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THE SZEMERÉDI PROPERTY IN ERGODIC W*-DYNAMICAL SYSTEMS

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ABSTRACT. We study weak mixing of all orders for asymptotically abelian weakly mixing state preserving C^*-dynamical systems, where the dynamics is given by the action of an abelian second countable locally compact group which contains a Følner sequence satisfying the Tempelman condition. For a smaller class of groups (which include $\mathbb{Z}^q$ and $\mathbb{R}^q$) this is then used to show that an asymptotically abelian ergodic W*-dynamical system either has the "Szemerédi property" or contains a nontrivial subsystem (a "compact factor") that does. A van der Corput lemma for Hilbert space valued functions on the group is one of our main technical tools.

KEYWORDS: *C^*- and* W*-*dynamical systems, weak mixing, compact dynamical systems, Szemerédi property.*

MSC (2000): 46L55.

1. INTRODUCTION

Consider a measure preserving dynamical system (X, Σ, ν, T), i.e. T is an invertible measure preserving transformation of a probability space (X, Σ, ν), namely a set X with σ-algebra Σ on which ν is a measure with $\nu(X) = 1$. Furstenberg [11], [12] proved that for any such system

$$\liminf_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} \nu(A \cap T^{-n}(A) \cap T^{-2n}(A) \cap \cdots \cap T^{-kn}(A)) > 0$$

if $\nu(A) > 0$. We will refer to this as a *Szemerédi property*; also see [14]. To prove this result requires (among other things) a structure theory in terms of so-called weakly mixing systems and compact systems.

Weak mixing is an important notion in ergodic theory, introduced by Koopman and von Neumann [15] in 1932 for actions of the group $\mathbb{R}$. Iterates of T above can be viewed as an action of the group $\mathbb{Z}$, and in this case the system above is called weakly mixing if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\nu(A \cap T^{-n}(B)) - \nu(A)\nu(B)| = 0$$

for all $A, B \in \Sigma$. Under this assumption Furstenberg proved that the system is in fact weakly mixing of all orders, namely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\nu(A_0 \cap T^{-m_1 n}(A_1) \cap \cdots \cap T^{-m_k n}(A_k)) - \nu(A_0)\nu(A_1)\cdots\nu(A_k)| = 0 \tag{1.1}$$

for all $A_0, \ldots, A_k \in \Sigma$, all $m_1, \ldots, m_k \in \mathbb{N}$ with $m_1 < m_2 < \cdots < m_k$, and all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, from which the Szemerédi property then follows easily for weakly mixing systems. On the other hand, the system is called compact if the orbit $\{f \circ T^n : n \in \mathbb{Z}\}$ of every $f \in L^2(\nu)$ is relatively compact in $L^2(\nu)$. Such systems can also be shown to have the Szemerédi property. As one might expect, these and related ideas have been studied for actions of more general groups; see for example [8], [2] (Section 4), [3] and [4].

The results of this paper form part of a programme to extend the structure theorems developed by Furstenberg and others to the operator algebraic setting. In this paper we first study weak mixing of all orders in a non-commutative C^*-algebraic setting where (X, Σ, ν, T) is replaced by a C^*-dynamical system (A, ω, τ) where ω is a state on the unital C^*-algebra A, and τ a group of $*$-automorphisms of A keeping ω invariant. This problem has also been studied by Niculescu, Ströh, and Zsidó [18] for actions of $\mathbb{Z}$, however we allow more general groups, namely abelian second countable locally compact groups which contains a Følner sequence satisfying certain conditions. The role of a Følner sequence is to replace the sequence of sets $\{1, \ldots, n\}$ appearing in the averages in the expressions above. We then proceed to the Szemerédi property for compact C^*-dynamical systems (Section 5). Finally, in Section 6, we use the results of the previous sections to study ergodic W*-dynamical systems (where A above is a σ-finite von Neumann algebra), however we only show that an asymptotic abelian ergodic system either has the Szemerédi property, or has a subsystem (called a factor) that has this property. This final result (Theorem 6.10) is proved for a smaller class of groups which however still contains $\mathbb{Z}^q$ and $\mathbb{R}^q$. Many of the intermediate results hold for more general groups or semigroups, as we will indicate. The asymptotic abelianness we refer to here is of a relatively weak form, namely "in the average" or "in density" as defined in Section 4. Asymptotic abelianness is needed to handle the weakly mixing case, while the compact case works without it, however in the latter we assume ω to be tracial while in the former we do not. So a certain level of commutativity is always present.

The largest part of the paper is devoted to the weakly mixing case. One of the technical tools we use in this case is a so-called van der Corput lemma which we discuss in Section 2. This type of lemma and related inequalities, inspired by the classical van der Corput difference theorem and van der Corput inequality, have been used by Bergelson et al. [1], [3], Furstenberg [13], Niculescu, Ströh, and Zsidó [18], and others, to study polynomial ergodic theorems, nonconventional ergodic averages, and noncommutative recurrence, for example. We extend the van der Corput lemma to more general groups, namely second countable amenable locally compact groups. The main result of this section is given by Theorem 2.6. After some preliminaries on weak mixing in Section 3, we devote Section 4 to showing how weak mixing implies weak mixing of all orders. The form of weak mixing of all orders we prove, involves replacing the multiplication with $m_1, \ldots, m_k$ in (1.1), by homomorphisms of the group over which we work, and this motivates why we incorporate such homomorphisms in a generalized definition of weak mixing in Section 3. The main result of Section 4 is Theorem 4.6.

2. A VAN DER CORPUT LEMMA

This section is devoted to proving a van der Corput lemma, stated in Theorem 2.6. Our proof of the van der Corput lemma will roughly follow that of [13] over the group $\mathbb{Z}$.

For (Y, μ) a measure space and $\mathfrak{H}$ a Hilbert space, consider a bounded function $f : \Lambda \to \mathfrak{H}$ with $\Lambda \subset Y$ measurable and $\mu(\Lambda) < \infty$, and $\langle f(\cdot), x\rangle$ measurable for every $x \in \mathfrak{H}$. Define $\int_\Lambda f \mathrm{d}\mu \in \mathfrak{H}$ by requiring

$$\left\langle \int_\Lambda f \mathrm{d}\mu, x \right\rangle := \int_\Lambda \langle f(y), x\rangle \mathrm{d}\mu(y) \tag{2.1}$$

for all $x \in \mathfrak{H}$. We will often use the notation $\int_\Lambda f(y)\mathrm{d}y = \int_\Lambda f \mathrm{d}\mu$, since there will be no ambiguity in the measure being used. Iterated integrals (when they exist) will be written as $\int_B \int_A f(y,z)\mathrm{d}y\mathrm{d}z$, which of course simply means $\int_B \left[\int_A f(y,z)\mathrm{d}y\right]\mathrm{d}z$, and similarly for triple integrals.

In a group G we will use the notations $Vg := \{vg : v \in V\}$, $VW := \{vw : v \in V, w \in W\}$, $V^{-1} := \{v^{-1} : v \in V\}$, etc. for any $V, W \subset G$ and $g \in G$, and we will use multiplicative notation even when working in an abelian group.

In Sections 2 to 4 of this paper G denotes an abelian second countable locally compact group with identity e, and regular Haar measure μ. Since G is abelian, it is amenable. In this section and the next the abelianness of G is in fact not crucial; the proofs go through even if G is not abelian but still amenable, and μ is right invariant (but see the remarks just before Theorem 2.6). Unfortunately in Section 4 this is not the case.

Since G is second countable and locally compact, it is σ-compact and hence its amenability (even for a nonabelian group) is equivalent to the existence of a Følner sequence (Λ_n) in G defined as follows:

DEFINITION 2.1. A *Følner sequence* in G is a sequence (Λ_n) of compact subsets of G such that $0 < \mu(\Lambda_n)$ for all n, and, for all $g \in G$,

$$\lim_{n\to\infty} \frac{\mu(\Lambda_n \Delta(\Lambda_n g))}{\mu(\Lambda_n)} = 0. \tag{2.1.1}$$

Refer to Theorem 4 in [9] and Theorems 1 and 2 in [10] for a very clear exposition of this. In fact, these papers show that we can choose a Følner sequence with stronger properties than those in Definition 2.1, but our definition will suffice for this paper. Furthermore, Theorem 3 in [9] shows that Definition 2.1 implies uniform convergence of (2.1.1) on compact sets, i.e.

$$\lim_{n\to\infty} \sup_{g\in K} \frac{\mu(\Lambda_n \Delta(\Lambda_n g))}{\mu(\Lambda_n)} = 0$$

for any non-empty compact $K \subset G$. We will have occasion to use this important fact later on. Throughout Sections 2 to 4, (Λ_n) will denote a Følner sequence in G. At the end of Section 4, we briefly consider simple examples of such sequences in $\mathbb{Z}^q$ and $\mathbb{R}^q$.

We assume second countability, since for second countable topological spaces X, Y, and their Borel σ-algebras S, T, the product σ-algebra obtained from S, T is the same as the Borel σ-algebra of the topological space $X \times Y$. This is needed in order to apply Fubini's theorem, which requires measurability in the product σ-algebra.

PROPOSITION 2.2. *Consider a bounded $f : G \to \mathfrak{H}$ with $\mathfrak{H}$ a Hilbert space, such that $\langle f(\cdot), x\rangle$ is Borel measurable for every $x \in \mathfrak{H}$. Then, for every n,*

$$\lim_{m\to\infty} \left\| \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m} f \mathrm{d}\mu - \frac{1}{\mu(\Lambda_m)} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_m} \int_{\Lambda_n} f(gh) \mathrm{d}h \mathrm{d}g \right\| = 0.$$

Proof. By (2.1) and Fubini's theorem

$$\int_{\Lambda_m} \int_{\Lambda_n} f(gh) \mathrm{d}h \mathrm{d}g = \int_{\Lambda_n} \int_{\Lambda_m} f(gh) \mathrm{d}g \mathrm{d}h$$

and in particular these iterated integrals exists. From this and the fact that μ is a right invariant measure, we have

$$\begin{aligned}
&\left\| \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m} f \mathrm{d}\mu - \frac{1}{\mu(\Lambda_m)} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_m} \int_{\Lambda_n} f(gh) \mathrm{d}h \mathrm{d}g \right\| \\
&= \left\| \frac{1}{\mu(\Lambda_n)} \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_n} \left[\int_{\Lambda_m} f(g) \mathrm{d}g - \int_{\Lambda_m} f(gh) \mathrm{d}g \right] \mathrm{d}h \right\|
\end{aligned}$$

$$= \Big\| \frac{1}{\mu(\Lambda_n)} \frac{1}{\mu(\Lambda_m)} \int\limits_{\Lambda_n} \Big[\int\limits_{\Lambda_m} f(g)\mathrm{d}g - \int\limits_{\Lambda_m h} f(g)\mathrm{d}g \Big] \mathrm{d}h \Big\|$$

$$= \Big\| \frac{1}{\mu(\Lambda_n)} \frac{1}{\mu(\Lambda_m)} \int\limits_{\Lambda_n} \Big[\int\limits_{\Lambda_m \setminus (\Lambda_m \cap (\Lambda_m h))} f(g)\mathrm{d}g - \int\limits_{(\Lambda_m h) \setminus (\Lambda_m \cap (\Lambda_m h))} f(g)\mathrm{d}g \Big] \mathrm{d}h \Big\|.$$

But if $b \in \mathbb{R}$ is an upper bound for $\|f(G)\|$, we have

$$\Big\| \int\limits_{\Lambda_m \setminus (\Lambda_m \cap (\Lambda_m h))} f(g)\mathrm{d}g - \int\limits_{(\Lambda_m h) \setminus (\Lambda_m \cap (\Lambda_m h))} f(g)\mathrm{d}g \Big\| \leqslant b \sup_{h \in \Lambda_n} \mu(\Lambda_m \Delta(\Lambda_m h))$$

therefore

$$\Big\| \frac{1}{\mu(\Lambda_m)} \int\limits_{\Lambda_m} f\mathrm{d}\mu - \frac{1}{\mu(\Lambda_m)} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_m} \int\limits_{\Lambda_n} f(gh)\mathrm{d}h\mathrm{d}g \Big\| \leqslant \frac{1}{\mu(\Lambda_m)} b \sup_{h \in \Lambda_n} \mu(\Lambda_m \Delta(\Lambda_m h)) \to 0$$

as $m \to \infty$. ∎

LEMMA 2.3. *Let $\mathfrak{H}$ be a Hilbert space, (Y, μ) a measure space, and $\Lambda \subset Y$ a measurable set with $\mu(\Lambda) < \infty$. Consider an $f : \Lambda \to \mathfrak{H}$ with $\|f(\cdot)\|$ measurable, and $\langle f(\cdot), x\rangle$ measurable for every $x \in \mathfrak{H}$, and with $\int_\Lambda \|f(y)\|\mathrm{d}y < \infty$ (which means $\int_\Lambda f\mathrm{d}\mu$ exists). Then*

$$\Big\| \int\limits_{\Lambda} f\mathrm{d}\mu \Big\|^2 \leqslant \mu(\Lambda) \int\limits_{\Lambda} \|f(y)\|^2 \mathrm{d}y.$$

Proof. From the definition of $\int_\Lambda f\mathrm{d}\mu$,

$$\Big\| \int\limits_{\Lambda} f\mathrm{d}\mu \Big\|^2 = \int\limits_{\Lambda} \Big[\int\limits_{\Lambda} \mathrm{Re}\langle f(y), f(z)\rangle \mathrm{d}z \Big] \mathrm{d}y \leqslant \frac{1}{2} \int\limits_{\Lambda} \Big[\int\limits_{\Lambda} (\|f(y)\|^2 + \|f(z)\|^2)\mathrm{d}z \Big] \mathrm{d}y. \quad ∎$$

PROPOSITION 2.4. *Consider the situation in Proposition 2.2. Assume furthermore that $F : G \times G \to \mathbb{C} : (g, h) \mapsto \langle f(g), f(h)\rangle$ is Borel measurable, and that $\Lambda_1, \Lambda_2 \subset G$ are Borel sets with $\mu(\Lambda_j) < \infty$. Then*

$$\Big\| \int\limits_{\Lambda_2} \int\limits_{\Lambda_1} f(gh)\mathrm{d}h\mathrm{d}g \Big\|^2 \leqslant \mu(\Lambda_2) \int\limits_{\Lambda_1} \int\limits_{\Lambda_1} \int\limits_{\Lambda_2} \langle f(gh_1), f(gh_2)\rangle \mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2$$

and in particular these integrals exist.

