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LIMITS OF PURE FUNCTIONALS OF C^*-ALGEBRAS

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ABSTRACT. It is shown that, for a C^*-algebra A, every (weak*-) limit of pure functionals is a multiple of a pure functional if and only if every limit of pure states is a multiple of pure states (a condition previously studied by Glimm). On the other hand, it is shown that the set of pure states $P(A)$ being closed does not force the set of pure functionals $G(A)$ to be closed. The conditions $\overline{G(A)} = G(A)$ and $\overline{G(A)} = G(A) \cup \{0\}$ are characterised in terms of sums of homogeneous C^*-algebras.

KEYWORDS: *Pure state, pure functional, irreducible representation, C^*-algebra, spectrum.*

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INTRODUCTION

Let A be a C^*-algebra and let $\widehat{A}$ be the spectrum of A, the space of all (equivalence classes of) irreducible representations of A. We write $A \in [FIN]$ to mean that all irreducible representations of A are on finite dimensional spaces. Let $G(A)$ and $P(A)$ be the sets of pure functionals (extreme points of the unit ball in the Banach dual A^* of A) and pure states of A respectively. In Theorem 4.1 of [3] Archbold and Shah have investigated when the weak*-closure $\overline{G(A)}$ is as large as it can be, that is, $\overline{G(A)} = A^*_1$, the unit ball in A^*. In this paper we investigate $\overline{G(A)}$ and show how small it can be. We start our investigation by showing that, for a C^*-algebra A every limit of pure functionals is a multiple of a pure functional if and only if every limit of pure states is a multiple of a pure state. Glimm showed that the second condition holds if and only if A is liminal, $\widehat{A}$ Hausdorff and at singular points $\pi \in \widehat{A}$, $\pi(A)$ is one dimensional [8]. In the special case where A is a unital (respectively non-unital) homogeneous C^*-algebra we have $\overline{G(A)} = G(A)$ (respectively $\overline{G(A)} = G(A) \cup \{0\}$). Hence we prove that if a C^*-algebra A is the finite direct sum (respectively c_0-sum) of homogeneous C^*-algebras then $\overline{G(A)} = G(A)$ (respectively $\overline{G(A)} = G(A) \cup \{0\}$, that is, $G(A)$ is nearly closed).

These results might suggest that $G(A)$ behaves in much the same way as $P(A)$ from the point of view of closure. However, there is a significant difference: $P(A)$ can be weak*-closed when $G(A)$ is not (see example after the proof of Theorem 2.8). Indeed, we show that, for a unital C^*-algebra A, the following are equivalent:

(i) $\overline{P(A)} = P(A)$;

(ii) $A \in [FIN]$, $\widehat{A}$ Hausdorff and at singular points $\pi \in \widehat{A}$, $\pi(A)$ is one dimensional;

(iii) $\overline{G(A)} = G(A) \cup \{\lambda\varphi : |\lambda| \leqslant 1, \varphi \in P(A)$ with GNS representation π_φ singular$\}$;

(iv) $\widehat{A}$ is Hausdorff and there exists a two sided closed ideal J of A such that J is a (finite or c_0) direct sum of homogeneous C^*-algebras with A/J abelian and its spectrum $\widehat{(A/J)}$ is the set of singular points of $\widehat{A}$ (Theorem 2.8).

In the non-unital case we will show that the following conditions are equivalent:

(i) $\overline{P(A)} \subseteq P(A) \cup \{0\} \cup \{\lambda\varphi : \lambda \in [0,1], \varphi \in P(A)$ with GNS representation π_φ singular$\}$;

(ii) $\overline{G(A)} = G(A) \cup \{0\} \cup \{\lambda\varphi : |\lambda| \leqslant 1, \varphi \in G(A)$ with GNS representation $\pi_{|\varphi|}$ singular$\}$;

(iii) $\widehat{A}$ is Hausdorff and there exists a closed two sided ideal J of A such that J is a (finite or c_0) direct sum of homogeneous C^*-algebras with A/J abelian and its spectrum $\widehat{(A/J)}$ is the set of singular points of $\widehat{A}$ (Theorem 2.10).

The above theorems illustrate how singular points in $\widehat{A}$ have greater impact on $\overline{G(A)}$ than on $\overline{P(A)}$. They, also, lead to the following corollaries:

(i) $\overline{G(A)} = G(A) \Rightarrow A$ is a finite direct sum of unital homogeneous C^*-algebras (Corollary 2.9).

(ii) $\overline{G(A)} = G(A) \cup \{0\} \Rightarrow A$ is a (finite or c_0) direct sum of homogeneous C^*-algebras A_n (where either at least one A_n is non-unital or infinitely many A_n are nonzero) (Corollary 2.11).

These are converses for the results mentioned at the end of the first paragraph. Finally, we give an example of a C^*-algebra A for which $\overline{P(A)} = P(A) \cup \{0\}$ but $\overline{G(A)}$ strictly contains $G(A) \cup \{0\}$.

We give some definitions and known results for the reader's convenience. The symbols $B(H)$ and $K(H)$ denote, respectively, the C^*-algebras of bounded linear and compact linear operators acting on a Hilbert space H with adjoint as involution and operator norm. A C^*-algebra A is said to be *liminal* if $\pi(A) = K(H_\pi)$ for every $\pi \in \widehat{A}$, where H_π is the Hilbert space associated with π. A C^*-algebra A is said to be *homogeneous of degree n* if every irreducible representation of A is of the same finite dimension n. A *point* $\pi \in \widehat{A}$ is said to be *Fell-regular (or a Fell-point)* if there exists an $a \in A^+$ (the set of positive elements of A) and a neighbourhood V of π such that $\sigma(a)$ is a rank-one projection for all $\sigma \in V$. A *point* $\pi \in \widehat{A}$ is said to be *Glimm-regular* if whenever (e, U) is a pair such that $e \in A$ and U is a neighbourhood of π and (i) $\sigma(e)$ is a projection for all $\sigma \in U$, (ii) $\pi(e)$ is a rank-one projection, then there exists a neighbourhood U_0 of π with $U_0 \subseteq U$ such that $\sigma(e)$ is a rank-one for all $\sigma \in U_0$. It is known [8], [10] that the notions agree if A is liminal with $\widehat{A}$ Hausdorff. For more details see [9]. Hence a *point* $\pi \in \widehat{A}$ is said to be *singular* if it is not Glimm-regular. From 4.5.3, 4.5.4 of [6] we know that if A is a C^*-algebra with continuous trace, then A is liminal, $\widehat{A}$ Hausdorff and every point of $\widehat{A}$ is Fell-regular; conversely, A is a C^*-algebra with continuous trace if $\widehat{A}$ is Hausdorff and every point of $\widehat{A}$ is Fell-regular.

For the following definitions we refer the reader to [2], [4]. If φ and ψ are pure states of a C^*-algebra A and p, q are their respective support projections in A^{**}, then the transition probability between φ and ψ is denoted by $\langle \varphi, \psi \rangle$ and is defined by $\langle \varphi, \psi \rangle = \varphi(q) = \psi(p)$. If φ and ψ are unitarily equivalent, there will be an irreducible representation $\pi : A \to B(H)$ and unit vectors $\xi, \eta \in H$ such that for every $a \in A$ we have $\varphi(a) = \langle \pi(a)\xi, \xi \rangle$, and $\psi(a) = \langle \pi(a)\eta, \eta \rangle$. Hence, the transition probability between φ and ψ is given by $\langle \varphi, \psi \rangle = |\langle \xi, \eta \rangle|^2$. If φ and ψ are inequivalent (that is, their respective GNS irreducible representations are not unitarily equivalent) then $\langle \varphi, \psi \rangle = 0$. We shall use the symbol $QS(A)$ to denote the convex and weak*-compact subset $\{\varphi \in A_1^* : \varphi \geqslant 0\}$ of A_1^*.

