J. OPERATOR THEORY
63:2(2010), 403–415

NEARLY INVARIANT SUBSPACES FOR BACKWARDS SHIFTS ON VECTOR-VALUED HARDY SPACES

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Communicated by Florian-Horia Vasilescu

ABSTRACT. This paper characterizes the subspaces of a vector-valued Hardy space that are nearly invariant under the backward shift, providing a vectorial generalization of a result of Hitt. Compact perturbations of shifts that are pure isometries are studied, and the unitary operators that are compact perturbations of restricted shifts are described, extending a result of Clark. The classification of nearly invariant subspaces gives information on the simply shift-invariant subspaces of the vector-valued Hardy space of an annulus. Finally, as a generalization of a result of Aleman and Richter it is proved that any such subspace has index bounded by the dimension of the vector space in which the functions take their values.

KEYWORDS: *Nearly invariant subspace, invariant subspace, vector-valued Hardy space, shift operator, annulus, Toeplitz operator.*

MSC (2000): Primary 46E20, 47B38; Secondary 30D55, 47A15.

1. INTRODUCTION

The purpose of this paper is to study the shift operator S (multiplication by the independent variable) on certain Hardy spaces, consisting of vector-valued analytic functions on the unit disc $\mathbb{D}$ and the annulus $A = \{r_0 < |z| < 1\}$, where r_0 is a positive real number less than unity.

We shall consider questions to do with simply S-invariant subspaces of $H^2(A, \mathbb{C}^m)$ and nearly S^*-invariant subspaces of $H^2(\mathbb{D}, \mathbb{C}^m)$, which are defined below.

In the scalar case, nearly S^*-invariant subspaces of $H^2(\mathbb{D})$ were introduced by Hitt [10] as a tool for classifying the simply S-invariant subspaces of $H^2(A)$, and studied further by Sarason [17], who used them to study the kernels of Toeplitz operators. There are many other significant contributions to the theory of invariant subspaces of $H^2(A)$, including those of Sarason [16], Royden [15], Yakubovich [19] and Aleman–Richter [2].

The vectorial case has not been much considered, and presents difficulties of its own. For the classification of reducing subspaces of $L^2(A,\mathbb{C}^m)$ (i.e., invariant under S and S^*) as well as doubly-invariant subspaces of $H^2(A,\mathbb{C}^m)$ (i.e., invariant under S and S^{-1}) and singly generated invariant subspaces in $L^2(A,\mathbb{C}^m)$ or $H^2(A,\mathbb{C}^m)$ (i.e., invariant under S), see [5].

We now introduce some necessary definitions and notation, after which we shall be able to summarise the main contributions of the paper.

The boundary ∂A of A consists of two circles $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $r_0\mathbb{T}$. For $1 \leqslant p < \infty$, let $L^p(\partial A)$ be the complex Banach space of Lebesgue measurable functions f on ∂A that are pth-power integrable with respect to Lebesgue measure, the norm of f being defined by

$$\|f\|_p = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}t})|^p \mathrm{d}t + \frac{1}{2\pi}\int_0^{2\pi} |f(r_0\mathrm{e}^{\mathrm{i}t})|^p \mathrm{d}t\Big)^{1/p}.$$

The complex Banach space $L^\infty(\partial A)$ is the set of bounded Lebesgue measurable functions f on ∂A. Obviously, for $1 \leqslant p \leqslant \infty$, we have:

$$L^p(\partial A) = L^p(\mathbb{T}) \oplus L^p(r_0\mathbb{T}),$$

where $L^p(\mathbb{T})$ and $L^p(r_0\mathbb{T})$ are endowed with normalized Lebesgue measure. For $1 \leqslant p < \infty$ the Hardy space $H^p(\partial A)$ denotes the closure in $L^p(\partial A)$ of $R(A)$, the set of rational functions with poles off $\overline{A} = \{z \in \mathbb{C} : r_0 \leqslant |z| \leqslant 1\}$. These functions have a natural analytic extension to A, and so we shall also use the notation $H^p(A)$ when we wish to emphasise this.

We employ an analogous notation for subspaces of $L^2(\partial A,\mathbb{C}^m)$; in general we use lower case letters for scalar functions and capital letters for vector-valued functions.

Recall that an operator T on $\mathcal{H}$ belongs to the class $C_{\cdot 0}$ if for all $x \in \mathcal{H}$, $\lim_{n\to\infty} \|T^{*n}x\| = 0$.

We write $v \otimes w$ for the operator defined by $(v \otimes w)(x) = \langle x, w\rangle v$.

A closed subspace $\mathcal{M}$ is simply S-invariant if $S\mathcal{M} \subset \mathcal{M}$ and doubly S-invariant if, in addition, $S^{-1}\mathcal{M} \subset \mathcal{M}$.

A subspace $\mathcal{F}$ of $H^2(\mathbb{D},\mathbb{C}^m)$ is said to be nearly S^*-invariant if $\mathcal{F}$ is closed, and if every element $f \in \mathcal{F}$ with $f(0) = 0$ satisfies $S^*f \in \mathcal{F}$.

