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THE HYPERCYCLICITY CRITERION AND HYPERCYCLIC SEQUENCES OF MULTIPLES OF OPERATORS

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ABSTRACT. Let T be a linear continuous operator acting on a Banach space X and $\{\lambda_n\}$ a sequence of non-zero complex numbers satisfying $\frac{\lambda_{n+1}}{\lambda_n} \to 1$. In this article we look at sequences of operators of the form $\{\lambda_n T^n\}$. In earlier work we showed that under the assumption that T is hypercyclic, if for some $x \in X$ the set $\{\lambda_n T^n x : n \in \mathbb{N}\}$ is somewhere dense then it is everywhere dense, a Bourdon–Feldman type theorem. In this article we show that this result fails to hold if the assumption of hypercyclicity for T is removed. A condition for the sequence $\{\lambda_n\}$ under which an Ansari type theorem holds, namely, if $\{\lambda_n T^n\}$ is hypercyclic then $\{\lambda_n T^{kn}\}$ is hypercyclic for $k = 2, 3, \ldots$, is given. We show that if this condition is not satisfied, the result may fail to hold. Furthermore, we establish equivalences to the hypercyclicity criterion for this class of operator sequences.

KEYWORDS: *Hypercyclic operators, hypercyclicity criterion.*

MSC (2000): 47A16.

1. INTRODUCTION

Let X be an infinite dimensional topological vector space either over the field $\mathbb{C}$ or $\mathbb{R}$ and $T_n : X \to X$ be a sequence of continuous linear operators. We say that the sequence $\{T_n\}$ is *hypercyclic* if there exists $x \in X$ such that the sequence $\{T_n x : n = 0, 1, 2, \ldots\}$ is dense in X. Such a vector x is called hypercyclic for the sequence $\{T_n\}$ and the set of hypercyclic vectors for $\{T_n\}$ is denoted $HC(\{T_n\})$. In the case the sequence $\{T_n\}$ comes from the iterates of a single operator T, i.e. $T_n = T^n$ for $n = 0, 1, 2, \ldots$ we say that T is hypercyclic and the set of hypercyclic vectors for T is denoted $HC(T)$. We refer the reader to the review articles [20], [9], [26], [31] and [21] for examples and background theory on hypercyclicity.

From now on when we refer to an operator T, we always assume that T is linear and continuous. Let $\{\lambda_n\}$ be a sequence of non-zero complex numbers satisfying $\frac{\lambda_{n+1}}{\lambda_n} \to 1$. Throughout this article we will be looking at sequences of operators of the form $\{\lambda_n T^n\}$. The reason for looking at sequences of this form is that their dynamics tend to resemble, in a sense, those of a sequence of iterates of a single operator. As stated in the abstract, we have already shown in [13] that if T is hypercyclic and for some $x \in X$ the set $\{\lambda_n T^n x : n \in \mathbb{N}\}$ is somewhere dense, then it is everywhere dense, a Bourdon–Feldman type theorem (see [11]). This result is further explored and examples are given where the theorem fails to hold when T is not hypercyclic. The crucial element in achieving this is the construction of a sequence of non-zero complex numbers $\{\lambda_n\}$ such that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ and the set $\{\lambda_n : n \in \mathbb{N}\}$ is dense in a bounded set with non-empty interior. In another direction, we give sufficient conditions under which a version of Ansari's Theorem (see [1]) holds in this setting, namely, if $\{\lambda_n T^n\}$ is hypercyclic then $\{\lambda_n T^{kn}\}$ is hypercyclic for every positive integer k. We stress here that the conditions to this version of Ansari's Theorem do not require T to be hypercyclic.

Until very recently it was an open problem whether every hypercyclic operator satisfies the hypercyclicity criterion (see Section 2). In [29] de la Rosa and Read gave the first example of a hypercyclic operator acting on a certain Banach space which does not satisfy the hypercyclicity criterion. Subsequently, Bayart and Matheron in [4], see also [3], showed that this is possible even in classical Banach or Hilbert spaces, i.e. $l^p(\mathbb{N})$ spaces, $1 \leqslant p < +\infty$. We would like to point out that for sequences of operators this problem is much easier. It is well known that for a wide class of Banach spaces X, there exists a sequence $\{T_n\}$ of operators on X being hypercyclic which fails the hypercyclicity criterion. In fact Bernal-González has shown the existence of a hypercyclic sequence $\{T_n\}$ which has a hereditarily hypercyclic subsequence $\{T_{n_k}\}$ and yet fails to satisfy the hypercyclicity criterion (see [6]). In the other direction, several equivalent forms to the hypercyclicity criterion have appeared (see [7], [8], [17], [19], [23], [25], [28], [16]). In this work we provide equivalent forms to the hypercyclicity criterion for sequences of operators of the form $\{\lambda_n T^n\}$ following the work of Grivaux (see [19]) and Peris and Saldivia (see [28]). In particular we introduce a new equivalent form, extending the work of Grivaux in [19]. This allows us to establish the Ansari type theorem for this class of operator sequences.

2. TOPOLOGICAL RESULTS ON HYPERCYCLIC OPERATORS

DEFINITION 2.1. A sequence of complex numbers $\{\lambda_n\}$ with $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ is called *admissible* if one of the following holds:

(i) There exist $m, M > 0$ and $n_0 \in \mathbb{N}$ such that $m < |\lambda_n| < M \quad \forall n \geqslant n_0$.

(ii) $|\lambda_n| \to 0$ as $n \to +\infty$.

(iii) $|\lambda_n| \to +\infty$ as $n \to +\infty$.

The next proposition will be used frequently throughout this paper.

PROPOSITION 2.2. *Let $\{\lambda_n\}$ be a sequence in $\mathbb{C} \setminus \{0\}$ such that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ and $T : X \to X$ be an operator. Assume that the sequence $\{\lambda_n T^n\}$ is hypercyclic and fix $x \in HC(\{\lambda_n T^n\})$.*

(i) *$\sigma_{\mathrm{p}}(T^*) \cap D = \emptyset$ and $\sigma_{\mathrm{p}}(T^*) \cap (\mathbb{C} \setminus \overline{D}) = \emptyset$, where D denotes the open unit disc. Hence, for every $\lambda \in D \cup (\mathbb{C} \setminus \overline{D})$ the operator $T - \lambda I$ has dense range. In addition, if P is a non-zero polynomial of degree q such that $P(T) = (T - \alpha_1 I)(T - \alpha_2 I) \cdots (T - \alpha_q I)$ and $|\alpha_j| \neq 1$ for all $j = 1, 2, \ldots, q$ then $P(T)x \in HC(\{\lambda_n T^n\})$.*

(ii) *If $\{\lambda_n\}$ is an admissible sequence then the point spectrum $\sigma_{\mathrm{p}}(T^*)$ is empty, hence for every $\lambda \in \mathbb{C}$ the operator $T - \lambda I$ has dense range. In addition, for every non-zero polynomial P, $P(T)x \in HC(\{\lambda_n T^n\})$.*

