ is nilpotent of order k, and ∞ if a is Drazin invertible, but $a(1-ab)$ is not nilpotent.

A bounded linear operator A on a Banach space X has a generalized Drazin inverse (1.2) if and only if 0 is an isolated (possibly removable) singularity of the resolvent $R(\lambda; A) = (\lambda I - A)^{-1}$ of A ([20], Theorems 4.2 and 5.1). The operator A has the conventional Drazin inverse if and only if 0 is at most a pole of the resolvent of A; this occurs if and only if for some $m \in \mathbb{N}$, $\mathcal{R}(A^{m+1}) = \mathcal{R}(A^m)$ and $\mathcal{N}(A^{m+1}) = \mathcal{N}(A^m)$.

The conventional Drazin inverse of a closed linear operator A was defined by Nashed and Zhao [30] for the case that 0 is a pole of the resolvent of A. The definition was later extended by Koliha and Tran [21], [22] to include the case when 0 is an isolated singularity of the resolvent. A special case of the Drazin inverse is the group inverse, that is, the Drazin inverse of index one. The group inverse of a closed linear operator A is defined as a bounded linear operator on X:

DEFINITION 1.1. [[21], Definition 2.1 and [30], Definition 2.1] An operator $A \in \mathcal{C}(X)$ is *group invertible* with the *group inverse* $A^{\mathsf{d}} \in \mathcal{B}(X)$ if

$$\mathcal{R}(A^{\mathsf{d}}) \cup \mathcal{R}(I - AA^{\mathsf{d}}) \subset \mathcal{D}(A)$$

and, for all $x \in \mathcal{D}(A)$,

$$AA^{\mathsf{d}}x = A^{\mathsf{d}}Ax, \quad A^{\mathsf{d}}AA^{\mathsf{d}} = A^{\mathsf{d}}, \quad AA^{\mathsf{d}}Ax = Ax. \tag{1.3}$$

The group inverse is often written as $A^{\sharp}$. Equivalently, A is group invertible (see [21]) if and only if

$$X = \mathcal{R}(A) \oplus \mathcal{N}(A); \tag{1.4}$$

(1.4) is equivalent to 0 being at most a simple pole of the resolvent of A.

In the case that A has a nonempty resolvent set, condition (1.4) is equivalent to

$$\mathcal{R}(A^2) = \mathcal{R}(A) \quad \text{and} \quad \mathcal{N}(A^2) = \mathcal{N}(A). \tag{1.5}$$

The operator A^{d} is unique if it exists, and is given by

$$A^{\mathsf{d}} = (A+P)^{-1}(I-P), \tag{1.6}$$

where P is the spectral projection of A at 0.

The foregoing definition is too restrictive for the generalized inverse of an infinitesimal generator of an operator semigroup or a cosine operator function. In this paper we generalize the concept of the group inverse to closed operators that obey less restrictive conditions than (1.4) and (1.5). The a-Drazin inverse A^{ad} of a closed linear operator A defined in this paper is in general a *closed unbounded* linear operator, which acts on a closed A-invariant subspace X_0 given by

$$X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A),$$

and satisfies milder conditions

$$\overline{\mathcal{R}(A^2)} = \overline{\mathcal{R}(A)} \quad \text{and} \quad \mathcal{N}(A^2) = \mathcal{N}(A).$$

One of the main reasons for our interest in the a-Drazin inverse A^{ad} of the infinitesimal generator A of an operator semigroup is that, at least in the case of holomorphic semigroups, A^{ad} acts as the infinitesimal generator for an associated semigroup (see Theorem 2.2 of [15]).

2. a-DRAZIN INVERSE

For basic concepts of operator theory of closed linear operators we refer the reader to Taylor and Lay's monograph [36]. Let X be a complex Banach space and $\mathcal{C}(X)$ the set of all closed linear operators A with the domain $\mathcal{D}(A) \subset X$ and the range $\mathcal{R}(A) \subset X$. We define inductively

$$\mathcal{D}_{n+1}(A) = \{f \in \mathcal{D}_n(A) : Af \in \mathcal{D}(A)\}, \quad n = 1, 2, \dots,$$

with $\mathcal{D}_1(A) = \mathcal{D}(A)$, and

$$\mathcal{N}(A^n) = \{f \in \mathcal{D}_n(A) : A^n f = 0\}, \quad \mathcal{R}(A^n) = \{A^n f : f \in \mathcal{D}_n(A)\}.$$

The set of all operators $A \in \mathcal{C}(X)$ with $\mathcal{D}(A) = X$ will be denoted by $\mathcal{B}(X)$; by the closed graph theorem the operators in $\mathcal{B}(X)$ are bounded. We say that a subspace $U \subset X$ is *invariant under* $A \in \mathcal{C}(X)$ if

$$A(U \cap \mathcal{D}(A)) \subset U.$$

We observe that, for any $n \in \mathbb{N}$, the subspaces $\mathcal{N}(A^n)$ and $\mathcal{R}(A^n)$ are invariant under A. The space $\overline{\mathcal{R}(A)}$ is also invariant under A.

PROPOSITION 2.1. *Let $A \in \mathcal{C}(X)$ be such that $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$ and that the space $X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ is closed in X, and let P be the projection of X_0 onto $\mathcal{N}(A)$ along $\overline{\mathcal{R}(A)}$. Then the following are true:*

(i) $P \in \mathcal{B}(X_0)$.

(ii) $APf = PAf = 0$ *if* $f \in (\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A)$.

(iii) $A + P$ *is an injective closed linear operator in* X_0 *with the domain*

$$\mathcal{D}(A+P) = (\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A)$$

and the range

$$\mathcal{R}(A+P) = A(\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A).$$

Proof. We observe that $(\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A) = X_0 \cap \mathcal{D}(A)$.

(i) P is defined on all of X_0 and closed since both spaces $\overline{\mathcal{R}(A)}$ and $\mathcal{N}(A)$ are closed. Hence $P \in \mathcal{B}(X_0)$.

(ii) The space $\mathcal{N}(A)$ is invariant under A. If $f \in X_0 \cap \mathcal{D}(A)$, then $Af \in \mathcal{R}(A) \subset \overline{\mathcal{R}(A)}$, so that $\overline{\mathcal{R}(A)}$ is also invariant under A. Hence $PAf = APf$; since $Pf \in \mathcal{N}(A)$, we have $APf = 0$.

(iii) We have $\mathcal{D}(A+P) = \mathcal{D}(P) \cap \mathcal{D}(A) = X_0 \cap \mathcal{D}(A)$ and $\mathcal{R}(A+P) \subset A(\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A)$. If $g = Au + v \in A(\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A)$, then $g = Af$ for $f = u + v \in X_0 \cap \mathcal{D}(A)$.

Let $f \in X_0 \cap \mathcal{D}(A)$ satisfy $(A+P)f = 0$. Then $w = Af = -Pf$. So $w \in \mathcal{R}(A) \cap \mathcal{N}(A)$, and hence $w = 0$. From $Pf = 0$ and $Af = 0$ we get $f \in \overline{\mathcal{R}(A)} \cap \mathcal{N}(A)$, which implies $f = 0$. ∎

CONVENTION 2.2. Let $A \in \mathcal{C}(X)$ be such that $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$ and that the space $\overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ is closed in X. In this paper we will consistently use the following notation:

(i) X_0 for the space $\overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$.

(ii) A_0 for the restriction of A to the space X_0 (note that X_0 is invariant under A).

(iii) $\mathcal{D}(A_0) = X_0 \cap \mathcal{D}(A) = (\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A)$, $\mathcal{R}(A_0) = A(\overline{\mathcal{R}(A)} \cap \mathcal{D}(A))$, $\mathcal{N}(A_0) = \mathcal{N}(A)$.

(iv) $P = P_{\mathcal{N}(A),\overline{\mathcal{R}(A)}}$ for the projection of X_0 onto $\mathcal{N}(A)$ along $\overline{\mathcal{R}(A)}$.

PROPOSITION 2.3. *Under the hypotheses of Proposition* 2.1, *define B by*

$$Bf = (A+P)^{-1}(I-P)f, \quad f \in \mathcal{R}(A_0) \oplus \mathcal{N}(A_0). \tag{2.1}$$

Then the following are true:

(i) *B is a closed linear operator in X_0 with*

$$\mathcal{D}(B) = \mathcal{R}(A_0) \oplus \mathcal{N}(A_0) \quad and \quad \mathcal{R}(B) = \mathcal{D}(A_0) \cap \mathcal{N}(P) = \overline{\mathcal{R}(A)} \cap \mathcal{D}(A).$$

(ii) *B is completely determined by the following relations*:

$$Bf = g, \quad where\ f = Ag + Pf, g \in X_0, Pg = 0. \tag{2.2}$$

(iii) $B^nA^nf = (I-P)f$ *for all* $f \in \mathcal{D}_n(A_0)$ *and all* $n \in \mathbb{N}$.
(iv) $A^nB^nf = (I-P)f$ *for all* $f \in \mathcal{D}_n(B)$ *and all* $n \in \mathbb{N}$.
(v) $ABAf = Af$ *for all* $f \in \mathcal{D}(A_0)$.
(vi) $BABf = Bf$ *for all* $f \in \mathcal{D}(B)$.
(vii) *If* $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^2)}$, *then B is densely defined in* X_0.

Proof. (i) By Proposition 2.1, the operator $A+P$ is closed and injective with

$$\mathcal{D}(A+P) = \mathcal{D}(A_0) \quad \text{and} \quad \mathcal{R}(A+P) = \mathcal{R}(A_0) \oplus \mathcal{N}(A_0).$$

Hence $B = (A+P)^{-1}(I-P) = (I-P)(A+P)^{-1}$ is a closed linear operator with the domain $\mathcal{R}(A_0) \oplus \mathcal{N}(A_0)$ and the range $(I-P)\mathcal{D}(A_0) = \overline{\mathcal{R}(A)} \cap \mathcal{D}(A)$.

(ii) Suppose first that $f = Ag + Pf$ with $Pg = 0$. Since Ag exists and $Pg = 0$, we have $g \in \mathcal{D}(A) \cap \overline{\mathcal{R}(A)}$ and $f \in \mathcal{R}(A_0) \oplus \mathcal{N}(A_0)$. Then $Ag = f - Pf = (I-P)f$, and

$$Bf = (A+P)^{-1}(I-P)f = (A+P)^{-1}Ag = (A+P)^{-1}(A+P)g = g. \tag{2.3}$$

Conversely, suppose that $f \in \mathcal{R}(A_0) \oplus \mathcal{N}(A_0)$. Then $f = A_0h + Pf$, where $h \in \mathcal{D}(A_0) = (\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A)$. Let $g = (I-P)h$. Then $Ag = A(I-P)h = Ah - APh = Ah$ by Proposition 2.1 (ii), and $Pg = P(I-P)h = 0$. Hence (2.3) holds.