The paper is organized as follows. In Section 2 we discuss compact perturbations of shifts that are still pure isometries. In particular we provide a large class of explicit finite rank perturbations of pure isometries that remain $C_{\cdot 0}$ or even pure isometries. We apply these ideas in Section 3 to classify the unitary operators that are compact perturbations of restricted shifts, extending a result of Clark [7]. In Theorem 3.2 we discuss vectorial restricted shifts and their compact unitary perturbations. In Section 4, using the results of Section 2 concerning finite rank perturbations of shifts that remain $C_{\cdot 0}$ contractions, we obtain the characterization of nearly S^*-invariant subspaces of $H^2(\mathbb{D},\mathbb{C}^m)$, providing the vectorial generalization of Hitt's result in [10]. Finally, in Section 5, we discuss simply S-invariant subspaces $\mathcal{M}$ in $H^2(A,\mathbb{C}^m)$, with $m \geqslant 2$. Those subspaces are linked to nearly S^*-invariant subspaces $\mathcal{N}$ in $H^2(\mathbb{D},\mathbb{C}^m)$, but contrary to the scalar case, it is not clear whether they can be recovered using the subspaces $\mathcal{N}$. Nevertheless, we obtain a generalization of a result of Aleman and Richter in [2]: we prove that any S-invariant subspace $\mathcal{M}$ in $H^2(A,\mathbb{C}^m)$ satisfies $\dim(\mathcal{M} \ominus (S-\lambda\mathrm{Id})\mathcal{M}) \leqslant m$ for all $\lambda \in r_0\mathbb{D}$.

2. COMPACT PERTURBATION OF PURE ISOMETRIES

The aim of this section is to discuss compact perturbations of shifts that keep the nice properties of pure isometries. Recall that an isometry T on $\mathcal{H}$ is pure whenever T is completely non-unitary, or equivalently whenever $\bigcap\limits_{n\geqslant 0} T^n\mathcal{H} = \{0\}$. Moreover, a (pure) isometry is said to have finite multiplicity if the dimension of $\mathcal{H} \ominus T(\mathcal{H})$ is finite.

We shall see how useful this is in the description of nearly S^*-invariant subspaces.

PROPOSITION 2.1. *Let $V \in \mathcal{L}(\mathcal{H})$ be a pure isometry, $(w_n)_{n\geqslant 1}$ an orthonormal sequence of vectors in $\mathcal{H}$ and $(\alpha_n)_{n\geqslant 1}$ a sequence in $\mathbb{D}\cup\{1\}$ tending to 1. Then it follows that:*

(i) *the operator $V + \sum\limits_{n\geqslant 1}(\alpha_n - 1)w_n \otimes V^*w_n$ is a completely non-unitary contraction;*

(ii) *if V is a pure isometry of finite multiplicity, then for all $N \geqslant 1$, $V + \sum\limits_{n=1}^{N}(\alpha_n - 1)w_n \otimes V^*w_n$ is $C_{\cdot 0}$.*

Proof. Note that $T := V + \sum\limits_{n\geqslant 1}(\alpha_n - 1)w_n \otimes V^*w_n$ satisfies

$$\mathrm{Id} - T^*T = \sum_{n\geqslant 1}(1-|\alpha_n|^2)V^*(w_n)\otimes V^*(w_n) \geqslant 0, \tag{2.1}$$

and therefore T is contractive. Moreover, the closed subspace $\mathcal{H}_{\mathrm{u}}$ corresponding to the unitary part of T is included in the orthogonal complement of the range of $\mathrm{Id} - T^*T$. Indeed,

$$\begin{aligned}((\mathrm{Id}-T^*T)\mathcal{H})^{\perp} &= \{y \in \mathcal{H} : \langle(\mathrm{Id}-T^*T)(x), y\rangle = 0,\ x \in \mathcal{H}\}\\ &= \{y \in \mathcal{H} : \langle x, y\rangle = \langle T(x), T(y)\rangle,\ x \in \mathcal{H}\}.\end{aligned}$$

Now, using (2.1), it follows that $\mathcal{H}_{\mathrm{u}}$ is included in $\mathcal{M}$, where $\mathcal{M}$ is the closed subspace of $\mathcal{H}$ generated by $\{V^*(w_n) : n \geqslant 1\}$. Since $T_{|\mathcal{M}^{\perp}} = V_{|\mathcal{M}^{\perp}}$ and since V is completely non-unitary, it follows that $\mathcal{H}_{\mathrm{u}} = \{0\}$, proving that T is completely non-unitary.

For the second assertion, we have already proved that T is contractive and the range of $\mathrm{Id} - T^*T$ is generated by $\{V^*(w_n) : 1 \neq n \geqslant N\}$. Using Proposition 3.2 in [3], since V is of finite multiplicity and since the range $\mathcal{M}$ of $\mathrm{Id} - T^*T$ is of finite dimension, while $T = V$ on $\mathcal{M}^\perp$, T is also $C_{\cdot 0}$. ∎

It is also possible to characterize all the compact perturbations of isometries that are isometric.

PROPOSITION 2.2 (Corollary 2.5 in [18]). *Let $V \in \mathcal{L}(\mathcal{H})$ be an isometry and $K \in \mathcal{L}(\mathcal{H})$ a compact operator. Then $V + K$ is an isometry if and only if there is an orthonormal sequence of vectors $(e_n)_{n \geqslant 1} \in \mathcal{H}$ and $(\alpha_n)_{n \geqslant 1} \subset \mathbb{T}$ such that $\lim_{n\to\infty} \alpha_n = 1$ and*

$$K = \sum_{n \geqslant 1} (\alpha_n - 1) e_n \otimes V^*(e_n).$$

Nevertheless the description of pure isometries that remain pure isometries is much more complicated. Nakamura [11] solved the case of rank one perturbations. He proved the following result.

