

## FOURIER TRANSFORM ON LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. The notion of Fourier transform is among the most important tools in analysis, which has been generalized in abstract harmonic analysis to the level of abelian locally compact groups. The aim of this paper is to further generalize the Fourier transform. Motivated by some recent works by Van Daele in the multiplier Hopf algebra framework, and by using the Haar weights, we define here the (generalized) Fourier transform and the inverse Fourier transform, at the level of locally compact quantum groups. We then consider the analogues of the Fourier inversion theorem, Plancherel theorem, and the convolution product. Along the way, we also obtain an alternative description of the dual pairing map between a quantum group and its dual.

KEYWORDS: *Locally compact quantum group, Haar weight, Fourier transform.*

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### 1. INTRODUCTION

The Fourier transform has been known for quite some time, and is among the very powerful tools in classical analysis. In abstract harmonic analysis (see [4], [3]), the theory was generalized to the case of abelian locally compact groups.

Let us briefly summarize: Given a locally compact abelian (LCA) group  $G$ , its dual object  $\widehat{G}$  is the group of continuous,  $\mathbb{T}$ -valued characters on  $G$ . The dual group  $\widehat{G}$  can be given an appropriate topology, making it into an LCA group again. By Pontryagin duality, it is known that the dual of  $\widehat{G}$  is isomorphic to  $G$ . Using the Haar measure on  $G$ , it is then possible to define the Fourier transform of a continuous function having compact support  $f \in C_c(G)$  (or, even a Schwartz function), obtaining  $\widehat{f} \in C_0(\widehat{G})$ . Of course, one can begin with the functions on  $\widehat{G}$ , and define the inverse Fourier transform. The Fourier inversion theorem holds, as well as Plancherel's theorem. One by-product is that we obtain in this way a Hilbert space isomorphism  $L^2(G) \cong L^2(\widehat{G})$ . Refer to the standard textbooks on the theory, for instance, [3].

If the group  $G$  is non-abelian, we no longer can define  $\widehat{G}$  as the dual group. Tannaka, Krein, and others have been able to define a dual object to a non-abelian group, from which the original group can be recovered. However, with the dual object not being a group, the Pontryagin duality does not hold anymore, and it is not possible to define the Fourier transform and the inverse Fourier transform between  $G$  and  $\widehat{G}$ . A modified version of the Fourier transform does exist, and is being used, in the representation theory of non-abelian, compact groups. Still, from the duality point of view, it is not really satisfactory.

The environment is now better, with the recent development of the theory of locally compact quantum groups [6], [7], [8], [10]. We now know that Pontryagin duality can be naturally extended to the wider setting of locally compact quantum groups. Indeed, given a locally compact quantum group  $(M, \Delta)$ , its dual object  $(\widehat{M}, \widehat{\Delta})$  is also a locally compact quantum group, and moreover, the dual of  $\widehat{M}$  is isomorphic to  $M$ .

On the other hand, a workable notion of Fourier transform at the quantum group level has been lacking so far. Among the challenges is that because of the way the dual quantum group is constructed, the Hilbert spaces for  $M$  and  $\widehat{M}$  are identical, and the Fourier transform tends to be “hidden”. (It is essentially like identifying the spaces  $L^2(G) = L^2(\widehat{G})$ , making the Fourier transform irrelevant.)

In his recent works, Van Daele has been working to improve this situation. In particular, in his preprint [11] (see also [9]), he proposes a definition for the generalized Fourier transform in the setting of multiplier Hopf algebras and algebraic quantum groups. He suggests in the paper that more can be done at the operator algebra level of locally compact quantum groups, and indeed, some indications of these new developments do appear in his other recent papers (for instance, [10]). However, at the time of writing the present paper, the author does not know of any place where his results are actually written out.

The present paper grew out in the hope of filling this gap. For the theory of locally compact quantum groups and the harmonic analysis on them to be further developed, it is certainly desirable to have the notion of Fourier transform clarified and refined. In this paper, by using Van Daele’s approach in [11] as a guideline, and by taking advantage of the generalized Pontryagin duality, we define and explore the notion of Fourier transform in the setting of locally compact quantum groups. The point we wish to make is that even though the Fourier transform may seem hidden, it is still there, and we are proposing a way to harness its usefulness. This generalized Fourier transform will be shown to satisfy many of the familiar results from classical analysis.

Due to the technical differences between the setting of multiplier Hopf algebras and the framework of locally compact quantum groups, the proofs and the details are quite different, while the end-results may look similar. Nevertheless, the author can never downplay the motivation and the strong influence Van Daele’s works gave him in preparing the present paper.

Here is how this paper is organized. We begin, in Section 2, by recalling the definition and some main results on locally compact quantum groups. We chose to work with the von Neumann algebra approach (as in [7]), which is known to be equivalent to the  $C^*$ -algebraic framework. Special attentions are given to Haar weights, the multiplicative unitary operator, and the antipode maps.

In Section 3, we give the definition of the generalized Fourier transform, sending certain elements of  $M$  to elements in  $\widehat{M}$ . We will also define the inverse Fourier transform, and observe the analogues of the Fourier inversion theorem and the Plancherel theorem. A formula for the “convolution product” will be also obtained.

In Section 4, we will use the Fourier transform to give an alternative (and apparently new) description of the dual pairing between certain dense subalgebras of  $M$  and  $\widehat{M}$ . This will be useful in our future works. Finally, we added a brief Appendix (Section 5), where we observe how all this is reflected in the special case of an ordinary locally compact group.

## 2. PRELIMINARIES: VON NEUMANN ALGEBRAIC QUANTUM GROUPS AND HAAR WEIGHTS

We will use the standard notations from the theory of weights. The weights we will be working with are normal, semi-finite faithful weights (“n.s.f. weights”, for short) on von Neumann algebras. For an n.s.f. weight  $\varphi$  on a von Neumann algebra  $M$ , we write:

- $\mathfrak{M}_\varphi^+ = \{x \in M^+ : \varphi(x) < \infty\}$ .
- $\mathfrak{N}_\varphi = \{x \in M : x^*x \in \mathfrak{M}_\varphi^+\}$ .
- $\mathfrak{M}_\varphi = \left\{ \sum_{i=1}^n y_i^* x_i : x_1, \dots, x_n, y_1, \dots, y_n \in \mathfrak{N}_\varphi \right\}$ .

