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THE ISOMETRIC REPRESENTATION THEORY OF A PERFORATED SEMIGROUP

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ABSTRACT. We consider the additive subsemigroup $\Sigma := \mathbb{N} \setminus \{1\}$ of $\mathbb{N}$, and study representations of Σ by isometries on Hilbert space with commuting range projections. Our main theorem says that each such representation is unitarily equivalent to the direct sum of a unitary representation, a multiple of the Toeplitz representation on $\ell^2(\Sigma)$, and a multiple of a representation by shifts on $\ell^2(\mathbb{N})$. We consider also the C^*-algebra $C^*(\Sigma)$ generated by a universal isometric representation with commuting range projections, and use our main theorem to identify the faithful representations of $C^*(\Sigma)$ and prove a structure theorem for $C^*(\Sigma)$.

KEYWORDS: *Perforated semigroup, isometric representation, Wold decomposition.*

MSC (2000): 47D05, 47B35.

INTRODUCTION

Coburn proved in 1967 that all C^*-algebras generated by non-unitary isometries are canonically isomorphic [1]. Coburn's result can be viewed as a theorem about the isometric representations of the semigroup $\mathbb{N}$, and this theorem has been generalised to other semigroups: to the positive cones of ordered subgroups of $\mathbb{R}$ by Douglas [2], to the positive cones of totally ordered abelian groups by Murphy [5], and to amenable quasi-lattice ordered groups by Nica [6] and Laca–Raeburn [4].

On the other hand, Murphy [5] and Jang [3] have proved that this theorem does not hold for the semigroup $\Sigma := \mathbb{N} \setminus \{1\}$, by writing down explicit isometric representations S on $\ell^2(\mathbb{N})$ and T on $\ell^2(\Sigma)$ such that $C^*(S)$ is not canonically isomorphic to $C^*(T)$. Here we explore this phenomenon by analysing the isometric representations of Σ, and investigating the structure of the C^*-algebras they generate. Our main result says that every isometric representation of Σ with commuting range projections is equivalent to a direct sum of a unitary representation, a multiple of S, and a multiple of T. The assumption that the range projections commute is a standard one in the area: it is automatic for the positive cones of total orders, and for quasi-lattice ordered groups it is a consequence of the Nica covariance condition used in [6] and [4].

We begin in Section 1 by describing the class of isometric representations of interest to us. We set up our conventions, particularly concerning the two main examples S and T, and establish some basic properties of isometric representations. We prove our main theorem in Section 2. Our main strategy is to analyse how the isometry V_3 interacts with the Wold decomposition of the isometry V_2. In Section 3 we consider the C^*-algebra $C^*(\Sigma)$ generated by a universal isometric representation with commuting range projections. We use our main theorem to obtain criteria which ensure that a given representation of $C^*(\Sigma)$ is faithful, and describe the structure of $C^*(\Sigma)$ in terms of the usual Toeplitz algebra $\mathcal{T} = C^*(\mathbb{N})$.

1. ISOMETRIC REPRESENTATIONS OF Σ

Throughout this paper, $\mathbb{N}$ denotes the additive semigroup of non-negative integers (including 0), and Σ denotes the subsemigroup $\mathbb{N} \setminus \{1\}$. An *isometric representation of Σ on a Hilbert space $\mathcal{H}$ with commuting range projections* is a map $V : \Sigma \to B(\mathcal{H})$ such that each V_n is an isometry, such that $V_{m+n} = V_m V_n$, and such that the range projections $V_n V_n^*$ commute with each other.

We have two main examples in mind.

EXAMPLE 1.1. Let $\{ e_{\Sigma,p} : p \in \Sigma \}$ be the usual orthonormal basis for $\ell^2(\Sigma)$. For each $n \in \Sigma$, the set $\{ e_{\Sigma,n+p} : p \in \Sigma \}$ is also orthonormal, and hence there is an isometry T_n on $\ell^2(\Sigma)$ such that $T_n e_{\Sigma,p} = e_{\Sigma,n+p}$. It is easy to check that $T_m T_n = T_{m+n}$, and that the range projections commute. We call T the *Toeplitz representation* of Σ.

EXAMPLE 1.2. Let R be the unilateral shift on $\ell^2(\mathbb{N})$, and define $S : \Sigma \to B(\ell^2(\mathbb{N}))$ by $S_n = R^n$. In terms of the usual orthonormal basis $\{ e_{\mathbb{N},p} \}$, S_n is characterised by $S_n e_{\mathbb{N},p} = e_{\mathbb{N},n+p}$. Then S is an isometric representation with commuting range projections. (The letter S reminds us that the operators S_n are shifts.)

Murphy and Jang observed that these two representations are not unitarily equivalent. To see this, we just need to note that

$$T_3^*(1 - T_2 T_2^*) T_3(e_{\Sigma,0}) = e_{\Sigma,0},$$

so that $T_3^*(1 - T_2 T_2^*) T_3$ is non-zero, whereas $S_3^*(1 - S_2 S_2^*) S_3 = 0$. (In the proof of Theorem 2.1 it will become clear why we looked at this operator.)

We now investigate general properties of an isometric representation $V : \Sigma \to B(\mathcal{H})$ with commuting range projections. The first and crucial property is that $V_3^2 = V_2^3$, because both are equal to V_6.

For $m, n \in \Sigma$ such that $m - n$ is also in Σ, the relation $V_m = V_n V_{m-n}$ allows us to cancel $V_n^* V_m = V_{m-n}$ and $V_m^* V_n = V_{m-n}^*$. While we cannot expect to cancel expressions like $V_n^* V_{n+1}$, there are interesting and useful relationships among these elements. We often use the next lemma without comment.

LEMMA 1.3. *We have $V_3^* V_2^2 = V_2^* V_3$ and $V_2^{*2} V_3 = V_3^* V_2$.*

Proof. Since the second equation is the adjoint of the first, it suffices to compute

$$V_3^* V_2^2 = V_3^*(V_3^* V_3) V_2^2 = V_3^{*2} V_3 V_2^2 = V_2^{*3} V_2^2 V_3 = V_2^* V_3. \quad \blacksquare$$

The assumption that the range projections commute implies that there are many other commuting projections around. For example:

LEMMA 1.4. *For every $k, n \in \Sigma$, $V_k^* V_n V_n^* V_k$ is a projection which commutes with every range projection $V_m V_m^*$.*

Proof. The elements $V_k^* V_n V_n^* V_k$ are certainly self-adjoint, and

$$(V_k^* V_n V_n^* V_k)^2 = V_k^*(V_n V_n^*) V_k V_k^* V_n V_n^* V_k = V_k^*(V_k V_k^*)(V_n V_n^*)^2 V_k = V_k^* V_n V_n^* V_k,$$

so they are projections. Then

$$\begin{aligned}(V_k^* V_n V_n^* V_k)(V_m V_m^*) &= V_k^* V_n V_n^* V_{m+k} V_{m+k}^* V_k = V_k^* V_{m+k} V_{m+k}^* V_n V_n^* V_k \\ &= (V_m V_m^*)(V_k^* V_n V_n^* V_k). \quad \blacksquare\end{aligned}$$

Since the semigroup Σ is generated by 2 and 3, it is natural to ask which pairs of isometries W_2 and W_3 generate an isometric representation of Σ.