THEOREM 2.3 (Theorem 1 in [11]). *Let $V \in \mathcal{L}(\mathcal{H})$ be a pure isometry with a cyclic vector g, $\|g\| = 1$. Let $F = (\alpha - 1) g \otimes V^*(g)$ where $|\alpha| = 1$. Define w on $\mathbb{D}$ by*

$$(1 - w(z))^{-1} = \langle (\mathrm{Id} - zV^*)^{-1} g, g \rangle.$$

Then $V + F$ is a pure isometry if and only if

$$\log(1 - |w|^2) \in L^1(\mathbb{T}) \quad \textit{and} \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |w(\mathrm{e}^{\mathrm{i}t})|^2}{|1 - \bar{\alpha} w(\mathrm{e}^{\mathrm{i}t})|^2} \, \mathrm{d}t = 1. \tag{2.2}$$

By iterating Theorem 2.3 we obtain conditions for finite rank perturbations to be pure, but in practice it is difficult to satisfy the cyclicity condition. Nevertheless there is a less direct way to obtain finite rank perturbations that are pure isometries.

COROLLARY 2.4. *Let $V \in \mathcal{L}(\mathcal{H})$ be a pure isometry with a cyclic vector g_1, $\|g_1\| = 1$. For $n \geqslant 2$, let $g_2, \dots, g_n$ in $\mathcal{H}$ such that $(g_k)_{k=1}^n$ is an orthonormal sequence and $g_k \perp V(g_1)$ for all $k = 2, \dots, n$. Let $F = \sum_{k=1}^{n} (\alpha_k - 1) g_k \otimes V^*(g_k)$ where $|\alpha_k| = 1$. Define w on $\mathbb{D}$ by*

$$(1 - w(z))^{-1} = \langle (\mathrm{Id} - zV^*)^{-1} g_1, g_1 \rangle.$$

Then $V + F$ is a pure isometry if and only if (2.2) *holds.*

Proof. The operator $V' = V + \sum_{k=2}^{n} (\alpha_k - 1) g_k \otimes V^*(g_k)$ is an isometry and g_1 is cyclic for V' since $g_k \perp V(g_1)$ for all $k = 2, \dots, n$. Using Theorem 2.3, we get the desired conclusion. ∎

In the particular case where V is the shift on the Hardy space $H^2(\mathbb{D})$, since the cyclic vectors are the outer functions, the above corollary provides a large number of explicit finite rank perturbations that remain pure isometries.

See also Theorem 5.5 in [4], asserting that in "most cases", finite rank perturbation of pure isometries that are isometric remain pure.

3. RESTRICTED SHIFTS

It is a well-known consequence of Beurling's theorem that the S^*-invariant subspaces on $H^2(\mathbb{D})$ have the form $K_\theta = H^2 \ominus \theta H^2$, where θ is an inner function. The *restricted shift* S_θ is defined to be the operator $S_\theta = P_{K_\theta} S_{|K_\theta}$, and it plays a key role in the Sz.-Nagy–Foias model theory (see, for example, [12]).

Clark [7] classified the rank-one unitary perturbations of S_θ; in the simplest case, when $\theta(0) = 0$, they are parametrized by the formula

$$U_\alpha = S_\theta + \alpha 1 \otimes S^*\theta,$$

where $\alpha \in \mathbb{T}$; a recent account of this work can be found in Section 8.9 of [6]. The literature on this subject is large, and we refer also to the survey papers [8], [14] for related work.

The class of unitary operators that are compact perturbations of S_θ may be described, as follows.

PROPOSITION 3.1. *Let $V \in \mathcal{L}(K_\theta)$ be a unitary operator such that $V - S_\theta$ is compact. Then there is an orthonormal sequence of vectors $(e_n)_{n\geqslant 1} \in K_\theta$ and $(\alpha_n)_{n\geqslant 1} \subset \mathbb{T}$ such that $\lim\limits_{n\to\infty} \alpha_n = 1$ and*

$$V - U_\alpha = \sum_{n\geqslant 1} (\alpha_n - 1) e_n \otimes U_\alpha^*(e_n). \tag{3.1}$$

Equivalently,

$$V - S_\theta = \alpha 1 \otimes S^*\theta + \sum_{n\geqslant 1} (\alpha_n - 1) e_n \otimes (S_\theta^*(e_n) + \overline{\alpha}\langle e_n, 1\rangle S^*\theta). \tag{3.2}$$

Further, all operators V given by the above expression are unitary and compact perturbations of S_θ. Moreover, $V - S_\theta$ has finite rank if and only if the above sums have finitely many terms.

Proof. Proposition 2.2 shows that any isometry $V \in \mathcal{L}(K_\theta)$ which is itself a compact perturbation of U_α can be written as $V = UU_\alpha$, where $U \in \mathcal{L}(K_\theta)$ is unitary and $U - \mathrm{Id}_{K_\theta}$ is compact. Thus all such perturbations V are themselves unitary. Expression (3.1) now follows since $V - U_\alpha$ is also compact.