Proof. If not there exist $\lambda \in \mathbb{C}$ and $x^* \in X^* \setminus \{0\}$ such that $T^* x^* = \lambda x^*$. Let $x \in HC(\{\lambda_n T^n\})$. Then the sequence $\{\langle \lambda_n T^n x, x^* \rangle : n = 1, 2, \ldots\}$ is dense in $\mathbb{C}$. Observe that $\langle \lambda_n T^n x, x^* \rangle = \lambda_n \overline{\lambda}^n \langle x, x^* \rangle$. Therefore the sequence $\{|\lambda_n||\lambda|^n\}$ is dense in $[0, +\infty)$. If $\{\lambda_n\}$ is such that $m < |\lambda_n| < M \; \forall n \geqslant n_0$ for some $m, M > 0$ and $n_0 \in \mathbb{N}$, since $\{|\lambda|^n\}$ is a geometric sequence, the sequence $\{|\lambda_n||\lambda|^n\}$ cannot be dense in $[0, +\infty)$ which is a contradiction. If $\{\lambda_n\}$ is such that $|\lambda_n| \to 0$ we consider the cases $|\lambda| \leqslant 1$ and $|\lambda| > 1$ separately. If $|\lambda| \leqslant 1$ then the sequence $\{\lambda_n \lambda^n\}$ is a null sequence and so the respective moduli cannot be dense in $[0, +\infty)$. If $|\lambda| > 1$ then there exists $N \in \mathbb{N}$ such that $\left|\frac{\lambda_{n+1}}{\lambda_n}\right| > \frac{1}{|\lambda|} \; \forall n \geqslant N$. Hence, for every $m \geqslant 1$ we get $\|\lambda_{N+m} \lambda^{N+m}| > |\lambda_N \lambda^N|$, which implies that the sequence $\{|\lambda_n||\lambda|^n\}$ cannot be dense in $[0, +\infty)$. In case $|\lambda_n| \to +\infty$ we consider the cases $|\lambda| < 1$ and $|\lambda| \geqslant 1$ separately. The details are left to the reader. The proof follows along the same lines as the corresponding proof for hypercyclic operators due to Bourdon (see [10]). ∎

PROPOSITION 2.3. *There exist sequences $\{z_n\}, \{\lambda_n\}$ of complex numbers such that:*

(i) *$|z_{n+1} - z_n| \to 0$ and $\overline{\{z_n : n = 1, 2, \ldots\}} = \mathbb{C}$;*

(ii) *$\frac{\lambda_{n+1}}{\lambda_n} \to 1$ and $\overline{\{\lambda_n : n = 1, 2, \ldots\}} = \mathbb{C}$.*

Proof. Let us first prove (i). Define recursively the sequences $\{z_n = \alpha_n + \mathrm{i}\beta_n\}$, $\{l_n\}$, $\{r_n\}$, $\{u_n\}$, $\{d_n\}$, $\{s_n\}$ and $\{h_n\}$ by $\alpha_1 = -1$, $\beta_1 = -1$, $l_1 = -1$, $r_1 = 1$, $d_1 = -1$, $u_1 = 1$, $s_1 = 1$, $h_1 = \frac{1}{2}$ and

$$\alpha_{i+1} = \alpha_i + h_i; \quad \beta_{i+1} = \begin{cases} \beta_i & \alpha_{i+1} \neq r_i \quad \text{and} \quad \alpha_{i+1} \neq l_i, \\ \beta_i + s_i|h_i| & \alpha_{i+1} = r_i \quad \text{or} \quad x_{i+1} = l_i; \end{cases}$$

$$l_{i+1} = d_{i+1} = \begin{cases} l_i & \alpha_{i+1} \neq l_i + h_i \quad \text{or} \quad \beta_{i+1} \neq u_i, \\ l_i - 1 & \alpha_{i+1} = l_i + h_i \quad \text{and} \quad \beta_{i+1} = u_i; \end{cases}$$

$$r_{i+1} = u_{i+1} = \begin{cases} r_i & \alpha_{i+1} \neq l_i \quad \text{or} \quad \beta_{i+1} \neq d_i, \\ r_i + 1 & \alpha_{i+1} = l_i \quad \text{and} \quad \beta_{i+1} = d_i; \end{cases} \qquad s_{i+1} = \begin{cases} -1 & \beta_{i+1} = u_i, \\ +1 & \beta_{i+1} = d_i, \\ s_i & \text{otherwise}; \end{cases}$$

$$h_{i+1} = \begin{cases} h_i & \alpha_{i+1} \neq l_{i+1} \quad \text{and} \quad \alpha_{i+1} \neq r_{i+1}, \\ -h_i & \alpha_{i+1} = r_{i+1}, \\ -h_i & \alpha_{i+1} = l_{i+1} \quad \text{and} \quad \beta_{i+1} \neq d_{i+1}, \\ -\frac{h_i}{2} & \alpha_{i+1} = l_{i+1} \quad \text{and} \quad \beta_{i+1} = d_{i+1}. \end{cases}$$

The sequence $\{z_n\}$ can be seen to trace out all numbers of the form $\frac{p}{2^q} + \mathrm{i}\frac{l}{2^m}$, $p, l \in \mathbb{Z}$, $q, m \in \mathbb{N}$ which is dense in $\mathbb{C}$. Also $h_n \to 0$ and so $|z_{n+1} - z_n| \to 0$. To prove (ii) define $\lambda_n = \mathrm{e}^{z_n}$ for $n = 1, 2, \ldots$ where $\{z_n\}$ is the sequence constructed in (i). It is clear that the sequence $\{\lambda_n\}$ has the desired properties. ∎

PROPOSITION 2.4. *Fix an operator S acting on a complex Banach space Y such that S is hereditarily hypercyclic for the whole sequence of natural numbers, i.e. for every sequence $\{n_k\}$ of positive integers the sequence $\{S^{n_k}\}$ is hypercyclic. Define the operator $T = I_{\mathbb{C}} \oplus S : \mathbb{C} \oplus Y \to \mathbb{C} \oplus Y$, where $I_{\mathbb{C}}$ denotes the identity operator on $\mathbb{C}$. Then for every sequence $\{\lambda_n\}$ of non-zero complex numbers such that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ and $\overline{\{\lambda_n : n = 1, 2, \ldots\}} = \mathbb{C}$ the following hold:*

(i) *The sequence $\{\lambda_n T^n\}$ is hypercyclic.*

(ii) *The sequence $\{\lambda_n T^n\}$ does not satisfy the hypercyclicity criterion.*

(iii) $\sigma_{\mathrm{p}}(T^*) = \{1\}$.