PROPOSITION 1.5. *Suppose that W_2 and W_3 are commuting isometries on $\mathcal{H}$ such that $W_2^3 = W_3^2$ and $W_2 W_2^*$ commutes with $W_3 W_3^*$. Then there is an isometric representation $V : \Sigma \to B(\mathcal{H})$ with commuting range projections such that $V_2 = W_2$ and $V_3 = W_3$.*

Proof. It is straightforward to check that the formula $V_{2p+3j} = W_2^p W_3^j$ gives a well-defined map of Σ into $B(\mathcal{H})$ such that each V_n is an isometry and $V_m V_n = V_{m+n}$. So we have to prove that the range projections commute. We begin by showing that $V_4 V_4^* = V_2^2 V_2^{*2}$ commutes with $V_3 V_3^*$:

$$\begin{aligned}(V_2^2 V_2^{*2})(V_3 V_3^*) &= V_2^*(V_2^3 V_2^{*3})(V_2 V_3) V_3^* = V_2^*(V_3^2 V_3^{*2})(V_3 V_2) V_3^* \\ &= V_2^* V_3^2 V_3^* V_2 V_3^*(V_2^* V_2) = V_2^* V_3 (V_3 V_3^*)(V_2 V_2^*) V_3^* V_2 \\ &= V_2^* V_3 (V_2 V_2^*)(V_3 V_3^*) V_3^* V_2 = (V_2^* V_2) V_3 V_2^* V_3 V_3^{*2} V_2 \\ &= V_3 V_2^*(V_3^* V_3) V_3 V_3^{*2} V_2 = V_3 (V_2^* V_3^*) V_3^2 V_3^{*2} V_2 \\ &= V_3 V_3^* V_2^* (V_3^2 V_3^{*2}) V_2 = V_3 V_3^* V_2^* (V_2^3 V_2^{*3}) V_2 \\ &= (V_3 V_3^*)(V_2^2 V_2^{*2}).\end{aligned}$$

Now fix $m, n \in \Sigma$, and assume without loss of generality that $m > n > 0$. If $m - n$ belongs to Σ then ordinary cancellation shows that

$$(V_m V_m^*)(V_n V_n^*) = V_m V_m^* = (V_n V_n^*)(V_m V_m^*).$$

We are left to handle the case where $m = n + 1$, and we deal with the cases $n = 2p$ and $n = 2p + 1$ separately. For $n = 2p$, we have $m = 2p + 1$, and

$$\begin{aligned}(V_m V_m^*)(V_n V_n^*) &= V_{2p+1} V_{2p+1}^* V_{2p} V_{2p}^* = (V_{2(p-1)} V_3)(V_3^* V_{2(p-1)}^*) V_{2p} V_{2p}^* \\ &= V_{2(p-1)} V_3 V_3^* (V_{2(p-1)}^* V_{2p})(V_2^* V_{2(p-1)}^*) \\ &= V_{2(p-1)} (V_2 V_2^*)(V_3 V_3^*) V_{2(p-1)}^* \\ &= V_{2p} V_2^* V_3 V_{2p+1}^* = V_{2p} V_2^* (V_{2(p-1)}^* V_{2(p-1)}) V_3 V_{2p+1}^* \\ &= V_{2p} V_{2p}^* V_{2p+1} V_{2p+1}^* = (V_n V_n^*)(V_m V_m^*).\end{aligned}$$

For $n = 2p + 1$, we have $m = 2(p + 1)$, and we use the result in the first paragraph:

$$\begin{aligned}(V_m V_m^*)(V_n V_n^*) &= V_{2(p+1)} V_{2(p+1)}^* V_{2p+1} V_{2p+1}^* \\ &= V_{2(p+1)} V_{2(p+1)}^* (V_{2(p-1)} V_3)(V_3^* V_{2(p-1)}^*) \\ &= (V_{2(p-1)} V_2^2)(V_{2(p+1)}^* V_{2(p-1)}) V_3 V_3^* V_{2(p-1)}^* \\ &= V_{2(p-1)} (V_2^2 V_2^{*2})(V_3 V_3^*) V_{2(p-1)}^* \\ &= V_{2(p-1)} (V_3 V_3^*)(V_2^2 V_2^{*2}) V_{2(p-1)}^* \\ &= V_{2p+1} V_3^* V_2^2 V_{2(p+1)}^* = V_{2p+1} V_3^* (V_{2(p-1)}^* V_{2(p-1)}) V_2^2 V_{2(p+1)}^* \\ &= V_{2p+1} V_{2p+1}^* V_{2(p+1)} V_{2(p+1)}^* = (V_n V_n^*)(V_m V_m^*). \quad \blacksquare\end{aligned}$$

REMARKS 1.6. (i) The subsemigroup Σ is the positive cone for the partial order on $\mathbb{Z}$ defined by $m \geqslant n \iff m - n \in \Sigma$. The pair $(\mathbb{Z}, \Sigma)$, however, is not quasi-lattice ordered in the sense of Nica [6]: while 5 is a common upper bound for 2 and 3, and is the smallest in the usual order on $\mathbb{Z}$, it is not a least upper bound in $(\mathbb{Z}, \Sigma)$ because 6 is a common upper bound which is not $\geqslant 5$ in $(\mathbb{Z}, \Sigma)$. So the general theory of [6] and [4] does not apply.

(ii) Since Σ is generated by the two elements 2 and 3, the map $\phi : (p, j) \mapsto 2p + 3j$ is a surjection of $\mathbb{N}^2$ onto Σ. If V is an isometric representation of Σ with commuting range projections, then $V \circ \phi$ is also a semigroup homomorphism. One might suspect that our "commuting range projections" hypothesis would imply that $V \circ \phi$ is a Nica covariant representation of $(\mathbb{Z}^2, \mathbb{N}^2)$ (which is equivalent to saying that $V_2^* = (V \circ \phi(1,0))^*$ and $V_3 = V \circ \phi(0,1)$ commute). However, this is not the case: when $V = S$, for example, the operator $S_2^* S_3$ is the unilateral shift, and hence is injective, whereas $S_3 S_2^*$ is not (for example, $S_3 S_2^*(e_{\mathbb{N},0}) = 0$). One consequence of our main theorem is that $V \circ \phi$ is only Nica covariant when every V_n is unitary (see Corollary 2.8).

2. THE DECOMPOSITION THEOREM

Suppose that V and W are isometric representations of a semigroup P on Hilbert spaces $\mathcal{H}_V$ and $\mathcal{H}_W$. We say that V *is a multiple of* W if there are a Hilbert space $\mathcal{H}$ and a unitary isomorphism $U : \mathcal{H}_V \to \mathcal{H}_W \otimes \mathcal{H}$ such that $UV_pU^* = W_p \otimes 1$ for $p \in P$. For our concrete representations S and T we can identify the tensor products $\ell^2(\mathbb{N}) \otimes \mathcal{H}$ and $\ell^2(\Sigma) \otimes \mathcal{H}$ with $\ell^2(\mathbb{N}, \mathcal{H})$ and $\ell^2(\Sigma, \mathcal{H})$, and we move freely from one realisation to the other.