Next, since $U_\alpha^* = S_\theta^* + \overline{\alpha} S^*\theta \otimes 1$, we easily obtain the equivalence between (3.1) and (3.2). The final remark is obvious. ∎

It is possible to drop the assumption that $\theta(0) = 0$. To do this, define

$$\mathcal{M} = \mathbb{C}(S^*\theta) = \Big\{ x\frac{\theta(z) - \theta(0)}{z} : x \in \mathbb{C} \Big\},$$

and let $\mathcal{N} = K_\theta \ominus \mathcal{M}$. Then we claim that

$$S_\theta(xS^*\theta + w) = x(\theta\overline{\theta(0)} - 1)\theta(0) + Sw,$$

for $x \in \mathbb{C}$ and $w \in \mathcal{N}$. Indeed, the orthogonal projection of $\theta - \theta(0)$ onto K_θ is easily seen to be $\theta\overline{\theta(0)}\theta(0) - \theta(0)$, and $Sw \in K_\theta$ already, since if $h \in H^2(\mathbb{D})$ then w is clearly orthogonal to each of $\theta(z)(h - h(0))/z$, $(\theta - \theta(0))h(0)/z$ and $\theta(0)h(0)/z$, and hence to their sum, namely $\theta(z)h(z)/z$.

We may define a unitary perturbation of S_θ by a rank-one operator, by the formula

$$U_\alpha(xS^*\theta + w) = \alpha x(\theta\overline{\theta(0)} - 1) + Sw,$$

where α is any complex number of modulus one. The extension of Proposition 3.1 to this case is straightforward, and will not be given separately.

Let us now consider the general case of the shift S on $H^2(\mathbb{D}, \mathbb{C}^m)$. Then, by the Beurling–Lax theorem, its S-invariant subspaces are given by the expression $\mathcal{M}_\Theta = \Theta H^2(\mathbb{D}, \mathbb{C}^r)$, where $0 \leqslant r \leqslant m$, and $\Theta \in H^\infty(\mathcal{L}(\mathbb{C}^r, \mathbb{C}^m))$ is *inner*, that is, $\Theta(\mathrm{e}^{\mathrm{i}t})$ is an isometry for almost all $t \in [0, 2\pi]$ (see, for example, Section 3.1 of [13]). By analogy with the scalar case, we write $K_\Theta = H^2(\mathbb{D}, \mathbb{C}^m) \ominus \Theta H^2(\mathbb{D}, \mathbb{C}^r)$, which is the general form of an S^*-invariant subspace of $H^2(\mathbb{D}, \mathbb{C}^m)$, and let S_Θ be the compression of S to K_Θ.

In order to prove a generalization of Proposition 3.1, we need to begin with a finite-rank perturbation of S_Θ that is unitary. For simplicity we keep the assumption that $\Theta(0) = 0$. We also consider only the non-degenerate case, $r = m$.

Let

$$\mathcal{M} = (S^*\Theta)\mathbb{C}^m = \Big\{ \frac{\Theta(z)}{z}\, x : x \in \mathbb{C}^m \Big\},$$

noting that $\mathcal{M}$ is an m-dimensional subspace of K_Θ, and let $\mathcal{N} = K_\Theta \ominus \mathcal{M}$. Then we claim that

$$S_\Theta(S^*\Theta x + w) = Sw,$$

for $x \in \mathbb{C}^r$ and $w \in \mathcal{N}$. This follows because the orthogonal projection of Θx onto K_Θ is 0, and $Sw \in K_\Theta$ already, since if $H \in H^2(\mathbb{D}, \mathbb{C}^m)$ then w is clearly orthogonal to each of $\Theta(z)(H(z) - H(0))/z$ and $\Theta(z)H(0)/z$, and hence to their sum, namely $\Theta(z)H(z)/z$.

We may define a rank-m perturbation of S_Θ by

$$U_A(S^*\Theta x + w) = Ax + Sw, \tag{3.3}$$

where $A \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)$ is any unitary matrix. We may use U_A to parametrize unitary perturbations of S_Θ, as follows.

THEOREM 3.2. *Let $\Theta \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^m))$ be an inner function with $\Theta(0) = 0$, let $K_\Theta = H^2(\mathbb{D}, \mathbb{C}^m) \ominus \Theta H^2(\mathbb{D}, \mathbb{C}^m)$, and let $S_\Theta = P_{K_\Theta} S_{|K_\Theta}$. An operator $V \in \mathcal{L}(K_\Theta)$ is a unitary operator with $V - S_\Theta$ compact, if and only if there is an orthonormal sequence of vectors $(e_n)_{n\geqslant 1} \in K_\Theta$ and $(\alpha_n)_{n\geqslant 1} \subset \mathbb{T}$ such that $\lim\limits_{n\to\infty} \alpha_n = 1$ and*

$$V - U_A = \sum_{n\geqslant 1} (\alpha_n - 1) e_n \otimes U_A^*(e_n), \tag{3.4}$$

where U_A is the operator defined in (3.3). *The operator $V - S_\theta$ has finite rank if and only if the sum in* (3.4) *has finitely many terms.*

Proof. We claim that U_A is unitary. It is an isometry, because it maps an orthogonal direct sum to an orthogonal direct sum, and it is isometric on each summand; it is surjective since any function F in K_θ decomposes into $F(0) + zS^*F$, where $F(0) \in \mathbb{C}^m$ and $S^*F \in K_\theta$. Hence it is unitary. The result now follows from Proposition 2.2, because any compact (respectively, finite-rank) perturbation of S_Θ is a compact (respectively, finite-rank) perturbation of U_A. ∎

4. NEARLY INVARIANT SUBSPACES IN $H^2(\mathbb{D}, \mathbb{C}^m)$

Recall that for $F \in H^2(\mathbb{D}, \mathbb{C}^m)$, the adjoint of the shift S is given by:

$$S^*(F)(z) := \frac{F(z) - F(0)}{z}.$$

DEFINITION 4.1. A subspace $\mathcal{F}$ of $H^2(\mathbb{D}, \mathbb{C}^m)$ is said to be nearly S^*-invariant if $\mathcal{F}$ is closed, and if every element $F \in \mathcal{F}$ with $F(0) = 0$ satisfies $S^*F \in \mathcal{F}$.