The space  $\mathfrak{M}_\varphi$  is a  $*$ -subalgebra of  $M$ , which is the “definition domain” of the weight  $\varphi$ . Observe that  $\mathfrak{M}_\varphi$  is obtained as the linear span of  $\mathfrak{M}_\varphi^+$  in  $M$ .

Let us begin with the definition of a von Neumann algebraic locally compact quantum group, as given by Kustermans and Vaes [7]. This definition is known to be equivalent to the definition in the  $C^*$ -algebra setting [6], [8]. As in the case of the  $C^*$ -algebraic quantum groups, the existence of Haar (invariant) weights is assumed as a part of the definition. The noticeable difference between the two approaches is the absence of the density conditions in the von Neumann algebra setting: it turns out that they follow automatically from the other conditions. Refer also to the recent paper by Van Daele [10], which gives an improved approach to the subject and is more natural.

**DEFINITION 2.1.** Let  $M$  be a von Neumann algebra, together with a unital normal  $*$ -homomorphism  $\Delta : M \rightarrow M \otimes M$  such that the “coassociativity condition” holds:  $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$ . Furthermore, we assume the existence of a left invariant weight and a right invariant weight, as follows:

(i)  $\varphi$  is an n.s.f. weight on  $M$  that is left invariant:

$$\varphi((\omega \otimes \text{id})(\Delta x)) = \varphi(x)\omega(1), \quad \text{for all } \omega \in M_*^+, x \in \mathfrak{M}_\varphi^+.$$

(ii)  $\psi$  is an n.s.f. weight on  $M$  that is right invariant:

$$\psi((\text{id} \otimes \omega)(\Delta x)) = \psi(x)\omega(1), \quad \text{for all } \omega \in M_*^+, x \in \mathfrak{M}_\psi^+.$$

Then we call  $(M, \Delta)$  a *von Neumann algebraic quantum group*. It can be shown that the Haar weights are unique, up to scalar multiplication.

Let us fix  $\varphi$ . By means of the GNS-construction  $(\mathcal{H}, \iota, \Lambda)$  for  $\varphi$ , we view  $M$  as a subalgebra of the operator algebra  $\mathcal{B}(\mathcal{H})$ , such as  $M = \iota(M) \subseteq \mathcal{B}(\mathcal{H})$ . So we will have:  $\langle \Lambda(x), \Lambda(y) \rangle = \varphi(y^*x)$  for  $x, y \in \mathfrak{N}_\varphi$ , and  $x\Lambda(y) = \Lambda(xy)$  for  $y \in \mathfrak{N}_\varphi, x \in M$ . As in standard weight theory, we can also consider the modular conjugation and the modular automorphism group of  $\varphi$ .

Meanwhile, there exists a certain unitary operator  $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ , called the *multiplicative unitary operator* for  $(M, \Delta)$ . It is defined by  $W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)((\Delta y)(x \otimes 1))$ , for  $x, y \in \mathfrak{N}_\varphi$ . It satisfies the pentagon equation of Baaq and Skandalis [2],  $W_{12}W_{13}W_{23} = W_{23}W_{12}$ , and one can check that  $\Delta x = W^*(1 \otimes x)W$ , for all  $x \in M$ . It is essentially the “left regular representation” associated with  $\varphi$ , and it also gives us the following useful characterization of  $M$ :

$$(2.1) \quad M = \overline{\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^w (\subseteq \mathcal{B}(\mathcal{H})),$$

where  $-^w$  denotes the von Neumann algebra closure (for instance, the closure under  $\sigma$ -weak topology).

There are other possible (and useful) characterizations of  $(M, \Delta)$ . See [7]. Meanwhile, if we take the norm closure in equation (2.1), instead of the weak closure, we obtain a  $C^*$ -algebra  $A$ . It turns out that by restricting  $\Delta$  to  $A$ , we obtain a reduced  $C^*$ -algebraic quantum group  $(A, \Delta)$ . Going the other way, we could begin with a  $C^*$ -algebraic quantum group and obtain a von Neumann algebraic quantum group by taking the weak closure of the underlying  $C^*$ -algebra in the GNS Hilbert space of a left Haar weight.

The main point in all these is that the above Definition 2.1 is a valid definition of a locally compact quantum group. For instance, the existence of other quantum group structure maps like “antipode” can be proved from the defining axioms.

Constructing the antipode is quite technical (it uses the right Haar weight), and we refer the reader to the main papers [6], [7]. See also an improved treatment given in [10], where the antipode is defined in a more natural way by means of Tomita–Takesaki theory. For our purposes, we will work with the following useful characterization of the antipode map  $S$ :

$$(2.2) \quad S((\text{id} \otimes \omega)(W)) = (\text{id} \otimes \omega)(W^*).$$

In fact, the subspace consisting of the elements  $(\text{id} \otimes \omega)(W)$ , for  $\omega \in \mathcal{B}(\mathcal{H})_*$ , is dense in  $M$ , and forms a core for  $S$ . Meanwhile, there exist a unique  $*$ -anti-automorphism  $R$  (called the “unitary antipode”) and a unique continuous one-parameter group  $\tau$  on  $M$  (called the “scaling group”) such that we have the following polar decomposition:  $S = R\tau_{-i/2}$ .

Since  $(R \otimes R)\Delta = \Delta^{\text{cop}}R$ , where  $\Delta^{\text{cop}}$  is the co-opposite comultiplication (i. e.  $\Delta^{\text{cop}} = \chi \circ \Delta$ , for  $\chi$  the flip map on  $M \otimes M$ ), the weight  $\varphi \circ R$  is right invariant. So for convenience, we will from now on choose  $\psi$  to equal  $\varphi \circ R$ . The GNS map for  $\psi$  will be written as  $\Gamma$ .

Next, let us consider the dual quantum group. Working with the other leg of the multiplicative unitary operator  $W$  than in equation (2.1), we define:

$$(2.3) \quad \widehat{M} = \overline{\{(\omega \otimes \text{id})(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^w \ (\subseteq \mathcal{B}(\mathcal{H})).$$

This is shown to be a von Neumann algebra. The comultiplication on it is defined by  $\widehat{\Delta}(y) = \Sigma W(y \otimes 1)W^*\Sigma$ , for all  $y \in \widehat{M}$  (here,  $\Sigma$  is the flip on  $\mathcal{H} \otimes \mathcal{H}$ ). The general theory assures that  $(\widehat{M}, \widehat{\Delta})$  is again a von Neumann algebraic quantum group, together with appropriate Haar weights  $\widehat{\varphi}$  and  $\widehat{\psi}$ . The operator  $\Sigma W^*\Sigma$  is the multiplicative unitary for  $(\widehat{M}, \widehat{\Delta})$ . It can be also shown that  $W \in M \otimes \widehat{M}$  and  $\Sigma W^*\Sigma \in \widehat{M} \otimes M$ .

