

A CHARACTERIZATION OF SCATTERED C^* -ALGEBRAS AND ITS APPLICATION TO C^* -CROSSED PRODUCTS

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Communicated by Kenneth R. Davidson

ABSTRACT. It is well known that any scattered C^* -algebra is of type I and AF. We give conditions for C^* -algebras being of type I or AF to be scattered. In particular, it is shown that a C^* -algebra A is scattered if and only if A is a type I C^* -algebra satisfying that the center of A is scattered. As an application to a C^* -dynamical system (A, G, α) with a compact abelian group G , it is shown that the fixed point algebra A^α of A under the action α is scattered if and only if so is the C^* -crossed product $A \times_\alpha G$.

KEYWORDS: *Scattered C^* -algebra, AF C^* -algebra, C^* -crossed product.*

MSC (2000): 46L05.

1. INTRODUCTION

Recall that a topological space is called *scattered* (or *dispersed*) if every non-empty subset necessarily contains an isolated point. For a compact Hausdorff space Ω , it is shown in [12] that Ω is scattered if and only if every Radon measure on Ω is atomic. As a non-commutative generalization of a scattered compact Hausdorff space, the notion of a scattered C^* -algebra was introduced independently by Jensen [7] and Rothwell [13]. We say that a C^* -algebra A is *scattered* if every positive linear functional on A is the countable sum of pure positive linear functionals on A , or equivalently, A is of type I and the spectrum \hat{A} of A is a scattered topological space equipped with the Jacobson topology ([8], Corollary 3 and [13], Theorem 3.8). The reader is referred to [2], [6], [7], [8], [13] for other equivalent conditions on scattered C^* -algebras. On the other hand, any scattered C^* -algebra is AF ([10], Lemma 5.1). Thus we have reached the question of whether every AF C^* -algebra of type I is scattered. But there are AF C^* -algebras of type I which are not scattered. For example, the C^* -algebra $C(2^\omega)$ of all continuous functions on the Cantor set 2^ω is one of such C^* -algebras. In fact, $C(2^\omega)$ is AF because 2^ω is totally disconnected. But the Cantor set, which is identified with

the spectrum of $C(2^\omega)$, is not scattered as a topological space. The purpose of this paper is to study conditions for C^* -algebras being of type I or AF to be scattered.

In Section 2, we will give conditions for an AF C^* -algebra of type I to be scattered in terms of the center of the C^* -algebra considered. More precisely, we prove that a C^* -algebra A is scattered if and only if A is a type I C^* -algebra satisfying that the center $Z(A)$ of A is scattered, or equivalently, A is an AF C^* -algebra of type I satisfying that the center $Z(A)$ of A is AF and that $C(2^\omega)$ can not be embedded into $Z(A_1)$ as a C^* -algebra, where A_1 denotes the C^* -algebra obtained from A by adjunction of an identity.

In Section 3, we consider the problem of when the C^* -crossed product of a scattered C^* -algebra becomes scattered. Let (A, G, α) be a C^* -dynamical system. We suppose that G is a compact abelian group. In [3], Chu showed that if a C^* -algebra A is scattered, then so is also the C^* -crossed product $A \times_\alpha G$. We remark that conversely, the scatteredness of $A \times_\alpha G$ does not necessarily imply that of A in general (see the last paragraph in Section 3). In Section 3, we elaborate Chu's result above, that is, we will show that the fixed point algebra A^α of A under the action α is scattered if and only if so is the C^* -crossed product $A \times_\alpha G$. Thus scatteredness of the C^* -crossed product can be completely characterized by the fixed point algebra. If A is scattered, then it is obvious that A^α is scattered. Hence our result refines Chu's result that the C^* -crossed product of a scattered C^* -algebra by a compact abelian group is scattered, and our main theorem in Section 3 could be considered to be a crucial result for the question of when the C^* -crossed product of a C^* -algebra by a compact abelian group is scattered.