A preliminary question concerning a nearly S^*-invariant subspace is whether there exists $F \in \mathcal{F}$ such that $F(0) \neq 0$. Indeed, if $\mathcal{F} \subset zH^2(\mathbb{D}, \mathbb{C}^m)$ then by definition $\mathcal{F}$ is nearly S^*-invariant if and only if $\mathcal{F}$ is S^*-invariant.

LEMMA 4.2. *Let $\mathcal{F}$ be a nearly S^*-invariant subspace with $\mathcal{F} \subset zH^2(\mathbb{D}, \mathbb{C}^m)$. Then $\mathcal{F} = \{0\}$.*

Proof. Let $F \in \mathcal{F}$ have Taylor series $F(z) = \sum\limits_{n\geqslant 0} a_n z^n$, with $(a_n) \in l^2(\mathbb{N}_0, \mathbb{C}^m)$, where $\mathbb{N}_0$ denotes the nonnegative integers. Since $F \in zH^2(\mathbb{D}, \mathbb{C}^m)$, we have $a_0 = 0$. As $\mathcal{F}$ is nearly S^*-invariant, the function G defined by $G(z) = F(z)/z = \sum\limits_{n\geqslant 0} a_{n+1} z^n$, belongs to $\mathcal{F}$. We conclude that $a_1 = 0$, and by induction we see that $a_n = 0$ for every $n \geqslant 0$, and hence $\mathcal{F} = \{0\}$. ∎

This lemma has an important consequence for what follows:

COROLLARY 4.3. *If $\mathcal{F}$ is a nearly S^*-invariant subspace of $H^2(\mathbb{D}, \mathbb{C}^m)$ and $\mathcal{F} \neq \{0\}$, then*

$$1 \leqslant \dim\,(\mathcal{F} \ominus (\mathcal{F} \cap zH^2(\mathbb{D}, \mathbb{C}^m))) \leqslant m.$$

Proof. By Lemma 4.2, $\mathcal{F}$ contains functions that do not vanish at 0. Thus the space $\mathcal{W} = \mathcal{F} \ominus (\mathcal{F} \cap (zH^2(\mathbb{D}, \mathbb{C}^m))$ is nontrivial. Let $r = \dim \mathcal{W}$. For $i \in \{1, \ldots, m\}$, let $F_i = P_{\mathcal{F}}(k_0 e_i)$, where $P_{\mathcal{F}}$ is the orthogonal projection onto $\mathcal{F}$ and k_0 is the reproducing kernel at 0. The functions F_i are elements of $\mathcal{F} \ominus (\mathcal{F} \cap zH^2(\mathbb{D}, \mathbb{C}^m))$, which generate this subspace. Indeed, if $G \in \mathcal{F}$ is orthogonal to all the F_i, then $G(0) = 0$, and so $G \in \mathcal{F} \cap zH^2(\mathbb{D}, \mathbb{C}^m))$. Thus $\dim \mathcal{F} \ominus (\mathcal{F} \cap zH^2(\mathbb{D}, \mathbb{C}^m)) \leqslant m$. ∎

In the remainder of this section, we are going to obtain a full description of nearly S^*-invariant subspaces. To do this we link them with S^*-invariant subspaces.

The following theorem is an adaptation to the vectorial case of Hitt's algorithm [10] and the operatorial version of Sarason [17].

THEOREM 4.4. *Let $\mathcal{F}$ be a nearly S^*-invariant subspace of $H^2(\mathbb{D}, \mathbb{C}^m)$ and let $(W_1, \ldots, W_r)$ be an orthonormal basis of $\mathcal{W} := \mathcal{F} \ominus (\mathcal{F} \cap zH^2(\mathbb{D}, \mathbb{C}^m))$.*

Let F_0 be the $m \times r$ matrix whose columns are $W_1, \ldots, W_r$. Then there exists an isometric mapping

$$J : \mathcal{F} \longrightarrow \mathcal{F}' \quad \text{given by } F_0 G \longmapsto G, \tag{4.1}$$

where $\mathcal{F}' := \{G \in H^2(\mathbb{D}, \mathbb{C}^r) : \exists F \in \mathcal{F},\ F = F_0 G\}$. Moreover $\mathcal{F}'$ is S^*-invariant.

Proof. Let $P_{\mathcal{W}}$ denote the orthogonal projection of $\mathcal{F}$ onto $\mathcal{W}$. If $F \in \mathcal{F}$, then $P_{\mathcal{W}}(F)$ has the form $P_{\mathcal{W}}(F)(z) = a_{0,1} W_1(z) + \cdots + a_{0,r} W_r(z)$, and so

$$\forall z \in \mathbb{D}, \quad F(z) = P_{\mathcal{W}}(F)(z) + F_1(z) = F_0 \begin{pmatrix} a_{0,1} \\ \vdots \\ a_{0,r} \end{pmatrix} + F_1(z),$$

with $F_1 \in \mathcal{F} \cap \mathcal{W}^\perp$. Moreover, since the family $\{W_i\}_{i=1,\ldots,r}$ forms an orthonormal basis of $\mathcal{W}$, we obtain the following identity between norms:

$$\|F\|^2 = |a_{0,1}|^2 + \cdots + |a_{0,r}|^2 + \|F_1\|^2 = \|A_0\|^2 + \|F_1\|^2,$$

where $A_0 = (a_{0,1}, \ldots, a_{0,r})^{\mathrm{t}}$.