(iii) Let $f \in \mathcal{D}(A_0)$. Then $Af \in \mathcal{R}(A_0) \subset \mathcal{D}(B)$, and

$$BAf = (A+P)^{-1}(I-P)Af = (A+P)^{-1}(A+P)(I-P)f = (I-P)f.$$

For the induction assume that for some $n \in \mathbb{N}$, $\mathcal{R}(A_0^n) \subset \mathcal{D}_n(B)$ and $B^nA^nf = (I-P)f$ for all $f \in \mathcal{D}_n(A_0)$. Let $f \in \mathcal{D}_{n+1}(A_0)$. Then $A^nf \in \mathcal{D}(A_0)$, $A^{n+1}f \in \mathcal{R}(A_0) \subset \mathcal{D}(B)$, and

$$BA^{n+1}f = BA(A^nf) = (I-P)A^nf = A^nf.$$

By the induction hypothesis, $A^nf \in \mathcal{D}_n(B)$ and $B^nA^nf = (I-P)f$. Hence

$$B^{n+1}A^{n+1}f = B^n(BA^{n+1}f) = B^nA^nf = (I-P)f.$$

(iv) Let $f \in \mathcal{D}(B)$. Then $Bf \in \mathcal{R}(B) \subset \mathcal{D}(A_0)$ and

$$ABf = A(I-P)(A+P)^{-1}f = (I-P)(A+P)(A+P)^{-1}f = (I-P)f.$$

The inductive step is proved similarly as in the preceding paragraph.

(v) and (vi) follow from (iii) and (iv) and from equations $APf = 0$ ($f \in \mathcal{D}(A_0)$) and $BPf = 0$ ($f \in \mathcal{D}(B)$).

(vii) We note that $\mathcal{R}(A_0) \subset \mathcal{R}(A)$, and $\overline{\mathcal{R}(A_0)} \subset \overline{\mathcal{R}(A)}$. Conversely,

$$\mathcal{R}(A^2) = A(\mathcal{R}(A) \cap \mathcal{D}(A)) \subset A(\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) = \mathcal{R}(A_0),$$

and $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^2)} \subset \overline{\mathcal{R}(A_0)}$. Thus $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A_0)}$, and

$$X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A) = \overline{\mathcal{R}(A_0)} \oplus \mathcal{N}(A_0) = \overline{\mathcal{R}(A_0) \oplus \mathcal{N}(A_0)} = \overline{\mathcal{D}(B)}.$$

This completes the proof. ∎

REMARK 2.4. In the setting of uniformly bounded powers of an operator A and of operator semigroups Butzer and Westphal [7], [8] defined an inverse operator B by

$$Bf = g, \quad \text{where } f = Ag + Pf, Pg = 0$$

under hypotheses implying those of Proposition 2.1. This definition was later extended to the setting of cosine operator functions (see, for instance, p. 319 of [3]). In the preceding proposition we proved that this definition is equivalent to

$$Bf = (A+P)^{-1}(I-P)f.$$

We are ready to give the definition of the a-Drazin inverse.

DEFINITION 2.5. Let $A \in \mathcal{C}(X)$. Then A is *a-Drazin invertible* if

(i) $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$ and the space $X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ is closed in X,

(ii) $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^2)}$.

The *a-Drazin inverse* (or *a-group inverse*) of A is an operator A^{ad} defined by

$$A^{\mathsf{ad}}f = (I-P)(A+P)^{-1}f, \tag{2.4}$$

$$f \in \mathcal{D}(A^{\mathsf{ad}}) = \mathcal{R}(A_0) \oplus \mathcal{N}(A_0) = A(\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A). \tag{2.5}$$

We summarize basic properties of A^{ad}.

THEOREM 2.6. *Let $A \in \mathcal{C}(X)$ be a-Drazin invertible. Then*:

(i) *A^{ad} is a closed linear operator in X_0 with the domain $\mathcal{D}(A^{\mathsf{ad}}) = \mathcal{R}(A_0) \oplus \mathcal{N}(A_0)$ and the range $\mathcal{R}(A^{\mathsf{ad}}) = \mathcal{D}(A_0) \cap \mathcal{N}(P) = \overline{\mathcal{R}(A)} \cap \mathcal{D}(A)$.*

(ii) *$\overline{\mathcal{R}(A^2)} = \overline{\mathcal{R}(A)}$ and $\mathcal{N}(A^2) = \mathcal{N}(A)$.*

(iii) *$\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A_0)}$.*

(iv) *The a-Drazin inverse A^{ad} is densely defined in $X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$.*

(v) *If $f \in \mathcal{D}(A^{\mathsf{ad}})$ and $g \in \mathcal{D}(A_0)$, then*

$$AA^{\mathsf{ad}}f=(I-P)f, \quad A^{\mathsf{ad}}Ag=(I-P)g, \quad A^{\mathsf{ad}}AA^{\mathsf{ad}}f=A^{\mathsf{ad}}f, \quad AA^{\mathsf{ad}}Ag=Ag. \tag{2.6}$$

Proof. (i) This follows from Proposition 2.3.

(ii) The first equality follows from $\mathcal{R}(A^2) \subset \mathcal{R}(A)$ and condition (ii) of Definition 2.5. For the second equality assume that $f \in \mathcal{N}(A^2)$. Then $f \in \mathcal{D}_2(A)$, and $A(Af) = 0$. Hence $Af \in \mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$, and $Af = 0$, that is, $f \in \mathcal{N}(A)$.

(iii), (iv) and (v) follow from Proposition 2.3 and its proof. ∎

We show that if an operator $A \in \mathcal{C}(X)$ is group invertible in the sense of Koliha and Tran [21] and Nashed and Zhao [30], then it is also a-Drazin invertible, and the two inverses agree.

THEOREM 2.7. *If $A \in \mathcal{C}(X)$ is group invertible in the sense of Definition* 1.1, *then A is a-Drazin invertible and $A^{\mathsf{ad}} = A^{\mathsf{d}} \in \mathcal{B}(X)$.*

Proof. Let A be group invertible. Then $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$, $\mathcal{R}(A)$ is closed by Theorem IV.5.10 of [36], 0 is at most a simple pole of the resolvent of A, and $\mathcal{R}(A^2) = \mathcal{R}(A)$. Hence the conditions of Definition 2.5 are fulfilled, and A is a-Drazin invertible. Further,

$$\mathcal{D}(A^{\mathsf{ad}}) = A(\mathcal{R}(A) \cap \mathcal{D}(A)) \oplus \mathcal{N}(A) = \mathcal{R}(A^2) \oplus \mathcal{N}(A) = \mathcal{R}(A) \oplus \mathcal{N}(A) = X,$$

which implies $A^{\mathsf{ad}} \in \mathcal{B}(X)$. Finally, from (1.6) and the definition of A^{ad} we conclude that $A^{\mathsf{ad}} = (A+P)^{-1}(I-P) = A^{\mathsf{d}}$. ∎

In general, the a-Drazin inverse of a closed operator A is unbounded. If it is bounded, it agrees with the group inverse previously defined by Koliha and Tran [21] and Nashed and Zhao [30]; in particular, in this case A^{ad} is defined on all of X:

THEOREM 2.8. *Let $A \in \mathcal{C}(X)$ be an a-Drazin invertible operator with a nonempty resolvent set. Then A^{ad} is bounded on $\mathcal{D}(A^{\mathsf{ad}})$ if and only if the range of A is closed. In this case $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$, $A^{\mathsf{ad}} \in \mathcal{B}(X)$ and the a-Drazin inverse A^{ad} coincides with the group inverse A^{d} of Definition* 1.1.

The proof of the theorem is obtained from the following proposition which is an important result in its own right:

PROPOSITION 2.9. *Under the hypotheses of Theorem* 2.8 *the following conditions are equivalent*:

(i) A^{ad} *is bounded on* $\mathcal{D}(A^{\mathsf{ad}})$.
(ii) $\mathcal{D}(A^{\mathsf{ad}}) = X_0$.
(iii) $\mathcal{R}(A_0)$ *is closed.*
(iv) $\mathcal{R}(A)$ *is closed.*
(v) $\mathcal{R}(A^2)$ *is closed.*
(vi) $\mathcal{R}(A) = \mathcal{R}(A^2)$.
(vii) $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$.

Proof. (i) $\Longrightarrow$ (ii). Let $f \in X_0$. Since A^{ad} is densely defined in X_0, there exists a sequence (f_n) in $\mathcal{D}(A^{\mathsf{ad}})$ with $f_n \to f$. Since A^{ad} is bounded on $\mathcal{D}(A^{\mathsf{ad}})$, the sequence $(A^{\mathsf{ad}} f_n)$ is Cauchy in X_0, and $A^{\mathsf{ad}} f_n \to g$ for some $g \in X_0$. By the closed graph theorem, $f \in \mathcal{D}(A^{\mathsf{ad}})$ and $g = A^{\mathsf{ad}} f$. Hence $\mathcal{D}(A^{\mathsf{ad}}) = X_0$.

(ii) $\Longrightarrow$ (iii). Let $f \in \overline{\mathcal{R}(A_0)} = \overline{A(X_1)}$, where $X_1 = \overline{\mathcal{R}(A)} \cap \mathcal{D}(A)$. Since $\mathcal{D}(A^{\mathsf{ad}}) = X_0$, $A^{\mathsf{ad}} f$ is defined, and belongs to $\mathcal{D}(A)$ (Theorem 2.6 (i)). By Theorem 2.6 (v), $f = f - Pf = (I-P)f = AA^{\mathsf{ad}} f \in A(X_1) = \mathcal{R}(A_0)$.

(iii) $\Longrightarrow$ (iv). Since $\mathcal{R}(A_0)$ is closed, according to Theorem 2.6 (iii) we have

$$\mathcal{R}(A_0) \subset \mathcal{R}(A) \subset \overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A_0)} = \mathcal{R}(A_0),$$

that is, $\mathcal{R}(A) = \mathcal{R}(A_0)$ is closed.