2. A CHARACTERIZATION OF SCATTERED C^* -ALGEBRAS

In this section, the term AF C^* -algebra has a slightly wider sense than the usual one which is assumed to be separable and to have identity. We will ease those restrictions. In fact, we say that a C^* -algebra A is an *approximately finite-dimensional* C^* -algebra (or simply AF) if A is a C^* -algebra in which any finite set of elements in A can be approximated arbitrarily closely in norm by elements of a finite-dimensional C^* -algebra. In this definition, any scattered C^* -algebra is AF ([10], Lemma 5.1).

For a C^* -algebra A , we always denote by $Z(A)$ the center of A , and by A_1 the C^* -algebra obtained from A by adjunction of an identity. Let \widehat{A} be the spectrum of A which is the set of all equivalence classes of nonzero irreducible representations equipped with the Jacobson topology. As is well known, \widehat{A} is locally compact, but it is not necessarily a Hausdorff space. If A is unital, \widehat{A} is compact. If I is a closed ideal of A , then \widehat{I} is regarded as an open subset of \widehat{A} . The reader is referred to [11] for background material on the spectra of C^* -algebras.

We will repeatedly use the result that if an abelian C^* -algebra is AF in our sense above, then its spectrum with the Jacobson topology is totally disconnected

([1], Proposition 3.1). Although such a result is given in [1] for separable C^* -algebras, the separability is not needed in order to obtain the result. In fact, the result follows from the fact that the projections in an abelian C^* -algebra correspond to the open-closed subsets in its spectrum and every AF C^* -algebra is generated by the projections.

For a locally compact space Ω , we denote by $C_0(\Omega)$ the C^* -algebra of all continuous functions on Ω vanishing at infinity. If Ω is compact, we denote by $C(\Omega)$ the C^* -algebra of all continuous functions on Ω . Throughout this section, we denote by 2^ω the Cantor set. Note that the Cantor set 2^ω is totally disconnected and is not scattered. Hence $C(2^\omega)$ is AF, but it is not scattered.

Here we recall that a topological space X is called *0-dimensional* if each point of X has a neighborhood base consisting of open-closed sets. Equivalently, X is 0-dimensional if and only if for each point $x \in X$ and each closed set F not containing x , there is an open and closed set containing x and not meeting F .

For a C^* -algebra A , we denote by A'' the enveloping von Neumann algebra of A , which is identified with the second dual of A as a Banach space. The following lemma is known for von Neumann algebras.

LEMMA 2.1. *Let A be a C^* -algebra and let e be a projection in the center $Z(A'')$ of A'' . Then we have*

$$Z(Ae) = Z(A)e.$$

Proof. We may assume that A is universally represented on the universal Hilbert space. Since it is obvious that $Z(Ae) \supset Z(A)e$, we will prove that $Z(Ae) \subset Z(A)e$. For this, we take any $ze \in Z(Ae)$ with $z \in A$. We have only to show that $z \in A'e$.

For any $a \in A$, we see that $(z(1 - e))(ae) = za(1 - e)e = 0$ and $(ae)(z(1 - e)) = aze(1 - e) = 0$, so that $(z(1 - e))(ae) = (ae)(z(1 - e))$. Hence $z(1 - e) \in (Ae)' = A'e$. (It is known that the last equality $(Ae)' = A'e$ holds if A is a $*$ -algebra on the Hilbert space.) Since we have

$$z = ze + z(1 - e) \in Z(Ae) + (Ae)' \subset (Ae)' = A'e,$$

we see that z has the form of $z = a'e$ with some $a' \in A'$, so that $z \in A'$. ■

A local characterization of a scattered C^* -algebra A is that each self-adjoint element $h \in A$ has a scattered spectrum $\text{Sp}_A(h)$ in A ([6]). We will use this characterization to prove the implication (v) \Rightarrow (i) in the following theorem. Furthermore, it follows from the local characterization of a scattered C^* -algebra or from Theorem 1 and Theorem 3 of [2] that every C^* -subalgebra of a scattered C^* -algebra is also scattered. This result will also be used repeatedly in the proof of the next theorem.