Proof. The double integral exists as in Proposition 2.2's proof. Let's now consider the triple integral. Since F is Borel measurable and G's product is continuous, $(g, h_1) \mapsto \langle f(gh_1), f(gh_2)\rangle$ is Borel measurable on $G \times G = G^2$ and hence measurable in the product σ-algebra on G^2. By Fubini's theorem we have

$$\int\limits_{\Lambda_1} \int\limits_{\Lambda_2} \langle f(gh_1), f(gh_2)\rangle \mathrm{d}g\mathrm{d}h_1 = \int\limits_{\Lambda_1 \times \Lambda_2} \langle f(gh_1), f(gh_2)\rangle \mathrm{d}(h_1, g)$$

and in particular the iterated integral exists. Furthermore, $G\times G^2 \to G^2 : (h_2, h_1, g) \mapsto (gh_1, gh_2)$ is continuous, so $G \times G^2 \to \mathbb{C} : (h_2, h_1, g) \mapsto \langle f(gh_1), f(gh_2)\rangle$ is measurable in the product σ-algebra of G and G^2. Hence by Fubini's theorem

$$\int\limits_{\Lambda_1}\int\limits_{\Lambda_1\times\Lambda_2} \langle f(gh_1), f(gh_2)\rangle \mathrm{d}(h_1, g)\mathrm{d}h_2 = \int\limits_{\Lambda_1\times\Lambda_1\times\Lambda_2} \langle f(gh_1), f(gh_2)\rangle \mathrm{d}(h_2, h_1, g)$$

and in particular, the triple integral exists, and we can do the three integrals in any order. By Lemma 2.3 it follows that

$$\begin{aligned}\left\|\int\limits_{\Lambda_2}\int\limits_{\Lambda_1} f(gh)\mathrm{d}h\mathrm{d}g\right\|^2 &\leqslant \mu(\Lambda_2)\int\limits_{\Lambda_2}\left\|\int\limits_{\Lambda_1} f(gh)\mathrm{d}h\right\|^2 \mathrm{d}g \\ &= \mu(\Lambda_2)\int\limits_{\Lambda_1}\int\limits_{\Lambda_1}\int\limits_{\Lambda_2} \langle f(gh_1), f(gh_2)\rangle \mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2\end{aligned}$$

and note in particular that the last equality proves that $g \mapsto \left\|\int\limits_{\Lambda_1} f(gh)\mathrm{d}h\right\|^2$ is measurable (and therefore its square root too), which means that Lemma 2.3 does indeed apply to this situation. ∎

PROPOSITION 2.5. *Consider the situation in Proposition* 2.2. *Assume that* $F : G \times G \to \mathbb{C} : (g, h) \mapsto \langle f(g), f(h)\rangle$ *is Borel measurable. Then* $\int\limits_{\Lambda}\langle f(g), f(gh)\rangle \mathrm{d}g$ *exists for all measurable* $\Lambda \subset G$ *with* $\mu(\Lambda) < \infty$*, and all* $h \in G$*. Assume that*

$$\gamma_h := \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}\langle f(g), f(gh)\rangle \mathrm{d}g$$

exists for all $h \in G$*. Then*

$$\lim_{m\to\infty}\frac{1}{\mu(\Lambda_m)}\int\limits_{\Lambda_n}\int\limits_{\Lambda_n}\int\limits_{\Lambda_m}\langle f(gh_1), f(gh_2)\rangle \mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2 = \int\limits_{\Lambda_n}\int\limits_{\Lambda_n}\gamma_{h_1^{-1}h_2}\mathrm{d}h_1\mathrm{d}h_2$$

for all n*, and in particular these integrals exist.*

Proof. The triple integral exists by Proposition 2.4. Let b be an upper bound for $(g, h) \mapsto |\langle f(g), f(h)\rangle|$, which exists since f is bounded. Fix any $m \in \mathbb{N}$, and set

$$A_n(h_1, h_2) := \frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}\langle f(gh_1), f(gh_2)\rangle \mathrm{d}g$$

for all $h_1, h_2 \in \Lambda_m$ and all n. Note that $A_n(h_1, h_2)$ exists and is a measurable function of (h_1, h_2) by Fubini's Theorem. Since F is Borel, and $G \to G^2 : g \mapsto (g, gh)$ is continuous, the map $G \to \mathbb{C} : g \mapsto \langle f(g), f(gh)\rangle$ is Borel for every

$h \in G$. Now,

$$
\begin{aligned}
|A_n(h_1,h_2) - \gamma_{h_1^{-1}h_2}| \leqslant \frac{1}{\mu(\Lambda_n)}\Big| \int_{\Lambda_n} \langle f(gh_1), f(gh_2)\rangle \mathrm{d}g - \int_{\Lambda_n} \langle f(g), f(gh_1^{-1}h_2)\rangle \mathrm{d}g \Big| \\
+ \Big| \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle f(g), f(gh_1^{-1}h_2)\rangle \mathrm{d}g - \gamma_{h_1^{-1}h_2}\Big|
\end{aligned}
$$

for all $h_1 \in G$ and $h_2 \in G$. But since μ is a right invariant measure

$$
\begin{aligned}
&\frac{1}{\mu(\Lambda_n)}\Big| \int_{\Lambda_n} \langle f(gh_1), f(gh_2)\rangle \mathrm{d}g - \int_{\Lambda_n} \langle f(g), f(gh_1^{-1}h_2)\rangle \mathrm{d}g \Big| \\
&\quad = \frac{1}{\mu(\Lambda_n)}\Big| \int_{\Lambda_n h_1} \langle f(g), f(gh_1^{-1}h_2)\rangle \mathrm{d}g - \int_{\Lambda_n} \langle f(g), f(gh_1^{-1}h_2)\rangle \mathrm{d}g \Big| \\
&\quad \leqslant \frac{1}{\mu(\Lambda_n)}\Big[\int_{(\Lambda_n h_1)\setminus\Lambda_n} |\langle f(g), f(gh_1^{-1}h_2)\rangle| \mathrm{d}g + \int_{\Lambda_n\setminus(\Lambda_n h_1)} |\langle f(g), f(gh_1^{-1}h_2)\rangle| \mathrm{d}g \Big] \\
&\quad \leqslant \frac{\mu(\Lambda_n \Delta(\Lambda_n h_1))}{\mu(\Lambda_n)} b
\end{aligned}
$$

for all $h_1 \in G$. Hence $\lim_{m\to\infty} A_n(h_1,h_2) = \gamma_{h_1^{-1}h_2}$.

Furthermore, $|A_n(h_1,h_2)| \leqslant \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} b\mathrm{d}g = b$, which implies that the sequence A_n is dominated by $B : \Lambda_m \times \Lambda_m \to \mathbb{R} : (h_1,h_2) \mapsto b$. Hence $\Lambda_m \times \Lambda_m \ni (h_1,h_2) \mapsto \gamma_{h_1^{-1}h_2}$ is in $L^1(\Lambda_m \times \Lambda_m, \mu \times \mu)$ and

$$
\lim_{n\to\infty} \int_{\Lambda_m\times\Lambda_m} A_n(h_1,h_2)\mathrm{d}(h_1,h_2) = \int_{\Lambda_m\times\Lambda_m} \gamma_{h_1^{-1}h_2}\mathrm{d}(h_1,h_2)
$$

by Lebesgue's dominated convergence theorem. The proposition now follows by Fubini's Theorem. ∎

Now we can finally state a van der Corput type lemma. It is worth pointing out again that the following result continues to hold even if G is not abelian, but still amenable, and μ is right invariant. The placing of the g in (2.1.1) is then of course important. Lemma 2.7 and Proposition 2.8 however require μ to be left rather than right invariant when G is not abelian, while they do not directly use property (2.1.1).

THEOREM 2.6. *Consider a bounded $f : G \to \mathfrak{H}$, with $\mathfrak{H}$ a Hilbert space, such that $\langle f(\cdot), x\rangle$ and $\langle f(\cdot), f(\cdot)\rangle : G \times G \to \mathbb{C}$ are Borel measurable (for all $x \in \mathfrak{H}$). Assume*

$$
\gamma_h := \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle f(g), f(gh)\rangle \mathrm{d}g
$$

exists for all $h \in G$. Also assume that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)^2} \int_{\Lambda_n} \int_{\Lambda_n} \gamma_{h_1^{-1}h_2} \mathrm{d}h_1 \mathrm{d}h_2 = 0 \tag{2.6.1}$$

(note that the integral exists by Proposition 2.5). Then

$$\lim_{m\to\infty} \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m} f \mathrm{d}\mu = 0.$$

Proof. By Proposition 2.2 and Proposition 2.4 we only have to show that for any $\varepsilon > 0$ there is an n and m_0 such that $|A_{n,m}| < \varepsilon$ for all $m > m_0$ where

$$A_{n,m} := \frac{1}{\mu(\Lambda_n)^2} \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_n} \int_{\Lambda_n} \int_{\Lambda_m} \langle f(gh_1), f(gh_2) \rangle \mathrm{d}g \mathrm{d}h_1 \mathrm{d}h_2.$$

But this follows from Proposition 2.5 and our assumptions, namely

$$\lim_{n\to\infty} \lim_{m\to\infty} A_{n,m} = \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)^2} \int_{\Lambda_n} \int_{\Lambda_n} \gamma_{h_1^{-1}h_2} \mathrm{d}h_1 \mathrm{d}h_2 = 0. \quad \blacksquare$$

We still need a few refinements regarding condition (2.6.1):

LEMMA 2.7. *Let $\Lambda \subset G$ be Borel and $\mu(\Lambda) < \infty$, and $S \subset G$ Borel such that $\Lambda^{-1}\Lambda \subset S$. For a Borel $f : G \to \mathbb{R}^+$ we then have*

$$\int_{\Lambda} \int_{\Lambda} f(h_1^{-1}h_2) \mathrm{d}h_1 \mathrm{d}h_2 \leqslant \mu(\Lambda) \int_S f \mathrm{d}\mu.$$

Proof. Let χ denote characteristic functions, and set $\varphi : \Lambda \times \Lambda \to G : (h_1, h_2) \mapsto h_1^{-1}h_2$. Then $f \circ \varphi$ is Borel on $\Lambda \times \Lambda$, and therefore measurable in the product σ-algebra on $\Lambda \times \Lambda$ obtained from Λ 's Borel σ-algebra, since φ is continuous. Let $Y \subset \Lambda^{-1}\Lambda$ be Borel in G. For $W \subset G \times G$, let $W_g := \{h : (g,h) \in W\}$. Then, since $\varphi^{-1}(Y)$ is Borel in $\Lambda \times \Lambda$ and hence Borel in $G \times G$, it follows that $\varphi^{-1}(Y)$ is in the product σ-algebra on $G \times G$, hence we can consider $(\mu \times \mu)(\varphi^{-1}(Y)) = \int_{\Lambda} \mu(\varphi^{-1}(Y)_g) \mathrm{d}g$. Now

$$\varphi^{-1}(Y) = \{(g, gh) : h \in Y, g \in \Lambda \cap (\Lambda h^{-1})\} \subset \{(g, gh) : h \in Y, g \in \Lambda\} =: V$$

but $V_g = gY$, therefore $\mu(\varphi^{-1}(Y)_g) \leqslant \mu(V_g) = \mu(gY) = \mu(Y)$, since μ is left invariant. Hence

$$\int_{\Lambda\times\Lambda} \chi_Y \circ \varphi \mathrm{d}(\mu \times \mu) = (\mu \times \mu)(\varphi^{-1}(Y)) \leqslant \mu(\Lambda)\mu(Y) = \mu(\Lambda) \int_S \chi_Y \mathrm{d}\mu.$$

There is an increasing sequence $f_n : S \to \mathbb{R}^+$ of simple functions converging pointwise to f. From the above we know that

$$\int_{\Lambda\times\Lambda} f_n \circ \varphi \mathrm{d}(\mu \times \mu) \leqslant \mu(\Lambda) \int_S f_n \mathrm{d}\mu$$

and by applying Lebesgue's monotone convergence first on the right and then of the left of this inequality, we obtain

$$\int_{\Lambda}\int_{\Lambda} f(h_1^{-1}h_2)\mathrm{d}h_1\mathrm{d}h_2 = \int_{\Lambda\times\Lambda} f\circ\varphi\mathrm{d}(\mu\times\mu) \leqslant \mu(\Lambda)\int_S f\mathrm{d}\mu$$

as required, where we have used Fubini's theorem, which holds in this case, since f is non-negative. ∎

PROPOSITION 2.8. *Consider a Borel measurable function $\gamma : G \to \mathbb{C}$ and let $\gamma_h := \gamma(h)$ for all $h \in G$. Also assume that*

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n^{-1}\Lambda_n}|\gamma_h|\mathrm{d}h = 0.$$

Then

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)^2}\int_{\Lambda_n}\int_{\Lambda_n}\gamma_{h_1^{-1}h_2}\mathrm{d}h_1\mathrm{d}h_2 = 0$$

if the iterated integral exists for all $n \geqslant n_0$ for some n_0.