(iv) $\Longrightarrow$ (v). First we show that if the range of A is closed, A^{ad} is bounded on $\mathcal{D}(A)$. From the open maping theorem it follows that there exists a positive constant ρ such that for each $h \in \mathcal{D}(A)$ we can find $g \in \mathcal{D}(A)$ such that $Ag = Ah$ and $\|g\| \leqslant \rho\|Ah\|$. Let $f \in \mathcal{D}(A^{\mathsf{ad}})$. Then $f = Ah + Pf$, where $h \in \mathcal{R}(A) \cap \mathcal{D}(A)$. Let $g \in \mathcal{D}(A)$ be such that $Ag = Ah$ and $\|g\| \leqslant \rho\|Ah\|$. We observe that $g - h \in \mathcal{N}(A)$ and $g = h + (g-h) \in \mathcal{R}(A) \oplus \mathcal{N}(A) = X_0$. Then $g \in \mathcal{D}(A_0)$, and $A^{\mathsf{ad}}Ag = (I-P)g$ by Theorem 2.6 (v). Therefore

$$\begin{aligned}\|A^{\mathsf{ad}}f\| &= \|A^{\mathsf{ad}}Ah + A^{\mathsf{ad}}Pf\| = \|A^{\mathsf{ad}}Ag\| = \|(I-P)g\| \\ &\leqslant \rho\|I-P\|\|Ah\| = \rho\|I-P\|\|(I-P)f\| \leqslant \rho\|I-P\|^2\|f\|.\end{aligned}$$

Hence A^{ad} is bounded on $\mathcal{D}(A^{\mathsf{ad}})$. Note that we have proved the equivalence of (i)–(iv). Then $\mathcal{R}(A^2) = A(\mathcal{R}(A) \cap \mathcal{D}(A)) = \mathcal{R}(A_0)$ is closed.

(v) $\Longrightarrow$ (vi). If $\mathcal{R}(A^2)$ is closed, then

$$\mathcal{R}(A^2) \subset \mathcal{R}(A) \subset \overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^2)} = \mathcal{R}(A^2),$$

and $\mathcal{R}(A) = \mathcal{R}(A^2)$.

(vi) $\Longrightarrow$ (vii). Since $\mathcal{N}(A^2) = \mathcal{N}(A)$ by Theorem 2.6 (ii), A has a finite ascent and descent (less than or equal to 1). Since the resolvent set of A is nonempty, Theorem V.6.2 of [36] implies that $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$ with $\mathcal{R}(A)$ closed.

(vii) $\Longrightarrow$ (i). Follows from Theorem 2.7. ∎

Observe that the assumption that the resolvent set of A is nonempty is used only in the proof that (vi) implies (vii).

3. EXAMPLES OF a-DRAZIN INVERSES

We start with an example of a bounded a-Drazin inverse.

EXAMPLE 3.1. We consider the space $X = \ell^1 \oplus \ell^1$, ℓ^1 being the space of all complex sequences $x = (\xi_1, \xi_2, \dots)$ such that $\sum_{n=1}^{\infty} |\xi_n| < \infty$. The right shift operator defined by $S(\xi_1, \xi_2, \xi_3, \dots) = (0, \xi_1, \xi_2, \dots)$ on ℓ^1 is injective. Let T be the algebraic inverse of the restriction of S from ℓ^1 to the range of S, and let $A = T \oplus 0$; then A is a closed linear operator in X with the domain $\mathcal{D}(A) = \{(\xi_i) \in \ell^1 \colon \xi_1 = 0\} \oplus \ell^1$. We observe that

$$\mathcal{N}(A) = \{0\} \oplus \ell^1 \quad \text{and} \quad \mathcal{R}(A) = \ell^1 \oplus \{0\},$$

which implies $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$.

Then A has the group inverse A^{d} in the sense of Definition 1.1, and according to Theorem 2.7 it has also the a-Drazin inverse A^{ad}. To calculate A^{ad}, we note that the projection $P = P_{\mathcal{N}(A),\mathcal{R}(A)}$ is given by $P = 0 \oplus I$. Then

$$A^{\mathsf{ad}} = (I-P)(A+P)^{-1} = (I \oplus 0)(T \oplus I)^{-1} = (I \oplus 0)(S \oplus I) = S \oplus 0 \in \mathcal{B}(X).$$

The following example presents an unbounded a-Drazin inverse of a closed linear operator. It demonstrates that the new inverse is different from previously defined Drazin type inverses, and that it fills a gap in the theory of generalized inverses as the operator under consideration is an a-Drazin invertible differential operator, which has no Drazin type inverse previously defined. For the background to this example see [3].

EXAMPLE 3.2. The family of all bounded, uniformly continuous complex-valued functions on an interval I will be denoted by $\mathsf{UCB}(I)$. More generally, $\mathsf{UCB}^k(I)$ is the set of all k times differentiable functions in $\mathsf{UCB}(I)$ whose derivatives belong to $\mathsf{UCB}(I)$. Let X be the space $X = \mathsf{UCB}(\mathbb{R})$ equipped with the uniform norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. We consider the operator $A = \mathrm{d}^2/\mathrm{d}x^2$ on X with the domain

$$\mathcal{D}(A) = \mathsf{UCB}^2(\mathbb{R}).$$

The kernel $\mathcal{N}(A)$ of A is the set of all constant functions on $\mathbb{R}$ (any such function belongs to $\mathsf{UCB}(\mathbb{R})$). For any $\xi > 0$ we define the operator P_ξ that assigns to each $f \in X$ the constant function

$$P_\xi f := \frac{1}{2\xi} \int_{-\xi}^{\xi} f(u)\mathrm{d}u \quad \text{for } \xi > 0. \tag{3.1}$$

Then P_ξ is linear with $\|P_\xi\| \leqslant 1$, and $P_\xi h = h$ if and only if $h \in \mathcal{N}(A)$. We show that

$$\operatorname*{s-lim}_{\xi\to\infty} P_\xi f = 0 \quad \text{if } f \in \overline{\mathcal{R}(A)}. \tag{3.2}$$

Let first $f \in \mathcal{R}(A)$. Then there exists $g \in \mathsf{UCB}^2(\mathbb{R})$ such that $f = g''$ and

$$\|P_\xi f\| = \|P_\xi g''\| = \Big\| \frac{1}{2\xi} \int_{-\xi}^{\xi} g''(u)\mathrm{d}u \Big\| \leqslant \frac{1}{\xi}\|g'\|$$

in the $\mathsf{UCB}(\mathbb{R})$ norm, which implies $\|P_\xi f\| \to 0$ as $\xi \to \infty$. Let $f \in \overline{\mathcal{R}(A)}$ and let $\varepsilon > 0$. Then there exists $g \in \mathcal{R}(A)$ with $\|g - f\| < \varepsilon$, and

$$\|P_\xi f\| \leqslant \|P_\xi g\| + \|P_\xi\|\|g - f\| \leqslant \|P_\xi g\| + \varepsilon,$$

so that $\limsup_{\xi\to\infty} \|P_\xi f\| \leqslant \varepsilon$. Since ε was arbitrary, $\lim_{\xi\to\infty} \|P_\xi f\| = 0$.

Suppose that $f \in \overline{\mathcal{R}(A)} \cap \mathcal{N}(A)$. Then $\lim_{\xi\to\infty} \|P_\xi f\| = 0$ while $P_\xi f = f$ for each $\xi > 0$. Hence $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$. We show that the space $X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ is closed in $\mathsf{UCB}(\mathbb{R})$. Since $P_\xi f$ converges for all $f \in X_0$ and $\|P_\xi\| \leqslant 1$ for all $\xi > 0$, we can define an operator P on $\overline{X_0}$ by

$$Pf = \operatorname*{s-lim}_{\xi\to\infty} P_\xi f = \lim_{\xi\to\infty} \frac{1}{2\xi}\int_{-\xi}^{\xi} f(u)\,\mathrm{d}u, \quad f \in \overline{X_0}.$$

Then $P \in \mathcal{B}(X_0)$ and $\|P\| \leqslant 1$. Let

$$f \in \overline{X_0} = \overline{\mathcal{R}(A) \oplus \mathcal{N}(A)}$$

and let $\varepsilon > 0$. Then there exists $g = u + v \in \mathcal{R}(A) \oplus \mathcal{N}(A)$ such that $\|f - g\| < (1/2)\varepsilon$. We observe that $g - Pg = g - Pu - Pv = g - v = u \in \mathcal{R}(A)$, and

$$\|(f - Pf) - (g - Pg)\| \leqslant \|f - g\| + \|P(f - g)\| \leqslant 2\|f - g\| < \varepsilon.$$

Thus $f - Pf \in \overline{\mathcal{R}(A)}$, and $f = (f - Pf) + Pf \in \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A) = X_0$. It is now clear that P is the projection of X_0 onto $\mathcal{N}(A)$ along $\overline{\mathcal{R}(A)}$.

In order to prove that $A = \mathrm{d}^2/\mathrm{d}x^2$ is a-Drazin invertible, we must show that $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^2)}$. We will derive this inclusion in the context of operator cosine functions in Section 5 (see Application 6.2). For the moment we shall assume this result to derive an explicit formula for A^{ad}.