THEOREM 2.1. *Suppose that $V : \Sigma \to B(\mathcal{H})$ is an isometric representation of $\Sigma := \mathbb{N} \setminus \{1\}$ with commuting range projections. Then there is a unique direct-sum decomposition $\mathcal{H} = \mathcal{H}_U \oplus \mathcal{H}_T \oplus \mathcal{H}_S$ such that $\mathcal{H}_U$, $\mathcal{H}_T$ and $\mathcal{H}_S$ are reducing for V, such that $V|_{\mathcal{H}_U}$ consists of unitary operators, such that $V|_{\mathcal{H}_T}$ is a multiple of T, and such that $V|_{\mathcal{H}_S}$ is a multiple of S.*

Since 2 and 3 generate Σ, the representation V is determined by the two isometries V_2 and V_3. Our strategy is to apply the following version of the Wold decomposition to the single isometry V_2, and to analyse how V_3 interacts with this decomposition.

PROPOSITION 2.2 (Wold Decomposition). *Let Z be an isometry on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_U := \bigcap_{n=0}^{\infty} Z^n(\mathcal{H})$ and $\mathcal{H}_0 := Z(\mathcal{H})^{\perp}$. Then $\mathcal{H}_U$ is a reducing subspace of $\mathcal{H}$ for Z with complement $\mathcal{H}_U^{\perp} = \overline{\operatorname{span}}\Big\{ \bigcup_{n=0}^{\infty} Z^n(\mathcal{H}_0) \Big\}$, $Z|_{\mathcal{H}_U}$ is unitary, and there is a unitary isomorphism $W\colon \mathcal{H}_U^{\perp} \to \ell^2(\mathbb{N}, \mathcal{H}_0)$ such that $WZW^*(\{k_n\}_{n=0}^{\infty}) = \{0, k_0, k_1, k_2, \ldots\}$ for all $\{k_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}, \mathcal{H}_0)$.*

As motivation for our argument, we apply the Wold decomposition to S_2 and T_2. For both isometries we have $\mathcal{H}_U = \{0\}$, and both

$$S_2(\ell^2(\mathbb{N}))^{\perp} = \operatorname{span}\{e_{\mathbb{N},0}, e_{\mathbb{N},1}\} \quad \text{and} \quad T_2(\ell^2(\Sigma))^{\perp} = \operatorname{span}\{e_{\Sigma,0}, e_{\Sigma,3}\}$$

are 2-dimensional. Sending

$$e_{\mathbb{N},i} \mapsto \begin{cases} e_{j0} & \text{if } i = 2j, \\ e_{j1} & \text{if } i = 2j+1, \end{cases} \quad \text{and} \quad e_{\Sigma,i} \mapsto \begin{cases} e_{j0} & \text{if } i = 2j, \\ e_{j1} & \text{if } i = 2j+3, \end{cases}$$

gives unitary isomorphisms of $\ell^2(\mathbb{N})$ and $\ell^2(\Sigma)$ onto $\ell^2(\mathbb{N} \times \{0,1\})$ which carry S and T into the representations determined on

$$f := \begin{pmatrix} f_{01} & f_{11} & f_{21} & f_{31} & \cdots \\ f_{00} & f_{10} & f_{20} & f_{30} & \cdots \end{pmatrix}$$

by

$$S_2 f = \begin{pmatrix} 0 & f_{01} & f_{11} & f_{21} & \cdots \\ 0 & f_{00} & f_{10} & f_{20} & \cdots \end{pmatrix} \quad \text{and} \quad S_3 f = \begin{pmatrix} 0 & f_{00} & f_{10} & f_{20} & \cdots \\ 0 & 0 & f_{01} & f_{11} & \cdots \end{pmatrix} \tag{2.1}$$

and

$$(2.2)\quad T_2f=\begin{pmatrix}0 & f_{01} & f_{11} & f_{21} & \cdots\\ 0 & f_{00} & f_{10} & f_{20} & \cdots\end{pmatrix} \text{ and } T_3f=\begin{pmatrix}f_{00} & f_{10} & f_{20} & f_{30} & f_{40} & \cdots\\ 0 & 0 & 0 & f_{01} & f_{11} & \cdots\end{pmatrix}.$$

We now turn to the proof of Theorem 2.1. Applying the Wold decomposition to the isometry V_2 gives a reducing subspace $\mathcal{H}_U$ such that $V_2|_{\mathcal{H}_U}$ is unitary, and since $V_3^2 = V_2^3$ it follows that V_3 and every other $V_{2p+3j} = V_2^pV_3^j$ are also unitary on $\mathcal{H}_U$. The Wold decomposition also tells us that the complement $\mathcal{H}_U^\perp$ can be identified with $\ell^2(\mathbb{N},\mathcal{H}_0)$ for $\mathcal{H}_0 := V_2(\mathcal{H})^\perp = \ker(V_2V_2^*)$. Our goal is to identify the subspaces $\mathcal{H}_{00}$ and $\mathcal{K}_{00}$ of $\mathcal{H}_0$ consisting of vectors which behave under V_3 as the vector $e_{00} \in \ell^2(\mathbb{N}\times\{0,1\})$ does under T_3 and S_3. The crucial property we isolate is that T_3e_{00} belongs to $\mathcal{H}_0 = \ker T_2T_2^*$, whereas S_3e_{00} belongs to $S_2(\mathcal{H})$ and is orthogonal to $S_2^2(\mathcal{H})$.

With this motivation, we define:

$$(2.3)\qquad \mathcal{H}_{00} := \{\, h\in\mathcal{H}_0 : V_3h\in\mathcal{H}_0 \,\}, \text{ and}$$

$$(2.4)\qquad \mathcal{K}_{00} := \{\, h\in\mathcal{H}_0 : V_3h\in V_2(\mathcal{H})\ominus V_2^2(\mathcal{H}) \,\}.$$

For the rest of the proof, we write $P_n := V_2^nV_2^{*n} - V_2^{n+1}V_2^{*n+1}$, which is the projection of $\mathcal{H}$ onto the complement of $V_2^{n+1}(\mathcal{H})$ in $V_2^n(\mathcal{H})$. With this notation,

$$\mathcal{H}_{00} = \{\, h\in\mathcal{H}_0 : P_0(V_3h) = V_3h \,\} \quad\text{and}\quad \mathcal{K}_{00} = \{\, h\in\mathcal{H}_0 : P_1(V_3h) = V_3h \,\}.$$

PROPOSITION 2.3. *We have a direct-sum decomposition*

$$(2.5)\qquad \mathcal{H}_0 = V_2(\mathcal{H})^\perp = \mathcal{H}_{00}\oplus V_3(\mathcal{H}_{00})\oplus\mathcal{K}_{00}\oplus V_2^*V_3(\mathcal{K}_{00})$$

in which the orthogonal projections on the summands are given by:

(i) *the projection on $\mathcal{H}_{00}$ is $P_{00} := V_3^*P_0V_3 = V_3^*P_0V_3P_0$;*

(ii) *the projection on $V_3(\mathcal{H}_{00})$ is $V_3P_{00}V_3^* = V_3^*P_3V_3P_0$;*

(iii) *the projection on $\mathcal{K}_{00}$ is $Q_{00} := V_3^*P_1V_3P_0$;*

(iv) *the projection on $V_2^*V_3(\mathcal{K}_{00})$ is $V_2^*V_3Q_{00}V_3^*V_2 = V_3^*P_2V_3P_0$.*

To compute some of these projections we need the following straightforward lemma.