1. MULTIPLES OF PURE FUNCTIONALS

Glimm in Theorem 6 of [8] has shown that if A is a C^* -algebra, then $\overline{P(A)} \subseteq \{\lambda\varphi : \lambda \in [0,1], \varphi \in P(A)\} \Leftrightarrow A$ is liminal, $\widehat{A}$ is Hausdorff and at singular points $\pi \in \widehat{A}, \pi(A)$ is one dimensional. In the following theorem, we extend this result to the space of pure functionals $\overline{G(A)}$ of a C^* -algebra A with the weak*-topology.

THEOREM 1.1. *Let A be a C^*-algebra. Then the following are equivalent:*
(i) $\overline{G(A)} \subseteq \{\lambda g : |\lambda| \leqslant 1, g \in G(A)\}$;
(ii) $\overline{P(A)} \subseteq \{\mu f : \mu \in [0,1], f \in P(A)\}$.

Proof. (i) $\Rightarrow$ (ii) Suppose $\overline{G(A)} \subseteq \{\lambda g : |\lambda| \leqslant 1, g \in G(A)\}$. Let $\rho \in \overline{P(A)}$; then since $\overline{P(A)} \subseteq \overline{G(A)}, \rho \in \overline{G(A)}$. Hence there exists $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$ and $g \in G(A)$ such that $\rho = \lambda g$. If $\lambda = 0$ then $\rho = 0 \in \{\mu f : \mu \in [0,1], f \in P(A)\}$. So let $\lambda \neq 0$. Then $\rho = (r\mathrm{e}^{\mathrm{i}\theta})g$ $(0 < r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi)$. Or $(\mathrm{e}^{\mathrm{i}\theta})g = (1/r)\rho \geqslant 0$. So $(\mathrm{e}^{\mathrm{i}\theta})g \in G(A) \cap S(A) = P(A)$, therefore $\rho = r(\mathrm{e}^{\mathrm{i}\theta}g) \in \{\mu f : \mu \in [0,1], f \in P(A)\}$. That is $\overline{P(A)} \subseteq \{\mu f : \mu \in [0,1], f \in P(A)\}$.

(ii) $\Rightarrow$ (i) Suppose $\overline{P(A)} \subseteq \{\mu f : \mu \in [0,1], f \in P(A)\}$. Let $\varphi \in \overline{G(A)}$ then there exists a net (φ_α) in $G(A)$ such that $\varphi_\alpha \to \varphi$. We have two cases: $\varphi = 0$ and $\varphi \neq 0$.

Case (i). Suppose $\varphi = 0$. Then φ = 0 = zero times a pure functional of A .

Case (ii). Suppose $\varphi \neq 0$. Then $(|\varphi_\alpha|)$ is a net of pure states of A in $QS(A)$ which is compact so by passing to a subnet we can assume that $|\varphi_\alpha| \to \rho$ for some $\rho \in QS(A)$. But ρ is the limit of pure states, so $\rho \in \overline{P(A)}$ and by our assumption $\rho = \mu\psi$ where $\mu \in [0,1], \psi \in P(A)$. By 3.3 of [7],

$$|\varphi_\alpha(a)|^2 \leqslant |\varphi_\alpha|(a^*a), \quad \forall a \in A. \tag{1.1}$$

Now because $\varphi_\alpha \to \varphi$ and $|\varphi_\alpha| \to \rho = \mu\psi$, the inequality (1.1) is preserved for the limits so that $|\varphi(a)|^2 \leqslant \mu\psi(a^*a)$, $\forall a \in A$. By page 400 last 4 lines of [7], there exists a vector $\eta \in H_{\pi_\psi}$ with $\|\eta\| \leqslant \sqrt{\mu} \leqslant 1$ such that, for $a \in A$, $\varphi(a) = \langle \pi_\psi(a)\xi_\psi, \eta\rangle = \|\eta\|\langle \pi_\psi(a)\xi_\psi, \eta/\|\eta\|\rangle$. Note that because $\varphi \neq 0, \|\eta\| \neq 0$ and so $\varphi = \lambda g$, where$|\lambda| \leqslant 1, g \in G(A)$. ∎

COROLLARY 1.2. *Let A be a C^*-algebra. Then the following are equivalent:*
(i) $\overline{G(A)} \subseteq \{\lambda g : |\lambda| \leqslant 1, g \in G(A)\}$;
(ii) *A is liminal, $\widehat{A}$ is Hausdorff and at singular points $\pi \in \widehat{A}, \pi(A)$ is one dimensional.*

This follows immediately from Theorem 2.1 and Theorem 6 of [8].

2. LIMITS OF PURE FUNCTIONALS AND HOMOGENEOUS C^*-ALGEBRAS

In the following two theorems we investigate $\overline{G(A)}$ when A is a unital (respectively non-unital) homogeneous C^*-algebra.

THEOREM 2.1. *Suppose A is a unital homogeneous C^*-algebra. Then $\overline{G(A)} = G(A)$.*

Proof. Suppose A is a homogeneous C^*-algebra of degree n, i.e, $\dim(H_\pi) = n$ for all $\pi \in \widehat{A}$. Let $\varphi \in \overline{G(A)}$, then there exists a net (φ_α) in $G(A)$ such that φ_α converges to φ (weak*). Since $\varphi_\alpha \in G(A)$ there exist an irreducible represention $\pi_\alpha \in \widehat{A}$ and unit vectors $\xi_\alpha, \eta_\alpha \in H_{\pi_\alpha}$ such that $\varphi_\alpha = \langle \pi_\alpha(\cdot)\xi_\alpha, \eta_\alpha\rangle$. Since A is unital, $\widehat{A}$ is compact, so there exists a subnet (π_μ) of (π_α) such that π_μ converges to some π in $\widehat{A}$. Now as A is homogeneous and $\pi \in \widehat{A}$, by Section 5 paragraph 3 of [1] there exists an open neighbourhood U of π and a subset $\{e_{ij} : 1 \leqslant i,j \leqslant n\}$ of A such that:

(i) for each $\sigma \in U$, $\{\sigma(e_{ij}) : 1 \leqslant i,j \leqslant n\}$ is a system of $n \times n$ matrix units for $\sigma(A)$ and hence for each $x \in \sigma(A)$ there is a unique matrix $[\beta_{ij}(x)] \in M_n(\mathbb{C})$ such that $x = \sum\limits_{i,j=1}^{n} \beta_{ij}(x)\sigma(e_{ij})$;

(ii) for each $a \in A$ the mapping $\theta_a : U \to M_n(\mathbb{C})$ defined by $\theta_a(\sigma) = [\beta_{ij}(\sigma(a))]$ is continuous on U.

Since $\pi_\mu \to \pi$ there exists μ_0 such that for $\mu \geqslant \mu_0$, $\pi_\mu \in U$. Let $a \in A$ and $\sigma \in U$, then $\sigma(a) = \sum_{i,j=1}^{n} \beta_{ij}(\sigma(a))\sigma(e_{ij})$ where $\beta_{ij}(\sigma(a)) \in \mathbb{C}$. For $\sigma \in U$, let $\Phi_\sigma : \sigma(A) \to M_n(\mathbb{C})$ be defined by $\Phi_\sigma(x) = [\beta_{ij}(x)]$ $(x \in \sigma(A))$. Then clearly it is a $*$-isomorphism. Note that $\theta_a(\sigma) = \Phi_\sigma(\sigma(a))$ $(a \in A, \sigma \in U)$ and so, in particular, $\|\theta_a(\sigma)\| \leqslant \|a\|$. There is a unique $\psi_\mu \in G(\pi_\mu(A))$ such that $\psi_\mu(\pi_\mu(a)) = \phi_\mu(a)$ $(a \in A)$ and so $\psi_\mu \circ \pi_\mu = \varphi_\mu$. For $\mu \geqslant \mu_0$, define $\rho_\mu = \psi_\mu \circ \Phi_{\pi_\mu}^{-1}$. Then $\rho_\mu \in G(M_n(\mathbb{C}))$. Since every irreducible representation of $M_n(\mathbb{C})$ is unitarily equivalent to the standard representation on $\mathbb{C}^n$, there exist unit vectors $\xi_\mu, \eta_\mu \in \mathbb{C}^n$ such that $\rho_\mu([\beta_{ij}]) = \langle [\beta_{ij}]\xi_\mu, \eta_\mu \rangle$.