By the definition of a nearly S^*-invariant subspace, the function $G_1 := S^* F_1$ lies in $\mathcal{M}$.

Thus we have $F = F_0 a_0 + S G_1$ and $\|F\|^2 = \|A_0\|^2 + \|G_1\|^2$. Noticing that $SG_1 = F - F_0 a_0 = \Big(\mathrm{Id} - \sum\limits_{j=1}^{r} W_j \otimes W_j\Big)(F)$, we deduce that

$$G_1 = S^* \Big(\mathrm{Id} - \sum_{j=1}^{r} W_j \otimes W_j\Big)(F) = R_{F_0}(F),$$

where $R_{F_0} = S^* \Big(\mathrm{Id} - \sum\limits_{j=1}^{r} W_j \otimes W_j\Big)$. It is clear that $\|R_{F_0}\| \leqslant 1$.

We proceed with G_1 as with F. By iterating we obtain

$$F(z) = F_0(A_0 z + A_1 z + \cdots + A_k z^k) + S^{k+1} R_{F_0}^{k+1}(F), \tag{4.2}$$

with the following identity (using successive orthogonal projections):

$$\|F\|^2 = \sum_{j=0}^{k} \|A_j\|^2 + \|R_{F_0}^{k+1}(F)\|^2. \tag{4.3}$$

The adjoint of R_{F_0} is $S - \sum_{j=1}^{r} W_j \otimes S^* W_j$, which is of class $C_{\cdot 0}$, applying the second assertion of Proposition 2.1.

From (4.3), we have

$$\sum_{j=0}^{N} \|A_j\|^2 \leqslant \|F\|^2 \quad \text{for all } N \in \mathbb{N}.$$

Let $G \in H^2(\mathbb{D}, \mathbb{C}^r)$ be given by $G(z) = \sum_{k=0}^{\infty} A_k z^k$. From equation (4.2) we have:

$$F_0(z)G(z) - F_0(z)(A_0 + A_1 z + \cdots + A_n z^n) = S^{n+1} R_{F_0}^{n+1}(F)(z).$$

Since R_{F_0} is $C_{\cdot 0}$, we see that $S^n R_{F_0}^{n+1}(F) \longrightarrow 0$ in $H^2(\mathbb{D}, \mathbb{C}^m)$ and hence in $H^1(\mathbb{D}, \mathbb{C}^m)$. Thus $F_0 G = F$ in $H^1(\mathbb{D}, \mathbb{C}^m)$. By the uniqueness of the limit, since F is in $H^2(\mathbb{D}, \mathbb{C}^m)$, we deduce that $F_0 G = F$ in $H^2(\mathbb{D}, \mathbb{C}^m)$. We then have $F_0 G = F$ with $G \in H^2(\mathbb{D}, \mathbb{C}^r)$. On the other hand, the expression (4.3) implies that

$$\|F\|^2 = \sum_{k=0}^{\infty} \|A_k\|^2 = \|G\|^2.$$

The mapping J is an isometry by the norm identity given above, and so $\mathcal{F}'$ is a closed subspace of $H^2(\mathbb{D}, \mathbb{C}^r)$, being the isometric image of a closed subspace.

It remains to show that $\mathcal{F}'$ is S^*-invariant in $H^2(\mathbb{D}, \mathbb{C}^r)$.

Let $G \in \mathcal{F}'$. We know that there exists $F \in \mathcal{F}$ such that $F = F_0 G$. Then, by construction, $F(z) = F_0(z) A_0 + A_1(z)$ and $A_0 = G(0)$. We obtain:

$$\frac{1}{z}(G(z) - G(0)) = \frac{G_1(z)}{z} \quad \text{where } F_0 F_1 = G_1.$$

Now $F_1 \in \mathcal{F} \cap z H^2(\mathbb{D})$, and then $F_1(0) = 0$. By the definition of a nearly S^*-invariant subspace, $z \mapsto F_1(z)/z \in \mathcal{F}$. Since $G_1(z)/z = S^* G(z)$, we have $S^* G \in \mathcal{F}'$, which proves that $\mathcal{F}'$ is S^*-invariant. ∎

COROLLARY 4.5. *Let $\mathcal{F}$ be a nearly S^*-invariant subspace of $H^2(\mathbb{D}, \mathbb{C}^m)$ and let $(W_1, \ldots, W_r)$ be an orthonormal basis of $\mathcal{W} := \mathcal{F} \ominus (\mathcal{F} \cap z H^2(\mathbb{D}, \mathbb{C}^m))$.*

Let F_0 be the $m \times r$ matrix with columns $W_1, \ldots, W_r$. Then there exists an inner function $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^{r'}, \mathbb{C}^r))$, which is unique up to unitary equivalence and vanishes at zero, such that

$$\mathcal{F} = F_0(H^2(\mathbb{D}, \mathbb{C}^r) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^{r'})).$$

Proof. By the preceding theorem, we have $\mathcal{F} = F_0\mathcal{F}'$ with $\mathcal{F}'$ an S^*-invariant subspace. By the corollary of the Beurling–Lax theorem, there exists $r' \leqslant r$ and $\Phi = (\phi_{ij}) \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^{r'}, \mathbb{C}^r))$ inner, unique up to conjugation by a unitary matrix, such that

$$\mathcal{F}' = H^2(\mathbb{D}, \mathbb{C}^r) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^{r'}).$$