Proof. Proposition 2.3 implies the existence of a sequence $\{\lambda_n\}$ satisfying the properties mentioned above. Since S is hypercyclic, $\sigma_{\mathrm{p}}(S^*) = \varnothing$ and hence it is obvious that $\sigma_{\mathrm{p}}(T^*) = \{1\}$.

Fix a countable dense set $\{x_j\}$ in Y and a countable dense set $\{\alpha_l\}$ in $\mathbb{C}$. For $j, l, s \in \{1, 2, \ldots\}$ and $n \in \{0, 1, 2, \ldots\}$ define the open sets

$$A(l,s) = \Big\{ n \in \mathbb{N} : |\lambda_n - \alpha_l| < \frac{1}{s} \Big\}, \quad V(j,s,n) = \Big\{ x \in Y : \|\lambda_n S^n x - x_j\| < \frac{1}{s} \Big\}.$$

Set

$$G = \bigcap_{j,l,s} \bigcup_{n \in A(l,s)} V(j,s,n).$$

We shall prove that G is dense in Y. From Baire's Category Theorem, it suffices to show that each set $\bigcup_{n \in A(l,s)} V(j,s,n)$ is dense. Fix $j, l, s \in \{1, 2, \ldots\}$, $y \in Y$ and $\varepsilon > 0$. We seek $x \in \bigcup_{n \in A(l,s)} V(j,s,n)$ such that $\|y - x\| < \varepsilon$. There exists an increasing sequence $\{n_k\}$ of positive integers such that $\lambda_{n_k} \to \alpha_l$. Since S is hereditarily hypercyclic for the whole sequence of natural numbers, there exists $x \in HC(\{S^{n_k}\})$ such that $\|y - x\| < \varepsilon$. Hence we may find a subsequence $\{n_{k_p}\}$ of $\{n_k\}$ such that $S^{n_{k_p}} x \to x_j$. Observe also that $\lambda_{n_{k_p}} \to \alpha_l$. Choosing $n = n_{k_p}$ for p sufficiently large we get $|\lambda_n - \alpha_l| < \frac{1}{s}$ and $\|\lambda_n S^n x - x_j\| < \frac{1}{s}$.

Let us show that $\{\lambda_n T^n\}$ is hypercyclic. Take any $x \in G$. Clearly, for every $\alpha \in \mathbb{C} \setminus \{0\}$, the vector $\alpha \oplus x$ is hypercyclic for $\{\lambda_n T^n\}$.

To show (ii) observe that because of the form of the first component of T, that is, $\lambda_n I_{\mathbb{C}}$, it is easy to see that T does not satisfy the hypercyclicity criterion. ∎

Recently, see Proposition 4.2 in [13], we established the following version of Bourdon and Feldman's theorem: *Let $\{\lambda_n\}$ be a sequence of positive (or complex) numbers such that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$. Suppose that T is a hypercyclic operator acting on a complex Banach space X. If for some vector $x \in X$, the set $\{\lambda_n T^n x\}$ is somewhere dense then it is everywhere dense.* In the next proposition we show that this result fails to hold in general if T is not hypercyclic.

PROPOSITION 2.5. *There exist a sequence of complex numbers $\{\lambda_n\}$ for which $\frac{\lambda_{n+1}}{\lambda_n} \to 1$, an operator T acting on a complex Banach space X and $x \in X$ such that the sequence $\{\lambda_n T^n x\}$ is somewhere dense but not everywhere dense.*

Proof. Take the closed unit square R centered at 0 and sides parallel to the axes in the complex plane $\mathbb{C}$. One may easily construct, under minor modification to the procedure used for constructing the sequence $\{z_n\}$ in Proposition 2.3, a sequence $\{z_n\}$ in $\mathbb{C}$ such that $|z_{n+1} - z_n| \to 0$ and $\overline{\{z_n\}} = R$. Define the sequence $\{\lambda_n\}$ by $\lambda_n = \mathrm{e}^{z_n}$. It is plain that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ and $\overline{\{\lambda_n\}} = \{\mathrm{e}^z : z \in R\}$. Using the operator $T : \mathbb{C} \oplus Y \to \mathbb{C} \oplus Y$ as defined in Proposition 2.4 and working as in the proof of Proposition 2.4 we find $x \in X = \mathbb{C} \oplus Y$ so that $\overline{\{\lambda_n T^n x\}} = \{\mathrm{e}^z : z \in R\} \oplus Y$. Hence, $\{\lambda_n T^n x\}$ is somewhere dense but not everywhere dense. ∎

REMARK 2.6. Observe that the sequence $\{\lambda_n\}$ constructed in Proposition 2.5 is also bounded below and above in modulus and so it is an admissible sequence. However for admissible sequences satisfying either condition (ii) or (iii) of Definition 2.1 we don't know if an analogue to Bourdon–Feldman's Theorem holds. However an analogue to Ansari's Theorem for admissible sequences satisfying condition (ii) or (iii) of Definition 2.1 fails to hold in general (see Proposition 2.10.)

The validity of Herrero's conjecture for the iterates of a single operator has been established in [12], [27], see also [11]. Below we show that Herrero's conjecture fails in general for sequences of operators of the form $\{\lambda_n T^n\}$ with $\frac{\lambda_{n+1}}{\lambda_n} \to 1$.

PROPOSITION 2.7. *There exist a sequence $\{\lambda_n\}$ in $\mathbb{C}$ with $\frac{\lambda_{n+1}}{\lambda_n} \to 1$, an operator T acting on a complex Banach space X and vectors $x_1, x_2 \in X$ such that $\bigcup_{j=1}^{2} \overline{\{\lambda_n T^n x_j : n = 1, 2, \dots\}} = X$ and $\overline{\{\lambda_n T^n x_j : n = 1, 2, \dots\}} \neq X \ \forall j = 1, 2$.*

Proof. By modifying the construction of z_n's in Proposition 2.3 one may construct a sequence $\{z_n\}$ lying in the infinite strip $S := \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) < \pi\}$ such that $|z_{n+1} - z_n| \to 0$ and $\overline{\{z_n\}} = \overline{S}$. Define $\lambda_n = \mathrm{e}^{z_n}$. If exp is the exponential map, we have $\exp(S) = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. It follows that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ and $\overline{\{\lambda_n\}} = \{z \in \mathbb{C} : \mathrm{Re}(z) \geqslant 0\}$. Using the operator T as defined in Proposition 2.4, we find a vector $x = \alpha \oplus y \in X = \mathbb{C} \oplus Y$, $\mathrm{Re}(\alpha) > 0$ so that $\overline{\{\lambda_n T^n x\}} = \{z \in \mathbb{C} : \mathrm{Re}(z) \geqslant 0\} \oplus Y$. Taking as $x_1 = x$, $x_2 = -x$ we are done. ∎

We next establish an Ansari type theorem under certain conditions. These conditions turn out to be optimal, see Propositions 2.10 and 2.11. Our proof relies on a new equivalent form of the hypercyclicity criterion established in the next section, see Theorem 3.1.