Since $\mathbb{C}^n$ is finite dimensional the unit shell in $\mathbb{C}^n$ is compact, so by passing to successive subnets, if necessary, we can assume that there exist unit vectors ξ, η such that $\lim_\mu \|\xi_\mu - \xi\| = 0$, $\lim_\mu \|\eta_\mu - \eta\| = 0$. For $[\beta_{ij}] \in M_n(\mathbb{C})$ define $\rho([\beta_{ij}]) = \langle [\beta_{ij}]\xi, \eta \rangle$. Then $\rho \in G(M_n(\mathbb{C}))$. Let $a \in A$. Then we have $\varphi_\mu(a) = \psi_\mu(\pi_\mu(a)) = \langle \theta_a(\pi_\mu)\xi_\mu, \eta_\mu \rangle$. Since θ_a is continuous on U therefore, $\lim_\mu \|\theta_a(\pi_\mu) - \theta_a(\pi)\| = 0$ and hence $|\langle \theta_a(\pi_\mu)\xi_\mu, \eta_\mu r\rangle - \langle \theta_a(\pi)\xi, \eta \rangle| \to 0$. Thus $\lim_\mu \varphi_\mu(a) = (\rho \circ \Phi_\pi \circ \pi)(a)$ $\forall a \in A$ and so, $\varphi = \rho \circ (\Phi_\pi \circ \pi)$. Now $\Phi_\pi \circ \pi : A \to M_n(\mathbb{C})$ is a surjective $*$-homomorphism of C^*-algebras and ρ is a pure functional on $M_n(\mathbb{C})$, therefore, $\rho \circ (\Phi_\pi \circ \pi)$ is a pure functional of A. Hence $\varphi \in G(A)$ and so $\overline{G(A)} = G(A)$. ∎

THEOREM 2.2. *Suppose A is a non-unital homogeneous C^*-algebra. Then* $\overline{G(A)} = G(A) \cup \{0\}$.

Proof. Let $\varphi \in \overline{G(A)}$, then there exists a net (φ_α) in $G(A)$ such that $\varphi_\alpha \to \varphi$. Since $\varphi_\alpha \in G(A)$ therefore there exist an irreducible represention $\pi_\alpha \in \widehat{A}$ and unit vectors $\xi_\alpha, \eta_\alpha \in H_{\pi_\alpha}$ such that $\varphi_\alpha = \langle \pi_\alpha(\cdot)\xi_\alpha, \eta_\alpha \rangle$. Now either (i) there exists a compact set K in $\widehat{A}$ such that (π_α) is frequently in $\widehat{A}$, or (ii) for every compact set K in $\widehat{A}$ eventually $\pi_\alpha \notin K$.

Case (i). In this case (π_α) has a subnet in K, but K is compact so there exists a further subnet (π_μ) in K such that π_μ converges to some π in K. Now as A is homogeneous and $\pi \in \widehat{A}$, arguing as in the proof of Theorem 3.1, we find a $\rho \in G(\pi(A))$ such that $\varphi_\mu \to \rho \circ \pi$ and hence $\varphi = \rho \circ \pi \in G(A)$.

Case (ii). For every compact set $K \subseteq \widehat{A}$ eventually $\pi_\alpha \notin K$, that is, for every compact set $K \subseteq \widehat{A}$ there exists α_0 such that $\pi_\alpha \notin K$ for all $\alpha \geqslant \alpha_0$. Choose an $a \in A$ and an $\varepsilon > 0$. Then, since the set $\{\sigma \in \widehat{A} : \|\sigma(a)\| \geqslant \varepsilon\}$ is compact, there exists α_0 such that for $\alpha > \alpha_0$, $\|\pi_\alpha(a)\| < \varepsilon$. Now since, for $\alpha \geqslant \alpha_0$, $|\varphi_\alpha(a)| = |\langle \pi_\alpha(a)\xi_\alpha, \eta_\alpha \rangle| \leqslant \|\pi_\alpha(a)\| < \varepsilon$ therefore $\lim_\alpha \varphi_\alpha(a) = 0$ for all $a \in A$ and hence $\varphi(a) = 0$ for all $a \in A$, that is, $\varphi = 0$. Thus from cases (i) and (ii) we get $\overline{G(A)} \subseteq G(A) \cup \{0\}$. On the other hand since A is non-unital, by 2.12.13 of [6], $0 \in \overline{P(A)}$ $(\subseteq \overline{G(A)})$, and therefore $G(A) \cup \{0\} \subseteq \overline{G(A)}$. Consequently, we get $\overline{G(A)} = G(A) \cup \{0\}$. Hence the theorem follows. ∎

Let $A = \bigoplus_{n\geqslant 1} A_n$ (c_0-sum) where each A_n is a C^*-algebra. Define $\Phi_n : A \to A_n$ by $\Phi_n(\vec{a}) = a_n$ where $\vec{a} \in A$. Let $\varphi \in A_n^*$. Define $\widetilde{\varphi} : A \to \mathbb{C}$ by $\widetilde{\varphi}(\vec{a}) = \varphi(a_n)$, then $\widetilde{\varphi} = \varphi \circ \Phi_n$. Let $\widetilde{G(A_n)} := \{\widetilde{\varphi} : \varphi \in G(A_n)\}$ and $\overline{\widetilde{G(A_n)}} := \{\widetilde{\varphi} : \varphi \in \overline{G(A_n)}\}$. With these notations we have the following lemma:

LEMMA 2.3. *Let $A = \bigoplus_{n\geqslant 1} A_n$ (c_0-sum) where each A_n is a C^*-algebra. Then:*

(i) $G(A) = \bigcup_{n\geqslant 1} \widetilde{G(A_n)}$;

(ii) $\overline{\widetilde{G(A_n)}} = \overline{\widetilde{G(A_n)}}$.

Proof. (i) Let $\varphi \in G(A_n)$, then $\widetilde{\varphi} \in G(A)$. Since $\widetilde{G(A_n)} \subseteq G(A)\ \forall n \geqslant 1$, therefore $\bigcup_{n\geqslant 1} \widetilde{G(A_n)} \subseteq G(A)$. Conversely, suppose $\rho \in G(A)$, then there exists an irreducible representation $\pi \in \widehat{A}$ and unit vectors $\xi, \eta \in H_\pi$ such that $\rho = \langle \pi(\cdot)\xi, \eta\rangle$. Define $J_n = \{\vec{x} \in A : \vec{x} = (0, 0, \ldots, x_n, 0, 0, \ldots), x_n \in A_n\}$. Then J_n is a closed two sided ideal of A and for $m \neq n$, $J_m J_n = \{0\}$. Since $J_m J_n \subseteq \ker\pi$ and $\ker\pi$ is prime so either $\pi(J_m) = \{0\}$ or $\pi(J_n) = \{0\}$. This shows that there is at most one value of n such that $\pi(J_n) \neq \{0\}$. We show that there is a value of n such that $\pi(J_n) \neq \{0\}$. For this, suppose $\pi(J_n) = \{0\}$ for all $n \geqslant 1$ and choose $\vec{a} \in A$ and an $\varepsilon > 0$. Since A is the c_0-sum of A_n's there exist n_0 such that, for $n \geqslant n_0$, $\|a_n\| < \varepsilon$. Hence $\|\pi(\vec{a})\| = \max_{i\geqslant n_0+1} \|a_i\| \leqslant \varepsilon$ and so $\|\pi(\vec{a})\| = 0$ for all $\vec{a} \in A$. But this implies that $\pi = 0$, a contradiction to the fact that $\pi \neq 0$. Thus there exists precisely one value of n, say N, such that $\pi(J_N) \neq \{0\}$. Now let $K_N = \ker\Phi_N$. Then $K_N J_N = \{0\}$, and so $K_N \subseteq \ker\pi$, (since $\ker\pi$ is prime). Hence there exists an irreducible representation π_0 of A_N on H_π such that $\pi = \pi_0 \circ \Phi_N$. Define $\varphi = \langle \pi_0(\cdot)\xi, \eta\rangle$, then $\varphi \in G(A_N)$. Hence $\varphi \circ \Phi_N = \rho$. That is $\rho = \widetilde{\varphi}$ and so $\rho \in \widetilde{G(A_N)}$ and thus $\rho \in \bigcup_{n\geqslant 1} \widetilde{G(A_n)}$. Therefore $G(A) \subseteq \bigcup_{n\geqslant 1} \widetilde{G(A_n)}$.