Proof. Since Λ_n is compact, $\Lambda_n^{-1}\Lambda_n$ is Borel, and so, by Lemma 2.7,

$$\begin{aligned}\left|\frac{1}{\mu(\Lambda_n)^2}\int_{\Lambda_n}\int_{\Lambda_n}\gamma_{h_1^{-1}h_2}\mathrm{d}h_1\mathrm{d}h_2\right| &\leqslant \frac{1}{\mu(\Lambda_n)^2}\int_{\Lambda_n}\int_{\Lambda_n}|\gamma_{h_1^{-1}h_2}|\mathrm{d}h_1\mathrm{d}h_2 \\ &\leqslant \frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n^{-1}\Lambda_n}|\gamma_h|\mathrm{d}h. \quad ∎\end{aligned}$$

3. WEAK MIXING

In this section we define weak mixing, and study some of its characterizations using simple tools like density limits. This sets the stage for our study of weak mixing of all orders in the next section. As mentioned in Section 2, G need not be abelian in this section. In fact, even the properties of Følner sequences are not needed until Corollary 3.8 (this means that in Definition 3.2 below one could in principle work with an arbitrary sequence (Λ_n) of Borel sets in G with $0 < \mu(\Lambda_n) < \infty$). The material in this section is fairly standard, except that we work with the notion "M-weak mixing" (and "M -ergodicity"), which is important in Section 4.

DEFINITION 3.1. Let ω be a state on a unital $*$-algebra A, i.e. a linear functional on A such that $\omega(a^*a) \geqslant 0$ and $\omega(1) = 1$. Let τ_g be a $*$-automorphism of A for every $g \in G$ such that $\tau_g \circ \tau_h = \tau_{gh}$ for all $g, h \in G$, and such that τ_e is the identity on A and $G \to \mathbb{C} : g \mapsto \omega(a\tau_g(b))$ is Borel measurable for all $a, b \in A$. Then we will call (A, ω, τ, G) a $*$-*dynamical system*. If we further have that A is a C^*-algebra, and the state is preserved, i.e. $\omega \circ \tau_g = \omega$ for all g in G, then (A, ω, τ, G) is called a *C^*-dynamical system*.

Suppose that (A, ω, τ, G) is a $*$-dynamical system. Then so is $(\overline{A}, \overline{\omega}, \tau, G)$ where $\overline{\omega}(a) = \overline{\omega(a)}$, while $\overline{A}$ is the $*$-algebra A with the original scalar multiplication replaced by $\alpha \cdot a = \overline{\alpha} a$ for all $\alpha \in \mathbb{C}$ and $a \in A$. If $A \otimes \overline{A}$ denotes the algebraic tensor product of A with $\overline{A}$, then $(A \otimes \overline{A}, \omega \otimes \overline{\omega}, \tau \otimes \tau, G)$ is also a $*$-dynamical system, where $(\tau \otimes \tau)_g := \tau_g \otimes \tau_g$. When A is normed, we assign the same norm to $\overline{A}$, and on $A \otimes \overline{A}$ for our purposes any norm satisfying $\|a \otimes b\| \leqslant \|a\| \|b\|$ will do, for example the spatial C^*-norm when A is a C^*-algebra. However, even in the normed case, $A \otimes \overline{A}$ will denote the algebraic tensor product; we will not work with the completion in the norm.

For a group G, let $\mathrm{Hom}(G)$ denote the set of *all* group homomorphisms $G \to G$.

DEFINITION 3.2. Let (A, ω, τ, G) be a $*$-dynamical system and consider an $M \subset \mathrm{Hom}(G)$ such that $G \to \mathbb{C} : g \mapsto \omega(a\tau_{\varphi(g)}(b))$ is Borel measurable for all $\varphi \in M$.

(i) (A, ω, τ, G) is said to be *M-weakly mixing relative t*o (Λ_n), if, for all $a, b \in A$, and for all $\varphi \in M$,

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n} |\omega(a\tau_{\varphi(g)}(b)) - \omega(a)\omega(b)| \mathrm{d}g = 0.$$

(ii) (A, ω, τ, G) is said to be *M-ergodic relative t*o (Λ_n), if, for all $a, b \in A$, and for all $\varphi \in M$,

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n} \omega(a\tau_{\varphi(g)}(b)) \mathrm{d}g = \omega(a)\omega(b).$$

REMARKS ON DEFINITION 3.2. If (A, ω, τ, G) is a $*$-dynamical system, then so is $(A, \omega, \tau_{\varphi(\cdot)}, G)$ for any $\varphi \in M$. So essentially we are looking at a set of systems indexed by M, and one can therefore expect that the known properties of weakly mixing and ergodic systems will extend to the situation in Definition 3.2, as we will review in the rest of the section. In the case of $G = \mathbb{Z}$, $\Lambda_n = \{1, \ldots, n\}$ and with $M = \{\mathrm{id}_{\mathbb{Z}}\}$, Definition 3.2(i) corresponds to the usual definition of weak mixing for an action of the group $\mathbb{Z}$. Since all homomorphisms of $\mathbb{Z}$ are of the form $n \mapsto kn$ for some $k \in \mathbb{Z}$, one can then easily show for C^*-dynamical system that $\{\mathrm{id}_{\mathbb{Z}}\}$-weak mixing implies $\mathrm{Hom}(\mathbb{Z}) \setminus \{0\}$-weak mixing, where 0 is the homomorphism $n \mapsto 0$. Note more generally that if the homomorphism given by $\varphi_0(g) = e$ for all $g \in G$ is in M then the system cannot be expected to be M-weakly mixing, hence we would not want φ_0 to be in M. We mention this simply because φ_0 does appear in the theory to follow, but not as an element of M.

We now turn to a few technical tools which we will need in Section 4.

DEFINITION 3.3. (i) A set $R \subset G$ is said to have *density zero relative to* (Λ_n), and we write $D_{(\Lambda_n)}(R) = 0$, if and only if there exists a measurable set $S \subset G$, with $R \subset S$ such that

$$\lim_{n\to\infty} \frac{\mu(\Lambda_n \cap S)}{\mu(\Lambda_n)} = 0.$$

(ii) We say that $f : G \to L$, with L a real or complex normed space, has *density limit* $a \in L$ relative to (Λ_n), if and only if for each $\varepsilon > 0$, $D_{(\Lambda_n)}(S_\varepsilon) = 0$, where

$$S_\varepsilon := \{h \in G : \|f(h) - a\| \geqslant \varepsilon\},$$

and we write it as

$$D_{(\Lambda_n)}\text{-}\lim f = D_{(\Lambda_n)}\text{-}\lim_h f(h) = a.$$

Note that if R and S have density zero relative to (Λ_n) and $V \subset S$, then $R \cap S$, $R \cup S$ and V also have density zero relative to (Λ_n).

We now give a Koopman–von Neumann type lemma:

LEMMA 3.4. *Let $f : G \to [0,\infty)$ be bounded and measurable. Then the following are equivalent:*

(i) $D_{(\Lambda_n)}$-$\lim f = 0$.

(ii) $\displaystyle\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f \,\mathrm{d}\mu = 0.$

Proof. For every $\varepsilon > 0$, let $S_\varepsilon := \{h \in G : f(h) \geqslant \varepsilon\}$, which is a measurable set, since f is measurable.

(i) $\Rightarrow$ (ii) From (i) we have that each S_ε has density zero relative to (Λ_n). Given any $\varepsilon > 0$ and index α, consider the term

$$\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f \mathrm{d}\mu = \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap S_\varepsilon} f \,\mathrm{d}\mu + \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap S_\varepsilon^c} f \mathrm{d}\mu.$$

Since S_ε has density zero relative to (Λ_n)

$$0 \leqslant \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap S_\varepsilon} f \,\mathrm{d}\mu \leqslant \frac{\mu(\Lambda_n \cap S_\varepsilon)}{\mu(\Lambda_n)} \sup f(G) \to 0$$

as $n \to \infty$. Also,

$$0 \leqslant \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap S_\varepsilon^c} f \,\mathrm{d}\mu \leqslant \frac{\mu(\Lambda_n \cap S_\varepsilon^c)}{\mu(\Lambda_n)} \varepsilon \leqslant \varepsilon$$

hence

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f \mathrm{d}\mu = 0.$$

(ii) $\Rightarrow$ (i) Clearly $\varepsilon\chi_{S_\varepsilon} \leqslant f$. Also note that $D_{(\Lambda_n)}(S_\varepsilon) = 0$, since S_ε is measurable and

$$\varepsilon\frac{\mu(\Lambda_n \cap S_\varepsilon)}{\mu(\Lambda_n)} \leqslant \frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n} f\mathrm{d}\mu$$

which tends to zero as $n \to \infty$. ∎

COROLLARY 3.5. *Let $f : G \to \mathbb{R}$ be bounded and measurable. Then*

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}[f(h)]^2\,\mathrm{d}h = 0$$

if and only if

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}|f(h)|\,\mathrm{d}h = 0.$$

Proof. Given any $\varepsilon > 0$. Let

$$S_\varepsilon := \{h \in G : [f(h)]^2 \geqslant \varepsilon^2\} = \{h \in G : |f(h)| \geqslant \varepsilon\}.$$

Suppose that $\lim\limits_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}[f(h)]^2\,\mathrm{d}h = 0$, i.e. $D_{(\Lambda_n)}\text{-}\lim\limits_{h}[f(h)]^2 = 0$ by Lemma 3.4. By the definition of the density limit we have $D_{(\Lambda_n)}(S_\varepsilon) = 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $D_{(\Lambda_n)}\text{-}\lim|f| = 0$, and hence $\lim\limits_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}|f(h)|\,\mathrm{d}h = 0$ by Lemma 3.4. The converse follows similarly. ∎

As a result, the $|\cdot|$ in Definition 3.2(i) of M-weak mixing, can be replaced by $|\cdot|^2$, which is useful below and in Section 4.

LEMMA 3.6. *Let $f : G \to \mathbb{C}$ bounded and measurable. Let $\beta \in \mathbb{C}$. If*

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}f(h)\mathrm{d}h = \beta \quad \textit{and} \quad \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}|f(h)|^2\mathrm{d}h = |\beta|^2,$$

then

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}|f(h)-\beta|^2\mathrm{d}h = 0.$$

Proof. This follows immediately if we note that we have the following, as $n \to \infty$:

$$\begin{aligned}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}|f(h)-\beta|^2\mathrm{d}h &= \frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}(f(h)-\beta)(\overline{f(h)-\beta})\mathrm{d}h \\ &= \frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}(|f(h)|^2 - \beta\overline{f(h)} - \overline{\beta}f(h) + |\beta|^2)\mathrm{d}h \to 0.\end{aligned}$$

∎

Next we consider standard characterizations of weak mixing, that we will need.

PROPOSITION 3.7. *Let (A,ω,τ,G) be a $*$-dynamical system and $M\subset \mathrm{Hom}(G)$ such that $g\mapsto \omega(a\tau_{\varphi(g)}(b))$ is measurable for all $a,b\in A$ and all $\varphi\in M$. The following are equivalent:*

(i) (A,ω,τ,G) *is M-weakly mixing relative to (Λ_n).*

(ii) $(A\otimes\overline{A},\omega\otimes\overline{\omega},\tau\otimes\tau,G)$ *is M-weakly mixing relative to (Λ_n).*

(iii) $(A\otimes\overline{A},\omega\otimes\overline{\omega},\tau\otimes\tau,G)$ *is M-ergodic relative to (Λ_n).*

Proof. (ii) $\Rightarrow$ (iii) Follows immediately from Definition 3.2.

(iii) $\Rightarrow$ (i) Let $a,b\in A$ and $\varphi\in M$. We have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega(a\tau_{\varphi(g)}(b))\mathrm{d}g &= \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\otimes\overline{\omega}((a\otimes 1)(\tau\otimes\tau)_{\varphi(g)}(b\otimes 1))\mathrm{d}g\\ &= \omega\otimes\overline{\omega}(a\otimes 1)\omega\otimes\overline{\omega}(b\otimes 1)=\omega(a)\omega(b).\end{aligned}$$

Furthermore,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}|\omega(a\tau_{\varphi(g)}(b))|^2\mathrm{d}g &= \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\otimes\overline{\omega}((a\otimes a)(\tau\otimes\tau)_{\varphi(g)}(b\otimes b))\mathrm{d}g\\ &= \omega\otimes\overline{\omega}(a\otimes a)\omega\otimes\overline{\omega}(b\otimes b)=|\omega(a)\omega(b)|^2.\end{aligned}$$

Therefore by Lemma 3.6 we have that

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}|\omega(a\tau_{\varphi(g)}(b))-\omega(a)\omega(b)|^2\mathrm{d}g=0,$$

and it follows from Corollary 3.5 that (A,ω,τ,G) is M-weakly mixing relative to (Λ_n).