LEMMA 2.4. *Suppose that $S\in B(\mathcal{H})$ is a partial isometry and P is the orthogonal projection onto a closed subspace $\mathcal{K}$ of $S^*S(\mathcal{H})$. Then SPS^* is the orthogonal projection onto $S(\mathcal{K})$.*

Proof of Proposition 2.3. For every $h\in\mathcal{H}_0$ and $k\in\mathcal{H}$, we have

$$(V_3h \mid V_2^4k) = (V_3h \mid V_3^2V_2k) = (h \mid V_3V_2k) = (h \mid V_2V_3k) = 0,$$

and hence $V_3h \in V_2^4(\mathcal{H})^\perp = \bigoplus_{n=0}^{3} P_n\mathcal{H}$. Thus $V_3P_0 = \sum_{n=0}^{3} P_nV_3P_0$ and $P_0 = \sum_{n=0}^{3} V_3^*P_nV_3P_0$. Since $V_3^*P_nV_3$ is self adjoint and

$$(V_3^*P_nV_3)^2 = V_3^*P_n(V_3V_3^*)P_nV_3 = V_3^*(V_3V_3^*)P_n^2V_3 = V_3^*P_nV_3,$$

$V_3^*P_nV_3$ is a projection; since Lemma 1.4 implies that P_0 commutes with $V_3^*P_nV_3$, each $V_3^*P_nV_3P_0$ is also a projection. Since their sum P_0 is also a projection, the projections $V_3^*P_nV_3P_0$ have orthogonal ranges, and we have a direct-sum decomposition $\mathcal{H}_0 = \bigoplus_{n=0}^{3} V_3^*P_nV_3P_0(\mathcal{H})$. So it remains to check that the ranges of these projections are as claimed.

For $h \in \mathcal{H}_{00}$ we have

$$V_3^*P_0V_3h = V_3^*(P_0V_3h) = V_3^*(V_3h) = h,$$

so $V_3^*P_0V_3$ is the identity on $\mathcal{H}_{00}$. Next, note that

$$V_3^*P_0V_3P_0 = V_3^*P_0V_3(1 - V_2V_2^*) = V_3^*P_0V_3 - V_3^*P_0V_2V_3V_2^* = V_3^*P_0V_3 - 0,$$

which gives the last equality in (i) and implies that the range of $V_3^*P_0V_3$ is contained in $\mathcal{H}_0$. For every $h \in \mathcal{H}$ we have

$$P_0(V_3(V_3^*P_0V_3h)) = (V_3V_3^*)(P_0^2V_3h) = V_3(V_3^*P_0V_3h),$$

so the range of $V_3^*P_0V_3$ is contained in $\mathcal{H}_{00}$. Similar calculations show that Q_{00} is the identity on $\mathcal{K}_{00}$, and that every k of the form $k = Q_{00}h$ satisfies $P_0k = k$ and $P_1(V_3k) = V_3k$, hence is in $\mathcal{K}_{00}$. This gives (iii).

To establish (ii), we use Lemma 2.4 and part (i) to see that the projection on $V_3(\mathcal{H}_{00})$ is $V_3(V_3^*P_0V_3)V_3^* = V_3V_3^*P_0$. Then we compute

$$\begin{aligned} V_3^*P_3V_3P_0 &= V_3^*(V_2^3P_0V_2^{*3})V_3P_0 = V_3^*(V_3^2P_0V_3^{*2})V_3P_0 = V_3P_0V_3^*P_0 \\ &= (V_3V_3^* - V_3V_2V_2^*V_3^*)P_0 = (V_3V_3^* - V_3V_2V_3^*V_2^*)P_0, \end{aligned}$$

which reduces to $V_3V_3^*P_0$ because $V_2^*P_0 = 0$.

For (iv), we apply Lemma 2.4, and deduce that the projection on $V_2^*V_3(\mathcal{K}_{00})$ is

$$V_2^*V_3Q_{00}V_3^*V_2 = V_2^*V_3(V_3^*P_1V_3P_0)V_3^*V_2.$$

We now compute using Lemma 1.3:

$$\begin{aligned} V_2^*V_3(V_3^*P_1V_3P_0)V_3^*V_2 &= V_2^*P_1V_3V_3^*V_3P_0V_3^*V_2 = P_0V_2^*V_3P_0V_3^*V_2 \\ &= P_0V_3^*V_2^2P_0V_2^{*2}V_3 = P_0V_3^*P_2V_3, \end{aligned}$$

which is $V_3^*P_2V_3P_0$ because Lemma 1.4 implies that $V_3^*P_2V_3$ and P_0 commute. ∎

Applying the isometry V_2^n to the decomposition (2.5) of $\mathcal{H}_0 = P_0(\mathcal{H})$ gives decompositions

$$P_n(\mathcal{H}) = V_2^n(\mathcal{H}_0) = V_2^n(\mathcal{H}_{00}) \oplus V_2^nV_3(\mathcal{H}_{00}) \oplus V_2^n(\mathcal{K}_{00}) \oplus V_2^nV_2^*V_3(\mathcal{K}_{00}),$$

and since the spaces $P_n\mathcal{H}$ themselves give a direct-sum decomposition of $\mathcal{H}_U^{\perp}$, we have

$$\mathcal{H} = \mathcal{H}_U \oplus \Big(\bigoplus_{n=0}^{\infty}(V_2^n(\mathcal{H}_{00}) \oplus V_2^n V_3(\mathcal{H}_{00}) \oplus V_2^n(\mathcal{K}_{00}) \oplus V_2^n V_2^* V_3(\mathcal{K}_{00}))\Big).$$

So with

$$\mathcal{H}_T := \bigoplus_{n=0}^{\infty}(V_2^n(\mathcal{H}_{00}) \oplus V_2^n V_3(\mathcal{H}_{00})), \tag{2.6}$$

$$\mathcal{H}_S := \bigoplus_{n=0}^{\infty}(V_2^n(\mathcal{K}_{00}) \oplus V_2^n V_2^* V_3(\mathcal{K}_{00})), \tag{2.7}$$

we certainly have $\mathcal{H} = \mathcal{H}_U \oplus \mathcal{H}_T \oplus \mathcal{H}_S$. Notice also that if we start with a decomposition $\mathcal{H} = \mathcal{K}_U \oplus \mathcal{K}_S \oplus \mathcal{K}_T$ as in the theorem, then this process will yield $\mathcal{H}_T = \mathcal{K}_T$ and $\mathcal{H}_S = \mathcal{K}_S$, so the decomposition is unique.