Combining the two inclusions we get $G(A) = \bigcup_{n\geqslant 1} \widetilde{G(A_n)}$.

(ii) We show that $\overline{\widetilde{G(A_n)}} \subseteq \overline{\widetilde{G(A_n)}} \subseteq \overline{\widetilde{G(A_n)}}$. Let $\rho \in \overline{\widetilde{G(A_n)}}$, then there exists a net (φ_α) in $G(A_n)$ such that $\widetilde{\varphi}_\alpha \to \rho$. Let $\vec{x} \in \ker\Phi_n$ then since $\rho(\vec{x}) = \lim \widetilde{\varphi}_\alpha(\vec{x}) = \lim \varphi_\alpha(\Phi_n(\vec{x})) = 0$, we get $\rho(\ker\Phi_n) = \{0\}$. Hence there exists $\varphi \in A_n^*$ such that $\rho = \varphi \circ \Phi_n$. Therefore $\rho = \widetilde{\varphi}$, and $\varphi = \lim_\alpha \varphi_\alpha \in \overline{G(A_n)}$ and so $\rho \in \overline{\widetilde{G(A_n)}}$. Thus $\overline{\widetilde{G(A_n)}} \subseteq \overline{\widetilde{G(A_n)}}$. Conversely, suppose $\rho \in \overline{\widetilde{G(A_n)}}$, then there exists $\varphi \in \overline{G(A_n)}$ such that $\rho = \varphi \circ \Phi_n$, that is, $\rho = \widetilde{\varphi}$. Since $\varphi \in \overline{G(A_n)}$ there exists a net (φ_α) in $G(A_n)$ such that $\varphi_\alpha \to \varphi$. Let $\vec{a} \in A$ then since $\widetilde{\varphi}_\alpha(\vec{a}) - \widetilde{\varphi}(\vec{a}) = (\varphi_\alpha \circ \Phi_n)(\vec{a}) - (\varphi \circ \Phi_n)(\vec{a}) = \varphi_\alpha(a_n) - \varphi(a_n) \to 0$ we get $\widetilde{\varphi}_\alpha \to \widetilde{\varphi}$. Therefore $\widetilde{\varphi} \in \overline{\widetilde{G(A_n)}}$. Hence $r\overline{\widetilde{G(A_n)}} \subseteq \overline{\widetilde{G(A_n)}}$. ∎

In the remaining part of this section we consider the class of C^*-algebras A satisfying the equivalent conditions of Theorem 1.1. In particular, we consider those A such that $G(A)$ is closed, and those A such that $\overline{G(A)} = G(A) \cup \{0\}$. Such algebras turn out to be the finite (respectively c_0) direct sum of homogeneous C^*-algebras.

THEOREM 2.4. *Suppose a C^*-algebra A is the finite direct-sum of unital homogeneous C^*-algebras, that is there exists a positive integer N such that $A = \sum_{n=1}^{N} \oplus A_n$, where each A_n is a unital homogeneous C^*-algebra. Then $\overline{G(A)} = G(A)$.*

Proof. Suppose $A \cong \sum_{n=1}^{N} \bigoplus A_n$ where each A_n is a unital homogeneous C^*-algebra. Then by Lemma 2.3 and Theorem 2.1, we get $\overline{G(A)} = \overline{\bigcup_{n=1}^{N} \widetilde{G(A_n)}} = \bigcup_{n=1}^{N} \overline{\widetilde{G(A_n)}} = \bigcup_{n=1}^{N} \widetilde{\overline{G(A_n)}} = \bigcup_{n=1}^{N} \widetilde{G(A_n)} = G(A)$. ∎

THEOREM 2.5. *Let a C^*-algebra A be the c_0-sum of homogeneous C^*-algebras, that is, $A = \bigoplus_{n\geqslant 1} A_n$ where each A_n is either zero or a non-zero homogeneous C^*-algebra and either at least one A_n is non-unital or infinitely many A_n are non-zero. Then $\overline{G(A)} = G(A) \cup \{0\}$.*

Proof. A non-unital implies that $0 \in \overline{P(A)} \subseteq \overline{G(A)}$. Therefore we get $G(A) \cup \{0\} \subseteq \overline{G(A)}$. Conversely, suppose $\psi \in \overline{G(A)}$ then there exists a net (ψ_α) in $G(A)$ such that $\psi_\alpha \to \psi$. But by Lemma 2.3(i), $G(A) = \bigcup_{n\geqslant 1} \widetilde{G(A_n)}$, so $\psi_\alpha \in \widetilde{G(A_{n_\alpha})}$ for some $n_\alpha \in \mathbb{N}$ and hence there exist some $\varphi_\alpha \in G(A_{n_\alpha})$ such that $\psi_\alpha = \widetilde{\varphi}_\alpha$ (see Section 3.3?????where from?). Now (n_α) is a net in (the compact set) $\mathbb{N} \cup \{\infty\}$, so by passing to a subnet, say, (n_μ) we have: either (i) $n_\mu \to \infty$, or (ii) $n_\mu \to n$ for some $n \in \mathbb{N}$.

Case (i). In this case for $\vec{x} \in A$ and $\varepsilon > 0$ there exists some μ_0 such that for $\mu \geqslant \mu_0$, $\|\Phi_{n_\mu}(\vec{x})\| = \|x_{n_\mu}\| < \varepsilon$. Therefore $\|\Phi_{n_\mu}(\vec{x})\| \to 0$. Since $|\psi_\mu(\vec{x})| \leqslant \|\Phi_{n_\mu}(\vec{x})\|$, therefore $|\psi_\mu(\vec{x})| \to 0$ and hence $\psi_\mu \to 0$. Thus $\psi = 0$.

Case (ii). In this case there exists a μ_0 such that $n_\mu = n$ for $\mu \geqslant \mu_0$. Therefore $\Phi_{n_\mu}(\vec{x}) = \Phi_n(\vec{x})$, for $\mu \geqslant \mu_0$. Let $\vec{a} \in \ker \Phi_n$ then for $\mu \geqslant \mu_0$, $|\psi_\mu(\vec{a})| \leqslant \|\Phi_n(\vec{a})\|$ and so $\psi(\ker \Phi_n) = \{0\}$. Therefore there exists $\varphi \in A_n^*$ such that $\varphi \circ \Phi_n = \psi$ (and $\|\varphi\| = \|\psi\|$). Since for $\vec{a} \in A$ and $\mu \geqslant \mu_0$, $|\varphi_\mu(a_n) - \varphi(a_n)| = |\psi_\mu(\vec{a}) - \psi(\vec{a})| \to 0$, $\varphi \in \overline{G(A_n)}$. But A_n is homogeneous, so by Theorem 2.2 $\overline{G(A_n)} \subseteq G(A_n) \cup \{0\}$. Therefore $\varphi \in G(A_n) \cup \{0\}$ and hence $\psi \in \widetilde{G(A_n)} \cup \{0\}$. But by Lemma 2.3(i) $G(A) = \Big(\bigcup_{m\geqslant 1} \widetilde{G(A_m}\Big)$. Therefore $\psi \in G(A) \cup \{0\}$, and hence $\overline{G(A)} \subseteq G(A) \cup \{0\}$. ∎

LEMMA 2.6. *A is a continuous trace C^*-algebra and $\overline{P(A)} \subseteq P(A) \cup \{0\}$ if and only if $A \cong \bigoplus_{n \geqslant 1} A_n$ (c_0 -sum) where each A_n is either zero or non-zero homogeneous C^*-algebra.*

Proof. Suppose $A \cong \bigoplus_{n \geqslant 1} A_n$ (c_0-sum) where each A_n is either zero or non-zero homogeneous C^*-algebra. Since each A_n has continuous trace, so does A. By the proof of Theorem 2.5, $\overline{G(A)} \subseteq G(A) \cup \{0\}$. Since $P(A) \subseteq G(A)$ and $G(A) \cap QS(A) = P(A)$, so $\overline{P(A)} \subseteq P(A) \cup \{0\}$.