(i) $\Rightarrow$ (ii) Given any $\varphi\in M$ and $a,b\in A\otimes\overline{A}$, with $a=\sum_{j=1}^{n}a_j\otimes c_j$ and $b=\sum_{k=1}^{m}b_k\otimes d_k$ where $a_1,\dots,a_n,b_1,\dots,b_m\in A$ and $c_1,\dots,c_n,d_1,\dots,d_m\in\overline{A}$, then

$$\begin{aligned}&|\omega\otimes\overline{\omega}(a(\tau\otimes\tau)_{\varphi(g)}(b)-\omega\otimes\overline{\omega}(a)\omega\otimes\overline{\omega}(b))|\\ &\leqslant\sum_{j=1}^{n}\sum_{k=1}^{m}(|\omega(a_j\tau_{\varphi(g)}(b_k))\overline{\omega}(c_j\tau_{\varphi(g)}(d_k))-\omega(a_j\tau_{\varphi(g)}(b_k))\overline{\omega}(c_j)\overline{\omega}(d_k)|\\ &\quad+|\omega(a_j\tau_{\varphi(g)}(b_k))\overline{\omega}(c_j)\overline{\omega}(d_k)-\omega(a_j)\overline{\omega}(c_j)\omega(b_k)\overline{\omega}(d_k)|)\\ &\leqslant\sum_{j=1}^{n}\sum_{k=1}^{m}(\|a_j\|_\omega\|b_k\|_\omega|\omega(c_j\tau_{\varphi(g)}(d_k))-\omega(c_j)\omega(d_k)|\\ &\quad+|\overline{\omega}(c_j)\overline{\omega}(d_k)|\,|\omega(a_j\tau_{\varphi(g)}(b_k))-\omega(a_j)\omega(b_k)|)\end{aligned}$$

hence (ii) follows from Definition 3.2(i). ∎

We can now show that the definition of M-weak mixing relative to a Følner sequence, is independent of the Følner sequence being used:

COROLLARY 3.8. *If a state preserving $*$-dynamical system (A, ω, τ, G) is M-weakly mixing relative to some Følner sequence in G, then it is M-weakly mixing relative to every Følner sequence in G.*

Proof. By the mean ergodic theorem (see [7]) the M-ergodicity of a $*$-dynamical system is independent of the Følner sequence being used. Hence M-weak mixing is also independent of the Følner sequence by Proposition 3.7(i) and (iii). ∎

4. WEAK MIXING OF ALL ORDERS

For a state ω on a unital $*$-algebra A, we will denote any GNS representation of (A, ω) by (H, ι), which is a Hilbert space H and a linear mapping $\iota : A \to H$ such that $\langle \iota(a), \iota(b) \rangle = \omega(a^*b)$ for all $a, b \in A$ and with $\iota(A)$ dense in H. The corresponding cyclic vector will be denoted by $\Omega := \iota(1)$. Note that we use the convention where inner products are conjugate linear in the first slot. For elements of A we will use the notation $\prod_{j=1}^{k} a_j$ to denote the product $a_1 \cdots a_k$ in this specific order. Lastly we remind the reader that we are still using the group G as described in Section 2, and as was mentioned there the fact that G is abelian becomes essential in the current section.

DEFINITION 4.1. Let (A, ω, τ, G) be a $*$-dynamical system where A has a submultiplicative norm. Such a $*$-dynamical system is said to be *M-asymptotically abelian relative t*o (Λ_n), where $M \subset \mathrm{Hom}(G)$, if $G \to A : g \mapsto \tau_{\varphi(g)}(b)$ is continuous, and, for all $a, b \in A$, and for all $\varphi \in M$, where $[a, b] := ab - ba$,

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \|[a, \tau_{\varphi(g)}(b)]\| \mathrm{d}g = 0.$$

Under this assumption, we show that weak mixing implies weak mixing of all orders. Our approach is strongly influenced by that of [14] for the case of a measure theoretic dynamical system and the group $\mathbb{Z}$. The proof is by induction, two steps of which are given by the following:

PROPOSITION 4.2. *Given $M \subset \mathrm{Hom}(G)$, let (A, ω, τ, G) denote any C^*-dynamical system such that $G \to A : g \mapsto \tau_{\varphi(g)}(a)$ is continuous for all $a \in A$, and for all $\varphi \in M$. We are going to work with a collection of such systems, but with G, (Λ_n) and M fixed. Given $k \in \mathbb{N}$, let $\varphi_1, \ldots, \varphi_k$ denote elements of M and $a_0, \ldots, a_k$ elements of A. Set $\varphi_0(h) = e$ for all $h \in G$. Consider the following statements:*

$$1(k) \quad \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left| \omega\Big(\prod_{j=0}^{k} \tau_{\varphi_j(g)}(a_j) \Big) - \prod_{j=0}^{k} \omega(a_j) \right| \mathrm{d}g = 0;$$

$$2(k) \quad \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\Big(\prod_{j=0}^{k} \tau_{\varphi_j(g)}(a_j)\Big)\mathrm{d}g = \prod_{j=0}^{k} \omega(a_j);$$

$$3(k) \quad \lim_{n\to\infty} \Big\| \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \iota\Big(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\Big)\mathrm{d}g - \prod_{j=1}^{k} \omega(a_j)\Omega \Big\| = 0.$$

Then

(i) $1(k)$ *implies* $2(k)$.

(ii) *If* $3(k)$ *holds for all M-weakly mixing (A,ω,τ,G) which are M-asymptotically abelian relative to (Λ_n), all $a_1,\dots,a_k$ and all distinct $\varphi_1,\dots,\varphi_k$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j,l \in \{1,\dots,k\}$, then $1(k)$ also holds for all M-weakly mixing (A,ω,τ,G) which are M-asymptotically abelian relative to (Λ_n), all $a_0,\dots,a_k$ and all distinct $\varphi_1,\dots,\varphi_k$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j,l \in \{1,\dots,k\}$.*

Proof. (i) Trivial.

(ii) Let $\kappa := \prod_{j=1}^{k} \omega(a_j)$. By the assumption we have that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\Big(\prod_{j=0}^{k} \tau_{\varphi_j(g)}(a_j)\Big)\,\mathrm{d}g = \lim_{n\to\infty} \Big\langle \iota(a_0^*), \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \iota\Big(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\,\mathrm{d}g\Big)\Big\rangle$$
$$= \langle \iota(a_0^*), \kappa\Omega\rangle = \prod_{j=0}^{k} \omega(a_j).$$

First note that Proposition 3.7(i) and (ii) imply that the product system $(A \otimes \overline{A}, \omega\otimes\overline{\omega}, \tau\otimes\tau, G)$ is M-weakly mixing. Since $\|a\otimes b\| \leqslant \|a\|\|b\|$, it is also easy to verify that $(A \otimes \overline{A}, \omega\otimes\overline{\omega}, \tau\otimes\tau, G)$ is M-asymptotic abelian relative to (Λ_n). Hence by the equality above (which applies to all systems which are both M-weakly mixing and M-asymptotic abelian) we obtain

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \Big|\omega\Big(\prod_{j=0}^{k} \tau_{\varphi_j(g)}(a_j)\Big)\Big|^2 \mathrm{d}g = \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\otimes\overline{\omega}\Big(\prod_{j=0}^{k} (\tau\otimes\tau)_{\varphi_j(g)}(a_j\otimes a_j)\Big)\mathrm{d}g$$
$$= \prod_{j=0}^{k} \omega\otimes\overline{\omega}(a_j\otimes a_j) = \Big|\prod_{j=0}^{k} \omega(a_j)\Big|^2$$

proving $1(k)$ by Corollary 3.5 and Lemma 3.6. ∎

Note that the only property of M-weak mixing and M-asymptotic abelianness which is used in Proposition 4.2's proof, is that if a system is M-weakly mixing, then so is its product system, and similarly for M-asymptotic abelianness. Proposition 4.2 would still hold if we just considered M-weakly mixing systems for example, or systems with some abstract property, call it E, as long as the product system is again an E dynamical system. M-weak mixing and M-asymptotic abelianness will be used more directly in subsequent steps.

In order to complete the induction argument, we need $1(1)$, and that if $2(k-1)$ holds for all relevant systems, then the same is true for $3(k)$. The latter requires some more work, and we will need to specialize the M that we will allow. Firstly note that since the group G is abelian, we have for any homomorphisms φ_1 and φ_2 of G that the function $\varphi' : G \to G$ defined by the following is also a homomorphism of G

$$\varphi'(g) := \varphi_2(g)^{-1}\varphi_1(g). \tag{4.1}$$

DEFINITION 4.3. Let $M \subset \mathrm{Hom}(G)$. We call M *translational* if for all $\varphi_1, \varphi_2 \in M$ with $\varphi_1 \neq \varphi_2$, the homomorphism φ' defined by (4.1) is also in M.

PROPOSITION 4.4. *Let (A, ω, τ, G) be a C^*-dynamical system which is M-asymptotic abelian relative to (Λ_n), with M translational. Set $\varphi_0(g) = e$ for all $g \in G$. Assume that for some $k \in \mathbb{N}$*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega\Big(\prod_{j=0}^{k-1} \tau_{\varphi_j(g)}(a_j)\Big)\mathrm{d}g = \prod_{j=0}^{k-1} \omega(a_j) \tag{4.4.1}$$

for all $a_0, \dots, a_{k-1} \in A$ and $\varphi_1, \dots, \varphi_{k-1} \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \dots, k-1\}$, and in particular the existence of the limit is assumed. Now set

$$u_h := \iota\Big(\prod_{j=1}^{k} \tau_{\varphi_j(h)}(a_j)\Big) - \kappa\Omega$$

for all $h \in G$, where $\kappa := \prod_{j=1}^{k} \omega(a_j)$, for a given set of $a_j \in A$ and $\varphi_j \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \dots, k\}$. Then

$$\gamma_h := \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle u_g, u_{gh}\rangle \mathrm{d}g$$

exists (where $\langle\cdot,\cdot\rangle$ is taken in H), and, for all $h \in G$,

$$\gamma_h = \prod_{j=1}^{k} \omega(a_j^* \tau_{\varphi_j(h)}(a_j)) - |\kappa|^2.$$

Proof. We have

$$\langle u_g, u_{gh}\rangle = \omega\Big(\Big(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\Big)^* \Big(\prod_{j=1}^{k} \tau_{\varphi_j(gh)}(a_j)\Big)\Big)$$
$$- \overline{\kappa\omega\Big(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\Big)} - \bar{\kappa}\omega\Big(\prod_{j=1}^{k} \tau_{\varphi_j(gh)}(a_j)\Big) + |\kappa|^2.$$

We now consider each of the terms in the last expression separately:

Step 1. We see that

$$\omega\Big(\Big(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j)\Big)^*\Big(\prod_{j=1}^{k}\tau_{\varphi_j(gh)}(a_j)\Big)\Big) = \omega\Big(\prod_{j=-k}^{k}\tau_{\varphi_{|j|}(g)}(b_j)\Big)$$

where $b_j = \begin{cases} a_{|j|}^* & \text{if } -k \leqslant j \leqslant -1, \\ 1 & \text{if } j = 0, \\ \tau_{\varphi_j(h)}(a_j) & \text{if } 1 \leqslant j \leqslant k. \end{cases}$

For $T_{-k}, \ldots, T_k \in A$ with $\|T_j\| \leqslant c$, one has (see Lemma 7.4 in [18]) that

$$\Big\|\prod_{j=-k}^{k} T_j - T_0\Big(\prod_{j=1}^{k}(T_{-j}T_j)\Big)\Big\| \leqslant c^{2k-1}\sum_{r=1}^{k}\sum_{l=-(r-1)}^{r-1}\|[T_{-r}, T_l]\|$$

and applying this to $T_j = \tau_{\varphi_{|j|}(g)}(b_j)$ with $c := \max_{-k\leqslant j\leqslant k}\|b_j\|$, and keeping in mind that $\|\tau_g(a)\| = \|a\|$, it follows that

$$\Big\|\prod_{j=-k}^{k}\tau_{\varphi_{|j|}(g)}(b_j) - \prod_{j=1}^{k}\tau_{\varphi_j(g)}(b_{-j}b_j)\Big\| \leqslant c^{2k-1}\sum_{r=1}^{k}\sum_{l=-(r-1)}^{r-1}\|[b_{-r}, \tau_{\varphi_r(g)^{-1}\varphi_{|l|}(g)}(b_l)]\|.$$