The left Haar weight  $\widehat{\varphi}$  on  $(\widehat{M}, \widehat{\Delta})$  is uniquely characterized by the GNS data  $(\mathcal{H}, \iota, \widehat{\Lambda})$ , where the GNS map  $\widehat{\Lambda} : \mathfrak{N}_{\widehat{\varphi}} \rightarrow \mathcal{H}$  is given by the following formulas (see Proposition 8.14 of [6]):

$$(2.4) \quad \widehat{\Lambda}((\omega \otimes \text{id})(W)) = \xi(\omega) \quad \text{and} \quad \langle \xi(\omega), \Lambda(x) \rangle = \omega(x^*).$$

To be a little more precise, consider:

$$\mathcal{I} = \{\omega \in \mathcal{B}(\mathcal{H})_* : \exists L \geq 0 \text{ such that } |\omega(x^*)| \leq L\|\Lambda(x)\| \text{ for all } x \in \mathfrak{N}_{\varphi}\}.$$

Then for every  $\omega \in \mathcal{I}$ , we can find  $\xi(\omega) \in \mathcal{H}$  such that  $\omega(x^*) = \langle \xi(\omega), \Lambda(x) \rangle$  for all  $x \in \mathfrak{N}_{\varphi}$  (by Riesz theorem). The equation (2.4) above is to be understood as saying that the elements  $(\omega \otimes \text{id})(W)$ ,  $\omega \in \mathcal{I}$ , form a core for  $\widehat{\Lambda}$  and that  $\widehat{\Lambda}((\omega \otimes \text{id})(W)) = \xi(\omega)$ . See also [10], to learn more on this construction.

Meanwhile, analogously as in equation (2.2), with  $\Sigma W^*\Sigma$  now being the multiplicative unitary, the (dense) subspace of the elements  $(\omega \otimes \text{id})(W^*)$ , for  $\omega \in \mathcal{B}(\mathcal{H})_*$ , forms a core for the antipode  $\widehat{S}$ , and  $\widehat{S}$  is characterized by

$$(2.5) \quad \widehat{S}((\omega \otimes \text{id})(W^*)) = (\omega \otimes \text{id})(W).$$

The unitary antipode and the scaling group can be also found, giving us the polar decomposition of  $\widehat{S} = \widehat{R}\widehat{\tau}_{-i/2}$ . As before, we may fix the right Haar weight as  $\widehat{\psi} = \widehat{\varphi} \circ \widehat{R}$ , with the corresponding GNS map written as  $\widehat{\Gamma}$ .

Repeating the whole process beginning with  $(\mathcal{H}, \iota, \widehat{\Lambda})$ , we can further construct the dual  $(\widehat{\widehat{M}}, \widehat{\widehat{\Delta}})$  of  $(\widehat{M}, \widehat{\Delta})$ . The generalized Pontryagin duality result (see [6], [7]) says:  $(\widehat{\widehat{M}}, \widehat{\widehat{\Delta}}) = (M, \Delta)$ , with  $\widehat{\widehat{\varphi}} = \varphi$  and  $\widehat{\widehat{\Lambda}} = \Lambda$ . The other structure

maps for  $(\widehat{M}, \widehat{\Delta})$  are also identified with those of  $(M, \Delta)$ : For instance,  $\widehat{S} = S$ ,  $\widehat{\psi} = \psi$ , etc. One useful result is the following, similar to equation (2.4) above (see Proposition 8.30 of [6], with  $\pi$  now read as the embedding map  $\iota$  and  $\widehat{\Lambda} = \Lambda$ ):

$$(2.6) \quad \langle \Lambda((\text{id} \otimes \omega)(W^*)), \widehat{\Lambda}(y) \rangle = \omega(y^*).$$

Here again, we actually need to consider a set  $\widehat{\mathcal{I}}$  that is similarly defined as before, and the equation (2.6) is accepted with the understanding that the elements  $(\text{id} \otimes \omega)(W^*)$ ,  $\omega \in \widehat{\mathcal{I}}$ , form a core for  $\Lambda$ .

We wrap up the section here. For further details, we refer the reader to the fundamental papers on the subject: [2], [12], [6], [7], [8], [10].

### 3. THE GENERALIZED FOURIER TRANSFORM

Let us denote by  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$ , the dense subalgebras of  $M$  and  $\widehat{M}$ , similar to the ones given in equations (2.1) and (2.3):

$$\mathcal{A} = \{(\text{id} \otimes \omega)(W) : \omega \in \widehat{M}_*\} (\subseteq M), \quad \widehat{\mathcal{A}} = \{(\omega \otimes \text{id})(W) : \omega \in M_*\} (\subseteq \widehat{M}).$$

The fact that these are indeed subalgebras follows from the fact that  $W \in M \otimes \widehat{M}$  and that  $W$  is a multiplicative unitary operator (see [2], [12]). See the lemma below:

LEMMA 3.1. *By the multiplicativity of  $W$  (the pentagon equation), the following results hold:*

(i) *For  $\omega_1, \omega_2 \in M_*$ , we have  $(\omega_1 \otimes \text{id})(W)(\omega_2 \otimes \text{id})(W) = (\mu \otimes \text{id})(W)$ , where  $\mu \in M_*$  is such that  $\mu(x) = (\omega_1 \otimes \omega_2)(\Delta x)$ , for  $x \in M$ .*

(ii) *For  $\theta_1, \theta_2 \in \widehat{M}_*$ , we have  $(\text{id} \otimes \theta_1)(W)(\text{id} \otimes \theta_2)(W) = (\text{id} \otimes \nu)(W)$ , where  $\nu \in \widehat{M}_*$  is such that  $\nu(y) = (\theta_1 \otimes \theta_2)(\widehat{\Delta}^{\text{cop}}(y))$ , for  $y \in \widehat{M}$ .*

(iii) *For  $\theta_1, \theta_2 \in \widehat{M}_*$ , we have  $(\text{id} \otimes \theta_1)(W^*)(\text{id} \otimes \theta_2)(W^*) = (\text{id} \otimes \nu')(W^*)$ , where  $\nu' \in \widehat{M}_*$  is such that  $\nu'(y) = (\theta_1 \otimes \theta_2)(\widehat{\Delta}(y))$ , for  $y \in \widehat{M}$ .*

REMARK 3.2. By (i), we see that  $\widehat{\mathcal{A}}$  is a subalgebra of  $\widehat{M}$ , while (ii) shows that  $\mathcal{A}$  is a subalgebra of  $M$ . (iii) will be also useful later.