Conversely, suppose A has continuous trace and $\overline{P(A)} \subseteq P(A) \cup \{0\}$. Then, by page 106 line 8 of [6], the self-adjoint ideal

$$m(A) = \{x \in A : \sigma \to \mathrm{tr}(\sigma(x)) \text{ is finite and continuous on } \widehat{A}\}$$

is dense in A. Note that $A \in [FIN]$, that is, $\dim(\sigma) < \infty\ \forall \sigma \in \widehat{A}$. For, suppose $\dim(\sigma)$ is not finite for some $\sigma \in \widehat{A}$. Since $\overline{P(A)} \supseteq \overline{P(\sigma(A))} = \overline{P(K(H_\sigma))}$ and, by Glimm's Vector State Space Theorem, $\overline{P(K(H_\sigma))} = \{t\omega_\xi|_{K(H_\sigma)}: t \in [0,1], \|\xi\| = 1\}$ we get a contradiction because $\overline{P(A)} \subseteq P(A) \cup \{0\}$. For $n \geqslant 1$, define $U_n = \{\sigma \in \widehat{A} : \dim(\sigma) = n\}$. We show that U_n is clopen. It is enough to show that every U_n is open. Suppose some U_n is not open. Then, since $\bigcup_{i=1}^{n-1} U_i$ is closed by 3.6.3(i) of [6] there exist a $\pi \in U_n$ and a net (π_α) in $\widehat{A} \setminus \left(\bigcup_{i=1}^{n} U_i\right)$ with $\pi_\alpha \to \pi$. Since π is Fell-regular there exists $e \in A^+$ and a neighbourhood V of π in $\widehat{A}$ such that $\sigma(e)$ is a rank-one projection for all $\sigma \in V$. So there exists α_0 such that for $\alpha \geqslant \alpha_0$ $\pi_\alpha(e)$ is a rank-one projection (since eventually $\pi_\alpha \in V$).

Choose unit vectors $\xi_1^{(\alpha)} \in H_{\pi_\alpha}$ and $\xi_1 \in H_\pi$ such that $\pi_\alpha(e)\xi_1^{(\alpha)} = \xi_1^{(\alpha)}$, $\pi(e)\xi_1 = \xi_1$. Let $\varphi_1^{(\alpha)} = \langle \pi_\alpha(\cdot)\xi_1^{(\alpha)}, \xi_1^{(\alpha)}\rangle$, and $\varphi_1 = \langle \pi(\cdot)\xi_1, \xi_1\rangle$. We show that $\varphi_1^{(\alpha)} \to \varphi_1$. Let $x \in A$, then, since $\pi(e)$ is a one dimensional projection, $\pi(exe) = \varphi_1(x)\pi(e)$. Since $|\varphi_1^{(\alpha)}(x) - \varphi_1(x)| \leqslant \|\pi_\alpha(exe - \varphi_1(x)e)\| \to 0$ (as $\widehat{A}$ is Hausdorff) we get $\varphi_1^{(\alpha)} \to \varphi_1$. Now extend $\xi_1^{(\alpha)}$ to an orthonormal set $\{\xi_1^{(\alpha)}, \xi_2^{(\alpha)}, \dots, \xi_{n+1}^{(\alpha)}\}$ in H_{π_α} and define $\varphi_i^{(\alpha)} = \langle \pi_\alpha(\cdot)\xi_i^{(\alpha)}, \xi_i^{(\alpha)}\rangle$ ($2 \leqslant i \leqslant n+1$). The set $QS(A)$ is weak*-compact, so by passing to successive subnets we get $\varphi_i^{(\mu)} \to \varphi_i$ for some $\varphi_i \in QS(A)$ ($2 \leqslant i \leqslant n+1$). But φ_i is the limit of pure states so $\varphi_i \in \overline{P(A)} \subseteq P(A) \cup \{0\}$. Note that $\varphi_i(\ker \pi) = \{0\}$ for all i. Therefore there exist a $\varphi_i' \in P(\pi(A)) \cup \{0\}$ such that $\varphi_i = \varphi_i' \circ \pi$ ($2 \leqslant i \leqslant n+1$). Since π is finite-dimensional, $\varphi_i = \omega_{\xi_i} \circ \pi$ where either $\|\xi_i\| = 1$ or $\xi_i = 0$ ($2 \leqslant i \leqslant n+1$).

Suppose that $1 \leqslant i < j \leqslant n+1$ and that $\|\xi_i\| = \|\xi_j\| = 1$. We show that $\xi_i \perp \xi_j$. Since A has continuous trace, by Theorem 2.3 of [4] the (transition probability) map $\langle \cdot, \cdot \rangle : R(A) \to [0,1], (\varphi, \psi) \mapsto \langle \varphi, \psi \rangle$ (where $R(A) = \{(\varphi, \psi) \in P(A) \times P(A) : \varphi \text{ is equivalent } \psi\}$) is continuous. Therefore $(\varphi_i^{(\mu)}, \varphi_j^{(\mu)}) \to (\varphi_i, \varphi_j)$

implies $\langle \varphi_i^{(\mu)}, \varphi_j^{(\mu)} \rangle \to \langle \varphi_i, \varphi_j \rangle$. But $\langle \varphi_i^{(\mu)}, \varphi_j^{(\mu)} \rangle = |\langle \xi_i^{(\mu)}, \xi_j^{(\mu)} \rangle|^2 = 0$, therefore $\langle \varphi_i, \varphi_j \rangle = 0$. Hence $|\langle \xi_i, \xi_j \rangle|^2 = 0$, that is, $\xi_i \perp \xi_j$.

Since $\dim(\pi) = n$ there is space for only n orthogonal pure states. Therefore at least one $\varphi_j = 0$ $(j = 2, 3, \dots, n+1)$. We have shown that φ_1 is a pure state of A such that $\varphi_1^{(\alpha)} \to \varphi_1$, so we get $\varphi_1^{(\mu)} \to \varphi_1, \varphi_j^{(\mu)} \to 0$. Let $\zeta_\mu = \frac{\xi_1^{(\mu)} + \xi_j^{(\mu)}}{\sqrt{2}}$. Define $\psi_\mu = \langle \pi_\mu(\cdot)\zeta_\mu, \zeta_\mu \rangle \in P(A)$. Let $a \in A$, and consider

$$\begin{aligned}\psi_\mu(a) = \langle \pi_\mu(a)\zeta_\mu, \zeta_\mu \rangle &= \Big\langle \pi_\mu(a)\Big(\frac{\xi_1^{(\mu)} + \xi_j^{(\mu)}}{\sqrt{2}}\Big), \Big(\frac{\xi_1^{(\mu)} + \xi_j^{(\mu)}}{\sqrt{2}}\Big)\Big\rangle \\ &= \frac{1}{2}\{\varphi_1^{(\mu)}(a) + \varphi_j^{(\mu)}(a) + 2\mathrm{Re}\langle \pi_\mu(a)\xi_1^{(\mu)}, \xi_j^{(\mu)} \rangle\} \to \frac{\varphi_1(a)}{2}.\end{aligned}$$

This is because $|\langle \pi_\mu(a)\xi_1^{(\mu)}, \xi_j^{(\mu)} \rangle|^2 \leqslant \|\pi_\mu(a^*)\xi_j{}^{(\mu)}\|^2 = \varphi_j^{(\mu)}(aa^*) \to 0$. Thus $\psi_\mu \to \varphi_1/2 \in \overline{P(A)} \subseteq P(A) \cup \{0\}$ a contradiction as $\varphi_1/2$ is of norm equal to $1/2$. Therefore U_n is open in $\widehat{A}$ $\forall n \geqslant 1$ (but possibly empty).