PROPOSITION 2.5. *The subspaces $\mathcal{H}_U$, $\mathcal{H}_T$ and $\mathcal{H}_S$ are reducing for V.*

Proof. Since $V_3^* = V_3^{*2}V_3 = V_2^{*3}V_3$, to prove that a subspace $\mathcal{K}$ is reducing for V, it suffices to prove that $\mathcal{K}$ is invariant under V_2, V_2^* and V_3. It is obvious that each of our subspaces is invariant under V_2. Since $\mathcal{H}_U = \bigcap_{n=0}^{\infty} V_2^n(\mathcal{H}) = \bigcap_{n=1}^{\infty} V_2^n(\mathcal{H})$, it is invariant under V_2^*, and since $V_3(V_2^n(\mathcal{H})) = V_2^n(V_3(\mathcal{H})) \subset V_2^n(\mathcal{H})$, it is also invariant under V_3. We have

$$V_2^*(\textstyle\bigcup_{n\geqslant 1, j=0,1} V_2^n V_3^j(\mathcal{H}_{00})) = \displaystyle\bigcup_{n\geqslant 0, j=0,1} V_2^n V_3^j(\mathcal{H}_{00}) \subset \mathcal{H}_T,$$

and since $\mathcal{H}_{00}$ and $V_3(\mathcal{H}_{00})$ are contained in $\mathcal{H}_0 = V_2(\mathcal{H})^{\perp} = \ker V_2^*$, they are trivially invariant under V_2^*. Thus $\mathcal{H}_T$ is invariant under V_2^*, and the same argument shows that $\mathcal{H}_S$ is invariant under V_2^*.

It follows from the identity $V_3^2 = V_2^3$ that $\mathcal{H}_T$ is invariant under V_3. Since $V_3(\mathcal{K}_{00}) \subset V_2V_2^*(\mathcal{H})$, we have $V_3(V_2^n(\mathcal{K}_{00})) = V_2^n(V_2V_2^*V_3(\mathcal{K}_{00})) \subset \mathcal{H}_S$, and

$$\begin{aligned} V_3(V_2^n V_2^* V_3(\mathcal{K}_{00})) &= (V_2^*V_2)V_3V_2^nV_2^*V_3(\mathcal{K}_{00}) = V_2^*V_3V_2^{n+1}V_2^*V_3(\mathcal{K}_{00}) \\ &= V_2^*V_3V_2^n(V_2V_2^*V_3(\mathcal{K}_{00})) = V_2^*V_3V_2^nV_3(\mathcal{K}_{00}) \\ &= V_2^*V_2^nV_3^2(\mathcal{K}_{00}) = V_2^*V_2^nV_2^3(\mathcal{K}_{00}) = V_2^{n+2}(\mathcal{K}_{00}) \end{aligned}$$

is also contained in $\mathcal{H}_S$. ∎

We next show that $V|_{\mathcal{H}_T}$ is equivalent to $T \otimes 1_{\mathcal{H}_{00}}$. We identify $\ell^2(\mathbb{N}) \otimes \mathcal{H}_{00}$ with $\ell^2(\mathbb{N} \times \{0,1\}, \mathcal{H}_{00})$, so that on matrices

$$f := \begin{pmatrix} f_{01} & f_{11} & f_{21} & f_{31} & \cdots \\ f_{00} & f_{10} & f_{20} & f_{30} & \cdots \end{pmatrix}$$

with $f_{nj} \in \mathcal{H}_{00}$, $T_2 \otimes 1$ and $T_3 \otimes 1$ are defined by the same formulas (2.2) as T_2 and T_3. We now recall that the projection P_{00} on $\mathcal{H}_{00}$ is given by $P_{00} = V_3^*P_0V_3$, and define $W_T\colon \mathcal{H}_T \to \ell^2(\mathbb{N}\times\{0,1\},\mathcal{H}_{00})$ by

$$(W_T h)_{nj} = P_{00}V_3^{*j}V_2^{*n}h.$$

Since $V_3^{*j}V_2^{*n}$ is an isometry of $V_2^nV_3^j(\mathcal{H}_{00})$ onto $\mathcal{H}_{00}$, W_T is a unitary isomorphism of $\mathcal{H}_T$ onto $\ell^2(\mathbb{N}\times\{0,1\},\mathcal{H}_{00})$.

PROPOSITION 2.6. *We have* $W_T(V|_{\mathcal{H}_T})W_T^* = T\otimes 1$.

Proof. We prove that $W_TV_i|_{\mathcal{H}_T} = (T_i\otimes 1)W_T$ for $i=2$ and $i=3$. Let $h\in\mathcal{H}_T$. Then

$$W_TV_2h = \begin{pmatrix} P_{00}V_3^*V_2h & P_{00}V_3^*V_2^*V_2h & P_{00}V_3^*V_2^{*2}V_2h & P_{00}V_3^*V_2^{*3}V_2h & \cdots \\ P_{00}V_2h & P_{00}V_2^*V_2h & P_{00}V_2^{*2}V_2h & P_{00}V_2^{*3}V_2h & \cdots \end{pmatrix} \tag{2.8}$$

and

$$(T_2\otimes 1)W_Th = \begin{pmatrix} 0 & P_{00}V_3^*h & P_{00}V_3^*V_2^*h & P_{00}V_3^*V_2^{*2}h & \cdots \\ 0 & P_{00}h & P_{00}V_2^*h & P_{00}V_2^{*2}h & \cdots \end{pmatrix}; \tag{2.9}$$

since $P_{00}V_2 = 0$ and

$$P_{00}V_3^*V_2 = (V_3^*P_0V_3)V_3^*V_2 = V_3^*(V_3V_3^*)P_0V_2 = 0,$$

the right-hand sides of (2.8) and (2.9) are the same, and $W_TV_2|_{\mathcal{H}_T} = (T_2\otimes 1)W_T$. Similarly,

$$W_TV_3h = \begin{pmatrix} P_{00}V_3^*V_3h & P_{00}V_3^*V_2^*V_3h & P_{00}V_3^*V_2^{*2}V_3h & P_{00}V_3^*V_2^{*3}V_3h & \cdots \\ P_{00}V_3h & P_{00}V_2^*V_3h & P_{00}V_2^{*2}V_3h & P_{00}V_2^{*3}V_3h & \cdots \end{pmatrix} \tag{2.10}$$

and

$$(T_3\otimes 1)W_Th = \begin{pmatrix} P_{00}h & P_{00}V_2^*h & P_{00}V_2^{*2}h & P_{00}V_2^{*3}h & \cdots \\ 0 & 0 & 0 & P_{00}V_3^*h & \cdots \end{pmatrix}. \tag{2.11}$$