Since M is translational, and (A, ω, τ, G) is asymptotically abelian relative to (Λ_n), it follows that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\Big(\prod_{j=-k}^{k}\tau_{\varphi_{|j|}(g)}(b_j)\Big)\mathrm{d}g &= \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\Big(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(b_{-j}b_j)\Big)\mathrm{d}g \\
&= \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\Big(\tau_{\varphi_1(g)}\Big(\prod_{j=1}^{k}\tau_{\varphi_1(g)^{-1}\varphi_j(g)}(b_{-j}b_j)\Big)\Big)\mathrm{d}g \\
&= \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\Big(\prod_{j=0}^{k-1}\tau_{\varphi'_j(g)}(b_{-(j+1)}b_{j+1})\Big)\mathrm{d}g \\
&= \prod_{j=0}^{k-1}\omega(b_{-(j+1)}b_{j+1}) = \prod_{j=1}^{k}\omega(a_j^*\tau_{\varphi_j(h)}(a_j))
\end{aligned}$$

by using (4.4.1), where $\varphi'_j(g) := \varphi_1(g)^{-1}\varphi_{j+1}(g)$ for all $g \in G$ and $j = 0, \ldots, k-1$, so $\varphi'_j \in M$ for $j = 1, \ldots, k-1$ since M is translational. Note that $\varphi'_j \neq \varphi'_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k-1\}$ and $\varphi'_0(g) = e$ for all $g \in G$, as required in our assumption. Hence

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\Big(\Big(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j)\Big)^*\Big(\prod_{j=1}^{k}\tau_{\varphi_j(gh)}(a_j)\Big)\Big)\mathrm{d}g = \prod_{j=1}^{k}\omega(a_j^*\tau_{\varphi_j(h)}(a_j)).$$

Step 2. It follows by assumption as in Step 1 that

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\overline{\omega\Big(\prod_{j=1}^{k}\tau_{\varphi_j(g)}(a_j)\Big)}\mathrm{d}g = \overline{\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\Big(\prod_{j=0}^{k-1}\tau_{\varphi_j'(g)}(a_{j+1})\Big)\mathrm{d}g}$$
$$= \overline{\prod_{j=0}^{k-1}\omega(a_{j+1})} = \overline{\kappa}.$$

Step 3. Lastly, using similar arguments as before,

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\Big(\prod_{j=1}^{k}\tau_{\varphi_j(gh)}(a_j)\Big)\mathrm{d}g = \lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\omega\Big(\prod_{j=0}^{k-1}\tau_{\varphi_j'(g)}\Big(\tau_{\varphi_{j+1}(h)}(a_{j+1})\Big)\Big)\mathrm{d}g$$
$$= \prod_{j=0}^{k-1}\omega(\tau_{\varphi_{j+1}(h)}(a_{j+1})) = \prod_{j=0}^{k-1}\omega(a_{j+1}) = \kappa.$$

Step 4. From Steps 1–3

$$\gamma_h = \prod_{j=1}^{k}\omega(a_j^*(\tau_{\varphi_j(h)}(a_j))) - |\kappa|^2$$

and in particular γ_h exists. ∎

Next we prove that weak mixing implies weak mixing of all orders. This is where our van der Corput lemma is finally applied, along with Propositions 4.2 and 4.4. We need one more definition:

DEFINITION 4.5. A Følner sequence (Λ_n) in G is said to satisfy the *Tempelman condition* if there is a real number $c > 0$ such that, for all $n \in \mathbb{N}$,

$$\mu(\Lambda_n^{-1}\Lambda_n) \leqslant c\mu(\Lambda_n).$$

See [17] for some discussion and further references related to this condition.

THEOREM 4.6. *Assume that there exists a Følner sequence (Λ_n) in G, satisfying the Tempelman condition, and such that $(\Lambda_n^{-1}\Lambda_n)$ is also a Følner sequence in G. Let $M \subset \mathrm{Hom}(G)$ be translational. Let (A, ω, τ, G) be an M-weakly mixing C^*-dynamical system which is M-asymptotically abelian relative to (Λ_n). Assume furthermore that $G \to A : g \mapsto \tau_{\varphi(g)}(a)$ is continuous in the norm topology on A for all $\varphi \in M$ and all $a \in A$. Then*

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n}\Big|\omega\Big(\prod_{j=0}^{k}\tau_{\varphi_j(g)}(a_j)\Big) - \prod_{j=0}^{k}\omega(a_j)\Big|\mathrm{d}g = 0$$

for any $a_j \in A$ and any $\varphi_1, \ldots, \varphi_k \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k\}$, and with $\varphi_0(g) = e$ for all $g \in G$.

Proof. We need to complete the induction argument started in Proposition 4.2, and we will continue using its notation and that of Proposition 4.4. Since $G \to A : g \mapsto \tau_{\varphi(g)}(a)$ is continuous, so is $G \to A : g \mapsto \prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)$ in the norm-topology on A. We then also have that $G \to H : g \mapsto \iota\Big(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\Big)$ is continuous, since $\|\iota(a)\| \leqslant \|a\|$. It follows that

$$G \times G \to \mathbb{R} : (g,h) \mapsto \Big\langle \iota\Big(\prod_{j=1}^{k} \tau_{\varphi_j(g)}(a_j)\Big), \iota\Big(\prod_{j=1}^{k} \tau_{\varphi_j(h)}(a_j)\Big)\Big\rangle$$

is continuous and hence $G \times G \to \mathbb{C} : (g,h) \mapsto \langle u_g, u_h\rangle$ is continuous and therefore Borel measurable. Note that $g \mapsto \langle u_g, x\rangle$ is also Borel measurable for all $x \in H$. Furthermore, $G \to H : g \mapsto u_g$ is bounded. (We need these properties, since we will be applying Theorem 2.6 to the function $g \mapsto u_g$.) Since $\mu(\Lambda_n^{-1}\Lambda_n) \leqslant c\mu(\Lambda_n)$, and we have M-weak mixing relative to $(\Lambda_n^{-1}\Lambda_n)$ by Corollary 3.8, it follows that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n^{-1}\Lambda_n} |\omega(a\tau_{\varphi(g)}(b)) - \omega(a)\omega(b)|\mathrm{d}g = 0 \tag{4.6.1}$$

for all $a, b \in A$ and $\varphi \in M$. By Proposition 4.4, assuming $2[k-1]$ for all M-weakly mixing C^*-dynamical systems, which are M-asymptotically abelian relative to (Λ_n) (for the given G and (Λ_n)), and of course for all $a_0, \ldots, a_{k-1}$ and all $\varphi_1, \ldots, \varphi_{k-1} \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k-1\}$, we have

$$\gamma_h := \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n} \langle u_g, u_{gh}\rangle \mathrm{d}g = \prod_{j=1}^{k} \omega(a_j^*(\tau_{\varphi_j(h)}(a_j))) - \prod_{j=1}^{k} |\omega(a_j)|^2$$

for any $a_1, \ldots, a_k$ and all $\varphi_1, \ldots, \varphi_k \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k\}$, for all $h \in G$. Using the following identity (also see Section 4 in [14])

$$\prod_{j=1}^{k} c_j - \prod_{j=1}^{k} d_j = \sum_{j=1}^{k} \Big(\prod_{l=1}^{j-1} c_l\Big)(c_j - d_j)\Big(\prod_{l=j+1}^{k} d_l\Big) \tag{4.6.2}$$

which holds in any algebra and is easily verified by induction, it follows that

$$\int\limits_{\Lambda_m^{-1}\Lambda_m} |\gamma_h|\mathrm{d}h \leqslant \sum_{j=1}^{k} A_j \prod_{l=j+1}^{k} |\omega(a_l)|^2 \int\limits_{\Lambda_m^{-1}\Lambda_m} |\omega(a_j^*(\tau_{\varphi_j(h)}(a_j))) - |\omega(a_j)|^2|\mathrm{d}h$$

where $A_j := \sup_{h\in G} \Big|\prod_{l=1}^{j-1} \omega(a_l^*(\tau_{\varphi_l(h)}(a_l))\Big| \leqslant \prod_{l=1}^{j-1} \|a_l\|^2$.

Note that $\int\limits_{\Lambda_m^{-1}\Lambda_m} |\gamma_h| \mathrm{d}h$ exists, since the integrand is continuous. Hence

$$\lim_{m\to\infty} \frac{1}{\mu(\Lambda_m)} \int\limits_{\Lambda_m^{-1}\Lambda_m} |\gamma_h| \mathrm{d}h = 0$$

by (4.6.1). From Proposition 2.8 and Theorem 2.6 we then have

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n} u_g \mathrm{d}g = 0$$

i.e., $3(k)$ holds for all M-weakly mixing C^*-dynamical systems, which are M-asymptotically abelian relative to (Λ_n), and all $a_1, \ldots, a_k$ and all $\varphi_1, \ldots, \varphi_k \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k\}$. But $1(1)$ holds for all $a_0, a_1 \in A$ and all $\varphi \in M$ for all M-weakly mixing C^*-dynamical systems, which are M-asymptotically abelian relative to (Λ_n), by Definition 3.2(i), completing the induction argument started in Proposition 4.2, and proving $1(k)$ for all $k \in \mathbb{N}$. ∎

We now briefly consider simple examples of Følner sequences with the required properties.

In the case where $G = \mathbb{Z}$ with the counting measure μ, and $\Lambda_n = \{-n, \ldots, n\}$ which is Følner in $\mathbb{Z}$, we have $\Lambda_n^{-1}\Lambda_n = \{-2n, \ldots, 2n\}$ which is also Følner, and $\mu(\Lambda_n) \leqslant \mu(\Lambda_n^{-1}\Lambda_n) \leqslant 2\mu(\Lambda_n)$ for $n \geqslant 1$. Similarly for $\mathbb{Z}^q$.

As another example, let Λ_m be the closed ball of radius m in $\mathbb{R}^q$ for any positive integer q. Note that (Λ_m) is a Følner sequence in $\mathbb{R}^q$ with $\Lambda_m^{-1}\Lambda_m = \Lambda_{2m}$, and $\mu(\Lambda_m^{-1}\Lambda_m) = 2^q \mu(\Lambda_m)$.

Concerning the assumption that M is translational, a simple example would be of the following type: Use the group $G = \mathbb{R}^q$. Let M be all $q \times q$ non-zero diagonal real matrices acting as linear operators on $\mathbb{R}^q$. (We exclude the zero matrix simply because this would make M-weak mixing impossible.) Then M is a translational set of homomorphisms of $\mathbb{R}^q$. The same is true if we drop the condition that the matrices be diagonal. Similarly if we work with $\mathbb{Z}^q$ instead of $\mathbb{R}^q$ and use matrices over the integers. These examples work simply because if $\varphi_1, \varphi_2 \in M$ and $\varphi_1 \neq \varphi_2$ then $-\varphi_2(g) + \varphi_1(g) = (\varphi_1 - \varphi_2)(g)$ while $\varphi_1 - \varphi_2 \in M$.

5. COMPACT SYSTEMS

In this section we prove a Szemerédi type property for compact C^*-systems as defined in Definition 5.1 below. Again we follow the basic structure of the proof given in [14], but we have to take into account certain subtleties and technical difficulties arising from working with a noncommutative C^*-algebra rather than with the abelian algebra $L^\infty(\nu)$ used in [14], and with more general groups and semigroups than $\mathbb{Z}$ and $\mathbb{N}$. First some notation and terminology.

A linear functional ω on a $*$-algebra A is called positive if $\omega(a^*a) \geqslant 0$ for all $a \in A$. This allows us to define a seminorm $\|\cdot\|_\omega$ on A by

$$\|a\|_\omega := \sqrt{\omega(a^*a)}$$

for all $a \in A$, as is easily verified using the Cauchy–Schwarz inequality for positive linear functionals.

A set V in a pseudo metric space (X, d) is said to be *ε-separated*, where $\varepsilon > 0$, if $d(x, y) \geqslant \varepsilon$ for all $x, y \in V$ with $x \neq y$. A set $B \subset X$ is said to be *totally bounded in* (X, d) if for every $\varepsilon > 0$ there exists a finite set $M_\varepsilon \subset X$, called a finite *ε-net*, such that for every $x \in B$ there is a $y \in M_\varepsilon$ with $d(x, y) < \varepsilon$. It is then not difficult to show that for any $\varepsilon > 0$ there exists a maximal set (in the sense of cardinality, or number of elements) $V \subset B$ that is ε-separated, and furthermore, if $B \neq \varnothing$, then V is finite with $|V| > 0$.

DEFINITION 5.1. Let ω be a positive linear functional on a $*$-algebra A, K a semigroup, and $\tau_g : A \to A$ a linear map for each $g \in K$ such that

$$\tau_g \circ \tau_h = \tau_{gh}$$

and

$$\|\tau_g(a)\|_\omega = \|a\|_\omega$$

for all $g, h \in K$ and $a \in A$. Assume that the *orbit*

$$B_a := \{\tau_g(a) : g \in K\}$$

is totally bounded in $(A, \|\cdot\|_\omega)$ for each $a \in A$. Then we call (A, ω, τ, K) a *compact system*. If furthermore A is a C^*-algebra and $\|\tau_g(a)\| \leqslant \|a\|$ in A's norm for all $a \in A$ and $g \in K$, then we refer to (A, ω, τ, K) as a *compact C^*-system*.

In particular, if the orbits of the $*$-dynamical systems and C^*-dynamical systems in Definition 3.1 are totally bounded in $(A, \|\cdot\|_\omega)$, then those systems will be called *compact*. Using the GNS construction, it is not too difficult to see that the L^2 definition of compactness mentioned in Section 1 is a special case of Definition 5.1.