PROPOSITION 2.8. *Suppose $\{\lambda_n\}$ is a sequence in $\mathbb{C} \setminus \{0\}$ such that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ and $\lim\limits_{n\to\infty} \frac{\lambda_{mn}}{\lambda_n}$ exists and is a non-zero complex number for every $m = 1, 2, \ldots$. Let $T : X \to X$ be an operator. If $\{\lambda_n T^n\}$ satisfies the hypercyclicity criterion then $\{\lambda_n T^{kn}\}$ is hypercyclic for every $k = 1, 2, \ldots$.*

Proof. Fix a countable dense set $\{x_j\}$ and $k \in \mathbb{N}$. Define the open sets $E(j,s,n) = \{x \in X : \|\lambda_n T^{kn} x - x_j\| < \frac{1}{s}\}$ for $j, s = 1, 2, \ldots$, $n = 0, 1, 2, \ldots$. It is easy to check that $HC(\{\lambda_n T^{kn}\}) = \bigcap\limits_{j,s} \bigcup\limits_{n} E(j,s,n)$. In view of Baire's category theorem it suffices to show that the set $\bigcup\limits_{n} E(j,s,n)$ is dense for every $j, s = 1, 2, \ldots$. Fix $j, s = 1, 2, \ldots$, $y \in X$ and $\varepsilon > 0$. We seek an $x \in \bigcup\limits_{n} E(j,s,n)$ such that $\|x - y\| < \varepsilon$. Denote $\xi_k = \lim\limits_{n\to\infty} \frac{\lambda_{kn}}{\lambda_n}$. Applying Theorem 3.1(iii) for $U = \frac{1}{\xi_k} B(y, \varepsilon/2)$ and $V = B(x_j, \frac{1}{s})$, there exists a strictly increasing sequence $\{n_l\}$ such that

$$\lambda_{n_l+i} T^{n_l+i}(U) \bigcap V \neq \emptyset \quad \forall i = 0, 1, \ldots, k, \quad \forall l = 1, 2, \ldots. \tag{2.1}$$

Also, there exists $N \in \mathbb{N}$ such that

$$\frac{\lambda_{kn}}{\lambda_n} \frac{1}{\xi_k} B(y, \varepsilon/2) \subset B(y, \varepsilon) \quad \forall n \geqslant N. \tag{2.2}$$

Now choose $n_l > kN$ and note that as (2.1) holds for $k+1$ consecutive integers, there exists an integer in $\{n_l, n_l + 1, \ldots, n_l + k\}$ of the form $k\rho$ for some integer $\rho > N$. Hence, $\lambda_{k\rho} T^{k\rho}\big(\frac{1}{\xi_k} B(y, \varepsilon/2)\big) \cap B(x_j, 1/s) \neq \emptyset$ and by (2.2) we get that

$$\emptyset \neq \lambda_\rho T^{k\rho}\Big(\frac{\lambda_{k\rho}}{\lambda_\rho} \frac{1}{\xi_k} B(y, \varepsilon/2)\Big) \cap B(x_j, 1/s) \subset \lambda_\rho T^{k\rho} B(y, \varepsilon) \cap B(x_j, 1/s).$$

This completes the proof of the proposition. ∎

REMARK 2.9. The previous proposition applies for a broad family of sequences $\{\lambda_n\}$. In particular, it applies for sequences of the form $\lambda_n = n^\alpha$, $\alpha \in \mathbb{R}$. Observe that in this case, the stronger condition $\lambda_{mn} = \lambda_m \lambda_n$ holds for every $m, n = 1, 2, \ldots$. It also applies for sequences of the form $\lambda_n = p(n)$ where p is a polynomial, since $\frac{p(mn)}{p(n)} \to m^{\deg p}$ where $\deg p$ denotes the degree of the polynomial p. Other sequences include those of the form $\lambda_n = (n+1)^\alpha \log(n+1)$, $\alpha \in \mathbb{R}$ since $\frac{\lambda_{mn}}{\lambda_n} \to (m+1)^\alpha$.

We next show that the assumption of the existence of $\lim_{n\to\infty} \frac{\lambda_{mn}}{\lambda_n}$ being a non-zero complex number for every $m = 1, 2, \ldots$ in Proposition 2.8 is optimal. We first recall the notion of topological mixing. A sequence of operators $\{T_n\}$ where $T_n : X \to X$ is called *topologically mixing* if for every non-empty open sets U, V in X, there exists a positive integer N such that $T_n(U) \cap V \neq \emptyset$ for every $n \geqslant N$.

PROPOSITION 2.10. *There exist a sequence $\{\lambda_n\}$ of non-zero complex numbers and a unilateral weighted shift $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ such that the following hold:*

(i) $\lim_{n\to+\infty} |\frac{\lambda_{n+1}}{\lambda_n} - 1| = 0$.

(ii) $\lim_{n\to\infty} \frac{\lambda_{mn}}{\lambda_n} = +\infty$ *for every* $m = 2, 3, \ldots$.

(iii) *The sequence $\{\lambda_n T^n\}$ satisfies the hypercyclicity criterion.*

(iv) *The sequence $\{\lambda_n T^{2n}\}$ is not hypercyclic.*

Proof. Applying a similar argument as in [14], it follows that if T is a unilateral weighted shift with weight sequence $\{\alpha_n\}$ and $\{\lambda_n\}$ is a sequence in $\mathbb{C} \setminus \{0\}$ with $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ then

$$\{\lambda_n T^n\} \quad \text{is topologically mixing if and only if} \lim_{n\to+\infty} |\lambda_n| \prod_{i=1}^{n} \alpha_i = +\infty.$$

Following the work of Salas in [30] and León–Saavedra in [24], it can easily be shown that

$$\{\lambda_n T^{2n}\} \quad \text{is hypercyclic if and only if} \quad \limsup_{n\to+\infty} |\lambda_n| \prod_{i=1}^{2n} \alpha_i = +\infty.$$

For every positive integer n, set $\alpha_n = \frac{1}{n^{1/n}}$, $\lambda_n = \frac{\sqrt{n}}{\prod_{i=1}^{n} \alpha_i}$ and let T be the unilateral weighted shift with weight sequence $\{\alpha_n\}$. Observe that:

$$\frac{\lambda_{n+1}}{\lambda_n} = \frac{\sqrt{n+1}}{\sqrt{n}} \frac{1}{\alpha_{n+1}} \to 1, \quad \text{and} \quad \frac{\lambda_n}{\lambda_{mn}} = \frac{1}{\sqrt{m}} \prod_{i=n+1}^{mn} \alpha_i \leqslant \frac{1}{\sqrt{m}} \frac{1}{(mn)^{1-1/m}} \to 0.$$

Hence (i) and (ii) hold. Observe that $\lambda_n \prod_{i=1}^{n} \alpha_i = \sqrt{n} \to +\infty$, therefore $\{\lambda_n T^n\}$ is topologically mixing and so satisfies the hypercyclicity criterion. On the other hand the quantity $\lambda_n \prod_{i=1}^{2n} \alpha_i$ is bounded by $\frac{1}{\sqrt{2}}$ for every n. Thus $\{\lambda_n T^{2n}\}$ is not hypercyclic. ∎

We next show, as kindly pointed to us by the referee, that Proposition 2.8 becomes false when the sequence $\{\lambda_n T^n\}$ does not satisfy the hypercyclicity criterion.