Now we are in a position to give the main theorem in this section.

THEOREM 2.2. *Let A be a C^* -algebra. Then the following conditions (i)–(v) are equivalent:*

(i) A is scattered.

(ii) A is an AF C^* -algebra of type I, the center $Z(A)$ of A is AF, and $C(2^\omega)$ can not be embedded into the center $Z(A_1)$ of A_1 as a C^* -algebra.

(iii) A is a type I C^* -algebra and the center $Z(A)$ of A is scattered.

(iv) Every abelian C^* -subalgebra B of A is AF and $C(2^\omega)$ can not be embedded into B_1 as a C^* -algebra.

(v) Every separable abelian C^* -subalgebra B of A is AF and $C(2^\omega)$ can not be embedded into B_1 as a C^* -algebra.

Proof. (i) \Rightarrow (ii) Since A is scattered, it is of type I and AF. Since every C^* -subalgebra of a scattered C^* -algebra is scattered, so is $Z(A)$. In particular, $Z(A)$ is AF. Since A is scattered, we easily see that A_1 is also scattered. In particular, $Z(A_1)$ is scattered. Since $C(2^\omega)$ is not scattered, it can not be embedded into $Z(A_1)$ because every C^* -subalgebra of a scattered C^* -algebra is scattered.

(ii) \Rightarrow (iii) Since $Z(A)$ is AF and $Z(A_1) = Z(A) + \mathbb{C} \cdot 1$, $Z(A_1)$ is also AF. Hence its spectrum $\widehat{Z(A_1)}$ is a compact Hausdorff and totally disconnected space ([1]). Here recall that a locally compact Hausdorff space is 0-dimensional as a topological space if and only if it is totally disconnected. Hence $\widehat{Z(A_1)}$ is 0-dimensional. On the other hand, $Z(A_1)$ is isomorphic to $C(\widehat{Z(A_1)})$. If there exists a surjective continuous map π from $\widehat{Z(A_1)}$ onto 2^ω , then π induces the injective homomorphism from $C(2^\omega)$ into $C(\widehat{Z(A_1)})$ in a usual way. But this contradicts the assumption. Thus there is no surjective continuous map from $\widehat{Z(A_1)}$ onto 2^ω , from which it follows that $\widehat{Z(A_1)}$ is scattered (see 2.Main Theorem of [12]). Thus we see that $\widehat{Z(A)}$ is also scattered.

(iii) \Rightarrow (i) Assume condition (iii). Then A_1 is of type I and $Z(A_1)$ is also scattered. If A_1 is scattered, then so is A . Hence without loss of generality we may assume that A is unital.

Since A is of type I, A has a series (so-called a composition series) of closed ideals $I_0 = \{0\} \subset I_1 \subset I_2 \subset \cdots \subset I_\alpha \subset \cdots \subset I_\beta = A$ such that $I_{\alpha+1}/I_\alpha$ has continuous trace for each α and $I_\alpha = \bigcup_{\gamma < \alpha} I_\gamma$ for each limit ordinal α (see 6.2.11 of [11]). We may assume that the composition series $\{I_\alpha\}$ consists of uncountably many ideals. To show (i), we have only to show that every I_α is scattered (see Proposition 2.5 or Proposition 2.6 of [7] or Proposition 3.6 of [13]). To do this, we use transfinite induction on α .