In this paper weakly mixing systems and compact systems appear as part of the structure of ergodic systems, so concrete examples of weakly mixing systems and compact systems are not crucial for our goal. Nevertheless, it is interesting to look at an example of a compact C^*-dynamical system in which the C^*-algebra is noncommutative. To do this we need a few simple tools, which we now discuss.

First note that if a set in a C^*-algebra A is totally bounded in A (i.e. in terms of A's norm), then it is also totally bounded in $(A, \|\cdot\|_\omega)$ for any positive linear functional ω on A, since $\|\cdot\|_\omega \leqslant \|\omega\|^{1/2}\|\cdot\|$ (keep in mind that ω is bounded, since it is positive and A is a C^*-algebra). Hence, if we can prove that the orbits of a given C^*-dynamical system (A, ω, τ, K) are totally bounded in A, then it follows that the system is compact. Of course, this is then a stronger form of compactness, but Example 5.4 happens to possess this stronger property, and it turns out to be easier to prove this than to prove compactness directly in terms of $\|\cdot\|_\omega$, since A's norm is submultiplicative, which makes it easier to work with than $\|\cdot\|_\omega$.

In Lemma 5.2 and Proposition 5.3 below, we work with a C^*-algebra A, an arbitrary *set* K, and a $*$-homomorphism $\tau_g : A \to A$ for each $g \in K$. When we say that an "orbit" $(a_g) \equiv (a_g)_{g\in K}$ is *totally bounded* in a space, we mean that the set $\{a_g : g \in K\}$ is totally bounded in that space. For any subset $\mathfrak{V} \subset A$ we will denote the set of all polynomials over $\mathbb{C}$ generated by the elements of $\mathfrak{V}$ and their adjoints, by $p(\mathfrak{V})$, i.e. $p(\mathfrak{A})$ consists of all finite linear combinations of all finite products of elements of $\mathfrak{V} \cup \mathfrak{V}^*$ with $\mathfrak{V}^* := \{a^* : a \in \mathfrak{V}\}$. We will use the notation $XY := \{xy : x \in X, y \in Y\}$ whenever X and Y are sets for which this multiplication of their elements is defined.

LEMMA 5.2. *If $(\tau_g(a))$ is totally bounded in A for every a in some subset $\mathfrak{V}$ of A, then $(\tau_g(a))$ is totally bounded in A for every $a \in p(\mathfrak{V})$.*

Proof. Consider any $a, b \in A$ for which $(\tau_g(a))$ and $(\tau_g(b))$ are totally bounded in A, and any $\varepsilon > 0$. By the hypothesis there are finite sets $M, N \subset A$ such that for each $g \in K$ there is an $a_g \in M$ and a $b_g \in N$ such that $\|\tau_g(a) - a_g\| < \varepsilon$ and $\|\tau_g(b) - b_g\| < \varepsilon$. Clearly

$$\|\tau_g(a)\tau_g(b) - a_g b_g\| \leqslant \|\tau_g(a)\|\|\tau_g(b) - b_g\| + \|\tau_g(a) - a_g\|\|b_g\| \leqslant \varepsilon(\|\tau_g(a)\| + \|b_g\|)$$

but note that $\|\tau_g(a)\| \leqslant \|a\|$, since τ_g is a $*$-homomorphism and A is a C^*-algebra, while $\|b_g\| < \|\tau_g(b)\| + \varepsilon \leqslant \|b\| + \varepsilon$. Since MN is a finite subset of A, and $a_g b_g \in MN$, it follows that $(\tau_g(ab))$ is totally bounded in A. Similarly $(\tau_g(a^*))$ and $(\tau_g(\alpha a + \beta b))$ are totally bounded in A for any $\alpha, \beta \in \mathbb{C}$, and this is enough to prove the lemma. ∎

PROPOSITION 5.3. *Now assume that A is generated by a subset $\mathfrak{V} \subset A$ for which $\tau_g(\mathfrak{V}) \subset p(\mathfrak{V})$ for every $g \in K$. Also assume that $(\tau_g(a))$ is totally bounded in A for every $a \in \mathfrak{V}$. Then $(\tau_g(a))$ is totally bounded in A for every $a \in A$.*

Proof. Firstly it is easily shown that if Y is a dense subspace of a normed space X, $U_g : Y \to Y$ is linear with $\|U_g\| \leqslant 1$ for all $g \in K$, and $(U_g y)$ is totally bounded in X for every $y \in Y$ (or in Y for every $y \in Y$), then for the unique bounded linear extension $U_g : X \to X$ the "orbit" $(U_g x)$ is totally bounded in X for every $x \in X$. (We also used this fact when we discussed the GNS-construction above.)

Now simply set $X = A$, $Y = p(\mathfrak{V})$ and $U_g = \tau_g$, then by our assumptions and Lemma 5.2 all the requirements in the remark above are met. ∎

EXAMPLE 5.4. We consider a so-called rotation C^*-algebra, and use Proposition 5.3 to show that we obtain a compact C^*-dynamical system. As described in Chapter VI of [6], let $\mathfrak{H} := L^2(\mathbb{R}/\mathbb{Z})$ and define two unitary operators U and V on $\mathfrak{H}$ by

$$(Uf)(t) = f(t + \theta) \quad \text{and} \quad (Vf)(t) = e^{2\pi i t} f(t)$$

for $f \in \mathfrak{H}$, where $\theta \in \mathbb{R}$ (though the interesting case is $\theta \notin \mathbb{Q}$). These operators satisfy

$$UV = e^{2\pi i\theta} VU. \tag{5.4.1}$$

Let $\mathfrak{A}$ be the C^*-algebra generated by U and V. Note that $\mathfrak{A}$ is noncommutative because of (5.4.1). Then, as shown in Chapter VI of [6], there is a unique trace ω on $\mathfrak{A}$, i.e. a state with $\omega(ab) = \omega(ba)$. Define $\tau : \mathfrak{A} \to \mathfrak{A}$ by $\tau(a) = U^* a U$ for all $a \in \mathfrak{A}$, then τ is a $*$-isomorphism and therefore $\|\tau(a)\| = \|a\|$, since $\mathfrak{A}$ is a C^*-algebra. Also, since ω is a trace and U is unitary, $\|\tau(a)\|_\omega = \|a\|_\omega$ for all $a \in \mathfrak{A}$. Hence $(\mathfrak{A}, \omega, \tau, \mathbb{N})$ is a C^*-dynamical system, where by slight abuse of notation τ here denotes the function $n \mapsto \tau^n$ as well, to fit it into Definitions 3.1 and 5.1's notation.

We now show that $(\mathfrak{A}, \omega, \tau, \mathbb{N})$ is compact: It is trivial that $(\tau^n(U)) = (U)$ is totally bounded in $\mathfrak{A}$. Furthermore, $\tau^n(V) = (U^*)^n V U^n = e^{-2\pi i n\theta} V$ by (5.4.1). Since the unit circle is compact, it follows that $(\tau^n(V))$ is totally bounded in $\mathfrak{A}$. From Proposition 5.3 with $\mathfrak{V} = \{U, V\}$ we conclude that $(\tau^n(a))$ is totally bounded in $\mathfrak{A}$ for all $a \in \mathfrak{A}$. In particular $(\mathfrak{A}, \omega, \tau, \mathbb{N})$ is a compact C^*-system. Similarly $(\mathfrak{A}, \omega, \tau, \mathbb{Z})$ is a compact C^*-dynamical system.

Now we resume the general theory.

DEFINITION 5.5. Let K be a semigroup. We call a set $E \subset K$ *relatively dense in* K if there exist an $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in K$ such that, for all $g \in K$,

$$E \cap \{gg_1, \ldots, gg_r\} \neq \varnothing.$$

Strictly speaking one could call this *left* relative denseness, with the right hand case being defined similarly in terms of $g_j g$, but we will only work with Definition 5.5 in this paper. The usual definition of relative denseness of a subset E in $\mathbb{N}$ is in terms of "bounded gaps" (see [19] for example), and it is easy to check that in this special case the two definitions are equivalent.

PROPOSITION 5.6. *Let K be a semigroup, $(X, \|\cdot\|)$ a seminormed space, and $U_g : X \to X$ a linear map for each $g \in K$ such that $U_g U_h = U_{gh}$ and $\|U_g x\| \geqslant \|x\|$ for all $g, h \in K$ and $x \in X$. Suppose that $B_{x_0} := \{U_g x_0 : g \in K\}$ is totally bounded in $(X, \|\cdot\|)$ for some $x_0 \in X$. Then for each $\varepsilon > 0$, the following set is relatively dense in K:*

$$E := \{g \in K : \|U_g x_0 - x_0\| < \varepsilon\}.$$

Proof. Since B_{x_0} is totally bounded in $(X, \|\cdot\|)$, there is a maximal $V = \{U_{g_1} x_0, \ldots, U_{g_r} x_0\}$, with $U_{g_j} x_0 \neq U_{g_l} x_0$ whenever $j \neq l$, which is ε-separated. But $\|U_{g'gg_j} x_0 - U_{g'gg_l} x_0\| \geqslant \|U_{g_j} x_0 - U_{g_l} x_0\|$ for any $g, g' \in K$, hence $V_{g'g} := \{U_{g'gg_1} x_0, \ldots, U_{g'gg_r} x_0\}$ is ε-separated, with r elements. Since $V_{g'g} \subset B_{x_0}$, it is also maximally ε-separated in B_{x_0}. But $U_{g'} x_0 \in B_{x_0}$, therefore $\|U_{gg_j} x_0 - x_0\| \leqslant \|U_{g'gg_j} x_0 - U_{g'} x_0\| < \varepsilon$ for some $j \in \{1, \ldots, r\}$. The last inequality follows from the maximality of $V_{g'g}$ in the following way. Suppose this inequality wasn't true for some j, then $V_{g'g} \cup \{U_{g'}x_0\}$ would be ε-separated and contained in B_{x_0}, but with strictly greater cardinality than $V_{g'g}$, since it would have one more element, namely $U_{g'}x_0$ (which of course is not already in $V_{g'g}$ if the inequality doesn't hold for any j), contradicting maximality.

Hence, for each $g \in K$ there exists an $h \in \{gg_1, \dots, gg_r\}$ such that $\|U_h x_0 - x_0\| < \varepsilon$, i.e.

$$E \cap \{gg_1, \dots, gg_r\} \neq \varnothing$$

for all $g \in K$, and so E is relatively dense in K. ∎

COROLLARY 5.7. *Let (A, ω, τ, K) be a compact system and let $m_0, \dots, m_k \in \mathbb{N} \cup \{0\}$. For any $\varepsilon > 0$ and $a \in A$, the set*

$$E := \{g \in K : \|\tau_{g^{m_j}}(a) - a\|_\omega < \varepsilon \text{ for } j = 0, \dots, k\}$$

is then relatively dense in K, where we write $\tau_{g^0}(a) \equiv a$.

Proof. Without loss we can assume that none of the m_j 's are zero. Then the result follows from Proposition 5.6 with ε replaced by $\varepsilon / \max\{m_0, \dots, m_k\}$, since for every $j = 0, \dots, k$ we have

$$\begin{aligned}\|\tau_{g^{m_j}}(a) - a\|_\omega &\leqslant \|\tau_{g^{m_j}}(a) - \tau_{g^{m_j-1}}(a)\|_\omega + \|\tau_{g^{m_j-1}}(a) - \tau_{g^{m_j-2}}(a)\|_\omega + \cdots + \|\tau_g(a) - a\|_\omega \\ &= \|\tau_{g^{m_j-1}}[\tau_g(a) - a]\|_\omega + \|\tau_{g^{m_j-2}}[\tau_g(a) - a]\|_\omega + \cdots + \|\tau_g(a) - a\|_\omega \\ &= m_j \|\tau_g(a) - a\|_\omega < \varepsilon\end{aligned}$$

for all $g \in K$ for which $\|\tau_g(a) - a\|_\omega < \varepsilon / \max\{m_0, \dots, m_k\}$. ∎

A positive linear functional ω on a C^*-algebra A is bounded, and without loss we can assume that $\|\omega\| = 1$ (the case $\omega = 0$ being trivial), i.e. ω is a *state* on A. By the Cauchy–Schwarz inequality we have

$$|\omega(ab)| \leqslant \|a^*\|_\omega \|b\|_\omega \leqslant \sqrt{\|aa^*\|}\,\|b\|_\omega = \|a\|\,\|b\|_\omega .$$

A *trace* is defined to be a state ω on a C^*-algebra A such that $\omega(ab) = \omega(ba)$ for all $a, b \in A$. Note that from the previous inequality we then we have

$$|\omega(abc)| = |\omega(cab)| \leqslant \|a\|\,\|b\|_\omega \|c\|$$

for all $a, b, c \in A$. This fact is used in the proof of Lemma 5.8. The set of positive elements of A will be denoted by A^+.