To find the a-Drazin inverse A^{ad} of the operator $A = \mathrm{d}^2/\mathrm{d}x^2$ on $\mathsf{UCB}(\mathbb{R})$ first assume that

$$f \in \mathcal{D}(A^{\mathsf{ad}}) = \mathcal{R}(A_0) \oplus \mathcal{N}(A_0) \subset \mathsf{UCB}^2(\mathbb{R}) \oplus \mathcal{N}(A).$$

Hence $f = g'' + Pf$ for some $g \in \mathsf{UCB}^2(\mathbb{R})$ and $Pf \in \mathbb{C}$. Set $w = g - Pg + Pf$. Then $Aw = Ag = g''$, $Pw = Pg - Pg + Pf = Pf$. Hence

$$(A + P)w = Aw + Pw = g'' + Pf = f$$

and

$$A^{\mathsf{ad}} f = (I - P)(A + P)^{-1} f = (I - P)w = w - Pw = g - Pg. \tag{3.3}$$

Define $h : \mathbb{R} \to \mathbb{C}$ by

$$h(x) = \int_0^x \int_0^s (f(u) - Pf)\,\mathrm{d}u\,\mathrm{d}s.$$

Then h satisfies $h''(x) = f(x) - Pf$, and therefore $h(x) = g(x) + c_1 x + c_2$ for some $c_1, c_2 \in \mathbb{C}$. Since $g \in \mathsf{UCB}(\mathbb{R})$, $\lim_{|x|\to\infty} (g(x)/x) = 0$ and $\lim_{|x|\to\infty} (h(x)/x) = c_1$. For brevity let us write

$$Qh := \lim_{|x|\to\infty} \frac{h(x)}{x} \tag{3.4}$$

whenever the (finite) limit exists for $h : \mathbb{R} \to \mathbb{C}$. Write $w(x) := h(x) - (Qh)x = g(x) + c_2 \in (\overline{\mathcal{R}(A)} \cap \mathcal{D}(A)) \oplus \mathcal{N}(A)$. (Observe that h need not belong to $\mathsf{UCB}(\mathbb{R})$.) According to (3.3),

$$(I - P)w = (I - P)(g + c_2) = g + c_2 - Pg - Pc_2 = g - Pg = A^{\mathsf{ad}} f.$$

Thus we have an explicit formula for $A^{\mathsf{ad}} f$:

$$A^{\mathsf{ad}} f(x) = (I - P)(h(x) - (Qh)x), \quad h(x) = \int_0^x \int_0^s (f(u) - Pf) \mathrm{d}u \mathrm{d}s. \tag{3.5}$$

We show that the a-Drazin inverse A^{ad} of $A = \mathrm{d}^2/\mathrm{d}x^2$ in $\mathsf{UCB}(\mathbb{R})$ is unbounded. In Application 6.2 we show that $(0, \infty)$ belongs to the resolvent set of A. Hence A has a nonempty resolvent set, and if A^{ad} were bounded, the range of A would be closed and $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$ by Theorem 2.8. This means that every $f \in \mathsf{UCB}(\mathbb{R})$ would be of the form $f = g'' + c$, where $g \in \mathsf{UCB}^2(\mathbb{R})$ and $c \in \mathbb{C}$. However, the function $f(x) = (1 + x^2)^{-1}$ is in $\mathsf{UCB}(\mathbb{R})$, but has no decomposition of the form described above. Indeed, any g satisfying $f = g'' + c$ would be of the form $g(x) = x \arctan x - (1/2) \log(1 + x^2) - (1/2)cx^2 + c_1 x + c_2$, but for no choice of the constants c, c_1, c_2 would g belong to $\mathsf{UCB}(\mathbb{R})$.

4. SEMIGROUPS OF OPERATORS

An operator semigroup is associated with the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t), & t \in \mathbb{R}_+, \\ u(0) = f \in \mathcal{D}(A), \end{cases} \tag{4.1}$$

where A is a closed linear operator on X.

A function $T(\cdot) : \mathbb{R}_+ \to \mathcal{B}(X)$ is an *operator semigroup* (or C_0*-semigroup*) if the following conditions are satisfied:

(i) $T(t + s) = T(t)T(s)$ for all $t, s \in \mathbb{R}_+$.
(ii) $T(0) = I$.
(iii) For each $f \in X$, $t \mapsto T(t)f$ is strongly continuous on $\mathbb{R}_+$.

The *infinitesimal generator* A of an operator semigroup $T(\cdot)$ is defined by

$$Af = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 T(t)f = \operatorname*{s\text{-}lim}_{s \to 0} \frac{T(s)f - f}{s}, \quad f \in \mathcal{D}(A), \tag{4.2}$$

where $\mathcal{D}(A)$ is the set of all $f \in X$ for which the derivative (4.2) exists. We will often write $T_A(\cdot)$ for an operator semigroup with the generator A. The *type* $\omega = \omega_A$ of the semigroup $T_A(\cdot)$ is defined as

$$\omega_A = \lim_{t \to \infty} t^{-1} \log \|T_A(t)\|.$$

The abstract Cauchy problem (4.1) is well posed if and only if A generates an operator semigroup. In this case the solution is given by

$$u(t) = T(t)f, \quad t \in \mathbb{R}_+.$$

We define the *Cesàro means* $H(t)$ for an operator semigroup $T(\cdot)$ by

$$H(t)f = \frac{1}{t}\int_0^t T(u)f\,\mathrm{d}u, \quad f \in X, t > 0, \tag{4.3}$$

and the *Abel means* by

$$\lambda \int_0^\infty \mathrm{e}^{-\lambda t} T(t)f\,\mathrm{d}t, \quad f \in X, t > 0. \tag{4.4}$$

If $\mathrm{Re}\lambda > \omega_A$, then the resolvent is given by

$$R(\lambda; A)f = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)f\,\mathrm{d}t, \quad f \in X. \tag{4.5}$$

We define three families of operator semigroups, $\mathcal{T}^0$, $\mathcal{T}^1$, and $\mathcal{T}^\infty$, by specifying

$$\begin{aligned}
T_A(\cdot) \in \mathcal{T}^0 &\iff \sup_{t>0} \|T_A(t)\| < \infty,\\
T_A(\cdot) \in \mathcal{T}^1 &\iff \|T_A(t)\| = o(t), t \to \infty, \text{ and } \sup_{t>0} \|H(t)\| < \infty,\\
T_A(\cdot) \in \mathcal{T}^\infty &\iff (0,\infty) \subset \rho(A) \text{ and } \sup_{\lambda>0} \lambda\|R(\lambda; A)\| < \infty.
\end{aligned}$$

We note the important inclusions

$$\mathcal{T}^0 \subset \mathcal{T}^1 \subset \mathcal{T}^\infty;$$

the first relation is clear, the second follows from (4.5) and [26].

THEOREM 4.1. *Let $T(\cdot) \in \mathcal{T}^1$ be an operator semigroup on X. Then the generator A of $T(\cdot)$ is a-Drazin invertible and*

$$A^{\mathsf{ad}} f = \operatorname*{s\text{-}lim}_{\lambda\to 0+} (\lambda^{-1} P f - R(\lambda; A) f), \quad f \in \mathcal{D}(A^{\mathsf{ad}}). \tag{4.6}$$

Proof. It is well known (see, for instance, Section 1.1 of [1]) that if $T(\cdot)$ is an operator semigroup, then for any $f \in X$ and any $t > 0$, $H(t)f \in \mathcal{D}(A)$. Also,

$$AH(t)f = \frac{T(t)f - f}{t} \quad \text{for } t > 0, \tag{4.7}$$

and for any $f \in \mathcal{D}(A)$,

$$H(t)f - f = \int_0^t AH(u)f\,\mathrm{d}u = A\int_0^t H(u)f\,\mathrm{d}u \in \mathcal{R}(A); \tag{4.8}$$

further,

$$AH(t)f = H(t)Af. \tag{4.9}$$

According to [2], if $T(\cdot) \in \mathcal{T}^\infty$, then $\mathcal{N}(A) \cap \overline{\mathcal{R}(A)} = \{0\}$ and the space $X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ is closed. Again by [2], if $T(\cdot) \in \mathcal{T}^1$,

$$\operatorname*{s-lim}_{t\to\infty} H(t)f = Pf, \tag{4.10}$$

where P is the projection of X_0 onto $\mathcal{N}(A)$ along $\overline{\mathcal{R}(A)}$ and where the limit on the left exists if and only if $f \in X_0$. Also

$$\lim_{\lambda\to 0+} \lambda R(\lambda; A)f = Pf, \quad f \in X_0, \tag{4.11}$$

where the limit on the left exists if and only if $f \in X_0$.

To prove that $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^2)}$, let $f \in \mathcal{R}(A)$, that is, let $f = Ag$ for some $g \in \mathcal{D}(A)$. By (4.8), for each $t > 0$, $g - H(t)g \in \mathcal{R}(A)$, say $g - H(t)g = Ah_t$ with $h_t \in \mathcal{D}(A)$. Then, using (4.9), we get

$$\begin{aligned} f &= (f - H(t)f) + H(t)f = (Ag - H(t)Ag) + H(t)f \\ &= A(g - H(t)g) + H(t)f = A^2 h_t + H(t)f. \end{aligned}$$

Since $f \in \mathcal{R}(A)$, $\operatorname*{s-lim}_{t\to\infty} H(t)f = Pf = 0$. Hence $f = \operatorname*{s-lim}_{t\to\infty} A^2 h_t \in \overline{\mathcal{R}(A^2)}$.

Let $f \in \mathcal{D}(A^{\mathrm{ad}})$; then $f = Ag + Pf$ with $g \in \overline{\mathcal{R}(A)} \cap \mathcal{D}(A)$. For $\lambda > 0$ we have

$$\begin{aligned} \lambda^{-1}Pf = R(\lambda; A)f &= \lambda^{-1}(Pf - \lambda R(\lambda; A)f) \\ &= R(\lambda; A)(Pf - f) = -R(\lambda; A)g = g - \lambda R(\lambda; A)g. \end{aligned}$$

Equation (4.6) then follows from (4.11). ∎

Using the preceding theorem and Proposition 2.9 we obtain the following result from which we can recover the well known uniform ergodic theorem of Lin ([26], Theorem) as well as his discrete uniform ergodic theorem ([25], Theorem).

THEOREM 4.2. *Let $T_A(\cdot) \in \mathcal{T}^1$ be an operator semigroup on X. Then the following conditions are equivalent*:

(i) *$H(t)$ converges in the operator norm as $t \to \infty$.*
(ii) *$\mathcal{R}(A)$ is closed.*
(iii) *$\mathcal{R}(A^2)$ is closed.*
(iv) *$X = \mathcal{R}(A) \oplus \mathcal{N}(A)$.*
(v) *The a-Drazin inverse A^{ad} of A is bounded* (and defined on X).

Proof. The hypothesis $T(\cdot) \in \mathcal{T}^1$ guarantees that A is a-Drazin invertible. The equivalence of (ii)–(v) is delivered by Proposition 2.9. The interesting part is the equivalence of (i) and (v).