*Proof.* By the pentagon equation ( $W_{12}W_{13}W_{23} = W_{23}W_{12}$ ), we have:

$$\begin{aligned} (\omega_1 \otimes \text{id})(W)(\omega_2 \otimes \text{id})(W) &= (\omega_1 \otimes \omega_2 \otimes \text{id})(W_{13}W_{23}) \\ &= (\omega_1 \otimes \omega_2 \otimes \text{id})(W_{12}^*W_{23}W_{12}) = (\mu \otimes \text{id})(W), \end{aligned}$$

where  $\mu \in M_*$  is such that  $\mu(x) = (\omega_1 \otimes \omega_2)(W^*(1 \otimes x)W)$ . Remembering the definition of  $\Delta x = W^*(1 \otimes x)W$ , we obtain the result (i). See [2] for the same result. The proofs for the other two are similar. ■

As for the involution, the following lemma will be useful. We skip the proof, which is straightforward (see [10]). We mention the result here mainly to set the notation for later sections.

LEMMA 3.3. *We will write  $\omega \in M_*^\sharp$ , if  $\omega \in M_*$  is such that there exists an element  $\omega^\sharp \in M_*$ , given by:  $\omega^\sharp(x) = \overline{\omega}(S(x)) = \overline{\omega([S(x)]^*)}$ , for all  $x \in \mathcal{D}(S)$ . In that case, we have:*

$$((\omega \otimes \text{id})(W))^* = (\omega^\sharp \otimes \text{id})(W).$$

The choice of  $\omega^\sharp$  above is unique, in the sense that if there exists a  $\rho \in M_*$  satisfying  $((\omega \otimes \text{id})(W))^* = (\rho \otimes \text{id})(W)$ , then we have  $\rho = \omega^\sharp$ .

Meanwhile, the subspace  $M_*^\sharp (\subseteq M_*)$  is a dense subalgebra of  $M_*$  (in the sense of (i) of Lemma 3.1), and is closed under taking  $^\sharp$ .

For convenience, we will from now on use the notation  $\lambda(\omega)$ ,  $\omega \in M_*$ , to mean  $\lambda(\omega) = (\omega \otimes \text{id})(W) \in \widehat{\mathcal{A}} (\subseteq \widehat{M})$ . So the last part of Lemma 3.3 implies that  $\lambda(M_*^\sharp)$  is dense in  $\widehat{\mathcal{A}}$  (and  $\widehat{M}$ ). On the other hand, since  $S$  may be unbounded in general, it may not be all of  $\widehat{\mathcal{A}}$ .

Dually, let us introduce also the notation  $\widehat{\lambda}(\theta)$ ,  $\theta \in \widehat{M}_*$ , to mean  $\widehat{\lambda}(\theta) = (\theta \otimes \text{id})(\Sigma W^* \Sigma) = (\text{id} \otimes \theta)(W^*) \in M$ . We can define  $\widehat{M}_*^\sharp$  the same way as above. Namely, we say  $\theta \in \widehat{M}_*^\sharp$ , if  $((\text{id} \otimes \theta)(W^*))^* = (\text{id} \otimes \theta^\sharp)(W^*)$ , for  $\theta^\sharp = \bar{\theta} \circ \widehat{S}$ . Since we have  $(\text{id} \otimes \theta^\sharp)(W^*) = (\text{id} \otimes \bar{\theta})(W)$ , we see that  $\widehat{\lambda}(\widehat{M}_*^\sharp)$  is contained and dense in  $\mathcal{A}$  and  $M$ .

Since we will be formulating the Fourier transform in terms of the Haar weights, let us recall the definition of the spaces  $\mathcal{I}$  and  $\widehat{\mathcal{I}}$ , as mentioned in the previous section:

$$\begin{aligned} \mathcal{I} &= \{\omega \in M_* : \exists L \geq 0 \text{ such that } |\omega(x^*)| \leq L \|\Lambda(x)\|, \forall x \in \mathfrak{N}_\varphi\}, \\ \widehat{\mathcal{I}} &= \{\theta \in \widehat{M}_* : \exists L \geq 0 \text{ such that } |\theta(y^*)| \leq L \|\widehat{\Lambda}(y)\|, \forall y \in \mathfrak{N}_{\widehat{\varphi}}\}. \end{aligned}$$

In the below, we will often work with the spaces  $\lambda(\mathcal{I}) (\subseteq \widehat{M})$  and  $\widehat{\lambda}(\widehat{\mathcal{I}}) (\subseteq M)$ . As noted in Section 2 (see also [7]), it is known that the space  $\lambda(\mathcal{I})$  is a core for  $\widehat{\Lambda}$  in  $\widehat{M}$ , while the space  $\widehat{\lambda}(\widehat{\mathcal{I}})$  is a core for  $\Lambda$  in  $M$ .

Let us now turn our attention to our main goal of defining the Fourier transform. In [9] and [11], at the level of multiplier Hopf algebras, the Fourier transform of an element  $a$  is defined as the linear functional  $\omega = \varphi(\cdot a)$ . In fact, the dual multiplier Hopf algebra is characterized as the collection of such linear functionals. With this as motivation, and considering that the multiplicative unitary operator  $W$  provides the duality (see comments in Section 3 of [10], and see also equation (4.1) below), we propose to take the definition of the Fourier transform as  $\mathcal{F}(a) = (\omega \otimes \text{id})(W)$ , where  $\omega = \varphi(\cdot a)$ . See the formulation given in Definition 3.4 below.

DEFINITION 3.4. For  $a \in \widehat{\lambda}(\widehat{\mathcal{I}}) (\subseteq M)$ , define  $\mathcal{F}(a) \in \widehat{M}$ , such that

$$\mathcal{F}(a) := (\varphi \otimes \text{id})(W(a \otimes 1)).$$

Note that formally, we can write it as  $\mathcal{F}(a) = (\omega \otimes \text{id})(W)$ , where  $\omega = \varphi(\cdot a)$ . We will call  $\mathcal{F}(a)$ , the *Fourier transform* of  $a$ .