LEMMA 5.8. *Let A be a C^*-algebra and ω a trace on A. Suppose that $b \in A^+$, $\|b\| \leqslant 1$ and $\omega(b) > 0$. Let $k \in \mathbb{N} \cup \{0\}$, then $\omega(b^{k+1}) > 0$ so we can choose $\varepsilon > 0$ such that $\varepsilon < \omega(b^{k+1})$. Consider $c_0, \dots, c_k \in A$ such that $\|c_j\| \leqslant 1$ and $\|c_j - b\|_\omega < \varepsilon/(k+1)$ for $j = 0, \dots, k$. Then*

$$\left|\omega\Big(\prod_{j=0}^{k} c_j\Big)\right| > \omega(b^{k+1}) - \varepsilon > 0.$$

Proof. We clearly have $\omega(b^{k+1}) > 0$. Furthermore,

$$\left|\omega\Big(\prod_{j=0}^{k} c_j\Big) - \omega(b^{k+1})\right| = \left|\omega\Big(\prod_{j=0}^{k} c_j - \prod_{j=0}^{k} b\Big)\right| \leqslant \sum_{j=0}^{k} \left(\Big\|\prod_{l=0}^{j-1} c_l\Big\| \|c_j - b\|_\omega \|b^{k-j}\|\right) < \varepsilon.$$

where we've used (4.6.2). ∎

COROLLARY 5.9. *Let (A, ω, τ, K) be a compact C^*-system with ω a trace. Suppose that $a \in A^+$, and $\omega(a) > 0$. Take any $m_0, \dots, m_k \in \mathbb{N} \cup \{0\}$ and any $\varepsilon > 0$ with $\varepsilon < \omega(a^{k+1})$. Then there exists a relatively dense set E in K such that, for all $g \in E$,*

$$\left|\omega\Big(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\Big)\right| > \omega(a^{k+1}) - \varepsilon > 0.$$

Proof. Since $\omega(a) > 0$, $\|a\| > 0$, so we can set $b := a/\|a\|$. For $c_j := \tau_{g^{m_j}}(b)$ we have $\|c_j\| \leqslant \|b\| = 1$, so from Lemma 5.8 it follows that

$$\left|\omega\Big(\prod_{j=0}^{k} \tau_{g^{m_j}}(b)\Big)\right| > \omega(b^{k+1}) - \frac{\varepsilon}{\|a\|^{k+1}}$$

for every $g \in K$ for which $\|\tau_{g^{m_j}}(b) - b\|_\omega < \varepsilon/\|a\|^{k+1}(k+1)$ for all $j = 0, \dots, k$. By Corollary 5.7 this set of g 's is relatively dense in K. ∎

So far in this section we have not used Følner sequences. However, for the remainder of this section we again make use of such sequences, so let G and (Λ_n) be as in Section 2; in particular G is abelian. We can make a simple refinement without complicating the proofs, namely in the rest of this section let K be a Borel set in G which forms a (necessarily abelian) semigroup and contains each Λ_n. We then say that (Λ_n) is a Følner sequence in K. (A trivial example is $G = \mathbb{Z}$, $\Lambda_n = \{1, \dots, n\}$ and $K = \mathbb{N}$.) This is just to make clear that only semigroup structure is used in this section. In the next section, where Theorem 5.13 below is applied, we will of course take K to be G. Let Σ denote the σ-algebra of Borel sets of G that are contained in K.

REMARK. It is also interesting to note that the arguments below do not require the semigroup K to be abelian, however this would require the existence of a slightly different type of Følner sequence. Namely, if G were non-abelian, and μ right invariant, one would have to assume the existence of a sequence (or a net) of Borel sets (Λ_n) of G (which are contained in K) with $0 < \mu(\Lambda_n) < \infty$ such that $\lim_{n\to\infty} \mu(\Lambda_n \Delta(g\Lambda_n))/\mu(\Lambda_n) = 0$ or all $g \in K$. Note that g is to the left of the set despite μ being right invariant.

Before we reach the main result of this section, we have to discuss Følner sequences a bit further.

LEMMA 5.10. *Take any $g_n \in K$ for each n. Then the following sequence is also a Følner sequence in K:*

$$(\Lambda_n g_n).$$

Proof. Since K has the right cancellation property, we have $(Ag)\Delta(Bg) = (A\Delta B)g$ for all $A, B \subset K$ and $g \in K$. Hence, as $n \to \infty$,

$$\frac{\mu((\Lambda_n g_n)\Delta(g(\Lambda_n g_\alpha)))}{\mu(\Lambda_n g_n)} = \frac{\mu((\Lambda_n \Delta(g\Lambda_n))g_n)}{\mu(\Lambda_n g_n)} = \frac{\mu(\Lambda_n \Delta(g\Lambda_n))}{\mu(\Lambda_n)} \longrightarrow 0. \quad \blacksquare$$

DEFINITION 5.11. Let (Λ_n) be any Følner sequence in K. Consider any $V \in \Sigma$ and set

$$D_{(\Lambda_n)}(V) := \lim_{n\to\infty}\left[\inf\left\{\frac{\mu(\Lambda_m \cap V)}{\mu(\Lambda_m)} : m \geqslant n\right\}\right] \equiv \liminf_{n\to\infty}\frac{\mu(\Lambda_n \cap V)}{\mu(\Lambda_n)}.$$

If $D_{(\Lambda_n)}(V) > 0$, then we say that V has *positive lower density relative to* (Λ_n).

It is easily checked that $D_{(\Lambda_n)}(V)$ in this definition always exists.

LEMMA 5.12. *Let $E \in \Sigma$ be relatively dense in K. Then:*

(i) *There exists an $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in K$ such that the following holds: for each $B \in \Sigma$ with $\mu(B) < \infty$ there exists a $j \in \{1, \ldots, r\}$ such that $\mu((Bg_j) \cap E) \geqslant \mu(B)/r$.*

(ii) *E has positive lower density relative to some Følner sequence in K.*

(iiii) *Let $f : K \to \mathbb{R}$ a Σ-measurable function with $f \geqslant 0$. Assume that $f(g) \geqslant \alpha$ for some $\alpha > 0$ and all $g \in E \in \Sigma$. Then there exists a Følner sequence (Λ_n) in K such that*

$$\liminf_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int_{\Lambda_n} f\,\mathrm{d}\mu > 0.$$

Proof. (i) Let $g_1, \ldots, g_r$ be given by Definition 5.5. Set $B_j := \{b \in B : bg_j \in E\}$ for $j = 1, \ldots, r$, so $B_j g_j = (Bg_j) \cap E \in \Sigma$ and hence $B_j \in \Sigma$. Now, for any $b \in B$ we know from Definition 2.3 that $E \cap \{bg_1, \ldots, bg_r\} \neq \varnothing$. So $bg_j \in E$ for some $j \in \{1, \ldots, r\}$, i.e. $b \in B_j$. Hence $B = \bigcup_{j=1}^{r} B_j$ and therefore

$$\mu(B) = \mu\Big(\bigcup_{j=1}^{r} B_j\Big) \leqslant \sum_{j=1}^{r}\mu(B_j) = \sum_{j=1}^{r}\mu(B_j g_j) = \sum_{j=1}^{r}\mu((Bg_j) \cap E)$$

from which the conclusion follows.

(ii) Consider any Følner sequence (Λ_n) in K. Let $g_1, \ldots, g_r \in K$ be as in Definition 5.5. For each n it follows from (i) that there exists a $j(n) \in \{1, \ldots, r\}$ such that

$$\frac{\mu((\Lambda_n g_{j(n)}) \cap E)}{\mu(\Lambda_n g_{j(n)})} \geqslant \frac{1}{r}$$

where we also made use of $\mu(\Lambda_n g_{j(n)}) = \mu(\Lambda_n)$. But it follows from Lemma 5.10 that (Λ'_n) given by $\Lambda'_n := \Lambda_n g_{j(n)}$ is a Følner sequence in K. Furthermore,

$$D_{(\Lambda'_n)}(E) = \liminf_{n\to\infty} \frac{\mu(\Lambda'_n \cap E)}{\mu(\Lambda'_n)} \geqslant \lim_{n\to\infty} \frac{1}{r} = \frac{1}{r}.$$

(iii) By (ii) there exists a Følner sequence (Λ_n) in K such that

$$\liminf_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f \mathrm{d}\mu \geqslant \liminf_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n \cap E} \alpha \, \mathrm{d}g = \alpha D_{(\Lambda_n)}(E) > 0. \quad \blacksquare$$

Finally we reach the goal of this section, namely a Szemerédi type property for compact C^*-systems:

THEOREM 5.13. *Let (A, ω, τ, K) be a compact C^*-system with ω a trace and K a Borel measurable semigroup in G such that $\Lambda_n \subset K$ for every n, where G and (Λ_n) are as in Section 2. Let $a \in A^+$ with $\omega(a) > 0$. Take any $m_0, \dots, m_k \in \mathbb{N} \cup \{0\}$. Assume that $g \mapsto \omega\Big(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\Big)$ and $g \mapsto \|\tau_{g^{m_j}}(a) - a\|_\omega$ are Σ-measurable on K for $j = 0, 1, \dots, k$. Then there exists a Følner sequence (Λ'_n) in K such that*

$$\liminf_{n\to\infty} \frac{1}{\mu(\Lambda'_n)} \int_{\Lambda'_n} \Big|\omega\Big(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\Big)\Big| \mathrm{d}\mu(g) > 0.$$

Proof. This follows from Lemma 5.12(iii) and Corollary 5.9, since $E = \{g \in K : \|\tau_{g^{m_j}}(a) - a\|_\omega < \varepsilon \text{ for } j = 0, \dots, k\}$ is Σ-measurable. $\blacksquare$

Note that if for example we assume that $g \mapsto \tau_g(a)$ is continuous in A's norm, then both $g \mapsto \omega\Big(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\Big)$ and $g \mapsto \|\tau_{g^{m_i}}(a) - a\|_\omega$ are continuous and hence Borel measurable.

6. ERGODIC SYSTEMS

In measure theoretic ergodic theory it is well known that a system is weakly mixing if and only if it contains no non-trivial compact factors. In this section we show that this result can be extended to noncommutative ergodic theory, where the measure space (and its algebra of L^∞-functions) is replaced by a σ-finite von Neumann algebra and a faithful normal state. To avoid confusion we stress that the word "factor" as used in this paper does not refer to a von Neumann algebra which is a factor (i.e. has trivial center), but to a subsystem of a dynamical system as defined below. The two main ingredients of the proof are a so-called "proper value theorem" due to Størmer (Theorem 2.5 in [20]), and the splitting theorem of Jacobs–Deleeuw–Glicksberg (see Section 2.4 in [16]). Once this is done, we prove our final result regarding the Szemerédi property in ergodic systems.

In this section G is a completely arbitrary group for which we will state additional requirements (like being abelian or locally compact) as needed.

DEFINITION 6.1. A *W*-dynamical system* (A, ω, τ, G) consists of a von Neumann algebra A on which we have a faithful normal state ω, and where $\tau : G \to \mathrm{Aut}(A) : g \mapsto \tau_g$ is a representation of any abelian group G as $*$-automorphisms of A (i.e. $\tau_e = \mathrm{id}_A$ and $\tau_g \circ \tau_h = \tau_{gh}$), such that $\omega \circ \tau_g = \omega$ for all g in G.

Note that the existence of a faithful normal state on A in this definition implies that A is σ-finite (see for example Proposition 2.5.6 in [5]). It is convenient to work in the GNS representation of such a system and for certain intermediate results the group G need not be abelian, therefore we will mostly work with the following:

DEFINITION 6.2. A *represented system* $(R, \omega_\Omega, \alpha)$ consists of the following: Firstly a von Neumann algebra R on a Hilbert space H, a unit vector $\Omega \in H$ which is cyclic and separating for R, in terms of which we define a state ω_Ω on R by $\omega_\Omega(a) = \langle \Omega, a\Omega \rangle$. Furthermore we have a unitary representation $U : G \to B(H) : g \mapsto U_g$ of an arbitrary group G (i.e. U_g is a unitary operator, $U_e = 1$ and $U_g U_h = U_{gh}$), such that $U_g \Omega = \Omega$ and $U_g R U_g^* \subset R$ for all $g \in G$, and in terms of which $\alpha : G \to \mathrm{Aut}(R) : g \mapsto \alpha_g$ is defined by $\alpha_g(a) = U_g a U_g^*$.

The notation in these two definitions will be used consistently, for example reference to a represented system will imply the notation $(R, \omega_\Omega, \alpha)$, H, G and U, and throughout the rest of this section $(R, \omega_\Omega, \alpha)$ is a represented system. Note that the GNS representation (H, π, Ω) of a W*-dynamical system (A, ω, τ, G) gives us a *corresponding* represented system $(R, \omega_\Omega, \alpha)$ where $R = \pi(A)$ and $\alpha_g(\pi(a)) = \pi(\tau_g(a))$ in terms of which U is uniquely defined. Also keep in mind that π is faithful in this situation.

For a represented system an *eigenoperator* of α is an $a \in A\backslash\{0\}$ such that there exists a function $\lambda_a : G \to \mathbb{C}$ with $\alpha_g(a) = \lambda_a(g)a$ for all $g \in G$. Note that in this case $|\lambda_a(g)| = 1$ for all $g \in G$, i.e. λ_a is *unimodular* and hence a group homomorphism to the circle. Similarly an *eigenvector* of U is an $x \in H\backslash\{0\}$ such that there exists a function $\lambda_x : G \to \mathbb{C}$ with $U_g x = \lambda_x(g)x$ for all g. Again note that λ_x is unimodular and hence a group homomorphism to the circle.

For a represented system we will denote the Hilbert subspace of H spanned by the eigenvectors of U by H_0. The Hilbert subspace of H spanned by the eigenvectors x with $\lambda_x = 1$ will be denoted by H_1. Note that $\mathbb{C}\Omega \subset H_1 \subset H_0$, with equality allowed.