We claim that Φ vanishes at zero. Since for $i \in \{1, \dots, r\}$, $V_i \in \mathcal{F}$, we have $V_i = Fe_i$ where $(e_i)_i$ is the standard basis of $\mathbb{C}^r$. As $e_i \in \mathcal{F}' = H^2(\mathbb{D}, \mathbb{C}^r) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^{r'})$, the e_i are orthogonal to $\Phi H^2(\mathbb{D}, \mathbb{C}^{r'})$, and then $\langle e_i, \Phi u_j\rangle_{H^2(\mathbb{D},\mathbb{C}^r)} = 0$ with (u_j) the canonical basis of $\mathbb{C}^{r'}$. It follows that $\phi_{ij}(0) = 0$; then Φ vanishes at zero, which completes the proof of the corollary. ∎

REMARK 4.6. Note that nearly S^*-invariant subspaces are linked to the problem of injectivity of Toeplitz operators. Indeed, suppose that Ψ is a function in $L^\infty(\mathbb{T}, \mathbb{C}^{m\times m})$ not identically equal to zero, such that the Toeplitz operator T_Ψ on $H^2(\mathbb{D}, \mathbb{C}^m)$ has a nontrivial kernel. Then $\ker T_\Psi$ is a nontrivial nearly S^*-invariant subspace. Therefore, according to our previous result, it equals $T_{F_0}\mathcal{M}'$ where $\mathcal{M}'$ is a S^*-invariant subspace, containing the constant functions, and on which F_0 acts isometrically. In the scalar case, Sarason first noticed this link, providing an alternative proof of Hayashi's result [9].

REMARK 4.7. One further application of nearly S^*-invariant subspaces in the vectorial case is to adjoint multipliers. Let B be a finite Blaschke product; then we can decompose $H^2(\mathbb{D})$ as the orthogonal direct sum

$$H^2(\mathbb{D}) = K_B \oplus BK_B \oplus B^2K_B \oplus \cdots,$$

where $K_B := H^2(\mathbb{D}) \ominus BH^2(\mathbb{D})$ has finite dimension, say m. This is naturally unitarily equivalent to the orthogonal direct sum

$$H^2(\mathbb{D}, \mathbb{C}^m) \simeq \mathbb{C}^m \oplus \mathbb{C}^m \oplus \mathbb{C}^m \oplus \cdots,$$

and moreover, the shift on $H^2(\mathbb{D}, \mathbb{C}^m)$ now corresponds to the operator T of multiplication by B on H^2.

Clearly, the adjoint operator T^* is given by $T^*f = P_{H^2}(f/B)$ for $f \in H^2(\mathbb{D})$. Thus we can classify the nearly T^*-invariant subspaces (i.e., those $\mathcal{M}$ with the property that if $f \in \mathcal{M}$ and $f/B \in H^2(\mathbb{D})$ then $f/B \in \mathcal{M}$), by expressing them in terms of the nearly S^*-invariant subspaces, which we have already classified.

Note that $f \mapsto f/B$ is an m-pseudomultiplier in the sense of Agler and Young [1]. Adjoint multipliers on a more general $H^2(\mathbb{D}, \mathbb{C}^m)$ may be treated by a similar approach.

5. SIMPLY S-INVARIANT SUBSPACES OF $H^2(A, \mathbb{C}^m)$

Nearly S^*-invariant subspaces were first introduced in the study of simply S-invariant subspaces of $H^2(A)$ by Hitt, and also considered further by Sarason.

In the vectorial case there is the same link between the two concepts. If we let $\mathcal{M}$ be a simply S-invariant subspace, then for all $n \in \mathbb{Z}$, the subspace $\{z \mapsto G(r_0/z) : G \in z^{-n}\mathcal{M} \cap H^2(\widehat{\mathbb{C}} \backslash r_0\overline{\mathbb{D}}, \mathbb{C}^m)\}$ is nearly S^*-invariant.

In the scalar case Aleman and Richter [2] derived the significant fact that $\dim(\mathcal{M} \ominus (S - \lambda \mathrm{Id})\mathcal{M}) = 1$ for each simply S-invariant subspace $\mathcal{M} \subset H^2(A)$ and $\lambda \in r_0\mathbb{D}$. It is possible to extend this result to the vectorial case, as in the following theorem.

THEOREM 5.1. *Let $\mathcal{M}$ be a nonzero simply S-invariant subspace of $H^2(A, \mathbb{C}^m)$. Then, for all $\lambda \in r_0\mathbb{D}$, there exists $r \in \{1, \dots, m\}$ such that*

$$\dim(\mathcal{M} \ominus (S - \lambda \mathrm{Id})\mathcal{M}) = r.$$

Proof. For $H \in \mathcal{M}^\perp$, $F \in \mathcal{M}$ and $\lambda \in \mathbb{C} \setminus \partial A$, define

$$T_H(F)(\lambda) = \Big\langle \frac{F}{z - \lambda}, H \Big\rangle = \frac{1}{2\mathrm{i}\pi} \int_{\partial A} \frac{\langle F(z), H(z)\rangle_{\mathbb{C}^m}}{z - \lambda}\, \mathrm{d}z.$$

Obviously, $T_H(F)$ is analytic in $\mathbb{C} \setminus \partial A$. Moreover, for $|\lambda| > 1$ and $z \in A$, since $1/(z-\lambda) = -1/\lambda \sum_{n \geqslant 0} z^n/\lambda^n$, it follows that $F/(z-\lambda) \in \mathcal{M}$ and thus $T_H(F)(\lambda) = 0 \quad$ for all $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Using Proposition 2 and Lemma 14 of [15],

$$T_H(F) \in H^r(A) \quad \text{for all } r \in (0,1), \tag{5.1}$$

since it is a Cauchy integral of a $L^1(\partial A)$ function. It also follows that $T_H(F) \in H^r(r_0\mathbb{D})$ for all $r \in (0,1)$. Another interesting property of $T_H(F)$ (cf. Proposition 2 in [15]) is the following:

$$T_H(F)^+(\xi) - T_H(F)^-(\xi) = \langle F(\xi), H(\xi)\rangle_{\mathbb{C}^m} \quad \text{a.e.} \tag{5.2}$$

where $T_H(F)^+$ and $T_H(F)^-$ denote the limit from outside and from inside on each component of ∂A.