If (i) holds, then $\|H(t) - P\| \to 0$ as $t \to \infty$, where P is the projection onto $\mathcal{N}(A)$ along $\overline{\mathcal{R}(A)}$. Fix $t > 0$ for which $\|H(t) - P\| < 1$; then $H(t) - P - I$ is invertible in $\mathcal{B}(X)$. Let $f \in \mathcal{D}(A^{\mathrm{ad}})$; then $f = Ag + Pf$ for some $g \in \overline{\mathcal{R}(A)} \cap$

$\mathcal{D}(A)$, and $A^{\mathsf{ad}} f = g$. Noting that $Pg = 0$ and using (4.8), we get

$$t(H(t) - P - I)g = t \int_0^t H(u)Ag\,\mathrm{d}u = \int_0^t \Big(\int_0^u T(s)(f - Pf)\mathrm{d}s \Big)\mathrm{d}u,$$

and so A^{ad} is bounded in view of

$$\|A^{\mathsf{ad}} f\| = \|g\| \leqslant t^{-1} \|(H(t) - P - I)^{-1}\| \Big(\int_0^t \int_0^u \|T(s)\| \|I - P\| \mathrm{d}s\mathrm{d}u \Big) \|f\|.$$

Conversely, assume that A^{ad} is bounded. By Proposition 2.9, $\mathcal{D}(A^{\mathsf{ad}}) = X = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Then any $f \in X$ is of the form $f = Ag + Pf$ with $g \in \mathcal{D}(A)$; observe that $g = A^{\mathsf{ad}} f$. Remembering that $H(t)Pf = Pf$ and taking into account (4.7), for any $t > 0$ we obtain

$$\begin{aligned} \|H(t)f - Pf\| &= \|H(t)(f - Pf)\| = \|H(t)Ag\| \\ &= t^{-1} \|T(t)g - g\| = t^{-1} (\|T(t)\| + 1) \|A^{\mathsf{ad}}\| \|f\|, \end{aligned}$$

which shows that $H(t)$ converges to P in the operator norm. ∎

We remark that the uniform Cesàro convergence of the semigroup, that is, condition (i) of the preceding theorem, is equivalent to its uniform Abel convergence:

$$\Big\|\lambda \int_0^\infty \mathrm{e}^{-\lambda t} T(t)\mathrm{d}t - P\Big\| \to 0 \quad \text{as } \lambda \to \infty. \tag{4.12}$$

A proof may be found in [26].

5. COSINE OPERATOR FUNCTIONS

Cosine operator functions are associated with the solution of the second order Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} u(t) = Au(t) & t \in \mathbb{R}, \\ u^{(k)}(0) = f_k \in \mathcal{D}(A) & k = 0, 1, \end{cases} \tag{5.1}$$

where A is a closed linear operator on X.

We say that $C(\cdot) : \mathbb{R} \to \mathcal{B}(X)$ is a *cosine operator function* (see [17], [35]) if it satisfies the following conditions:

(i) $C(t + s) - C(t - s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$ (d'Alembert functional equation).

(ii) $C(0) = I$.

(iii) For each $f \in X$, $t \mapsto C(t)f$ is strongly continuous on $\mathbb{R}$.

The *infinitesimal generator* A of a cosine operator function $C(\cdot)$ is defined by

$$Af = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_0 C(t)f = \operatorname*{s\text{-}lim}_{s\to 0} \frac{2}{s^2}(C(s)f - f), \quad f \in \mathcal{D}(A), \tag{5.2}$$

where $\mathcal{D}(A)$ is the set of all $f \in X$ for which the second derivative (5.2) exists. It is often assumed that a cosine operator function $C(\cdot)$ is *nondegenerate*, that is,

$$C(t)f = 0 \text{ for all } t \neq 0 \implies f = 0.$$

We mention that the Cauchy problem (5.1) is well posed if and only if A generates a cosine operator function $C(\cdot)$, with the solution given by

$$u(t) = C(t)f_0 + \int_0^t C(u)f_1 \mathrm{d}u, \quad t \in \mathbb{R}.$$

For a nondegenerate cosine operator function $C(\cdot)$ we define the *Cesàro means* $E(t)$ for $t \neq 0$ by

$$E(t)f = \frac{2}{t^2}\int_0^t (t-u)C(u)f\mathrm{d}u = \frac{2}{t^2}\int_0^t \int_0^u C(v)f\mathrm{d}v\mathrm{d}u, \quad f \in X; \tag{5.3}$$

then

$$E(t)f \in \mathcal{D}(A), \quad AE(t) = \frac{2}{t^2}(C(t) - I), \quad t \neq 0. \tag{5.4}$$

(See Fundamental Lemma 2.14 of [35].) According to Lemma 3.1 of [6],

$$AE(t)f = E(t)Af \quad \text{for all } t \neq 0 \text{ and all } f \in \mathcal{D}(A). \tag{5.5}$$

We also define the *Abel means* by $\lambda \int_0^\infty \mathrm{e}^{-\lambda u}C(u)\mathrm{d}u$ as in the case of operator semigroups. If $\mathrm{Re}\lambda > \omega_A$, then

$$R(\lambda^2; A) = \lambda^{-1}\int_0^\infty \mathrm{e}^{-\lambda u}C(u)\mathrm{d}u. \tag{5.6}$$

We define three families of cosine operator functions, $\mathcal{C}^0$, $\mathcal{C}^2$, and $\mathcal{C}^\infty$, introduced in [3] by specifying

$$\begin{aligned} C_A(\cdot) \in \mathcal{C}^0 &\iff \sup_{t>0}\|C_A(t)\| < \infty, \\ C_A(\cdot) \in \mathcal{C}^2 &\iff \|C_A(t)\| = o(t^2), t \to \infty, \text{ and } \sup_{t>0}\|E(t)\| < \infty, \\ C_A(\cdot) \in \mathcal{C}^\infty &\iff (0,\infty) \subset \rho(A) \text{ and } \sup_{\lambda>0}\lambda\|R(\lambda; A)\| < \infty. \end{aligned}$$

We have the inclusions

$$\mathcal{C}^0 \subset \mathcal{C}^2 \subset \mathcal{C}^\infty;$$

they follow from the corresponding relations for semigroups and Proposition 5.2.

The following theorem is our main result on the a-Drazin invertibility of a generator of a cosine operator function.

THEOREM 5.1. *Let $C(\cdot) \in \mathcal{C}^2$ be a cosine operator function on X. Then the generator A of $C(\cdot)$ is a-Drazin invertible and*

$$A^{\mathsf{ad}} f = \operatorname*{s-lim}_{\lambda\to 0+}(\lambda^{-2}Pf - R(\lambda^2; A)f), \quad f \in \mathcal{D}(A^{\mathsf{ad}}). \tag{5.7}$$

Proof. According to Lemma 3.1 and 3.2 of [3], $\mathcal{N}(A) \cap \overline{\mathcal{R}(A)} = \{0\}$ and the space $X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ is closed. By Theorem 3.1 of [3],

$$\operatorname*{s-lim}_{t\to\infty} E(t)f = Pf \tag{5.8}$$

where P is the projection of X_0 onto $\mathcal{N}(A)$ along $\overline{\mathcal{R}(A)}$ and where the limit of the left exists if and only if $f \in X_0$. Further,

$$\lim_{\lambda\to 0+} \lambda^2 R(\lambda^2; A)f = Pf, \quad f \in X_0, \tag{5.9}$$

where the limit of the left exists if and only if $f \in X_0$.

We need to prove the inclusion $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^2)}$. Let $f \in \mathcal{R}(A)$, that is, $f = Ag$ for some $g \in \mathcal{D}(A)$, and let $t > 0$. According to Lemma 2.4 of [3], $g - E(t)g \in \mathcal{R}(A)$, say $g - E(t)g = Ah_t$, where $h_t \in \mathcal{D}(A)$. By (5.5), $E(t)Ag = AE(t)g$, and

$$\begin{aligned} f &= (f - E(t)f) + E(t)f = (Ag - E(t)Ag) + E(t)f \\ &= (Ag - AE(t)g) + E(t)f = A^2 h_t + E(t)f. \end{aligned}$$

Since $f \in \mathcal{R}(A)$, $\operatorname*{s-lim}_{t\to\infty} E(t)f = Pf = 0$. Then $f = \operatorname*{s-lim}_{t\to\infty} A^2 h_t \in \overline{\mathcal{R}(A^2)}$.

The proof of equation (5.7) is similar to the proof of (4.6) (see also Theorem 4.1 of [3]). ∎

Under the hypotheses of the preceding theorem let $A^{\mathsf{ad}} = (I - P)(A + P)^{-1}$ be the a-Drazin inverse of the generator A. Then A^{ad} is a closed linear operator densely defined in $X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$. According to Proposition 2.3, A^{ad} coincides with the operator B defined by Butzer and Gessinger in [3] by specifying relations which we recorded as equation (2.2).

Relations between cosine operator functions and operator C_0-semigroups relevant to the present paper were studied by Fattorini [12], [13], and especially in Gessinger's dissertation [14]. We need the following result — for part (a) see p. 92 of [12] and p. 140 of [16], for part (b) see Proposition 2.17 of [14] and Lemma 2.2 of [3]), for part (c) see Satz 3.4 of [14] and p. 481 of [15].

PROPOSITION 5.2. (a) *Let $C_A(\cdot)$ be a cosine operator function on a Banach space X with a generator A. Then A generates an operator semigroup $T_A(\cdot)$ given by*

$$T_A(t)f = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp(-s^2/4t) C_A(s) f \, ds. \tag{5.10}$$

(b) *The operator norms of* $C_A(\cdot)$ *and* $T_A(\cdot)$ *are connected by*:

(i) *If* $\|C_A(t)\| = o(t^2)$ *as* $|t| \to \infty$, *then* $\|T_A(t)\| = o(t)$ *as* $t \to \infty$.

(ii) *If* $\|C_A(t)\| = O(t^2)$ *as* $|t| \to \infty$, *then* $\|T_A(t)\| = O(t)$ *as* $t \to \infty$.

(c) *Let* $T_A(\cdot) \in \mathcal{T}^0$ *be an equibounded holomorphic semigroup of type* θ, *(see p. 33 of* [16]*) on* X. *Then the operator* $B = A^{\mathsf{ad}}$ *generates an equicontinuous holomorphic semigroup on* X_0 *of the same type, given by*

$$T_B(t)f = f + Pf - \sqrt{t}\int_0^\infty J_1(2\sqrt{tu})T_A(u)f\frac{\mathrm{d}u}{\sqrt{u}} \quad (f \in X_0), \tag{5.11}$$

$J_1(\cdot)$ *being the Bessel function of order* 1.

Using this result we can deduce the following counterpart of Theorem 4.2 (see also Theorem 3.3 of [3]).

THEOREM 5.3. *Let* $C_A(\cdot) \in \mathcal{C}^2$ *be a cosine operator function on* X. *Then the following conditions are equivalent*:

(i) $E(t)$ *converges in the operator norm as* $t \to \infty$.

(ii) $\mathcal{R}(A)$ *is closed.*

(iii) $\mathcal{R}(A^2)$ *is closed.*

(iv) $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$.