Since $V_3^*V_2^{*n}V_3 = V_2^{*n}V_3^*V_3 = V_2^{*n}$, the top rows of these last two matrices are the same. To see that the bottom rows are also the same, we compute:

$$\begin{aligned} P_{00}V_3 &= (V_3^*P_0V_3)V_3 = V_3^*P_0V_2^3 = 0, \\ P_{00}V_2^*V_3 &= P_{00}(V_2^*V_3) = V_3^*P_0V_3(V_3^*V_2^2) = V_3^*V_3V_3^*P_0V_2^2 = 0, \text{ and} \\ P_{00}V_2^{*2}V_3 &= (V_3^*P_0V_3)(V_3^*V_2) = V_3^*V_3V_3^*P_0V_2 = 0; \end{aligned}$$

and, for $n\geqslant 3$,

$$P_{00}V_2^{*n}V_3 = P_{00}(V_2^{*(n-3)}V_2^{*3})V_3 = P_{00}V_2^{*(n-3)}V_3^{*2}V_3 = P_{00}V_3^*V_2^{*(n-3)},$$

which tells us that the nth entry in the bottom rows of (2.10) and (2.11) agree. Thus $W_TV_3|_{\mathcal{H}_T} = (T_3\otimes 1)W_T$, and the result follows. ∎

To see that $V|_{\mathcal{H}_S}$ is a multiple of S, define $W_S\colon \mathcal{H}_S \to \ell^2(\mathbb{N}\times\{0,1\},\mathcal{K}_{00})$ by

$$(W_Sh)_{nj} = Q_{00}V_3^{*j}V_2^jV_2^{*n}h;$$

It follows from the direct sum decomposition (2.7) that W_S is a unitary isomorphism of $\mathcal{H}_S$ onto $\ell^2(\mathbb{N} \times \{0,1\}, \mathcal{K}_{00})$. On $\ell^2(\mathbb{N} \times \{0,1\}, \mathcal{K}_{00})$, $S \otimes 1$ is given by the formulas (2.1).

PROPOSITION 2.7. *We have* $W_S(V|_{\mathcal{H}_S})W_S^* = S \otimes 1$.

Proof. The proof follows the same strategy as that of Proposition 2.6, but some of the calculations are a bit trickier, so we include the details. We prove that $W_S V_i|_{\mathcal{H}_S} = (S_i \otimes 1)W_S$ for $i = 2$ and $i = 3$. Let $h \in \mathcal{H}_S$. Then

$$W_S V_2 h = \begin{pmatrix} Q_{00}V_3^*V_2V_2h & Q_{00}V_3^*V_2V_2^*V_2h & Q_{00}V_3^*V_2V_2^{*2}V_2h & \cdots \\ Q_{00}V_2h & Q_{00}V_2^*V_2h & Q_{00}V_2^{*2}V_2h & \cdots \end{pmatrix} \tag{2.12}$$

and

$$(S_2 \otimes 1)W_S h = \begin{pmatrix} 0 & Q_{00}V_3^*V_2h & Q_{00}V_3^*V_2V_2^*h & Q_{00}V_3^*V_2V_2^{*2}h & \cdots \\ 0 & Q_{00}h & Q_{00}V_2^*h & Q_{00}V_2^{*2}h & \cdots \end{pmatrix}. \tag{2.13}$$

Since $Q_{00}V_2 = V_3^*P_1V_3P_0V_2 = 0$ and

$$Q_{00}V_3^*V_2^2 = (P_0V_3^*P_1V_3)V_3^*V_2^2 = P_0V_3^*(V_3V_3^*)P_1V_2^2 = 0,$$

the right-hand sides of (2.12) and (2.13) are the same, and $W_SV_2 = (S_2 \otimes 1)W_S$ on $\mathcal{H}_S$. Next we compare

$$W_S V_3 h = \begin{pmatrix} Q_{00}V_3^*V_2V_3h & Q_{00}V_3^*V_2V_2^*V_3h & Q_{00}V_3^*V_2V_2^{*2}V_3h & \cdots \\ Q_{00}V_3h & Q_{00}V_2^*V_3h & Q_{00}V_2^{*2}V_3h & \cdots \end{pmatrix} \tag{2.14}$$

and

$$(S_3 \otimes 1)W_S h = \begin{pmatrix} 0 & Q_{00}h & Q_{00}V_2^*h & Q_{00}V_2^{*2}h & Q_{00}V_2^{*3}h & \cdots \\ 0 & 0 & Q_{00}V_3^*V_2h & Q_{00}V_3^*V_2V_2^*h & Q_{00}V_3^*V_2V_2^{*2}h & \cdots \end{pmatrix}. \tag{2.15}$$

The necessary three entries in (2.14) do indeed vanish:

$$\begin{aligned} Q_{00}V_3 &= P_0V_3^*P_1V_3^2 = P_0V_3^*P_1V_2^3 = 0, \\ Q_{00}V_3^*V_2V_3 &= Q_{00}V_3^*V_3V_2 = (V_3^*P_1V_3P_0)V_2 = 0, \text{ and} \\ Q_{00}V_2^*V_3 &= (P_0V_3^*P_1V_3)(V_3^*V_2^2) = P_0V_3^*(V_3V_3^*)P_1V_2^2 = 0. \end{aligned}$$

For $n \geqslant 1$, we expand the nth entry in the top row of (2.14) using the identity $P_1 = P_1V_2V_2^*$:

$$\begin{aligned} Q_{00}V_3^*V_2V_2^{*n}V_3h &= (P_0V_3^*P_1V_3)V_3^*V_2V_2^*V_2^{*(n-1)}V_3h \\ &= P_0V_3^*(V_3V_3^*)P_1V_2V_2^*V_2^{*(n-1)}V_3h \\ &= P_0V_3^*(V_3V_3^*)P_1V_2^{*(n-1)}V_3h = P_0V_3^*P_1(V_3V_3^*)V_2^{*(n-1)}V_3h \\ &= (P_0V_3^*P_1V_3)V_2^{*(n-1)}V_3^*V_3h = Q_{00}V_2^{*(n-1)}h, \end{aligned}$$

which is the nth entry in the top row of (2.15). Now we let $n \geqslant 2$, and work on the nth entry in the bottom row of (2.14), again using $P_1 = P_1 V_2 V_2^*$:

$$\begin{aligned}
Q_{00} V_2^{*n} V_3 h &= Q_{00} V_2^{*(n-2)} V_2^{*2} V_3 h = Q_{00} V_2^{*(n-2)} V_3^* V_2 h = Q_{00} V_3^* V_2^{*(n-2)} V_2 h \\
&= (P_0 V_3^* P_1 V_3) V_3^* V_2^{*(n-2)} V_2 h = P_0 V_3^* (V_3 V_3^*) P_1 V_2^{*(n-2)} V_2 h \\
&= P_0 V_3^* (V_3 V_3^*)(P_1 V_2 V_2^*) V_2^{*(n-2)} V_2 h \\
&= (P_0 V_3^* P_1 V_3) V_3^* V_2 V_2^{*(n-1)} V_2 h = Q_{00} V_3^* V_2 V_2^{*(n-2)} h,
\end{aligned}$$

which is the nth entry in the bottom row of (2.15). We have now proved that $W_S V_2 = (S_3 \otimes 1) W_S$ on $\mathcal{H}_S$, and the result follows. ∎

Proposition 2.7 completes the proof of Theorem 2.1.