The claim above that  $\mathcal{F}(a) \in \widehat{M}$  is an easy consequence of the fact that  $W \in M \otimes \widehat{M}$ . Actually, we can be a little more precise:

PROPOSITION 3.5. Let  $a \in \widehat{\lambda}(\widehat{\mathcal{I}})$ , and let  $\mathcal{F}(a)$  be the Fourier transform of  $a$ . Then we have  $\mathcal{F}(a) \in \lambda(\mathcal{I}) (\subseteq \widehat{M})$ . Moreover, we have:

$$\langle \widehat{\Lambda}(\mathcal{F}(a)), \Lambda(x) \rangle = \langle \Lambda(a), \Lambda(x) \rangle,$$

for any  $x \in \mathfrak{N}_\varphi$ . Since the vectors of the form  $\Lambda(x)$ ,  $x \in \mathfrak{N}_\varphi$ , are dense in the Hilbert space  $\mathcal{H}$ , this means that  $\widehat{\Lambda}(\mathcal{F}(a)) = \Lambda(a)$  in  $\mathcal{H}$ .

*Proof.* As noted in Definition 3.4, we may, at least formally, regard  $\mathcal{F}(a) = (\omega \otimes \text{id})(W)$ , where  $\omega = \varphi(\cdot a)$ . On the other hand, since  $a \in \widehat{\lambda}(\widehat{\mathcal{I}})$ , we know that  $a \in \mathfrak{N}_\varphi$ . In addition, we have:

$$|\omega(x^*)| = |\varphi(x^*a)| = |\langle \Lambda(a), \Lambda(x) \rangle| \leq L \|\Lambda(x)\|,$$

for any  $x \in \mathfrak{N}_\varphi$  and  $L = \|\Lambda(a)\|$ . This means that  $\omega \in \mathcal{I}$ , and we have  $\mathcal{F}(a) = \lambda(\omega) \in \lambda(\mathcal{I})$ . Moreover, from equation (2.4), we have:

$$\langle \widehat{\Lambda}(\mathcal{F}(a)), \Lambda(x) \rangle = \omega(x^*) = \varphi(x^*a) = \langle \Lambda(a), \Lambda(x) \rangle. \quad \blacksquare$$

Let us now define the inverse Fourier transform. A justification for this definition will be given in Theorem 3.8 below.

DEFINITION 3.6. For  $b \in \lambda(\mathcal{I}) (\subseteq \widehat{M})$ , define  $\mathcal{F}^{-1}(b) \in M$ , such that

$$\mathcal{F}^{-1}(b) := (\text{id} \otimes \widehat{\varphi})(W^*(1 \otimes b)).$$

We will call  $\mathcal{F}^{-1}(b)$ , the *inverse Fourier transform* of  $b$ .

Formally, it can be written as  $\mathcal{F}^{-1}(b) = (\text{id} \otimes \theta)(W^*) = \widehat{\lambda}(\theta)$ , where  $\theta = \widehat{\varphi}(\cdot b)$ . So by considering equation (2.2), it may be written (again, formally) as  $\mathcal{F}^{-1}(b) = S((\text{id} \otimes \theta)(W)) = (\text{id} \otimes (S\theta))(W)$ . In other words,  $\mathcal{F}^{-1}(b)$  may be considered as the linear functional  $S\theta = \theta(\widehat{S}^{-1}(\cdot)) = \widehat{\varphi}(\widehat{S}^{-1}(\cdot)b)$ . Compare this with the result in Lemma 2.1 of [11], in the multiplier Hopf algebra setting. The reason we have the left Haar weight  $\widehat{\varphi}$  here, instead of the right integral as in [11], could be attributed to the (“opposite”) way the comultiplication  $\widehat{\Delta}$  is defined in our operator algebra setting: See Remark 4.1, preceding Proposition 4.2.

PROPOSITION 3.7. Let  $b \in \lambda(\mathcal{I})$ , and let  $\mathcal{F}^{-1}(b)$  be the inverse Fourier transform of  $b$ . Then we have  $\mathcal{F}^{-1}(b) \in \widehat{\lambda}(\widehat{\mathcal{I}}) (\subseteq M)$ . Moreover, we have:

$$\langle \Lambda(\mathcal{F}^{-1}(b)), \widehat{\Lambda}(y) \rangle = \langle \widehat{\Lambda}(b), \widehat{\Lambda}(y) \rangle, \quad \text{for any } y \in \mathfrak{N}_{\widehat{\varphi}}.$$



Since the vectors of the form  $\widehat{\Lambda}(y)$ ,  $y \in \mathfrak{N}_{\widehat{\varphi}}$ , are dense in the Hilbert space  $\mathcal{H}$ , this means that  $\Lambda(\mathcal{F}^{-1}(b)) = \widehat{\Lambda}(b)$  in  $\mathcal{H}$ .

*Proof.* We can, at least formally, regard  $\mathcal{F}^{-1}(b) = (\text{id} \otimes \theta)(W^*)$ , where  $\theta = \widehat{\varphi}(\cdot b)$ . The proof that  $\theta \in \widehat{\mathcal{I}}$  goes in exactly the same way as before, so that we have  $\mathcal{F}^{-1}(b) = \widehat{\lambda}(\theta) \in \widehat{\lambda}(\widehat{\mathcal{I}})$ . Meanwhile, from equation (2.6), we have:

$$\langle \Lambda(\mathcal{F}^{-1}(b)), \widehat{\Lambda}(y) \rangle = \theta(y^*) = \widehat{\varphi}(y^*b) = \langle \widehat{\Lambda}(b), \widehat{\Lambda}(y) \rangle. \quad \blacksquare$$

Observe that the definition of  $\mathcal{F}^{-1}$  was obtained by imitating the definition of  $\mathcal{F}$ , changing  $\varphi$  into  $\widehat{\varphi}$ , and  $\lambda$  into  $\widehat{\lambda}$ . Still, its proper justification is provided by the following result, which says that  $\mathcal{F}^{-1}(\mathcal{F}(a)) = a$  and  $\mathcal{F}(\mathcal{F}^{-1}(b)) = b$ . This would be our “Fourier inversion theorem”:

**THEOREM 3.8.** *Let  $(M, \Delta)$  and  $(\widehat{M}, \widehat{\Delta})$  be a mutually dual pair of locally compact quantum groups, and let  $\widehat{\lambda}(\widehat{\mathcal{I}}) \subseteq M$  and  $\lambda(\mathcal{I}) \subseteq \widehat{M}$  be the (dense) subalgebras, on which the Fourier transform,  $\mathcal{F}$ , and the inverse Fourier transform,  $\mathcal{F}^{-1}$ , are defined. Then we have:*

- (i) For  $a \in \widehat{\lambda}(\widehat{\mathcal{I}})$ , we have  $\mathcal{F}^{-1}(\mathcal{F}(a)) = a$ .
- (ii) For  $b \in \lambda(\mathcal{I})$ , we have  $\mathcal{F}(\mathcal{F}^{-1}(b)) = b$ .