(v) *The a-Drazin inverse* A^{ad} *of* A *is bounded* (and defined on X).

Proof. We prove only the equivalence of conditions (i) and (v).

If (i) holds, then $\|E(t) - P\| \to 0$ as $t \to \infty$. Fixing $t > 0$ with $\|E(t) - P\| < 1$, we have $E(t) - P - I$ invertible in $\mathcal{B}(X)$. If $f \in \mathcal{D}(A^{\mathsf{ad}})$, then $f = Ag + Pf$ for some $g \in \overline{\mathcal{R}(A)} \cap \mathcal{D}(A)$, and $A^{\mathsf{ad}}f = g$. According to the proof of Theorem 3.3 of [3],

$$\frac{t^2}{2}[E(t) - P - I]g = \int_0^t\int_0^u\int_0^v\int_0^w C(s)(f - Pf)\mathrm{d}s\mathrm{d}w\mathrm{d}v\mathrm{d}u,$$

and

$$\begin{aligned}\|A^{\mathsf{ad}}f\| = \|g\| &\leqslant \|(E(t) - P - I)^{-1}\| \int_0^t\int_0^u\int_0^v\int_0^w C(s)\|f - Pf\|\mathrm{d}s\mathrm{d}w\mathrm{d}v\mathrm{d}u \\ &\leqslant \|(E(t) - P - I)^{-1}\|\frac{t^2}{2}\{1 + \|P\|\}\|f\|.\end{aligned}$$

Thus A^{ad} is bounded.

Conversely, if A^{ad} is bounded, then $\mathcal{D}(A^{\mathsf{ad}}) = \overline{\mathcal{D}(A^{\mathsf{ad}})} = X = \mathcal{R}(A) \oplus \mathcal{N}(A)$. So any $f \in X$ is of the form $f = Ag + Pf$ with $g \in \mathcal{D}(A)$, noting that $g = A^{\mathsf{ad}}f$. Since $E(t)Pf = Pf$ for all $f \in X$ and all $t \in \mathbb{R}$ (see Lemma 3.2 of [3]), we have by (5.4),

$$\begin{aligned}\|E(t)f-Pf\| &= \|E(t)(f-Pf)\| = \|E(t)Ag\|\\ &= \frac{2}{t^2}\|C(t)g-g\| \leqslant \frac{2}{t^2}(\|C(t)\|+1)\|A^{\mathsf{ad}}\|\|f\|.\end{aligned}$$

Hence $E(t)$ converges to P in the operator norm as $t\to\infty$. ∎

The uniform Cesàro convergence of the cosine operator function, that is, condition (i) of the preceding theorem, is equivalent to its uniform Abel convergence:

$$\Big\|\lambda\int_0^\infty \mathrm{e}^{-\lambda t}C(t)\mathrm{d}t - P\Big\| \to 0 \quad \text{as } t\to\infty. \tag{5.12}$$

(See Theorem 3.3 of [3].)

6. DIFFERENTIAL EQUATIONS AND THE a-DRAZIN INVERSE

This section is concerned with applications to differential equations involving operator cosine functions satisfying the conditions of Theorem 5.1, and describing the a-Drazin inverse of its generator.

APPLICATION 6.1. The wave equation in the space $L^2_{2\pi}$ (see Section 5.1 of [3]). Let $X = L^2_{2\pi} = L^2_{2\pi}(\mathbb{R})$ be the space of all complex valued 2π-periodic L^2-functions on $\mathbb{R}$ equipped with the norm

$$\|f\| = \|f\|_{L^2_{2\pi}} = \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\mathrm{d}x\Big)^{1/2}.$$

The equation is given by

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}w(x,t) = \frac{\partial^2}{\partial t^2}w(x,t),\\ &w(x,0) = f(x)\in L^2_{2\pi}, \quad \frac{\partial}{\partial t}w(x,t)\Big|_{t=0} = 0.\end{aligned} \tag{6.1}$$

Its solution is a cosine operator function

$$w(x,t) = C(t)f(x) := \frac{1}{2}[f(x+t)+f(x-t)] = \sum_{k\in\mathbb{Z}}\cos kt\,\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} \tag{6.2}$$

where

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\mathrm{e}^{-\mathrm{i}ku}\mathrm{d}u, \quad k\in\mathbb{Z},$$

acting on the space $L^2_{2\pi}$. According to [37], $C(\cdot)$ is a cosine operator function satisfying $\|C(t)\| = 1$ for all $x \in \mathbb{R}$ with the infinitesimal generator $A = \mathrm{d}^2/\mathrm{d}x^2$ whose domain is

$$\mathcal{D}(A) = \{f \in L^2_{2\pi} : f, f' \in \mathsf{AC}_{2\pi}, f'' \in L^2_{2\pi}\},$$

where $\mathsf{AC}_{2\pi} = \mathsf{AC}_{2\pi}(\mathbb{R})$ is the set of all complex valued absolutely continuous 2π-periodic functions in $\mathbb{R}$. The null space $\mathcal{N}(A)$ of A is the set of all complex valued 2π-periodic constant functions in $\mathbb{R}$. The cosine operator function $C(\cdot)$ satisfies the hypotheses of Theorem 5.1, which means that the space $X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ is closed and $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^2)}$. The operator $P = P_{\mathcal{N}(A),\mathcal{R}(A)}$ projects each function $f \in L^2_{2\pi}$ onto the constant function

$$Pf = \widehat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\mathrm{d}u.$$

Let $f \in \mathcal{D}(A^{\mathsf{ad}})$. Then $A^{\mathsf{ad}} f = (I-P)(A+P)^{-1} f = (I-P)g$, where g is a solution to the equation $(A+P)g = f$. To solve this equation we expand f and g in a Fourier series and obtain

$$(A+P)g(x) = -\sum_{k\in\mathbb{Z}} k^2\, \widehat{g}(k)\mathrm{e}^{\mathrm{i}kx} + \widehat{g}(0) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} = f(x).$$

From the uniqueness of Fourier coefficients we obtain

$$g(x) = -\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k^2}\, \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx} + \widehat{g}(0),$$

and

$$A^{\mathsf{ad}} f(x) = (I-P)g(x) = g(x) - \widehat{g}(0) = -\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k^2}\, \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}. \tag{6.3}$$

Observe that A has the Fourier series representation

$$Af(x) = -\sum_{k\in\mathbb{Z}} k^2\, \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}, \quad f \in \mathcal{D}(A). \tag{6.4}$$

From (6.3) we deduce $\|A^{\mathsf{ad}} f\| \leqslant \|f\|$, which proves that the a-Drazin inverse of A is bounded. Consequently, by Theorem 2.8, $L^2_{2\pi} = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Hence every $f \in L^2_{2\pi}$ has representation $f = g'' + \widehat{f}(0)$, where $g \in \mathcal{D}(A)$.

Utilising p. 343 of [3] we get another explicit representation for A^{ad}:

$$A^{\mathsf{ad}} f(x) = \int_0^x \int_0^u f(v)\mathrm{d}v\mathrm{d}u - \frac{x^2}{4\pi} \int_{-\pi}^{\pi} f(v)\mathrm{d}v - \frac{x}{2\pi} \int_{-\pi}^{\pi} \int_0^u f(v)\mathrm{d}v\mathrm{d}u$$
$$- \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^x \int_0^u f(v)\mathrm{d}v\mathrm{d}u\mathrm{d}x + \frac{\pi}{12} \int_{-\pi}^{\pi} f(v)\mathrm{d}v.$$

The associated heat equation

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2} w(x,t) = \frac{\partial}{\partial t} w(x,t) \quad (-\pi < x < \pi; t > 0),\\
&\operatorname*{s-lim}_{t\to 0+} w(x,t) = f(x), \quad w(-\pi,t) = w(\pi,t),\\
&w_x(-\pi,t) = w_x(\pi,t)
\end{aligned} \tag{6.5}$$

has a classical solution

$$W_A(t)f(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-tk^2}\, \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}$$

with A defined by (6.4). The existence of the semigroup associated with $B = A^{\mathsf{ad}}$ defined by (6.3) follows from Proposition 5.2 (c); it is given by

$$W_B(t)f(x) = \widehat{f}(0) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathrm{e}^{-t/k^2}\, \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}$$

(see p. 487 of [15]), where the k^2 in the exponential factor e^{-tk^2} of $W_A(t)$ is replaced by k^{-2}.

The existence of the cosine operator function associated with the bounded operator B, thus $C_B(\cdot)$, follows from the counterpart of Proposition 5.2 (c) for $C_B(\cdot)$ (see p. 483 of [15]), and is given by

$$\begin{aligned}
C_B(t)f(x) &= f(x) + Pf - \sqrt{t}\int_0^\infty J_1(2\sqrt{2u}) \sum_{k\in\mathbb{Z}} \cos ku\, \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\frac{\mathrm{d}u}{\sqrt{u}}\\
&= f(x) + \widehat{f}(0) - \sum_{k\in\mathbb{Z}} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\int_0^\infty \sqrt{\frac{t}{u}} J_1(2\sqrt{tu}) \cos ku \mathrm{d}u\\
&= \widehat{f}(0) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \cos(t/k)\, \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}.
\end{aligned}$$

Here we have made use of the fact that the expression

$$\int_0^\infty \sqrt{\frac{t}{u}} J_1(2\sqrt{ut}) \cos ku \mathrm{d}u$$

is equal to $1 - \cos(t/k)$ for $k > 0$ and to 1 for $k = 0$ (see I pp. 54, 110; II p. 20 of [11]), as well as the fact that the nth partial sum of the Fourier series of $f \in L^2_{2\pi}$ tends to f in the $L^2_{2\pi}$-norm for $n \to \infty$.

APPLICATION 6.2. The wave equation for $\mathsf{UCB}(\mathbb{R})$. Let $C(\cdot)$ be the translation operator

$$C(t)f(x) = \frac{1}{2}(f(x+t) + f(x-t)), \quad x,t \in \mathbb{R}, \tag{6.6}$$

in the space $X = \mathsf{UCB}(\mathbb{R})$ equipped with the uniform norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. It is known [29] that $C(\cdot)$ is a cosine function satisfying $\|C(t)\| = 1$ for all $t \in \mathbb{R}$, and that $C(\cdot)$ is a solution to the wave equation (6.1) with $f \in \mathsf{UCB}(\mathbb{R})$.