COROLLARY 2.8. *Define $\phi : \mathbb{N}^2 \to \Sigma$ by $\phi(p, j) = 2p + 3j$, and suppose V is an isometric representation of Σ on $\mathcal{H}$ with commuting range projections. If $V \circ \phi$ is a Nica covariant representation of $(\mathbb{Z}^2, \mathbb{N}^2)$, then every V_n is unitary.*

Proof. For $(\mathbb{Z}^2, \mathbb{N}^2)$, Nica covariance says that $V_2^* = (V \circ \phi(1,0))^*$ commutes with $V_3 = V \circ \phi(0,1)$. For both $V = S$ and $V = T$, we can write down elements of $\mathcal{H}$ which are in the kernel of $V_3 V_2^*$ but not in the kernel of $V_2^* V_3$. So for general V, if either $\mathcal{H}_S$ or $\mathcal{H}_T$ were non-zero, we could find elements of $\mathcal{H}_S$ or $\mathcal{H}_T$ with the same property. Thus $\mathcal{H} = \mathcal{H}_U$, and the result follows from Theorem 2.1. ∎

3. THE C^*-ALGEBRA OF Σ

Modifications of the standard arguments (as in [5], for example) show that there is a unital C^*-algebra $C^*(\Sigma)$ generated by an isometric representation $v : \Sigma \to C^*(\Sigma)$ with commuting range projections which is universal for such representations: for every isometric representation $V : \Sigma \to B$ with commuting range projections, there is a unique homomorphism $\pi_V : C^*(\Sigma) \to B$ such that $V = \pi_V \circ v$. In this section we describe conditions on V which ensure that π_V is faithful, and give a concrete description of $C^*(\Sigma)$ in terms of the usual Toeplitz algebra $\mathcal{T}$.

THEOREM 3.1. *Let $\Sigma := \mathbb{N} \setminus \{1\}$, and let $V \colon \Sigma \to B(\mathcal{H})$ be an isometric representation with commuting range projections. Then the representation π_V of $C^*(\Sigma)$ is faithful if and only if*

$$V_3^*(V_2 V_2^* - V_2^2 V_2^{*2}) V_3 (1 - V_2 V_2^*) \neq 0 \quad \textit{and} \quad V_3^*(1 - V_2 V_2^*) V_3 \neq 0. \tag{3.1}$$

Since $V_3^*(V_2 V_2^* - V_2^2 V_2^{*2}) V_3 (1 - V_2 V_2^*)$ and $V_3^*(1 - V_2 V_2^*) V_3$ are the projections on $\mathcal{K}_{00}$ and $\mathcal{H}_{00}$, (3.1) says that the subspaces $\mathcal{H}_T$ and $\mathcal{H}_S$ in the decomposition of Theorem 2.1 are both non-zero. So Theorem 3.1 implies in particular that $\pi_{T \oplus S}$ is faithful.

Proof. First notice that in the representation π_S, the operator

$$\pi_S\big(v_3^*(v_2v_2^* - v_2^2v_2^{*2})v_3(1 - v_2v_2^*)\big) = S_3^*(S_2S_2^* - S_2^2S_2^{*2})S_3(1 - S_2S_2^*)$$

fixes the vector $e_{\mathbb{N},0}$, and hence $v_3^*(v_2v_2^* - v_2^2v_2^{*2})v_3(1 - v_2v_2^*)$ is non-zero in $C^*(\Sigma)$. Similarly, $\pi_T(v_3^*(1 - v_2v_2^*)v_3)$ fixes $e_{\Sigma,0}$, and $v_3^*(1 - v_2v_2^*)v_3 \neq 0$. So if π_V is faithful, the images of both these elements of $C^*(\Sigma)$ must be non-zero, which is exactly what (3.1) says.

Now suppose V satisfies (3.1), and consider the decomposition $\mathcal{H} = \mathcal{H}_U \oplus \mathcal{H}_T \oplus H_S$ of Theorem 2.1, noticing that (3.1) implies that $\mathcal{H}_S$ and $\mathcal{H}_T$ are non-zero. Write $V_U := V|_{\mathcal{H}_U}$, $V_T := V|_{\mathcal{H}_T}$ and $V_S := V|_{\mathcal{H}_S}$, and fix $a \in C^*(\Sigma)$. Then we can check on generators that $\pi_V = \pi_{V_U} \oplus \pi_{V_T} \oplus \pi_{V_S}$, and hence we have

$$\|\pi_V(a)\| = \max\big\{\|\pi_{V_U}(a)\|,\ \|\pi_{V_T}(a)\|,\ \|\pi_{V_S}(a)\|\big\}. \tag{3.2}$$

Since $\mathcal{H}_T$ is non-zero and $V_T \sim T \otimes 1$, and we can check on generators that $\pi_{T\otimes 1} = \pi_T \otimes 1$, we have $\pi_{V_T} \sim \pi_T \otimes 1$. Similarly, $\pi_{V_S} \sim \pi_S \otimes 1$. Thus (3.2) implies that

$$\|\pi_V(a)\| = \max\big\{\|\pi_{V_U}(a)\|,\ \|\pi_T(a)\|,\ \|\pi_S(a)\|\big\}.$$

The operator $\pi_{V_U}(a) \oplus \pi_S(a)$ belongs to the C^*-algebra generated by $U_1 \oplus R$, where $U_1 = (V_U)_2^{-1}(V_U)_3$ is unitary and $R = S_2^*S_3$ is the unilateral shift, and hence the Lemma on page 724 of [1] implies that $\|\pi_{V_U}(a)\| \leqslant \|\pi_S(a)\|$. Thus

$$\|\pi_V(a)\| = \max\big\{\|\pi_T(a)\|,\ \|\pi_S(a)\|\big\}.$$

Since every C^*-algebra has a faithful representation and every representation of $C^*(\Sigma)$ has the form π_W, there is a faithful representation of the form π_W, and then $\mathcal{H}_T$ and $\mathcal{H}_S$ are both non-zero by the first part of the proof. We can then deduce from the argument of the previous paragraph that

$$\|a\| = \|\pi_W(a)\| = \max\big\{\|\pi_T(a)\|,\ \|\pi_S(a)\|\big\} = \|\pi_V(a)\|,$$

which since a is an arbitrary element of $C^*(\Sigma)$ implies that π_V is faithful. ∎

We can view the Toeplitz algebra $\mathcal{T}$ either as the C^*-subalgebra of $B(\ell^2(\mathbb{N}))$ generated by the unilateral shift, or as the C^*-subalgebra of $B(L^2(\mathbb{T}))$ generated by the Toeplitz operators T_ϕ with symbol $\phi \in C(\mathbb{T})$. In either realisation, $\mathcal{T}$ contains the algebra $\mathcal{K}$ of compact operators, and the quotient $\mathcal{T}/\mathcal{K}$ is naturally isomorphic to $C(\mathbb{T})$. In the proof of the following theorem we realise $\mathcal{T}$ as a subalgebra of $B(\ell^2(\mathbb{N}))$.