*Proof.* We saw earlier that for all  $a \in \widehat{\lambda}(\widehat{\mathcal{I}})$ , we have  $\widehat{\Lambda}(\mathcal{F}(a)) = \Lambda(a) \in \mathcal{H}$  (see Proposition 3.5). And, from Proposition 3.7, we have, for all  $b \in \lambda(\mathcal{I})$  that  $\Lambda(\mathcal{F}^{-1}(b)) = \widehat{\Lambda}(b) \in \mathcal{H}$ . Taking  $b = \mathcal{F}(a) \in \lambda(\mathcal{I})$ , and combining the two equations, we obtain

$$\Lambda(\mathcal{F}^{-1}(\mathcal{F}(a))) = \widehat{\Lambda}(\mathcal{F}(a)) = \Lambda(a),$$

in the Hilbert space  $\mathcal{H}$ .

Let us now write:  $a = \widehat{\lambda}(\omega) = (\text{id} \otimes \omega)(W^*)$  and  $\mathcal{F}^{-1}(\mathcal{F}(a)) = \widehat{\lambda}(\theta) = (\text{id} \otimes \theta)(W^*)$ , where  $\omega, \theta \in \widehat{\mathcal{I}}$ . By equation (2.6), and by the observation that  $\Lambda(\mathcal{F}^{-1}(\mathcal{F}(a))) = \Lambda(a)$ , we see that for any  $y \in \mathfrak{N}_{\widehat{\varphi}}$ , we have:

$$\omega(y^*) = \langle \Lambda(a), \widehat{\Lambda}(y) \rangle = \langle \Lambda(\mathcal{F}^{-1}(\mathcal{F}(a))), \widehat{\Lambda}(y) \rangle = \theta(y^*).$$

Since  $\mathfrak{N}_{\widehat{\varphi}}$  is dense in  $\widehat{M}$ , it follows that  $\theta = \omega$ , in  $\widehat{M}_*$ . From this, it follows that  $\widehat{\lambda}(\theta) = \widehat{\lambda}(\omega)$ , in  $M$ . In other words,  $\mathcal{F}^{-1}(\mathcal{F}(a)) = a \in \widehat{\lambda}(\widehat{\mathcal{I}}) (\subseteq M)$ . In exactly the same way, we can also prove that  $\mathcal{F}(\mathcal{F}^{-1}(b)) = b \in \lambda(\mathcal{I}) (\subseteq \widehat{M})$ .  $\blacksquare$

Below is our version of the “Plancherel formula”, in terms of the Haar weights. Since the vectors  $\Lambda(a)$ ,  $a \in \widehat{\lambda}(\widehat{\mathcal{I}})$ , and  $\widehat{\Lambda}(b)$ ,  $b \in \lambda(\mathcal{I})$ , are dense in the Hilbert space  $\mathcal{H}$ , the following proposition implies that the maps  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  can be considered as unitary maps on  $\mathcal{H}$  (as is to be expected).

**PROPOSITION 3.9.** *Let  $\widehat{\lambda}(\widehat{\mathcal{I}}) \subseteq M$  and  $\lambda(\mathcal{I}) \subseteq \widehat{M}$  be the (dense) subalgebras, as defined earlier. Then we have:*

- (i) For  $a \in \widehat{\lambda}(\widehat{\mathcal{I}}) (\subseteq M)$ , we have  $\widehat{\varphi}(\mathcal{F}(a)^* \mathcal{F}(a)) = \varphi(a^* a)$ .
- (ii) For  $b \in \lambda(\mathcal{I}) (\subseteq \widehat{M})$ , we have  $\varphi(\mathcal{F}^{-1}(b)^* \mathcal{F}^{-1}(b)) = \widehat{\varphi}(b^* b)$ .

*Proof.* Recall that for  $a \in \widehat{\lambda}(\widehat{\mathcal{I}})$ , we have  $\widehat{\Lambda}(\mathcal{F}(a)) = \Lambda(a) \in \mathcal{H}$  (see Proposition 3.5). It follows that:

$$\widehat{\varphi}(\mathcal{F}(a)^* \mathcal{F}(a)) = \langle \widehat{\Lambda}(\mathcal{F}(a)), \widehat{\Lambda}(\mathcal{F}(a)) \rangle = \langle \Lambda(a), \Lambda(a) \rangle = \varphi(a^* a).$$

The second equation can be proved in exactly the same way. ■

Meanwhile, remembering from classical analysis that the Fourier transform converts multiplication of functions to convolution product, and vice versa, we can use our Fourier transform to define the “convolution product” on the quantum group  $(M, \Delta)$ :

DEFINITION 3.10. For  $a, c \in \widehat{\lambda}(\widehat{\mathcal{I}}) (\subseteq M)$ , define their *convolution product*, written  $a * c$ , by

$$a * c := \mathcal{F}^{-1}(\mathcal{F}(a)\mathcal{F}(c)).$$

Since  $\mathcal{F}(a)$  and  $\mathcal{F}(c)$  are contained in  $\lambda(\mathcal{I})$ , and since  $\lambda(\mathcal{I})$  is a subalgebra of  $\widehat{M}$ , we have  $\mathcal{F}(a)\mathcal{F}(c) \in \lambda(\mathcal{I})$ . So by Proposition 3.7, the expression  $\mathcal{F}^{-1}(\mathcal{F}(a)\mathcal{F}(c))$  is well-defined, and is contained in  $\widehat{\lambda}(\widehat{\mathcal{I}})$ . While this is certainly all valid, some more discussion will be helpful for further understanding of the convolution product. See the result below.