Now take $(m+1)$ functions $H_1, \dots, H_{m+1}$ in $\mathcal{M}^\perp$ and $F_1, \dots, F_{m+1}$ in $\mathcal{M}$, and then consider the following analytic determinant

$$d(\lambda) = \det(T_{j,k}(\lambda)_{1 \leqslant j,k \leqslant m+1}),$$

where $T_{j,k} = T_{H_j}(F_k)$. Obviously, $d(\lambda) = 0$ for all $|\lambda| > 1$ since $T_{j,k}(\lambda) = 0$ for all $|\lambda| > 1$. Now, since $T^+_{j,k}(\xi) = 0$ a.e. on $\mathbb{T}$, by (5.2),

$$T^-_{j,k}(\xi) = -\langle F_k(\xi), H_j(\xi)\rangle_{\mathbb{C}^m} \quad \text{a.e. on } \mathbb{T}.$$

Note that, since the vectors $H_1(\xi), \dots, H_{m+1}(\xi)$ are linearly dependent in $\mathbb{C}^m$, it follows that $\det(\langle F_j(\xi), H_k(\xi)\rangle_{\mathbb{C}^m})_{1 \leqslant j,k \leqslant m+1} = 0$. Therefore $d^-(\lambda)$, the limit from inside of $d(\lambda)$, vanishes almost everywhere on $\mathbb{T}$. Using (5.1), the function d belongs at least to $\mathcal{N}(A)$, the Nevanlinna class of A, since it is a polynomial of functions in $H^r(A)$.

Using once more [15], p. 162 the function d is zero on A. Using similar arguments we show that the function d is also zero on $r_0\mathbb{D}$ for any choice of $H_1, \dots, H_{m+1}$ in $\mathcal{M}^\perp$ and $F_1, \dots, F_{m+1}$ in $\mathcal{M}$.

We obtain a contradiction if $\dim(\mathcal{M} \ominus (S - \lambda \mathrm{Id})\mathcal{M}) \geqslant m+1$ as follows. Choose $F_1 \perp (S - \lambda\mathrm{Id})\mathcal{M}$, $F_1 \neq 0$, and then, since $F_1/(z-\lambda) \notin \mathcal{M}$, take $H_1 \neq 0$ such that $T_{1,1}(\lambda) \neq 0$. Now take $F_2 \in \mathcal{M}$ such that $T_{1,2}(\lambda) = 0$. This choice is possible since $\dim(\mathcal{M} \ominus (S - \lambda\mathrm{Id})\mathcal{M}) \geqslant 2$. Then take $H_2 \in \mathcal{M}^\perp$ such that $T_{2,2}(\lambda) \neq 0$. The next step consists in choosing $F_3 \in \mathcal{M}$ such that $T_{1,3}(\lambda) = 0 = T_{2,3}(\lambda)$. This is possible since $\dim(\mathcal{M} \ominus (S - \lambda\mathrm{Id})\mathcal{M}) \geqslant 3$. Then take $H_3 \in \mathcal{M}^\perp$ such that $T_{3,3}(\lambda) \neq 0$. We continue this way until we obtain $T_{j,m+1}(\lambda) = 0$ for all $j < m+1$ and $T_{m+1,m+1}(\lambda) \neq 0$. Therefore $d(\lambda) = \prod\limits_{j=1}^{m+1} T_{j,j}(\lambda)$, since the matrix $(T_{j,k}(\lambda))_{1 \leqslant j,k \leqslant m+1}$ is upper triangular. We obtain this way $d(\lambda) \neq 0$, a contradiction. ∎

REMARK 5.2. Let $\mathcal{M}$ be a nonzero simply S-invariant subspace. Using Theorem 4.5 in [5], $D_S(\mathcal{M})$, the smallest closed subspace containing $\mathcal{M}$ and invariant for S and S^{-1} has the form $H^2(\partial A)W_1 \oplus^\perp \cdots \oplus^\perp H^2(\partial A)W_r$ for some $1 \leqslant r \leqslant m$ and $W_1, \dots, W_r$ in $H^\infty(\partial A, \mathbb{C}^m)$ such that for each k one has $\|W_k\|$ constant a.e. on each of $\mathbb{T}$ and $r_0\mathbb{T}$. In the particular case where $D_S(\mathcal{M})$ is singly generated, say $D_S(\mathcal{M}) = H^2(\partial A)W$, it follows that $\mathcal{M} = \mathcal{N}W$, where $\mathcal{N}$ is a simply S-invariant subspace in $H^2(\partial A)$. Explicit expressions of simply S-invariant subspace in $H^2(\partial A)$ can be found in [10] and [2].

Acknowledgements. The authors are grateful to the EPSRC for financial support.

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Received August 9, 2007; revised April 4, 2008.