THEOREM 3.2. *Let $\Sigma := \mathbb{N} \setminus \{1\}$ and let $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{K}$ be the quotient map. Then $C^*(\Sigma)$ is isomorphic to*

$$C := \{(A, B) \in \mathcal{T} \oplus \mathcal{T} : q(A) = q(B)\}.$$

For the proof, we need a lemma.

LEMMA 3.3. *Let $U\colon \ell^2(\mathbb{N}) \to \ell^2(\Sigma)$ be the unitary isomorphism such that $Ue_{\mathbb{N},0} = e_{\Sigma,0}$ and $Ue_{\mathbb{N},n} = e_{\Sigma,n+1}$ for $n \geqslant 1$. Then $U^*T_pU - S_p$ is a finite-rank operator on $\ell^2(\mathbb{N})$ for every $p \in \Sigma$.*

Proof. If $p = 0$ the result is trivial, so suppose $p \in \Sigma \setminus \{0\}$. We now compute, using the notation $h \otimes \overline{k}$ for the rank-one operator $g \mapsto (g \mid k)h$:

$$(U^*T_pU)e_{\mathbb{N},0} = e_{\mathbb{N},p-1} = (S_p + e_{\mathbb{N},p-1} \otimes \overline{e}_{\mathbb{N},0} - e_{\mathbb{N},p} \otimes \overline{e}_{\mathbb{N},0})e_{\mathbb{N},0},$$

and for $n \geqslant 1$,

$$(U^*T_pU)e_{\mathbb{N},n} = e_{\mathbb{N},n+p} = (S_p + e_{\mathbb{N},p-1} \otimes \overline{e}_{\mathbb{N},0} - e_{\mathbb{N},p} \otimes \overline{e}_{\mathbb{N},0})e_{\mathbb{N},n}.$$

Thus $U^*T_pU - S_p = e_{\mathbb{N},p-1} \otimes \overline{e}_{\mathbb{N},0} - e_{\mathbb{N},p} \otimes \overline{e}_{\mathbb{N},0}$. ∎

Proof of Theorem 3.2. Theorem 3.1 implies that $\pi_{S \oplus T} = \pi_S \oplus \pi_T$ is faithful. Take U as in Lemma 3.3, and define $\psi : C^*(\Sigma) \to B(\ell^2(\mathbb{N})) \oplus B(\ell^2(\mathbb{N}))$ by $\psi(a) = (\pi_S(a), U^*\pi_T(a)U)$. We claim that ψ is an isomorphism of $C^*(\Sigma)$ onto C. It is injective because $\pi_S \oplus \pi_T$ is. Since the operators $\pi_S(v_p) = S_p$ are all powers of the unilateral shift, and Lemma 3.3 implies that $U^*\pi_T(v_p)U = U^*T_pU$ differs from S_p by a finite-rank operator, ψ has range in C. So it remains to prove that every element of C is in the range of ψ.

Let $(A, A + K) \in C$. Since $S_2^*S_3 = \pi_S(v_2^*v_3)$ is the unilateral shift, π_S maps $C^*(\Sigma)$ onto $\mathcal{T}$. Thus there exists $a \in C^*(\Sigma)$ such that $\pi_S(a) = A$, and then

$$A + K = U^*\pi_T(a)U + (\pi_S(a) - U^*\pi_T(a)U) + K,$$

which is $U^*\pi_T(a)U + L$, say, where L is compact. So we need to show that $(0, L)$ is in the range of ψ, and to do this it suffices to show that every rank-one operator $(0, e_{\mathbb{N},i} \otimes \overline{e}_{\mathbb{N},j})$ is in the range of ψ. Computations show that

$$\begin{aligned} \psi(1 - (v_2^*v_3)^*(v_2^*v_3)) &= (0, e_{\mathbb{N},0} \otimes \overline{e}_{\mathbb{N},0}), \\ \psi(v_2v_2^*(1 - (v_2^*v_3)(v_2^*v_3)^*)) &= (0, e_{\mathbb{N},1} \otimes \overline{e}_{\mathbb{N},1}), \text{ and} \\ \psi(v_{i+1}v_{i+1}^*(1 - v_iv_i^*)) &= (0, e_{\mathbb{N},i} \otimes \overline{e}_{\mathbb{N},i}) \quad \text{for } i \geqslant 2, \end{aligned}$$

so for each i there exists $b_i \in C^*(\Sigma)$ such that $\psi(b_i) = (0, e_{\mathbb{N},i} \otimes \overline{e}_{\mathbb{N},i})$. Now some more calculations show that if $j \geqslant 1$, then

$$\begin{aligned} \psi(b_0v_{j+1}^*) &= (0, e_{\mathbb{N},0} \otimes \overline{e}_{\mathbb{N},j}), \text{ and} \\ \psi(b_iv_{i+1}v_{j+1}^*) &= (0, e_{\mathbb{N},i} \otimes \overline{e}_{\mathbb{N},j}) \quad \text{for every } i \geqslant 1; \end{aligned} \tag{3.3}$$

the adjoint of (3.3) shows that every $(0, e_{\mathbb{N},j} \otimes \overline{e}_{\mathbb{N},0})$ is also in the range of ψ. Thus every $(0, e_{\mathbb{N},i} \otimes \overline{e}_{\mathbb{N},j})$ is in the range of ψ, as required. ∎

REMARK 3.4. This structure theorem for $C^*(\Sigma)$, or more precisely the lemma used to prove it, has some interesting implications for Toeplitz operators. Let $e_n : z \mapsto z^n$ be the usual orthonormal basis for $L^2(\mathbb{T})$, let $H^2(\Sigma)$ be the closed span of $\{e_n : n \in \Sigma\}$, let P^Σ be the orthogonal projection of $L^2(\mathbb{T})$ on $H^2(\Sigma)$, and define the Toeplitz operator T_ϕ^Σ with symbol $\phi \in C(\mathbb{T})$ by $T_\phi^\Sigma(f) = P^\Sigma(\phi f)$. The usual Hardy space $H^2(\mathbb{T})$ is naturally isomorphic to $\ell^2(\mathbb{N})$, and the usual Toeplitz operator T_{e_n} is then equivalent to S_n; the same isomorphism carries $H^2(\Sigma)$ onto $\ell^2(\Sigma)$, and $T^{\Sigma}_{e_n}$ into T_n. Let $U : H^2(\mathbb{T}) \to H^2(\Sigma)$ be the unitary operator such that $Ue_0 = e_0$ and $Ue_n = e_{n+1}$ for $n \geqslant 1$. Then Lemma 3.3 implies that $U^*T^{\Sigma}_{e_n}U - T_{e_n}$ has finite rank, and we can deduce from the linearity and continuity of the maps $\phi \mapsto T^{\Sigma}_{\phi}$ and $\phi \mapsto T_{\phi}$ that $U^*T^{\Sigma}_{\phi}U - T_{\phi}$ is compact for every $\phi \in C(\mathbb{T})$. It follows that T^{Σ}_{ϕ} is Fredholm if and only if T_{ϕ} is Fredholm, that is, if and only if ϕ is non-vanishing, and the usual index theorem then gives

$$\operatorname{ind} T^{\Sigma}_{\phi} = \operatorname{ind} T_{\phi} = -\deg \phi.$$

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