PROPOSITION 2.11. *There exist a sequence $\{\lambda_n\}$ of non-zero complex numbers and an operator T acting on a Banach space X such that the following hold:*

(i) $\lim_{n\to+\infty} |\frac{\lambda_{n+1}}{\lambda_n} - 1| = 0$.

(ii) *The limit* $\lim_{n\to\infty} \frac{\lambda_{mn}}{\lambda_n}$ *exists for every* $m = 2, 3, \ldots$.
(iii) *The sequence* $\{\lambda_n T^n\}$ *is hypercyclic.*
(iv) *The sequence* $\{\lambda_n T^{2n}\}$ *is not hypercyclic.*

Proof. If S is as in Proposition 2.4 define $T = -I \oplus S$ and consider a sequence $\{\lambda_n\}$ of complex numbers with positive real part such that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$, $\frac{\lambda_{mn}}{\lambda_n} \to 1$ for every $m = 1, 2, \ldots$ and $\{\lambda_{2n}\}, \{\lambda_{2n+1}\}$ are dense in the right half-plane. Then it is easy to check that $\{\lambda_n T^n\}$ is hypercyclic but $\{\lambda_n T^{2n}\}$ is not. ∎

We establish below weak variants of Ansari and Bourdon–Feldman's theorems.

PROPOSITION 2.12. *Let* $\{\lambda_n\}$ *be a sequence in* $\mathbb{C} \setminus \{0\}$ *such that* $\frac{\lambda_{n+1}}{\lambda_n} \to 1$, *let* $T : X \to X$ *be an operator and* $\{\lambda_n T^n\}$ *be hypercyclic.*

(i) *Suppose that* $\frac{\lambda_{mn}}{\lambda_n} \to \xi_m$ *exists and is a non-zero complex number for every* $m = 1, 2, \ldots$, $\{\lambda_n T^n\}$ *satisfies the hypercyclicity criterion and* $\overline{\{\lambda_n T^n x\}} = X$ *for some* $x \in X$. *Then* $\overline{\{\lambda_m \lambda_n T^{kn} x : n, m = 1, 2, \ldots\}} = X$ *for every* $k = 1, 2, \ldots$.

(ii) *If* $\overline{\{\lambda_n T^n x\}}^{o} \neq \emptyset$ *for some* $x \in X$, *then* $\overline{\{\lambda_m \lambda_n T^n x : n, m = 1, 2, \ldots\}} = X$.

Proof. Throughout, for every $y \in X$ and an operator $S : X \to X$ we will be using the sets $F_y(S)$ defined by $F_y(S) = \{w \in X : \exists n_k \to +\infty, \lambda_{n_k} S^{n_k} y \to w\}$. Fix k a positive integer. We first show that

$$X = F_x(T) = F_{\xi_k x}(T^k) \cup F_{T(\xi_k x)}(T^k) \cup \cdots \cup F_{T^{k-1}(\xi_k x)}(T^k).$$

Observe that our hypothesis implies $X = F_x(T)$. Fix a $y \in F_x(T)$ and note that there exists a strictly increasing sequence of positive integers $\{n_l\}$ such that $\lambda_{n_l} T^{n_l} x \to y$. As each term in $\{n_l\}$ can be expressed as $n_l = kq_l + j_l$ for integers q_l and $j_l \in \{0, 1, \ldots, k-1\}$, there exists a subsequence $\{n_{l_s}\}$ where each term is of the form $n_{l_s} = kq_{l_s} + j$ for $j \in \{0, 1, \ldots, k-1\}$. So without loss of generality assume that $n_l = kq_l + j$ for some $j \in \{0, 1, \ldots, k-1\}$. Hence $\lambda_{kq_l+j} T^{kq_l+j} x \to y$. Since $\frac{\lambda_{kq_l}}{\lambda_{q_l}} \to \xi_k$ and $\frac{\lambda_{kq_l+j}}{\lambda_{kq_l}} \to 1$, we get that $\lambda_{q_l} T^{kq_l}(T^j \xi_k x) \to y$. Hence $y \in F_{T^j(\xi_k x)}(T^k)$ which proves our claim. It now follows that one of the sets in the union, call it $F_{T^r(\xi_k x)}(T^k)$ for some $r \in \{0, 1, \ldots, k-1\}$, must have non-empty interior. From Proposition 2.8 the sequence $\{\lambda_n T^{kn}\}$ is hypercyclic. Choose $z \in HC(\{\lambda_n T^{kn}\}) \cap F_{T^r(\xi_k x)}(T^k) \neq \emptyset$. There exists a strictly increasing sequence $\{t_n\}$ of positive integers such that $\lambda_{t_n} T^{kt_n}(T^r(\xi_k x)) \to z$. Since the set $F_{T^r(\xi_k x)}(T^k)$ is T^k-invariant, $\lambda_{t_n} T^{k(t_n+m)}(T^r(\xi_k x)) \to T^{km} z$, $\forall m = 1, 2, \ldots$. Therefore $\lambda_m \lambda_{t_n} T^{k(t_n+m)}(T^r(\xi_k x)) \to \lambda_m T^{km} z \quad \forall m = 1, 2, \ldots$. This means that the sequence $\{\lambda_m \lambda_n T^{kn}(T^r(\xi_k x)) : n, m = 1, 2, \ldots\}$ approximates every element of the sequence $\{\lambda_m T^{km} z\}$ which is dense in X and since T^r by Proposition 2.2 has dense range, the result follows.

Let us prove (ii). Since $\overline{\{\lambda_n T^n x\}}^o \neq \emptyset$ it follows that $F_x(T)^o \neq \emptyset$. Fix $z \in F_x(T)^o \cap HC(\{\lambda_n T^n\})$. There exists a strictly increasing sequence of positive integers $\{n_k\}$ such that $\lambda_{n_k} T^{n_k} x \to z$. From the T-invariance of $F_x(T)$ and since $\frac{\lambda_{n_k}}{\lambda_{n_k+m}} \to 1$, $m = 1, 2, \ldots$ it follows that $\lambda_m \lambda_{n_k+m} T^{n_k+m} x \to \lambda_m T^m z$, $m = 1, 2, \ldots$. Hence the sequence $\{\lambda_m \lambda_n T^n x : n, m = 1, 2, \ldots\}$ approximates every element of the dense sequence $\{\lambda_n T^n z : n = 1, 2, \ldots\}$. ∎

REMARK 2.13. In the above proposition the assumption that the sequence $\{\lambda_n T^n\}$ is hypercyclic cannot be removed. Indeed, consider the sequence $\{\lambda_n\}$, the operator $T : X \to X$ and $x \in X$ as defined in Proposition 2.5. We have $\overline{\{\lambda_n T^n x\}}^o \neq \emptyset$ and it easily follows that $\overline{\{\lambda_m \lambda_n T^n x : n, m = 1, 2, \ldots\}} \neq X$.