First we show that I_1 is scattered. Recall that $Z(A)$ is isomorphic to $C(\widehat{A})$ by the Dauns–Hofmann Theorem ([11], 4.4.8). Hence $C(\widehat{A})$ is scattered. Here we remark that \widehat{A} is not necessarily a Hausdorff space, that is, the spectrum of $C(\widehat{A})$ may not be \widehat{A} . Since I_1 has continuous trace, it is a liminary C^* -algebra with Hausdorff spectrum. Since $\widehat{I_1}$ is identified with an open subset of \widehat{A} and \widehat{A} is compact, $C_0(\widehat{I_1})$ is isomorphic to a closed ideal of $C(\widehat{A})$ (which is a scattered

C^* -algebra) in a canonical way. Hence $C_0(\widehat{I}_1)$ is scattered, which means that \widehat{I}_1 is scattered because it is a locally compact Hausdorff space. Since I_1 is of type I, it is scattered.

Next we will show that for any α , I_α is scattered. Assume that I_γ is scattered for all $\gamma < \alpha$. If α is not a limit ordinal, $I_\alpha/I_{\alpha-1}$ has continuous trace. Let p be the open projection in $Z(A'')$ satisfying that $I_{\alpha-1} = A''p \cap A$. Then $A/I_{\alpha-1} \cong A(1-p)$, and it follows from Lemma 2.1 that

$$Z(A/I_{\alpha-1}) \cong Z(A(1-p)) = Z(A)(1-p) \cong Z(A)/(Z(A) \cap I_{\alpha-1}).$$

Thus $Z(A/I_{\alpha-1})$ is scattered, since it is a quotient of a scattered C^* -algebra $Z(A)$ (see Proposition 2.4 of [7] or Proposition 3.6 of [13]). Since $I_\alpha/I_{\alpha-1}$ is a closed ideal with continuous trace of $A/I_{\alpha-1}$, with I_1 replaced by $I_\alpha/I_{\alpha-1}$ and with A by $A/I_{\alpha-1}$ in the discussion of the preceding paragraph it follows from the same discussion as in the paragraph that $I_\alpha/I_{\alpha-1}$ is scattered. Then scatteredness of both $I_{\alpha-1}$ and $I_\alpha/I_{\alpha-1}$ implies that I_α is scattered ([7], Proposition 2.4). If α is a limit ordinal, I_α is the norm closure of $\bigcup_{\gamma < \alpha} I_\gamma$. Since I_γ is scattered for all $\gamma < \alpha$, it follows from Proposition 2.5 of [7] that I_α is scattered.

(i) \Rightarrow (iv) Since A is scattered, every abelian C^* -subalgebra B of A is also scattered, hence AF. Since every C^* -subalgebra of a scattered C^* -algebra is scattered and $C(2^\omega)$ is not scattered, $C(2^\omega)$ can not be embedded into B_1 which is also scattered.

(iv) \Rightarrow (v) This is trivial.

(v) \Rightarrow (i) By adjunction of an identity to A if necessary, we may assume that A is unital. Take any self-adjoint element h from A . In order to show (i), it suffices to show that the spectrum $\text{Sp}_A(h)$ of h in A is scattered. Consider the C^* -algebra $C^*(h)$ generated by h and 1. Then $\widehat{C^*(h)}$ is homeomorphic to $\text{Sp}_A(h)$. Hence we have only to show that $\widehat{C^*(h)}$ is a scattered topological space.

Since $C^*(h)$ is an abelian C^* -subalgebra of A , by assumption it is AF. Hence $\widehat{C^*(h)}$ is a compact Hausdorff and totally disconnected space. Then $\widehat{C^*(h)}$ is 0-dimensional as a topological space. On the other hand, we see from the assumption that $C(2^\omega)$ can not be embedded into $C(\widehat{C^*(h)})$. Hence there is no surjective continuous map from $\widehat{C^*(h)}$ onto 2^ω . Thus it follows from 2.Main Theorem of [12] that $\widehat{C^*(h)}$ is scattered. ■

REMARK 2.3. We remark that every abelian C^* -subalgebra of an AF C^* -algebra is not necessarily AF. A well-known example is as follows (cf. III6, p. 95 in [4]). Consider $C(2^\omega)$ as an AF C^* -algebra. Then the abelian C^* -algebra $C([0, 1])$ consisting of all continuous functions on the closed interval $[0, 1]$ can be embedded into $C(2^\omega)$. Thus we regard $C([0, 1])$ as an abelian C^* -subalgebra of $C(2^\omega)$. But, $C([0, 1])$ is not AF because $[0, 1]$ is not totally disconnected.