PROPOSITION 3.11. *Suppose  $a, c \in \widehat{\lambda}(\widehat{\mathcal{I}}) \subseteq M$ , and let  $a * c \in \widehat{\lambda}(\widehat{\mathcal{I}})$  be their convolution product. Then we have the following alternative description:*

$$a * c = (\varphi \otimes \text{id})([(S^{-1} \otimes \text{id})(\Delta c)](a \otimes 1)).$$

*Proof.* For  $a, c \in \widehat{\lambda}(\widehat{\mathcal{I}})$ , we know from Proposition 3.5 that we can write  $\mathcal{F}(a) = (\omega_1 \otimes \text{id})(W)$  and  $\mathcal{F}(c) = (\omega_2 \otimes \text{id})(W)$ , where  $\omega_1 = \varphi(\cdot a) \in \mathcal{I}$  and  $\omega_2 = \varphi(\cdot c) \in \mathcal{I}$ . Then by Lemma 3.1 (see also [2]), we can write

$$\mathcal{F}(a)\mathcal{F}(c) = (\omega_1 \otimes \text{id})(W)(\omega_2 \otimes \text{id})(W) = (\mu \otimes \text{id})(W) \in \lambda(\mathcal{I}),$$

where  $\mu \in M_*$  is given by  $\mu(x) = (\omega_1 \otimes \omega_2)(\Delta(x))$ . So for any  $x \in \mathfrak{N}_\varphi$ , we have:

$$\mu(x^*) = (\omega_1 \otimes \omega_2)(\Delta(x^*)) = (\varphi \otimes \varphi)(\Delta(x^*)(a \otimes c)) = \varphi([( \text{id} \otimes \varphi)(\Delta(x^*)(1 \otimes c))])a.$$

Recall now the “strong” left invariance property (Proposition 5.40 of [6]), which says that for any  $x, c \in \mathfrak{N}_\varphi$ , it is known that  $(\text{id} \otimes \varphi)(\Delta(x^*)(1 \otimes c))$  is contained in the domain of the antipode  $S$ , and that

$$S((\text{id} \otimes \varphi)(\Delta(x^*)(1 \otimes c))) = (\text{id} \otimes \varphi)((1 \otimes x^*)(\Delta c)).$$

Combining this result together with the result we obtained above, we have, for  $a, c \in \widehat{\lambda}(\widehat{\mathcal{I}})$  and any  $x \in \mathfrak{N}_\varphi$ , the following:

$$\begin{aligned} \mu(x^*) &= \varphi([( \text{id} \otimes \varphi)(\Delta(x^*)(1 \otimes c))])a = \varphi([S^{-1}((\text{id} \otimes \varphi)((1 \otimes x^*)(\Delta c)))]a) \\ &= (\varphi \otimes \varphi)((1 \otimes x^*)[(S^{-1} \otimes \text{id})(\Delta c)](a \otimes 1)) \\ &= \varphi(x^*(\varphi \otimes \text{id})([(S^{-1} \otimes \text{id})(\Delta c)](a \otimes 1))). \end{aligned}$$

Letting  $z = (\varphi \otimes \text{id})([(S^{-1} \otimes \text{id})(\Delta c)](a \otimes 1))$ , we see that:  $\mu(x^*) = \varphi(x^*z)$ , for any  $x \in \mathfrak{N}_\varphi$ . Since  $\mu \in \mathcal{I}$ , the Fourier inversion theorem implies that in fact,  $z \in \widehat{\lambda}(\widehat{\mathcal{I}})$  and that  $\mathcal{F}(z) = (\mu \otimes \text{id})(W)$ . Or,  $\mathcal{F}(z) = \mathcal{F}(a)\mathcal{F}(c)$ .

Remembering Definition 3.10, we conclude that:

$$a * c = \mathcal{F}^{-1}(\mathcal{F}(a)\mathcal{F}(c)) = z = (\varphi \otimes \text{id})([(S^{-1} \otimes \text{id})(\Delta c)](a \otimes 1)). \quad \blacksquare$$

With only a slight change in the formulation and the proof, we can define the convolution product also on the dual quantum group. In the below is the corresponding result:

**PROPOSITION 3.12.** *For  $b, d \in \lambda(\mathcal{I}) \subseteq \widehat{M}$ , we can define their “convolution product”, written  $b * d$ , by*

$$b * d := \mathcal{F}(\mathcal{F}^{-1}(b)\mathcal{F}^{-1}(d)).$$

*Then we have the following alternative description:*

$$b * d = (\widehat{\varphi} \otimes \text{id})([\widehat{S}^{-1} \otimes \text{id})(\widehat{\Delta}d)](b \otimes 1).$$

We skip the proof here, since it is really no different from the one given in Proposition 3.11. For this, (iii) of Lemma 3.1 will be useful. Meanwhile, see Appendix (Section 5) for a discussion that these expressions for the convolution products are natural generalizations of the ordinary convolution product in classical analysis.

#### 4. THE DUAL PAIRING

For a finite dimensional Hopf algebra  $A$ , its dual object is none other than the dual vector space  $A'$ , equipped with the Hopf algebra structure obtained naturally from that of  $A$  [1]. In general, however, a typical quantum group would be infinite dimensional, and in that case, the dual vector space is too big to be given any reasonable structure. Though there are often ways to get around this problem to define a dual pairing map, things are more tricky for the analytical settings, where the quantum groups are required to have additional ( $C^*$ -algebra or von Neumann algebra) structure.

It turns out that in the locally compact quantum group framework, the dual pairing between a quantum group  $M$  and its dual  $\widehat{M}$  is only defined at a dense subalgebra level, using the multiplicative unitary operator. It is certainly a correct definition, being a natural generalization of the obvious dual pairing between  $A$  and  $A'$  in the finite-dimensional case. However, the way the pairing is defined makes it rather difficult to work with, and we often have to devise some indirect ways to get around this problem. In this section, we show an alternative description of the dual pairing using the Haar weight, motivated by the Fourier transform. This new description of the dual pairing may be useful in some future research projects.

Let us begin by recalling the definition of the dual pairing, at the level of the dense subalgebras  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  (see Section 3) of the quantum groups  $(M, \Delta)$  and  $(\widehat{M}, \widehat{\Delta})$ . That is, for  $a = (\text{id} \otimes \theta)(W) \in \mathcal{A}$  and  $b = (\omega \otimes \text{id})(W) \in \widehat{\mathcal{A}}$ , we have:

$$(4.1) \quad \langle b | a \rangle = \langle (\omega \otimes \text{id})(W) | (\text{id} \otimes \theta)(W) \rangle := (\omega \otimes \theta)(W) = \omega(a) = \theta(b).$$

The definition is suggested by [2]. The properties of this dual pairing map is given below in Proposition 4.2.