We next proceed by showing that for syndetic sequences $\{n_k\}$, we may have $HC(\{\lambda_n T^n\}) \neq HC(\{\lambda_{n_k} T^{n_k}\})$ which extends some results from [26], [28].

THEOREM 2.14. *Let $\{\lambda_n\}$ be a sequence in $\mathbb{C} \setminus \{0\}$ such that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ and let $T : X \to X$ be a linear continuous operator. Suppose there exists $x \in X$ so that $\overline{\{\lambda_n T^n x\}} = X$. Then there exists a syndetic sequence $\{n_k\}$ with $n_{k+1} - n_k \leqslant 2$ for every $k = 1, 2, \ldots$ such that $\overline{\{\lambda_{n_k} T^{n_k} x\}} \neq X$.*

Proof. Fix $y \in X$ so that $Ty \neq y$. Take $\varepsilon > 0$ such that $B(y, \varepsilon) \cap B(Ty, \varepsilon) = \emptyset$. Let N be the smallest positive integer such that $\lambda_N T^N x \in B(y, \varepsilon/2)$ and

$$\Big| \frac{\lambda_{n+1}}{\lambda_n} - 1 \Big| < \frac{\varepsilon}{2\|T\|(\|y\| + \varepsilon)}, \quad \forall n \geqslant N.$$

Define the set $U = B(y, \varepsilon) \cap T^{-1}(B(Ty, \varepsilon/2))$. Observe that U is non-empty since $y \in U$. Let us now define the following two subsets of $\mathbb{N}$: $\Lambda_1 = \{n \geqslant N : \lambda_n T^n x \in U\}$ and $\Lambda_2 = \{n \geqslant N : \lambda_n T^n x \notin U\}$. We shall show that if $n \in \Lambda_1$ then $n+1 \in \Lambda_2$. Indeed, let $n \in \Lambda_1$ which implies that $\lambda_n T^n x \in U$ and hence $\|\lambda_n T^n x\| \leqslant \|y\| + \varepsilon$. Notice that $T(U) \subset B(Ty, \varepsilon/2)$ and therefore $\|\lambda_n T^{n+1} x - Ty\| \leqslant \frac{\varepsilon}{2}$. Then we have

$$\begin{aligned}\|\lambda_{n+1} T^{n+1} x - Ty\| &\leqslant \|\lambda_{n+1} T^{n+1} x - \lambda_n T^{n+1} x\| + \|\lambda_n T^{n+1} x - Ty\| \\ &\leqslant \Big| \frac{\lambda_{n+1}}{\lambda_n} - 1 \Big| \|T\| \, \|\lambda_n T^n x\| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2\|T\|(\|y\| + \varepsilon)} \|T\| (\|y\| + \varepsilon) + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Since $B(y, \varepsilon) \cap B(Ty, \varepsilon) = \emptyset$ it follows that $\lambda_{n+1} T^{n+1} x \notin B(y, \varepsilon)$ which in turn implies that $\lambda_{n+1} T^{n+1} x \notin U$. Hence $n+1 \in \Lambda_2$. It is now evident that if we consider an enumeration n_k, $k = 1, 2, \ldots$ of the elements of the set Λ_2 such that $n_{k+1} < n_k$ for $k = 1, 2, \ldots$, the sequence $\{n_k\}$ serves our purposes. ∎

We complement the previous result following the work of Peris and Saldivia in [28].

THEOREM 2.15. *Suppose that $\{\lambda_n\}$ is a sequence in $\mathbb{C}\setminus\{0\}$ with $\frac{\lambda_{n+1}}{\lambda_n}\to 1$, $T: X\to X$ is an operator. If $\{\lambda_n T^n x\}$ is dense for some $x\in X$, then for every syndetic sequence $\{n_k\}$ of positive integers, the set $\{\lambda_{n_k}T^{n_k}x\}$ is somewhere dense.*

Proof. There exists a positive integer such that $n_{k+1}-n_k\leqslant m$ for every $k=1,2,\ldots$. Consider the sets $F=\{y\in X:\exists m_l\to+\infty \quad \text{such that} \quad \lambda_{m_l}T^{m_l}x\to y\}$ and $F_i=\{y\in X:\exists\{n_{m_l}\}\subset\{n_k\} \quad \text{such that} \quad \lambda_{n_{m_l}}T^{n_{m_l}-i}x\to y\}$ for all $i=0,1,\ldots,m$. It is easy to check that the sets F, F_i are closed, $F{=}X{=}\bigcup_{i=0}^{m}F_i$ and $T(F_i)\subset F_{i-1}$ for $i{=}1,\ldots,m$. Using the fact that T has dense range by Proposition 2.2, the rest of the proof is exactly the same with the proof of Lemma 3.1 in [28]. ∎

3. THE HYPERCYCLICITY CRITERION

In this section we, on one hand, extend Theorem 3.2 in [19] and, on the other hand, provide a new equivalent form of the hypercyclicity criterion.

THEOREM 3.1. *Suppose that $\{\lambda_n\}$ is a sequence of non-zero complex numbers such that $\frac{\lambda_{n+1}}{\lambda_n}\to 1$ and $T: X\to X$ is an operator. The following are equivalent:*

(i) *The sequence $\{\lambda_n T^n\oplus\lambda_n T^n\}$ is hypercyclic.*

(ii) *For every non-empty open subsets U, V of X there exists integer $n\in\mathbb{N}$ such that $\lambda_n T^n(U)\cap V\neq\emptyset$ and $\lambda_{n+1}T^{n+1}(U)\cap V\neq\emptyset$.*

(iii) *For every $m\in\mathbb{N}$ and for every non-empty open subsets U, V of X there exists $n\in\mathbb{N}$ such that $\lambda_{n+i}T^{n+i}(U)\cap V\neq\emptyset\ \forall i=0,1,\ldots,m$.*

(iv) *There exists $p\in\mathbb{N}$ such that for every U, V non-empty open subsets of X there exists $n\in\mathbb{N}$ such that $\lambda_n T^n(U)\cap V\neq\emptyset$ and $\lambda_{n+p}T^{n+p}(U)\cap V\neq\emptyset$.*

(v) *For every syndetic sequence $\{n_k\}$ the sequence $\{\lambda_{n_k}T^{n_k}\}$ is hypercyclic.*