3. AN APPLICATION TO C*-CROSSED PRODUCTS

Let (A, G, α) be a C*-dynamical system, that is, a triple (A, G, α) consisting of a C*-algebra A , a locally compact group G with left invariant Haar measure dt and a group homomorphism α from G into the automorphism group of A such that $G \ni t \rightarrow \alpha_t(x)$ is continuous for each x in A in the norm topology. Denote by $K(A, G)$ the linear space of all continuous functions from G into A with compact support and by $L^1(A, G)$ the completion of $K(A, G)$ by the L^1 -norm (see 7.6 of [11] for the Banach*-algebra structure of $L^1(A, G)$). Then the C*-crossed product $A \rtimes_{\alpha} G$ of A by G is the enveloping C*-algebra of $L^1(A, G)$. We denote by $Z(A \rtimes_{\alpha} G)$ the center of $A \rtimes_{\alpha} G$, and by $M(A \rtimes_{\alpha} G)$ the multiplier algebra of $A \rtimes_{\alpha} G$.

Suppose that G is abelian. Then the dual action $\widehat{\alpha}$ of the dual group \widehat{G} of G on $A \rtimes_{\alpha} G$ is defined by

$$\widehat{\alpha}_{\gamma}(x)(t) = \langle t, \gamma \rangle x(t) \quad \text{for } x \in L^1(A, G),$$

where $\langle t, \gamma \rangle$ denotes the value of γ at t .

Here we denote by λ the canonical unitary representation of G into $M(A \rtimes_{\alpha} G)$ satisfying $\alpha_t(\cdot) = \lambda_t \cdot \lambda_t^* = \text{Ad}\lambda_t(\cdot)$ for all $t \in G$, and we put

$$\lambda_f = \int_G \lambda_t f(t) dt \quad \text{for } f \in L^1(G).$$

LEMMA 3.1. *Let (A, G, α) be a C*-dynamical system. Suppose that G is a locally compact abelian group. Then $Z(A \rtimes_{\alpha} G)$ can be embedded into the C*-tensor product $A^{\alpha} \otimes C_0(\widehat{G})$ as a C*-algebra, where A^{α} is the fixed point algebra of A under α .*

Proof. Let \mathcal{B} be the C*-algebra generated by the set $\{z\lambda_t : z \in Z(A \rtimes_{\alpha} G), t \in G\}$ in $A \rtimes_{\alpha} G$. Since $Z(A \rtimes_{\alpha} G)$ is the center of $A \rtimes_{\alpha} G$, $Z(A \rtimes_{\alpha} G)$ is $\widehat{\alpha}$ -invariant. Since $\widehat{\alpha}_{\gamma}(\lambda_t) = \langle t, \gamma \rangle \lambda_t$, we see that \mathcal{B} is $\widehat{\alpha}$ -invariant. Furthermore, it is easy to verify that \mathcal{B} is a G -product (see 7.8.1 of [11] for the definition of a G -product). Hence it follows from 7.8.8 of [11] that $\mathcal{B} = B \rtimes_{\alpha} G$ with some α -invariant C*-subalgebra B of A . Since any element of $Z(A \rtimes_{\alpha} G)$ commutes with all elements of $M(A \rtimes_{\alpha} G)$ and since $\{\lambda_t : t \in G\} \subset M(A \rtimes_{\alpha} G)$, each element z of $Z(A \rtimes_{\alpha} G)$ commutes with any λ_t . Since G is abelian, we then see that $\lambda_t(z\lambda_s)\lambda_t^* = z\lambda_s$ for all $s, t \in G$. Hence every element of \mathcal{B} is a fixed point for $\text{Ad}\lambda$.