The infinitesimal generator of $C(\cdot)$ is the differential operator $A = \mathrm{d}^2/\mathrm{d}x^2$ with the domain $\mathcal{D}(A) = \mathsf{UCB}^2(\mathbb{R})$ which we studied in Example 3.2. Since the semigroup $C(\cdot)$ satisfies the hypotheses of Theorem 5.1, we conclude that $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^2)}$. This fills in the gap in Example 3.2, where we derived the explicit equation (3.5) for the a-Drazin inverse of the generator of our cosine operator function $C(\cdot)$. Recall that in the same example we also proved that A^{ad} is unbounded.

Another formula for A^{ad} can be obtained using (5.7). For this we first note that according to equation (5.10) of [3],

$$\lambda R(\lambda^2; A)f(x) = \frac{1}{2}\int_{-\infty}^{\infty} \exp(-\lambda|x-u|)f(u)\mathrm{d}u, \quad \lambda > 0,$$

so that

$$A^{\mathsf{ad}}f(x) = \operatorname*{s-lim}_{\lambda\to 0+}\Big(\frac{1}{\lambda^2}Pf - \frac{1}{2\lambda}\int_{-\infty}^{\infty} \exp(-\lambda|x-u|)f(u)\mathrm{d}u\Big). \tag{6.7}$$

The semigroup T_A associated with the operator A is given by

$$T_A(t)f(x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \exp(-(x-u)^2/4t)f(u)\mathrm{d}u = W_A(t)f(x) = w(x,t). \tag{6.8}$$

It solves the heat equation

$$\begin{aligned} &\frac{\partial^2}{\partial x^2}w(x,t) = \frac{\partial}{\partial t}w(x,t) \quad (-\infty < x < \infty, t > 0), \\ &\operatorname*{s-lim}_{t\to 0+} w(x,t) = f(x). \end{aligned} \tag{6.9}$$

In the following theorem we give a new description of the operator semigroup generated by A^{ad}. It is of unusual interest because its proof involves the interchange of order of two integrals which do not converge absolutely.

First we recall the definition of the generalized hypergeometric function

$${}_0F_2(a,b;z) := \sum_{k=0}^{\infty} \frac{1}{(a)_k(b)_k}\frac{z^k}{k!},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol.

THEOREM 6.3. *The semigroup T_B associated with the a-Drazin inverse $B = A^{\mathsf{ad}}$ of the infinitesimal generator $A = \mathrm{d}^2/\mathrm{d}x^2$ of the cosine operator function $C(\cdot)$ defined* *in* (6.6) *is of the form*

$$T_B(t)f(x) := W_B(t)f(x) = f(x) + Pf - \sqrt{\frac{t}{4\pi}} \int_{-\infty}^{\infty} f(x-v)H(t,v)\mathrm{d}v, \tag{6.10}$$

where H is given as the Mellin-type convolution integral

$$H(t,v) := \int_0^{\infty} J_1(2\sqrt{tu}) \exp(-v^2/4u) \frac{\mathrm{d}u}{u}, \tag{6.11}$$

explicitly evaluated in terms of the generalized hypergeometric function as

$$H(t,v) = 2\,{}_0F_2\Big(\frac{1}{2}, \frac{3}{2}; \frac{tv^2}{4}\Big) - |v|\sqrt{\pi t}\,{}_0F_2\Big(\frac{3}{2}, 2; \frac{tv^2}{4}\Big). \tag{6.12}$$

Proof. Starting with (5.11), we have

$$\begin{aligned}\sqrt{t}\int_0^{\infty} J_1(2\sqrt{tu})T_A(u)f(x)\frac{\mathrm{d}u}{\sqrt{u}} &= \sqrt{\frac{t}{4\pi}}\int_0^{\infty} J_1(2\sqrt{tu})\Big(\int_{-\infty}^{\infty} \frac{1}{\sqrt{u}} \exp(-v^2/4u)f(x-v)\mathrm{d}v\Big)\frac{\mathrm{d}u}{\sqrt{u}} \\ &= \sqrt{\frac{t}{4\pi}}\int_{-\infty}^{\infty} f(x-v)\Big(\int_0^{\infty} J_1(2\sqrt{tu}) \exp(-v^2/4u)\frac{\mathrm{d}u}{u}\Big)\mathrm{d}v \\ &= \sqrt{\frac{t}{4\pi}}\int_{-\infty}^{\infty} f(x-v)H(t,v)\mathrm{d}v \qquad (6.13)\end{aligned}$$

provided the order of integration can be interchanged. This part of the proof is nontrivial as it involves nonabsolute integration, and is demonstrated in detail in the Appendix.

Now we evaluate the function $H(t,v)$ in terms of the generalized hypergeometric function:

$$\begin{aligned}H(t,v) &= \int_0^{\infty} J_1(2\sqrt{tu}) \exp(-v^2/4u)\frac{\mathrm{d}u}{u} = \int_0^{\infty} J_1\Big(\frac{2\sqrt{t}}{\sqrt{y}}\Big) \exp(-v^2/4y)\frac{\mathrm{d}y}{y} \\ &= \mathcal{L}\Big[y^{-1}J_1\Big(\frac{2\sqrt{t}}{\sqrt{y}}\Big)\Big]\Big(\frac{v^2}{4}\Big) = 2\,{}_0F_2\Big(\frac{1}{2}, \frac{3}{2}; \frac{tv^2}{4}\Big) - |v|\sqrt{\pi t}\,{}_0F_2\Big(\frac{3}{2}, 2; \frac{tv^2}{4}\Big), \qquad (6.14)\end{aligned}$$

where $\mathcal{L}$ denotes the Laplace transform (see p. 266, Equation of [31]; the formula there contains a misprint in the sign, which has been corrected in the derivation of (6.14)). ∎

The proof that the order of integration in (6.13) may be inverted is relegated to the Appendix below. The reason is that

$$\int_0^{\infty}\int_{-\infty}^{\infty} \frac{J_1(2\sqrt{tu})}{u} \exp(-v^2/4u)f(x-v)\mathrm{d}u\mathrm{d}v$$

does not exist as a Lebesgue integral, but only as an improper Lebesgue integral, which can be viewed as a special case of the non-absolute generalized Riemann integral [27], also known as the Kurzweil–Henstock integral [23]. This necessitates the use of non-absolute Fubini's theorem.

REMARK 6.4. Since neither A nor its a-Drazin inverse A^{ad} are bounded, one cannot determine the cosine operator function $C_B(\cdot)$ generated by $B = A^{\mathsf{ad}}$.

7. APPENDIX

We justify the changing of the order of integration in (6.13). For notational convenience, we shall work with the simplified function

$$G(x;u,v) = \frac{J_1(\sqrt{u})}{u}\exp(-v^2/u)f(x-v), \tag{7.1}$$

where $0 < u < \infty$, $-\infty < v < \infty$, and f is a uniformly continuous bounded function on $(-\infty,\infty)$. Since $\exp(-v^2/u)$ is an even function of v, without a loss of generality we may restrict the interval for v to $(0,\infty)$, and consider G on the positive quadrant $(0,\infty)\times(0,\infty)$.

We observe that for a fixed x, $G(x;u,v)$ is not in general Lebesgue integrable on $(0,\infty)\times(0,\infty)$. Indeed, set $f \equiv 1$ and calculate the repeated integral

$$\begin{aligned}\int_0^\infty \mathrm{d}u \int_0^\infty |G(x;u,v)|\mathrm{d}v &= \int_0^\infty \frac{|J_1(\sqrt{u})|}{u}\mathrm{d}u \int_0^\infty \exp(-v^2/u)\mathrm{d}v \\ &= \frac{1}{2}\int_0^\infty \frac{|J_1(\sqrt{u})|}{u}\sqrt{\pi}\sqrt{u}\mathrm{d}u = \sqrt{\pi}\int_0^\infty |J_1(s)|\mathrm{d}s,\end{aligned}$$

which is infinite. However, the integral $\int_0^\infty J_1(s)\mathrm{d}s$ exists as an improper Lebesgue integral

$$\int_0^{\to\infty} J_1(s)\mathrm{d}s := \lim_{t\to\infty}\int_0^t J_1(s)\mathrm{d}s.$$

We need a theorem on the existence of the improper Lebesgue integral

$$\lim_{R_1\to\infty,\,R_2\to\infty}\int_0^{R_1}\int_0^{R_2} g(u,v)\mathrm{d}u\mathrm{d}v. \tag{7.2}$$

By a two dimensional Hake's theorem [28], the improper Lebesgue integral is a special case of the generalized Riemann integral, for which Fubini's theorem holds ([27], Section 6.4).

THEOREM 7.1. *Let g be a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ such that*

(i) $g \in L^1((0,R_1)\times(0,\infty)) \cap L^1((0,\infty)\times(0,R_2))$ *for all* $R_1, R_2 > 0$,

(ii) *each of the repeated integrals*

$$I_1(R_1,R_2) = \int_{R_1}^{\infty} \mathrm{d}u \int_0^{R_2} g(u,v)\mathrm{d}v, \quad I_2(R_1,R_2) = \int_0^{R_1} \mathrm{d}u \int_{R_2}^{\infty} g(u,v)\mathrm{d}v,$$

$$I_3(R_1,R_2) = \int_{R_1}^{\to\infty} \mathrm{d}u \int_{R_2}^{\infty} g(u,v)\mathrm{d}v,$$

converges to 0 *as* $R_1, R_2 \to \infty$.

Then g has an improper Lebesgue integral (7.2) *on* $(0,\infty)\times(0,\infty)$.