(vi) *For every sequence $\{n_k\}$ such that $n_{k+1}-n_k\leqslant 2$ for every $k=1,2,\ldots$, the sequence $\{\lambda_{n_k}T^{n_k}\}$ is hypercyclic.*

Proof. Let us first prove that (i) $\Rightarrow$ (ii). Consider (U,V) a pair of non-empty open subsets of X. Since $\frac{\lambda_{n+1}}{\lambda_n}\to 1$, there exist $x\in X\setminus\{0\}$, $\varepsilon>0$ and $n_0\in\mathbb{N}$ such that $B(x,\varepsilon)\subset U$, $\|x\|>\varepsilon$ and $B(x,\varepsilon/2)\subset\frac{\lambda_{n+1}}{\lambda_n}B(x,\varepsilon)$ for all $n\geqslant n_0$. Define $U_1=B(x,\varepsilon/2)$, $V_1=V$, $U_2=B(x,\varepsilon/2)$, $V_2=T^{-1}(V)$. Under the hypothesis that the sequence $\{\lambda_n T^n\oplus\lambda_n T^n\}$ is hypercyclic, there exists $n>n_0$ such that $\lambda_n T^n(U_1)\cap V_1\neq\emptyset$ and $\lambda_n T^n(U_2)\cap V_2\neq\emptyset$. Since $\lambda_n T^n(B(x,\varepsilon/2))\cap T^{-1}(V)\neq\emptyset$, applying T we get $\lambda_{n+1}T^{n+1}\big(\frac{\lambda_n}{\lambda_{n+1}}B(x,\varepsilon/2)\big)\cap V\neq\emptyset$. From the last we easily arrive at $\lambda_{n+1}T^{n+1}(U)\cap V\neq\emptyset$ which gives the desired result.

That (ii) $\Rightarrow$ (iv) is trivial.

We show that (iv) $\Rightarrow$ (i). Let U_1, U_2, V_1, V_2 be non-empty open subsets of X. Fix $v_1\in HC(\{\lambda_n T^n\})\cap V_1$. There exists $r_1\in\mathbb{N}$ such that $u_1:=\lambda_{r_1}T^{r_1}v_1\in U_1$. By Proposition 2.2, T^{r_1} has dense range. It follows that there exists $u_2\in U_2$ such that

$u_2 = \lambda_{r_1} T^{r_1} w_2$ for some $w_2 \in X$. Let $v_2 \in V_2$ and $\delta > 0$ such that $B(v_2, \delta) \subset V_2$ and $B(u_2, \delta) \subset U_2$. Fix any positive number $\alpha > 0$ with $\alpha \neq 1$. Then the operator $T^p - \alpha I$ can be decomposed as

$$T^p - \alpha I = (T - \alpha^{1/p} I)(T - \alpha^{1/p} e^{2\pi i/p} I) \cdots (T - \alpha^{1/p} e^{2\pi i(p-1)/p} I).$$

Since $|\alpha^{1/p} e^{2\pi i j/p}| = \alpha^{1/p} \neq 1$ for every $j = 1, 2, \dots p-1$, Proposition 2.2(i) implies that $(T^p - \alpha I)v_1 \in HC(\{\lambda_n T^n\})$. Hence there exists $q_1 \in \mathbb{N}$ such that

$$\|\lambda_{q_1} T^{q_1} (T^p - \alpha I) v_1 + (v_2 - w_2)\| < \frac{\delta}{2|\lambda_{r_1}| \|T\|^{r_1}}. \tag{3.1}$$

Since $v_1 \in HC(\{\lambda_n T^n\})$, there exists $p_1 \in \mathbb{N}$ such that

$$\|\lambda_{p_1} T^{p_1} v_1 - (v_2 - \alpha \lambda_{q_1} T^{q_1} v_1)\| < \min\Big\{ \frac{\delta}{2|\lambda_{r_1}| \|T\|^{r_1}}, \delta \Big\}. \tag{3.2}$$

Define the vectors $z_2 = \lambda_{p_1} T^{p_1} u_1 + \lambda_{q_1} T^{q_1 + p} u_1$, $y_2 = \lambda_{p_1} T^{p_1} v_1 + \alpha \lambda_{q_1} T^{q_1} v_1$ and by (3.1), (3.2) and the triangle inequality we have $\|z_2 - u_2\| < \delta$, $\|y_2 - v_2\| < \delta$. From now on, arguing as in Grivaux's proof of Theorem 3.2 in [19], assertion (i) follows.

It is obvious that (iii) $\Rightarrow$ (ii).

It remains to show that (ii) $\Rightarrow$ (iii). Since assertions (i), (ii) and (iv) are equivalent, it suffices to prove that (i) $\Rightarrow$ (iii). Consider $m \in \mathbb{N}$ and (U, V) a pair of non-empty open subsets of X. Since $\frac{\lambda_{n+1}}{\lambda_n} \to 1$, it is immediate that $\frac{\lambda_{n+i}}{\lambda_n} \to 1$ for every $i = 0, 1, \dots, m$. Hence there exist $x \in X \setminus \{0\}$, $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $B(x, \varepsilon) \subset U$, $\|x\| > \varepsilon$ and $B(x, \varepsilon/2) \subset \frac{\lambda_{n+i}}{\lambda_n} B(x, \varepsilon)$ for all $n \geqslant n_0$, for all $i = 0, 1, \dots, m$. Define $U_i = B(x, \varepsilon/2)$, $V_i = T^{-(i-1)}(V)$ for $i = 1, \dots, m+1$. Assertion (i) is equivalent to the hypercyclicity criterion. Hence applying Theorem 2.2 in [7] it follows that the sequence $\{\lambda_n T^n \oplus \cdots \oplus \lambda_n T^n\}$ (m-fold) is hypercyclic in X^m. Therefore there exists $n > n_0$ such that $\lambda_n T^n(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2, \dots, m$. Working as in the proof of (i) $\Rightarrow$ (ii) we obtain the desired result.