Take any $b \in B$. Since G is amenable, we identify $A \rtimes_{\alpha} G$ and $B \rtimes_{\alpha} G$ with the reduced C*-crossed products $A \rtimes_{\alpha, r} G$ and $B \rtimes_{\alpha, r} G$, respectively. Then $b\lambda_f$ with $f \in L^1(G)$ is an element of $\mathcal{B} = B \rtimes_{\alpha} G$. Since we have

$$\alpha_t(b)\lambda_f = \lambda_t b \lambda_t^* \lambda_f = \lambda_t (b \lambda_f) \lambda_t^* = \text{Ad}\lambda_t(b \lambda_f) = b \lambda_f$$

for all $t \in G$ and all $f \in L^1(G)$, we conclude that $\alpha_t(b) = b$. Thus we see that $B \subset A^{\alpha}$. Since G is amenable, we see that $B \rtimes_{\alpha} G \subset A^{\alpha} \rtimes_{\alpha} G$ (this inclusion is

not true in general unless G is amenable [9]). On the other hand, we easily see that $A^\alpha \times_\alpha G$ is isomorphic to $A^\alpha \otimes C^*(G) = A^\alpha \otimes C_0(\widehat{G})$, since α acts trivially on A^α . Since $Z(A \times_\alpha G) \subset \mathcal{B}$ by the definition of \mathcal{B} , we see that $Z(A \times_\alpha G) \subset A^\alpha \otimes C_0(\widehat{G})$. ■

As an application of Theorem 2.2, now we give the main theorem in this section.

THEOREM 3.2. *Let (A, G, α) be a C^* -dynamical system. Suppose that G is compact abelian. Then the following conditions are equivalent:*

- (i) *The fixed point algebra A^α of A is scattered.*
- (ii) *A^α is of type I and the center $Z(A^\alpha)$ of A^α is scattered.*
- (iii) *$A \times_\alpha G$ is scattered.*

Proof. (i) \Rightarrow (ii) This is trivial.

(ii) \Rightarrow (iii) Since G is compact and A^α is of type I, it follows from Theorem 3.2 of [5] that $A \times_\alpha G$ is of type I. Since A^α is scattered and \widehat{G} is discrete, $A^\alpha \otimes C_0(\widehat{G})$ is scattered ([3], Proposition 1). Since $Z(A \times_\alpha G)$ can be embedded into $A^\alpha \otimes C_0(\widehat{G})$ by Lemma 3.1, $Z(A \times_\alpha G)$ is scattered. Then Theorem 2.2 yields that $A \times_\alpha G$ is scattered.

(iii) \Rightarrow (i) It is well known that A^α is isomorphic to a hereditary C^* -subalgebra of $A \times_\alpha G$. This shows that A^α is scattered. ■

With the above notation, if A is scattered, so is A^α . Hence Theorem 3.2 shows that if A is scattered, then $A \times_\alpha G$ is also scattered. Thus Theorem 3.2 generalizes Chu's result. Remark that scatteredness of $A \times_\alpha G$ does not imply that of A . For example, consider the abelian C^* -algebra $C(\mathbb{T})$ of all continuous functions on the one-dimensional torus group \mathbb{T} . Since \mathbb{T} is not scattered, $C(\mathbb{T})$ is not scattered. Consider the group action τ of \mathbb{T} on $C(\mathbb{T})$ defined by $\tau_s(f)(t) = f(s^{-1}t)$ for $f \in C(\mathbb{T})$. Then the C^* -crossed product $C(\mathbb{T}) \times_\tau \mathbb{T}$ is scattered because we see that $C(\mathbb{T}) \times_\tau \mathbb{T}$ is isomorphic to the C^* -algebra $\mathcal{C}(L^2(\mathbb{T}))$ which consists of all compact operators on the Hilbert space $L^2(\mathbb{T})$.

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Received August 9, 2007.