Observe that $U_n = \widehat{A} \setminus \bigcup\limits_{m \neq n} U_m$. Then U_n is closed $\forall n \geqslant 1$ (and so clopen). So by 3.2.2 of [6] there exists a closed two sided ideal A_n of A such that $\widehat{A}_n = U_n$. Observe that $\widehat{A} = \bigcup\limits_{n \geqslant 1} \widehat{A}_n$ and $A_n \neq \{0\} \Leftrightarrow \widehat{A}_n \neq \emptyset$. If $A_n \neq \{0\}$ then, by definition of U_n, A_n is a homogeneous C^*-algebra of degree n. Let $\mathcal{A} := \bigoplus\limits_{n \geqslant 1} A_n$ (c_0-sum). Define $\chi_n : \widehat{A} \to [0,1]$ by $\chi_n(\pi) = \begin{cases} 1 & \text{if } \pi \in \widehat{A}_n, \\ 0 & \text{if } \pi \notin \widehat{A}_n. \end{cases}$ Clearly χ_n is a bounded continuous function on $\widehat{A}$. Temporarily fix $a \in A$. By the Dauns–Hoffman theorem there exists an $a_n \in A$ such that

$$\pi(a_n) = \chi_n(\pi)\pi(a) \quad (\pi \in \widehat{A}). \tag{2.1}$$

If $\pi \in \widehat{A} \setminus \widehat{A}_n$ then $\pi(a_n) = 0$ and so $a_n \in A_n$. Let $\varepsilon > 0$, then by 3.3.7 of [6] the set $K = \{\pi \in \widehat{A} : \|\pi(a)\| \geqslant \varepsilon\}$ is compact in $\widehat{A}$ and hence is covered by a finite number of open sets in the sequence $(\widehat{A}_n)$. Hence there exists a finite $m \in \mathbb{N}$ such that $K \subseteq \bigcup\limits_{n=1}^{m} \widehat{A}_n$. Let $n > m$ then $K \cap \widehat{A}_n = \emptyset$ so that, by (2.1), when $\pi \in \widehat{A}_n$ we have $\|\pi(a_n)\| = \|\pi(a)\| < \varepsilon$. Therefore, we get

$$\|a_n\| = \max_{\pi \in \widehat{A}_n} \|\pi(a_n)\| < \varepsilon \quad (n > m) \tag{2.2}$$

([6], 3.3.6). Define $\widehat{a} \in \mathcal{A}$ by $\widehat{a}(n) = a_n$. Now define $\theta : A \to \mathcal{A}$ by $\theta(a) = \widehat{a}$. It is routine to check that θ is a $*$-isomorphism from A onto $\mathcal{A}$. ∎

The following well-known lemma is the special case of Lemma 2.6. Recall that if $0 \notin \overline{P(A)}$ then A is unital ([6], 2.12.13).

LEMMA 2.7. *A is a continuous trace C^*-algebra and $\overline{P(A)} = P(A)$ if and only if $A = \sum_{n=1}^{N} \oplus A_n$, where each A_n is either zero or non-zero unital homogeneous C^*-algebra.*

THEOREM 2.8. *Let A be a unital C^*-algebra. Then the following are equivalent:*
(i) $\overline{P(A)} = P(A)$;
(ii) $A \in [FIN]$, $\widehat{A}$ *Hausdorff and at singular points* $\pi \in \widehat{A}$, $\pi(A)$ *is one dimensional;*
(iii) $\overline{G(A)} = G(A) \cup \{\lambda\varphi : |\lambda| \leqslant 1, \varphi \in P(A)$ *with* GNS *representation* π_φ *singular*$\}$;
(iv) $\widehat{A}$ *is Hausdorff and there exists a two sided closed ideal J of A such that J is a (finite or c_0) direct sum of homogeneous C^*-algebras with A/J abelian and its spectrum $\widehat{(A/J)}$ is the set of singular points of $\widehat{A}$.*

Proof. (i) $\Leftrightarrow$ (ii). This follows from Theorem 6 of [8] and the fact that A is unital.

(ii) $\Rightarrow$ (iv) Suppose that (ii) holds. The set U of Fell points of $\widehat{A}$ is always open and so $U = \widehat{J}$ for some closed two sided ideal J of A. By construction, $\widehat{(A/J)}$ is the set of singular popints of $\widehat{A}$. If $\pi \in \widehat{(A/J)}$ then $\pi(A/J)$ is one dimensional and hence A/J is abelian.

Since $\widehat{J}$ is a Hausdorff and every point of $\widehat{J}$ is a Fell point, J has continuous trace. Let $\varphi \in \overline{P(J)}$. Then there is a net (φ_α) in $P(J)$ such that $\varphi_\alpha \to \varphi$ in the $\sigma(J^*, J)$ topology. For each α there is a unique $\psi_\alpha \in P(A)$ such that $\psi_\alpha|_J = \varphi_\alpha$. Since $S(A)$ is compact, there is a subnet ($\psi_{\alpha(\mu)}$) of (ψ_α) convergent to some $\psi \in S(A)$. By (i), $\psi \in P(A)$. But $\varphi = \psi|_J$, so $\varphi \in P(J) \cup \{0\}$. Thus $\overline{P(J)} \subseteq P(J) \cup \{0\}$. By Lemma 2.6, J is a direct sum of homogeneous C^*-algebras.

(iv) $\Rightarrow$ (ii) Suppose that (iv) holds. If $\pi \in \widehat{J}$ then π is finite dimensional and if $\pi \in \widehat{(A/J)}$ then π is one dimensional. Hence $A \in [FIN]$. If $\pi \in \widehat{A}$ is singular then $\pi \in \widehat{(A/J)}$ and hence π is one dimensional because A/J is abelian.

(ii) $\Rightarrow$ (iii) Suppose $A \in [FIN]$, $\widehat{A}$ Hausdorff and at singular points $\pi \in \widehat{A}$, $\pi(A)$ is one dimensional. Let $\varphi \in \overline{G(A)}$ then there exists a net (φ_α) in $G(A)$ such that $\varphi_\alpha \to \varphi$. Since $\varphi_\alpha \in G(A)$ there exists $\pi_\alpha \in \widehat{A}$ and unit vectors $\xi_\alpha, \eta_\alpha \in H_{\pi_\alpha}$ such that $\varphi_\alpha = \langle \pi_\alpha(\cdot)\xi_\alpha, \eta_\alpha \rangle$. Since A is unital, $\widehat{A}$ is compact, so by passing to a subnet (π_μ) we get $\pi_\mu \to \pi$ for some $\pi \in \widehat{A}$. Note that $\varphi(\ker \pi) = \{0\}$, because $\widehat{A}$ is Hausdorff. Either (i) π is singular or (ii) π is not singular.

Case (i). Suppose π is singular. Then H_π is one dimensional, and there is a unique pure state ψ of A associated with π, such that $\varphi = \lambda\psi$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$.