Let us now show that (i) $\Rightarrow$ (v). Let $\{n_k\}$ be a syndetic sequence, that is, there exists $N \in \mathbb{N}$ such that $n_{k+1} - n_k \leqslant N$ for every $k = 1, 2, \dots$. By Theorem 2.2 in [7] the sequence of direct sums $\{\lambda_n T^n \oplus \lambda_n T^n \oplus \cdots \oplus \lambda_n T^n\}$ of N copies of $\lambda_n T^n$ is hypercyclic in X^N. Consider U_0, V_0 any two non-empty open sets of X. Take $x \in U_0$, $y \in V_0$ and $\delta > 0$ sufficiently small such that $B(x, 2\delta) \subset U_0$ and $B(y, 2\delta) \subset V_0$. Define $U = B(x, \delta)$, $V = B(y, \delta)$ and $U_i = T^{-(i-1)}(U)$ for $i = 1, \dots, N$. Then there exists an integer $k \geqslant N$ such that $\lambda_k T^k(U_i) \cap V \neq \emptyset$ for all $i = 1, \dots, N$. Hence, $\lambda_k T^{k-i+1}(U) \cap V \neq \emptyset \quad \forall i = 1, \dots, N$. Since $k-(N-1), k-(N-2), \dots, k-1, k$ are N consecutive integers, it is clear that inductively we may construct two sequences $\{k_l\}$ and $\{i_l\} \subset \{1, \dots, N\}$ such that $\{n_{k_l}\} \subset \{n_k\}$, $n_{k_l} < n_{k_{l+1}}$ for all $l = 1, 2, \dots$ and $\lambda_{n_{k_l} + i_l - 1} T^{n_{k_l}}(U) \cap V \neq \emptyset$ for all $l = 1, 2, \dots$. By the previous construction we conclude that $\frac{\lambda_{n_{k_l} + i_l - 1}}{\lambda_{n_{k_l}}} \to 1$. Therefore, it is now easy to show that there exists some positive integer l_0 such that $\lambda_{n_{k_l}} T^{n_{k_l}}(B(x,2\delta)) \cap B(y,2\delta) \neq \emptyset$ for all $l \geqslant l_0$, which in turn implies that $\lambda_{n_{k_l}} T^{n_{k_l}}(U_0) \cap V_0 \neq \emptyset$ for all $l \geqslant l_0$. Hence the sequence $\{\lambda_n T^n\}$ is hypercyclic. This proves that (i) $\Rightarrow$ (v).

That (v) $\Rightarrow$ (vi) is trivial.

Observe that Lemma 3.4 in [19] holds for sequences of operators (not only for iterates of a single operator). Combining the last observation with the fact that assertions (i)–(iv) are equivalent it readily follows that (vi) $\Rightarrow$ (i). This completes the proof of the theorem. ∎

REMARK 3.2. In the proof of the above theorem, the choice of $\alpha \neq 1$ was necessary since $\{\lambda_n\}$ was not necessarily admissible. In case $\{\lambda_n\}$ is admissible, a choice of $\alpha = 1$ will also work as in Grivaux's proof of Theorem 3.2 in [19].

The next theorem extends Proposition 4.1 in [19].

THEOREM 3.3. *Suppose that $\{\lambda_n\}$ is a sequence in $\mathbb{C} \setminus \{0\}$ such that $\frac{\lambda_{n+1}}{\lambda_n} \to 1$, $T : X \to X$ is an operator and the sequence $\{\lambda_n T^n\}$ is hypercyclic. The following are equivalent:*

(i) *The sequence $\{\lambda_n T^n \oplus \lambda_n T^n\}$ is hypercyclic.*

(ii) *$T \oplus T$ is cyclic.*

(iii) *For every non-empty open sets U_1, V_1, U_2, V_2 of X there exists a polynomial p such that $p(T)(U_1) \cap V_1 \neq \emptyset$ and $p(T)(U_2) \cap V_2 \neq \emptyset$.*

(iv) *For every pair U, V of non-empty open subsets of X and every neighborhood W of zero there is a polynomial p such that $p(T)(U) \cap W \neq \emptyset$ and $p(T)(W) \cap V \neq \emptyset$.*

Proof. That (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) is obvious.

To show (ii) $\Rightarrow$ (iii) it suffices to prove that the set of cyclic vectors for $T \oplus T$ is dense in $X \oplus X$. In view of Theorem 1 in [22] due to Herrero it is enough to show that $\sigma_{\mathrm{p}}(T^* \oplus T^*)^o = \emptyset$. Since $\{\lambda_n T^n\}$ is hypercyclic, Proposition 2.2 implies that $\sigma_{\mathrm{p}}(T^*) \subset \{z \in \mathbb{C} : |z| = 1\}$ which has empty interior in $\mathbb{C}$. Hence $\sigma_{\mathrm{p}}(T^* \oplus T^*)^o = \emptyset$. Using similar arguments as in the proof of Proposition 4.1 in [19], it is easy to show that (iv) $\Rightarrow$ (i). ∎

4. MISCELLANEOUS RESULTS ON TOPOLOGICALLY MIXING WEIGHTED SHIFTS AND SUMS OF HYPERCYCLIC OPERATORS

Topologically mixing operators have been extensively studied, see for example [2], [5],[15] and [16]. In [14] a characterization of topologically mixing unilateral and bilateral weighted shifts was obtained through the weight sequence. In a similar fashion one can show the following (see also the proof of Proposition 2.10 for an analogous statement for unilateral shifts).

THEOREM 4.1. *Fix a sequence $\{\lambda_n\}$ in $\mathbb{C} \setminus \{0\}$ with $\frac{\lambda_{n+1}}{\lambda_n} \to 1$.*

(i) *Let* $T : X \to X$ *be an operator. If* $\{\lambda_n T^n\}$ *satisfies the hypercyclicity criterion for some syndetic sequence* $\{n_k\}$ *then* $\{\lambda_n T^n\}$ *is topologically mixing.*

(ii) *Let* $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ *be a bilateral backward shift with weight sequence* $\{\alpha_i, i \in \mathbb{Z}\}$. *Then* $\{\lambda_n T^n\}$ *is topologically mixing if and only if*

$$|\lambda_n| \prod_{i=1}^{n} \alpha_i \to +\infty \quad \textit{and} \quad |\lambda_n| \prod_{i=1}^{n} \alpha_{-i} \to 0.$$

S. Grivaux has recently proved the following deep result: *Every continuous linear operator acting on an infinite dimensional separable complex Hilbert space can be written as a sum of two hypercyclic operators.* Following Grivaux's argument with minor modifications we obtain the following (the details are left to the reader).

THEOREM 4.2. *Let* H *be a complex separable infinite dimensional Hilbert space.*

(i) *Let* $T : H \to H$ *be an operator,* $H_+(T)$ *be the vector space spanned by the kernels* $\ker(T - \lambda I)$ *with* $|\lambda| > 1$ *and* $H_-(T)$ *be the vector space spanned by the kernels* $\ker(T - \lambda I)$ *with* $|\lambda| < 1$. *If both* $H_+(T)$, $H_-(T)$ *are dense then for every sequence* $\{\lambda_n\}$ *in* $\mathbb{C} \setminus \{0\}$ *so that* $\frac{\lambda_{n+1}}{\lambda_n} \to 1$, *the sequence* $\{\lambda_n T^n\}$ *satisfies the hypercyclicity criterion for the whole sequence of natural numbers.*

(ii) *Every operator* $T : H \to H$ *can be written as a sum of two operators* T_1, T_2, *i.e.* $T = T_1 + T_2$, *such that for every* $i = 1, 2$ *and every sequence of complex numbers* $\{\lambda_n\}$ *satisfying* $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ *the sequence* $\{\lambda_n T_i^n\}$ *is hypercyclic.*

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