Proof. Suppose that g satisfies the conditions on the theorem. Using Fubini's theorem for the Lebesgue integral, we obtain

$$\int_0^{\to\infty} \Big(\int_0^{\infty} g(u,v)\mathrm{d}v\Big)\mathrm{d}u$$

$$= \int_0^{R_1}\Big(\int_0^{R_2} g(u,v)\mathrm{d}v\Big)\mathrm{d}u + \int_{R_1}^{\infty}\Big(\int_0^{R_2} g(u,v)\mathrm{d}v\Big)\mathrm{d}u + \int_0^{R_1}\Big(\int_{R_2}^{\infty} g(u,v)\mathrm{d}v\Big)\mathrm{d}u + \int_{R_1}^{\to\infty}\Big(\int_{R_2}^{\infty} g(u,v)\mathrm{d}v\Big)\mathrm{d}u$$

$$= \int_0^{R_1}\int_0^{R_2} g(u,v)\mathrm{d}u\mathrm{d}v + I_1(R_1,R_2) + I_2(R_1,R_2) + I_3(R_1,R_2).$$

Hence by the hypotheses,

$$\int_0^{R_1}\int_0^{R_2} g(u,v)\mathrm{d}u\mathrm{d}v \to \int_0^{\to\infty} \Big(\int_0^{\infty} g(u,v)\mathrm{d}v\Big)\mathrm{d}u$$

as $R_1, R_2 \to \infty$, and the improper Lebesgue integral (7.2) of $g(u,v)$ on $(0,\infty)\times(0,\infty)$ exists and equals $\int_0^{\to\infty} \Big(\int_0^{\infty} g(u,v)\mathrm{d}v\Big)\mathrm{d}u$. ∎

We verify that our given function G satisfies the conditions of the theorem. First we show that $G \in L^1((0,\infty)\times(0,R_2))$. For this we have to check that the integral

$$\int_0^{\infty} \mathrm{d}u \int_0^{R_2} |G(x;u,v)|\mathrm{d}v$$

is finite for each UCB function f and each $R_2 > 0$. Clearly, G is Lebesgue measurable. Write $\|f\| = \sup_{-\infty<t<\infty} |f(t)|$. Then

$$\int_0^\infty \mathrm{d}u \int_0^{R_2} \frac{|J_1(\sqrt{u})|}{u} \exp(-v^2/u)|f(x-v)|\mathrm{d}v \leqslant \|f\| \int_0^\infty \frac{|J_1(\sqrt{u})|}{u}\mathrm{d}u \int_0^{R_2} \exp(-v^2/u)\mathrm{d}v$$
$$= \frac{\|f\|\sqrt{\pi}}{2}\int_0^\infty \frac{|J_1(\sqrt{u})|}{u}\mathrm{erf}\,(R_2/\sqrt{u})\sqrt{u}\mathrm{d}u$$

where erf is the error function. The integrand in the last integral satisfies

$$\frac{|J_1(u^{1/2})|}{u^{1/2}}\mathrm{erf}\,(R_2u^{-1/2}) \asymp \frac{1}{2} \quad \text{as } u \to 0+,$$
$$\frac{|J_1(u^{1/2})|}{u^{1/2}}\mathrm{erf}\,(R_2u^{-1/2}) = O(u^{-5/4}) \quad \text{as } u \to \infty;$$

hence the integral is finite.

Next we show that $G \in L^1((0,R_1) \times (0,\infty))$ for all UCB functions f and all $R_1 > 0$, that is, we show that the integral

$$\int_0^{R_1} \mathrm{d}u \int_0^\infty |G(x;u,v)|\mathrm{d}v$$

is finite:

$$\int_0^{R_1} \mathrm{d}u \int_0^\infty \frac{|J_1(\sqrt{u})|}{u} \exp(-v^2/u)|f(x-v)|\mathrm{d}v \leqslant \|f\| \int_0^{R_1} \frac{|J_1(\sqrt{u})|}{u}\mathrm{d}u \int_0^\infty \exp(-v^2/u)\mathrm{d}v$$
$$= \frac{\|f\|\sqrt{\pi}}{2} \int_0^{R_1} \frac{|J_1(\sqrt{u})|}{\sqrt{u}}\mathrm{d}u < \infty.$$

This proves that $G \in L^1((0,R_1) \times (0,\infty))$.

We verify that the repeated integral

$$I_3(R_1,R_2) = \int_{R_1}^{\to\infty} \mathrm{d}u \int_{R_2}^\infty \frac{J_1(\sqrt{u})}{u} \exp(-v^2/u)f(x-v)\mathrm{d}v$$

satisfies $I_3(R_1,R_2) \to 0$ as $R_1,R_2 \to \infty$.

Substituting $u = s^2$ and writing

$$E(s) = \int_{R_2}^\infty \exp(-v^2/s^2)f(x-v)\mathrm{d}v \tag{7.3}$$

we get

$$I_3(R_1,R_2)=2\int\limits_{\sqrt{R_1}}^{\to\infty} J_1(s)E(s)\frac{\mathrm{d}s}{s}.$$

Writing $\alpha=(1/4)\pi$, we obtain

$$J_1(s)=\sqrt{\frac{2}{\pi}}\Big(\frac{\cos(s+\alpha)}{s^{1/2}}+\frac{\Delta(s)}{s^{3/2}}\Big),\quad s\geqslant 1,$$

where $\Delta(s)$ is bounded on $[1,\infty)$.

Integrating by parts gives

$$\int\limits_{\sqrt{R_1}}^{\to\infty}\cos(s+\alpha)\frac{E(s)}{s^{3/2}}\mathrm{d}s=-\frac{\sin(\sqrt{R_1}+\alpha)E(\sqrt{R_1})}{R_1^{3/4}}-\int\limits_{\sqrt{R_1}}^{\to\infty}\sin(s+\alpha)\frac{\mathrm{d}}{\mathrm{d}s}\frac{E(s)}{s^{3/2}}\mathrm{d}s.$$

The derivative of $E(s)$ is obtained by differentiating under the integral sign in (7.3):

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{E(s)}{s^{3/2}}=\frac{1}{s^{3/2}}\int\limits_{R_2}^{\infty}\Big(\frac{2v^2}{s^2}-\frac{3}{2}\Big)\exp(-v^2/s^2)f(x-v)\frac{\mathrm{d}v}{s}.$$

Hence

$$\frac{\sqrt{\pi}}{2\sqrt{2}}I_3=-\frac{\sin(\sqrt{R_1}+\alpha)E(\sqrt{R_1})}{R_1^{3/4}}+\int\limits_{\sqrt{R_1}}^{\infty}D(s)\frac{\mathrm{d}s}{s^{3/2}},$$

where

$$D(s)=\int\limits_{R_2}^{\infty}\Big[-\sin(s+\alpha)\Big(\frac{2v^2}{s^2}-\frac{3}{2}\Big)+\Delta(s)\Big]\exp(-v^2/s^2)f(x-v)\frac{\mathrm{d}v}{s}.$$

Substituting $\lambda=v/s$, we get

$$|E(s)|\leqslant\|f\|s\int\limits_{R_2/s}^{\infty}\exp(-\lambda^2)\mathrm{d}\lambda,\quad |D(s)|\leqslant\|f\|\int\limits_{R_2/s}^{\infty}\Big(2\lambda^2+M_\Delta+\frac{3}{2}\Big)\exp(-\lambda^2)\mathrm{d}\lambda,$$

where $M_\Delta=\max\limits_{1\leqslant s<\infty}|\Delta(s)|$. Thus

$$|E(\sqrt{R_1})|\leqslant\|f\|R_1^{1/2}\int\limits_{0}^{\infty}\exp(-\lambda^2)\mathrm{d}\lambda=O(R_1^{1/2})\quad\text{as } R_1\to\infty$$

independently of $R_2\in(0,\infty)$, and

$$|D(s)|\leqslant\|f\|\int\limits_{0}^{\infty}\Big(2\lambda^2+M_\Delta+\frac{3}{2}\Big)\exp(-\lambda^2)\mathrm{d}\lambda=O(1)\quad\text{as } R_1\to\infty$$

again independently of $R_2 \in [1, \infty)$. Therefore

$$I_3(R_1, R_2) = O(R_1^{-1/4}) + O\Big(\int_{\sqrt{R_1}}^{\infty} \frac{ds}{s^{3/2}} \Big) = O(R_1^{-1/4}) + O((R_1^{1/2})^{-1/2}) = O(R_1^{-1/4})$$

as $R_1 \to \infty$ independently of $R_2 \geqslant 1$. This proves $I_3(R_1, R_2) \to 0$ as $R_1, R_2 \to \infty$.

A straightforward modification of the preceding argument shows that $I_1(R_1, R_2) = O(R_1^{-1/4})$ as $R_1 \to \infty$ independently of $R_2 \geqslant 1$, and $I_1(R_1, R_2) \to 0$ as $R_1, R_2 \to \infty$.

To show that $I_2(R_1, R_2) \to 0$ as $R_1, R_2 \to \infty$, we estimate $|I_2(R_1, R_2)|$ separately on the sets $\sqrt{R_1} \leqslant R_2^{1-\varepsilon}$ and $\sqrt{R_1} > R_2^{1-\varepsilon}$, where $\varepsilon > 0$ is fixed. For $\sqrt{R_1} \leqslant R_2^{1-\varepsilon}$ we get

$$|I_2(R_1, R_2)| = O(R_2^{2-3\varepsilon} \exp(-2\varepsilon R_2)) \quad \text{as } R_2 \to \infty$$

independently of $R_1 \geqslant 1$. For $\sqrt{R_1} > R_2^{1-\varepsilon}$ we split

$$\frac{1}{2} I_2 = \int_0^{R_2^{2-2\varepsilon}} J_1(s) E(s) \frac{ds}{s} + \int_{R_2^{2-2\varepsilon}}^{R_1} J_1(s) E(s) \frac{ds}{s}.$$

The first integral is $O(R_2^{2-3\varepsilon} \exp(-2\varepsilon R_2))$ as $R_2 \to \infty$ independently of $R_1 \geqslant 1$. We split the range of integration yet again in the second integral to obtain

$$\Big| \int_{R_2^{2-2\varepsilon}}^{R_1} (\cdot) \Big| \leqslant \Big| \int_{R_2^{2-2\varepsilon}}^{\infty} (\cdot) \Big| + \Big| \int_{R_1}^{\infty} (\cdot) \Big| = O(R_1^{-1/4}) + O(R_2^{-(2-2\varepsilon)/4}), \quad R_1, R_2 \to \infty.$$

Theorem 7.1 is now applicable, and (6.13) holds.

Acknowledgements. This research project was partially supported by a grant from the Graduiertenkolleg "Hierarchie und Symmetrie in mathematischen Modellen", RWTH, Aachen. The second author was partially supported by the University of Melbourne Grant MRDGS 425007.

The authors are indebted to Professors Tuo-Yeong Lee (Singapore) and Washek Pfeffer (Davis, California) for their suggestions on the use of Fubini's theorem for the Kurzweil–Henstock (generalized Riemann) integral, and to Professor W.W. Wood (Melbourne, Australia) for his help with the asymptotic analysis in the Appendix. The authors also thank Professor Anatoly Kilbas (Minsk, Belarus) for his independent proof of the corrected formula (6.14) as well as Professor Jean Mawhin (Louvain-la-Neuve, Belgium) for valuable suggestions.

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Received February 3, 2007.