Case (ii). Suppose π is not singular. Since A is liminal and $\widehat{A}$ is Hausdorff, by [10], π is Fell-regular. So $\pi \in \widehat{J}$ where J is as in the proof of (ii) $\Rightarrow$ (iv). As previously shown in (ii) $\Rightarrow$ (iv), J is a direct sum of homogeneous C^*-algebras and so $\overline{G(J)} \subseteq G(J) \cup \{0\}$ by Theorems 2.4 and 2.5. Eventually $\pi_\mu \in \widehat{J}$ and hence $\varphi_\mu|_J \in G(J)$. Hence $\varphi|_J \in G(J) \cup \{0\}$. If $\varphi|_J \in G(J)$ then $\varphi \in G(A)$. If $\varphi(J) = \{0\}$ then $\varphi(A) = \varphi(J + \ker \pi) = \{0\}$ since $\ker \pi$ is a maximal closed two sided ideal and does not contain J. This completes case (ii).

For the reverse inclusion we need only show that $\lambda\psi \in \overline{G(A)}$ where $\psi \in P(A)$, π_ψ is singular and $|\lambda| \leqslant 1$, $\lambda \in \mathbb{C}$. Let $\varphi = \lambda\psi$ where $\psi \in P(A)$ with π_ψ singular and $|\lambda| \leqslant 1$. Since $A \in [FIN]$ and $\widehat{A}$ is Hausdorff, singularity of π_ψ implies that there exists an element $e \in A^+$, a neighbourhood U of π_ψ and a net (π_α) in U convergent to π_ψ such that:

(a) $\pi_\psi(e)$ is the rank one projection fixing ξ_ψ,
(b) $\sigma(e)$ is a projection for all $\sigma \in U$, and
(c) $\operatorname{rank}(\pi_\alpha(e)) \geqslant 2 \ \forall\alpha$.

In fact $\pi_\psi(e)$ is the identity since π_ψ is one dimensional. By (c) choose, for each α, orthogonal unit vctors $\xi_\alpha, \eta_\alpha \in \pi_\alpha(e)H_{\pi_\alpha}$ and define

$$\varphi_\alpha = \left\langle \pi_\alpha(\cdot)\xi_\alpha, \overline{\lambda}\xi_\alpha + \sqrt{1-|\lambda|^2}\eta_\alpha \right\rangle \in G(A).$$

Let $a \in A$. Since $\pi_\psi(e)$ is the 1-dimensional projection fixing ξ_ψ, $\pi_\psi(eae) = \psi(a)\pi_\psi(e)$. Since $\widehat{A}$ is Hausdorff, the map $\widehat{A} \to \mathbb{R}^+$ defined by $\sigma \mapsto \|\sigma(a)\|$ is continuous for all $a \in A$. Therefore $\|\pi_\alpha(eae - \psi(a)e)\| \to \|\pi_\psi(eae - \psi(a)e)\| = 0$. Now since $\varphi_\alpha(e) = \lambda$ we have

$$|\varphi_\alpha(a) - \varphi(a)| = |\varphi_\alpha(eae - \psi(a)e)| \leqslant \|\pi_\alpha(eae - \psi(a)e)\| \to 0.$$

Hence $\varphi_\alpha \to \varphi \in \overline{G(A)}$. Thus the desired implication follows.

(iii) $\Rightarrow$ (i) Suppose that (iii) holds. Then by Theorem 1.1, every limit of pure states is a multiple of a pure state. Since A is unital, $\overline{P(A)} = P(A)$. This completes the proof. ∎

Here we give an example of a C^*-algebra A for which $P(A)$ is weak*-closed but $G(A)$ is not. Let $A = \{x = (x_n) : x_n \in M_2(\mathbb{C}), x_n \underset{n}{\to} \operatorname{diag}(\lambda(x), \lambda(x))\}$, i.e, A is the C^*-algebra of sequences of 2×2 complex matrices such that $x_n \underset{n}{\to} \operatorname{diag}(\lambda(x), \lambda(x))$. One can see that A is unital and satisfies all of the Glimm's conditions, therefore $\overline{P(A)} = P(A)$. But $\overline{G(A)} \nsubseteq G(A)$, for example $\lambda/\sqrt{2} \in \overline{G(A)}$ which is not a pure functional.

COROLLARY 2.9. *Suppose A is a C^*-algebra. If $\overline{G(A)} = G(A)$, then A is a finite direct-sum of unital homogeneous C^*-algebras.*

Proof. Suppose $\overline{G(A)} = G(A)$, then $\overline{P(A)} = P(A)$. Hence A is unital, for otherwise $0 \in \overline{P(A)}$ a contradiction. Part (ii) and (iii) of Theorem 2.8 implies that $\widehat{A}$ is Hausdorff and has no singular points. So A is a C^*-algebra with continuous trace and by Lemma 2.7, A is a finite direct-sum of unital homogeneous C^*-algebras. ∎

THEOREM 2.10. *Let A be a non-unital C^*-algebra. Then the following are equivalent:*

(i) $\overline{P(A)} \subseteq P(A) \cup \{0\} \cup \{\lambda\varphi : \lambda \in [0,1], \varphi \in P(A)$ *with* GNS *representation* π_φ *singular*$\}$*;*

(ii) $\overline{G(A)} = G(A) \cup \{0\} \cup \{\lambda\varphi : |\lambda| \leqslant 1, \varphi \in G(A)$ *with* GNS *representation* $\pi_{|\varphi|}$ *singular*$\}$*;*

(iii) $\widehat{A}$ *is Hausdorff and there exists a two sided closed ideal J of A such that J is a (finite or c_0) direct sum of homogeneous C^*-algebras with A/J abelian and its spectrum $\widehat{(A/J)}$ is the set of singular points of $\widehat{A}$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that (i) holds. Let $\psi \in \overline{G(A)}$ then there exists a net (ψ_α) in $G(A)$ such that $\psi_{\ \alpha} \to \psi$. We have two cases: (i) $\psi = 0$, and (ii) $\psi \neq 0$.

Case (i). Suppose $\psi = 0$. Then ψ belongs to the right hand side of (ii).

Case (ii). Suppose $\psi \neq 0$. By 1.1 of [5], $|\psi_\alpha|$ is a pure state of A. Now $(|\psi_\alpha|)$ is a net in $QS(A)$ which is compact so by passing to a subnet we can assume that $|\psi_\alpha| \to \rho$ for some $\rho \in QS(A)$. But ρ is the limit of pure states, so $\rho \in \overline{P(A)} \subseteq P(A) \cup \{0\} \cup \{\lambda\varphi : \lambda \in [0,1], \varphi \in P(A)$ with GNS representation π_φ singular$\}$.

Therefore $\rho \in P(A) \subseteq G(A)$ or $\rho = 0$ or $\rho = \lambda\varphi$ where $\lambda \in [0,1], \varphi \in P(A)$ with GNS π_φ singular. From 3.3 of [7] we have

$$|\psi_\alpha(a)|^2 \leqslant |\psi_\alpha|(a^*a), \quad \forall a \in A. \tag{2.3}$$

Now because $\psi_{\ \alpha} \to \psi$ and $|\psi_\alpha| \to \rho$, the inequality (2.3) is preserved for the limits:

$$|\psi(a)|^2 \leqslant \rho(a^*a), \quad \forall a \in A. \tag{2.4}$$

Now if $\rho \in P(A)$ then by (2.4), $|\psi(a)| \leqslant \|\pi_\rho(a)\xi_\rho\|$ hence by page 400 last 4 lines of [7] there exist a unit vector $\eta \in H_{\pi_\varphi}$ such that $\psi(a) = \langle \pi_\rho(a)\xi_\rho, \eta\rangle$ and so $\psi \in G(A)$. Since $\psi \neq 0$, $\rho \neq 0$ by (2.4). Finally, if $\rho = \lambda\varphi$ where $\lambda \in [0,1], \varphi \in P(A)$ with GNS π_φ singular, then by page 400 last 4 lines of [7] there exist a vector $\eta \in H_{\pi_\varphi}$ with $\|\eta\| \leqslant \sqrt{\lambda} \leqslant 1$ such that for $a \in A$,

$$\psi(a) = \langle \pi_\varphi(a)\xi_\varphi, \eta\rangle = \|\eta\|\langle \pi_\varphi(a)\xi_\varphi, \eta/\|\eta\|\rangle = \|\eta\|\omega(a)$$

where $\omega \in G(A)$ and $|\omega| = \varphi$, so that $\pi_{|\omega|}$ is singular. Note that because $\psi \neq 0$, $\|\eta\| \neq 0$. This completes the proof that $\overline{G(A)}$ is contained in the right hand side of (ii).