DEFINITION 6.3. A represented system is called *ergodic* (respectively *weakly mixing*) when $\dim H_1 = 1$ (respectively $\dim H_0 = 1$). A W*-dynamical system is called *ergodic* (respectively *weakly mixing*) when its corresponding represented system is ergodic (respectively weakly mixing).

When G is as in Section 2, then ergodicity and weak mixing of the dynamical system (A, ω, α) as given in Definition 6.3 are equivalent to $\{\mathrm{id}_G\}$-ergodicity and $\{\mathrm{id}_G\}$-weak mixing as given in Definition 3.2. For ergodicity this follows from the mean ergodic theorem, and for weak mixing it can be shown to follow from the general theory in Section 2.4 of [16].

For a represented system we define a norm $\|\cdot\|_\Omega$ on R by $\|a\|_\Omega = \omega_\Omega(a^*a)^{1/2} = \|a\Omega\|$ where $\|\cdot\|$ denotes the norm of H.

DEFINITION 6.4. A *factor* (N, ω, τ) of a W*-dynamical system (A, ω, τ, G) consists of a $*$-algebra $N \subset A$ and the restrictions of ω and τ_g to N, such that $\tau_g(N) \subset N$ for all g. Similarly a *factor* $(N, \omega_\Omega, \alpha)$ of a represented system $(R, \omega_\Omega, \alpha)$ consists of a $*$-algebra $N \subset R$ and the restrictions of ω_Ω and α_g to N, such that $\alpha_g(N) \subset N$ for all g. Such factors are called *compact* if respectively every orbit $\tau_G(a) = \{\tau_g(a) : g \in G\}$ is totally bounded in $(N, \|\cdot\|_\omega)$ or every orbit $\alpha_G(a) = \{\alpha_g(a) : g \in G\}$ is totally bounded in $(N, \|\cdot\|_\Omega)$. A factor will be called *nontrivial* if R strictly contains $\mathbb{C}1$.

For a represented system the orbit of $x \in H$ will be denoted by $U_G x = \{U_g x : g \in G\}$.

Let $B(H)$ denote the algebra of all bounded linear operators in the Hilbert space H, and let S' denote the commutant of a set $S \subset B(H)$.

Let B denote the set of all eigenoperators of α together with the zero operator, and let C be the $*$-algebra generated by B. Set $N = C''$, hence N is a von Neumann algebra contained in R.

PROPOSITION 6.5. *$(N, \omega_\Omega, \alpha)$ is a compact factor of $(R, \omega_\Omega, \alpha)$.*

Proof. It is easily seen that B is closed under adjoints, products and scalar multiples, hence

$$C = \left\{ \sum_{j=1}^{n} a_j : a_1, \ldots, a_n \in B, n > 0 \right\} \tag{6.5.1}$$

but for $a \in B$ we have $\alpha_h(\alpha_g(a)) = \lambda_a(h)\alpha_g(a)$, hence $\alpha_g(B) \subset B$ and $\alpha_g(C) \subset C$ for all g. For any $a \in N$ there exists a net (a_γ) in C such that $a_\gamma x \to ax$ for all $x \in H$ according to von Neumann's density theorem. Therefore $\langle x, \alpha_g(a_\gamma)y\rangle = \langle U_g^* x, a_\gamma U_g^* y\rangle \to \langle U_g^* x, a U_g^* y\rangle = \langle x, \alpha_g(a)y\rangle$ for all $x, y \in H$ and all $g \in G$. In other words $\alpha_g(a_\gamma)$ converges in the weak operator topology to $\alpha_g(a)$, but $\alpha_g(a_\gamma) \in C \subset N$ so by the bicommutant theorem $\alpha_g(a) \in N$. This proves that $\alpha_g(N) \subset N$, and therefore $(N, \omega_\Omega, \alpha)$ is a factor of $(R, \omega_\Omega, \alpha)$.

Next we show that $(N, \omega_\Omega, \alpha)$ is compact. First note that for $a \in B$ we have $\alpha_G(a) = \lambda_a(G)a$ or $\alpha_G(a) = \{0\}$, and both these orbits are totally bounded in B with the pseudo metric obtained by restricting $\|\cdot\|_\Omega$ to B, since $\lambda_a(G)$ is a subset of the unit circle (which is compact) in $\mathbb{C}$. From (6.5.1) we can then conclude that $\alpha_G(a)$ is totally bounded in $(C, \|\cdot\|_\Omega)$ for every $a \in C$. Again by von Neumann's density theorem for any $a \in N$ and $\varepsilon > 0$ there is $b \in C$ such that

$\|a-b\|_\Omega = \|a\Omega - b\Omega\| < \varepsilon$. Now, if E is a finite ε-net in $(C, \|\cdot\|_\Omega)$ for $\tau_G(b)$, then from

$$\|\alpha_g(a) - c\|_\Omega \leqslant \|\alpha_g(a) - \alpha_g(b)\|_\Omega + \|\alpha_g(b) - c\|_\Omega < 2\varepsilon$$

for some $c \in E$ we see that E is a finite 2ε-net for $\alpha_G(a)$, i.e. the latter is totally bounded. ∎

LEMMA 6.6. *If G is abelian, then H_0 is the set of all elements $x \in H$ whose orbits $U_G x$ in H are totally bounded.*

Proof. This is essentially a special case of general results proven in Section 2.4 of [16]. We show how it follows from those general results. Let H_k be the set of all elements of H with totally bounded orbits under U.

Let E be the set of all eigenvectors of U, so $H_0 = \overline{\text{span}E}$. Clearly $U_G x$ is totally bounded for every $x \in E$, since $\lambda_x(G)$ is a subset of the unit circle in $\mathbb{C}$. From this it follows that $U_G x$ is totally bounded for every $x \in H_0$. i.e. $H_0 \subset H_k$.

Now suppose that $H_0 \neq H_k$. It is straightforward to show that H_k is a Hilbert subspace of H, so it follows that there is an $x \in H_k \backslash \{0\}$ which is orthogonal to H_0. So $x \in H_v := H \ominus H_0$, but H_v is the space of so-called "flight vectors" which means that there is an S in the weak operator closure $\overline{U_G}$ of U_G in $B(H)$ such that $Sx = 0$. This is the essence of the splitting theorem in Section 2.4 of [16] as applied to a unitary group on a Hilbert space. However, it is easily seen that $U_G y$ is relatively weakly compact for every $y \in H$, since any closed ball in H is weakly compact, hence according to Lemma 2.4.2 in [16] $\overline{U_G}x = \overline{U_G x}^{\mathrm{w}}$ where $\overline{U_G x}^{\mathrm{w}}$ denotes the weak closure of $U_G x$ in H. The norm closure $\overline{U_G x}$ of the totally bounded set $U_G x$ is compact and therefore weakly compact and hence weakly closed, so $\overline{U_G x}^{\mathrm{w}} \subset \overline{U_G x}$. Putting all this together we have $0 = Sx \in \overline{U_G x}$ contradicting $x \neq 0$ and the fact that U is a unitary group and hence normpreserving. ∎

PROPOSITION 6.7. (i) *If $(R, \omega_\Omega, \alpha)$ is ergodic but not weakly mixing, then the factor $(N, \omega_\Omega, \alpha)$ is nontrivial.*

(ii) *If $(R, \omega_\Omega, \alpha)$ is weakly mixing and G is abelian, then every element $a \in M$ with a totally bounded orbit $\alpha_G(a)$ lies in $\mathbb{C}1$. In particular $(R, \omega_\Omega, \alpha)$ has no nontrivial compact factor.*

Proof. (i) According to Theorem 2.5 in [20] the map $a \mapsto a\Omega$ is a bijection from the set of eigenoperators of α to the set of eigenvectors of U, where we simultaneously note that our definition of ergodicity of $(R, \omega_\Omega, \alpha)$ is equivalent to that of [20] (namely $\alpha_g(a) = a$ for all g implies that $a \in \mathbb{C}1$), since Ω is cyclic and separating for R. Since H_0 strictly contains $\mathbb{C}\Omega$ by Definition 6.3, it follows that B strictly contains $\mathbb{C}1$, hence N strictly contains $\mathbb{C}1$. Therefore $(N, \omega_\Omega, \alpha)$ is indeed nontrivial.

(ii) Consider any $a \in R$ with totally bounded orbit, then from $\|\alpha_g(a) - b\|_\Omega = \|U_g a\Omega - b\Omega\|$ we see that $U_G a\Omega$ is totally bounded in H. So $a\Omega \in \mathbb{C}\Omega$

by Lemma 6.6 and Definition 6.3, but Ω is separating for R, hence $a \in \mathbb{C}1$. In particular, the $*$-algebra any compact factor of $(R, \omega_\Omega, \alpha)$ must be contained in $\mathbb{C}1$. ∎

Using these results, we can now prove

THEOREM 6.8. *Let (A, ω, τ, G) be an ergodic W*-dynamical system. Then (A, ω, τ, G) is weakly mixing if and only if it has no non-trivial compact factor.*

Proof. Let (H, π, Ω) be the GNS representation of (A, ω) and $(R, \omega_\Omega, \alpha)$ the corresponding represented system. Since π is faithful (giving a $*$-isomorphism $A \to R$), it is simple to show that a nontrivial compact factor in (A, ω, τ, G) gives one in $(R, \omega_\Omega, \alpha)$, and vice versa. The theorem then follows from Propositions 6.5 and 6.7. ∎

DEFINITION 6.9. A W*-dynamical system (A, ω, τ, G), with G as in Section 2, is said to have the *Szemerédi property* if there exists a Følner sequence (Λ_n) in G such that for any $k \in \mathbb{N}$ and $m_1, \ldots, m_k \in \mathbb{N}$ with $m_1 < \cdots < m_k$ and for all $a \in A^+$ with $\omega(a) > 0$,

$$\liminf_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left|\omega\left(a \prod_{j=1}^{k} \tau_{g^{m_j}}(a)\right)\right| \mathrm{d}g > 0.$$

THEOREM 6.10. *Suppose that (A, ω, τ, G) is a non-trivial (i.e. $A \neq \mathbb{C}$) ergodic W*-dynamical system, with G locally compact, second countable, (and abelian, by Definition 6.1), and containing a Følner sequence (Λ_n) satisfying the Tempelman condition and such that $(\Lambda_n^{-1}\Lambda_n)$ is also a Følner sequence. Let ω be a trace and $g \mapsto \tau_g(a)$ be continuous for every $a \in A$. Suppose that for each $m \in \mathbb{Z}\backslash\{0\}$ there exists a Følner sequence (Γ_n) and a $c > 0$ such that*

$$\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f(g^m)\, \mathrm{d}g \leqslant \frac{c}{\mu(\Gamma_n)} \int_{\Gamma_n} f(g)\mathrm{d}g$$

for all Borel measurable $f : G \to [0, \infty)$ and all n, and such that (A, ω, τ, G) is $\{\mathrm{id}_G\}$-asymptotically abelian relative to (Γ_n). Then the Szemerédi property holds for a nontrivial factor of (A, ω, τ, G).

Proof. Set $m_0 := 0$ and $g^0 := e$. Let $a \in A$ with $\omega(a) > 0$. From Theorem 6.8 it follows that (A, ω, τ, G) is either weakly mixing or it has a non-trivial compact factor. If (A, ω, τ, G) is weakly mixing then (A, ω, τ, G) is M-weakly mixing, where $M := \{\varphi_m : m \in \mathbb{Z}\backslash\{0\}\}$ and $\varphi_m : G \to G : g \mapsto g^m$. This can be seen from

$$\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\omega(a\tau_{g^m}(b)) - \omega(a)\omega(b)|\mathrm{d}g \leqslant \frac{c}{\mu(\Gamma_n)} \int_{\Gamma_n} |\omega(a\tau_g(b)) - \omega(a)\omega(b)|\mathrm{d}g \to 0$$

as $n \to \infty$. Similarly (A, ω, τ, G) is M-asymptotically abelian relative to (Λ_n). Since M is translational, it now follows from Theorem 4.6 that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n} \Big|\omega\Big(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\Big) - \omega(a)^{k+1}\Big| \mathrm{d}g = 0,$$

and therefore

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n} \Big|\omega\Big(\prod_{j=0}^{k} \tau_{g^{m_j}}(a)\Big)\Big| \mathrm{d}g = \omega(a)^{k+1} > 0.$$

If (A, ω, τ, G) is not weakly mixing, it has a nontrivial compact factor (N, ω, τ, G). Continuity of $g \mapsto \tau_g(a)$ implies that $g \mapsto \omega\Big(\prod\limits_{j=0}^{k} \tau_{g^{m_j}}(a)\Big)$ and $g \mapsto \|\tau_{g^{m_j}}(a) - a\|_\omega$ are also continuous. The Szemerédi property for (N, ω, τ, G) then follows directly from Theorem 5.13. ∎

It is easily seen that for $G = \mathbb{Z}^q$ and $G = \mathbb{R}^q$ such (Λ_n) and (Γ_n) exist: For $G = \mathbb{Z}^q$ let $\Lambda_n = \{-n, \dots, n\}^q$ and $\Gamma_n = \{-|m|n, -|m|n + 1, \dots, |m|n\}$. For $\mathbb{R}^q$, take $\Lambda_n = [-n, n]^q$ and $\Gamma_n = |m|\Lambda_n$.

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