Conversely, let $\psi = \lambda\varphi$ where $\varphi \in G(A)$ with $\pi_{|\varphi|}$ singular and $|\lambda| \leqslant 1$. As in the proof of Theorem 2.8 (ii) $\Rightarrow$ (iii) we obtain that $\lambda|\varphi| = \lim \varphi_\alpha$ where $\varphi_\alpha \in G(A)$. There exists a unitary element $u \in A + \mathbb{C}1$ such that $\varphi = |\varphi|(u\,\cdot)$. Hence $\lambda\varphi = \lim \varphi_\alpha(u\,\cdot) \in \overline{G(A)}$. This completes the proof that (ii) holds.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds. Then by Corollary 1.2, every singular element of $\widehat{A}$ is one dimensional. Let $\psi \in \overline{P(A)} \subseteq \overline{G(A)}$. By (ii), either $\psi \in G(A) \cap QS(A) = P(A)$, or $\psi = 0$ or $\psi = \lambda\varphi$ where $0 < |\lambda| \leqslant 1$, $\varphi \in G(A)$ and $\pi_{|\varphi|}$ is singular. In the latter case, $\varphi = \mu|\varphi|$ where $|\mu| = 1$ because $\pi_{|\varphi|}$ is one dimensional. So $\psi = \lambda\mu|\varphi|$. Since $\psi \geqslant 0$ and $\psi \neq 0$, we get $0 < \lambda\mu \leqslant 1$.

(i) $\Rightarrow$ (iii). Suppose that (i) holds. By Theorem 6 of [8] A is liminal, $\widehat{A}$ is Hausdorff and every singular element of $\widehat{A}$ is one dimensional. As in the proof of Theorem 2.8 (ii) $\Rightarrow$ (iv), we obtain a closed two sided ideal J of A with $\widehat{J}$ the set of Fell points of $\widehat{A}$, hence $\widehat{(A/J)}$ is the set of singular points of $\widehat{A}$, and A/J is abelian. Since $\widehat{J}$ is Hausdorff and each point of $\widehat{J}$ is Fell, J has continuous trace.

Let $\varphi \in \overline{P(J)}$ then (as in the proof of Theorem 2.8 (ii) $\Rightarrow$ (iv), $\varphi = \psi|_J$ for some $\psi \in \overline{P(A)}$. By (i), either $\psi \in P(A) \cup \{0\}$ (in which case $\varphi \in P(J) \cup \{0\}$) or $\psi = \lambda\rho$ where $0 < \lambda < 1$, $\rho \in P(A)$ and π_ρ is singular. In the latter case, $\pi_\rho \in \widehat{(A/J)}$ and so $\varphi = \lambda\, \rho|_J = 0$. Thus $\overline{P(J)} \subseteq P(J) \cup \{0\}$. By Lemma 2.6, J is a direct sum of homogeneous C^*-algebras.

(iii) $\Rightarrow$ (i). Suppose that (iii) holds. Suppose that $\varphi = \lim \varphi_\alpha$, where $\varphi_\alpha \in P(A)$ has GNS representation π_α. Arguing as in the proof of Theorem 2.2, we see that either $\varphi = 0$ or (by passing to a subnet) $\pi_\alpha \to \pi$ for some $\pi \in \widehat{A}$. In the latter case, $\varphi(\ker \pi) = \{0\}$ because $\widehat{A}$ is Hausdorff (note that if $a \in \ker \pi$ then $\|\pi_\alpha(a)\| \to \|\pi(a)\| = 0$).

Suppose that π is singular, i.e, $\pi \in \widehat{(A/J)}$. Then π is one dimensional because A/J is abelian. Thus φ is a multiple of a pure state which has a singular GNS representation.

Finally, suppose that $\pi \in \widehat{J}$. Since $\varphi_\alpha|_J \to \varphi|_J$, it follows from Theorems 2.4 and 2.5 that $\varphi|_J \in (G(J) \cup \{0\}) \cap QS(J) = P(J) \cup \{0\}$. If $\varphi|_J \in P(J)$ then $\varphi \in P(A)$. If $\varphi|_J = \{0\}$ then, since $J \nsubseteq \ker \pi$, $\varphi(A) = \varphi(J + \ker \pi) = \{0\}$ (by (iii), A is liminal and so $\ker \pi$ is a maximal closed two sided ideal of A). Thus $\varphi = 0$. ∎

REMARK 2.11. Note that condition (ii) of Theorem 2.10 is equivalent to

(ii′) $\overline{G(A)} = G(A) \cup \{0\} \cup \{\lambda\varphi : |\lambda| \leqslant 1,\ \varphi \in P(A) \text{ with } \pi_\varphi \text{ singular}\}$.

The reason for this is that both (ii) and (ii′) force any singular element of $\widehat{A}$ to be one dimensional (by Corollary 1.2).

COROLLARY 2.12. *Let A be a C^*-algebra. If $\overline{G(A)} = G(A) \cup \{0\}$ then A is the c_0-sum of homogeneous C^*-algebras, that is, $A = \bigoplus\limits_{n \geqslant 1} A_n$ where each A_n is either zero or a non-zero homogeneous C^*-algebra and either at least one A_n is non-unital or infinitely many A_n are non-zero.*

Proof. Suppose $\overline{G(A)} = G(A) \cup \{0\}$, then $\overline{P(A)} \subseteq P(A) \cup \{0\}$. So condition (i) of Theorem 2.10 holds and hence condition (ii) holds. These two conditions together with $\overline{G(A)} = G(A) \cup \{0\}$, tell us that $\widehat{A}$ has no singular points and so A is a continouous trace C^*-algebra. Therefore by Lemma 2.6 A is a direct sum of homogeneous C^*-algebras. If A is a finite direct sum of unital homogeneous C^*-algebras then, by Theorem 2.4, $\overline{G(A)} = G(A)$, a contradiction. Hence A has the required form. ∎

EXAMPLE 2.13. (i) Let A be the C^*-subalgebra of $C([0,1], M_2)$ consisting of those f for which $\lim\limits_{t\to 0} f(t) = 0$ and $\lim\limits_{t\to 1} f(t) = \operatorname{diag}(\rho(f), \rho(f))$ where $\rho(f) \in \mathbb{C}$.

Then ρ is a singular point of $\widehat{A}$, $\overline{P(A)} = P(A) \cup \{0\}$ (so the inclusion in Theorem 2.10(i) is strict) and $\overline{G(A)} = G(A) \cup \{\lambda\rho : |\lambda| \leqslant 1\}$.

(i) Let A be the C^*-subalgebra of $C([0,1], M_3)$ consisting of those f for which $\lim\limits_{t\to 1} f(t) = \operatorname{diag}(\rho(f), \rho(f), 0)$ where $\rho(f) \in \mathbb{C}$ Then ρ is a singular point of $\widehat{A}$, $\overline{P(A)} = P(A) \cup \{\lambda\rho : 0 \leqslant \lambda \leqslant 1\}$ (so that equality holds in Theorem 2.10(i)) and $\overline{G(A)} = G(A) \cup \{\lambda\rho : |\lambda| \leqslant 1\}$.

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