REMARK 4.1. Let us point out here the difference in conventions between pure algebra and the operator algebra settings. In purely algebraic frameworks (Hopf algebras, QUE algebras, and multiplier Hopf algebras), the dual comultiplication on  $A'$  is simply obtained by dualizing the product on  $A$  via the pairing map. On the other hand, in the setting of locally compact quantum groups, the definition of the dual comultiplication on  $\widehat{M}$  (as reviewed in Section 2) is actually “flipped”. This results in the dual pairing given in equation (4.1) to become a “skew” pairing, in the sense of item (ii) of Proposition 4.2 below, as well as the dual antipode in item (iii) below appearing with an inverse. This “opposite” way the comultiplication has been chosen on  $\widehat{M}$  also explains the appearance of  $\widehat{\varphi}$  (as opposed to  $\widehat{\psi}$ ) in Definition 3.6 earlier of the inverse Fourier transform.

PROPOSITION 4.2. *Let  $(M, \Delta)$  and  $(\widehat{M}, \widehat{\Delta})$  be the dual pair of locally compact quantum groups, and let  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  be their dense subalgebras, as defined earlier. Then the map  $\langle \cdot | \cdot \rangle : \widehat{\mathcal{A}} \times \mathcal{A} \rightarrow \mathbb{C}$ , given by equation (4.1), is a valid dual pairing. Moreover, we have:*

- (i)  $\langle b_1 b_2 | a \rangle = \langle b_1 \otimes b_2 | \Delta(a) \rangle$ , for  $a \in \mathcal{A}$ ,  $b_1, b_2 \in \widehat{\mathcal{A}}$ .
- (ii)  $\langle b | a_1 a_2 \rangle = \langle \widehat{\Delta}^{\text{cop}}(b) | a_1 \otimes a_2 \rangle$ , for  $a_1, a_2 \in \mathcal{A}$ ,  $b \in \widehat{\mathcal{A}}$ .
- (iii)  $\langle b | S(a) \rangle = \langle \widehat{S}^{-1}(b) | a \rangle$ , for  $a \in \mathcal{A}$ ,  $b \in \widehat{\mathcal{A}}$ .

*Proof.* Bilinearity of  $\langle \cdot, \cdot \rangle$  is obvious. So let us just prove the three equalities. Let  $a = (\text{id} \otimes \theta)(W) \in \mathcal{A}$ , and suppose that  $b_1 = (\omega_1 \otimes \text{id})(W) \in \widehat{\mathcal{A}}$  and  $b_2 = (\omega_2 \otimes \text{id})(W) \in \widehat{\mathcal{A}}$ . Then:

$$\begin{aligned} \langle b_1 b_2 | a \rangle &= \langle (\omega_1 \otimes \text{id})(W)(\omega_2 \otimes \text{id})(W) | (\text{id} \otimes \theta)(W) \rangle \\ &= \langle (\mu \otimes \text{id})(W) | (\text{id} \otimes \theta)(W) \rangle = (\mu \otimes \theta)(W) \\ &= \mu((\text{id} \otimes \theta)(W)) = (\omega_1 \otimes \omega_2)(W^*[\text{id} \otimes (\text{id} \otimes \theta)(W)]W) \\ &= (\omega_1 \otimes \omega_2)(\Delta a) = \langle b_1 \otimes b_2 | \Delta(a) \rangle. \end{aligned}$$

The second equality follows from the multiplicativity of  $W$  (see Lemma 3.1). The definition of  $\Delta a$  (for  $a \in \mathcal{A} \subseteq M$ ) was used in the next to last equality.

Now let  $a_1 = (\text{id} \otimes \theta_1)(W) \in \mathcal{A}$  and  $a_2 = (\text{id} \otimes \theta_2)(W) \in \mathcal{A}$ , while  $b = (\omega \otimes \text{id})(W) \in \widehat{\mathcal{A}}$ . Then:

$$\begin{aligned} \langle b | a_1 a_2 \rangle &= \langle (\omega \otimes \text{id})(W) | (\text{id} \otimes \theta_1)(W)(\text{id} \otimes \theta_2)(W) \rangle \\ &= \langle (\omega \otimes \text{id})(W) | (\text{id} \otimes \nu)(W) \rangle = (\omega \otimes \nu)(W) = \nu((\omega \otimes \text{id})(W)), \end{aligned}$$

where  $\nu \in \widehat{M}_*$  is such that  $\nu(y) := (\theta_1 \otimes \theta_2)(W(y \otimes 1)W^*)$ . This again follows from the multiplicativity of  $W$  (see again Lemma 3.1). So we have:

$$\langle b | a_1 a_2 \rangle = \nu(b) = (\theta_1 \otimes \theta_2)(\widehat{\Delta}^{\text{cop}}(b)) = \langle \widehat{\Delta}^{\text{cop}}(b) | a_1 \otimes a_2 \rangle.$$

To prove the third equation, let  $a = (\text{id} \otimes \theta)(W) \in \mathcal{A}$ , and suppose that  $b = (\omega \otimes \text{id})(W) \in \widehat{\mathcal{A}}$ . Then by equation (2.5), we know that  $(\omega \otimes \text{id})(W^*)$  is contained in  $\mathcal{D}(\widehat{S})$  and that  $\widehat{S}((\omega \otimes \text{id})(W^*)) = b$ . If, in particular,  $\omega \in M_*^\sharp$  and  $\theta \in \widehat{M}_*^\sharp$  (see Lemma 3.3), then we can further write

$$\widehat{S}^{-1}(b) = (\omega \otimes \text{id})(W^*) = ((\overline{\omega} \otimes \text{id})(W))^* = (\overline{\omega}^\sharp \otimes \text{id})(W),$$

and also  $S(a) = (\text{id} \otimes \theta)(W^*) = (\text{id} \otimes \overline{\theta}^\sharp)(W)$ . Then we have:

$$\begin{aligned} \langle \widehat{S}^{-1}(b) | a \rangle &= \langle (\overline{\omega}^\sharp \otimes \text{id})(W) | (\text{id} \otimes \theta)(W) \rangle = (\overline{\omega}^\sharp \otimes \theta)(W) \\ &= \theta((\overline{\omega}^\sharp \otimes \text{id})(W)) = \theta((\omega \otimes \text{id})(W^*)) = \omega((\text{id} \otimes \theta)(W^*)) \\ &= \omega[S[(\text{id} \otimes \theta)(W)]] = \omega(S(a)) = \langle b | S(a) \rangle. \end{aligned}$$

Since such elements are dense, we conclude that:  $\langle b | S(a) \rangle = \langle \widehat{S}^{-1}(b) | a \rangle$ , for all  $a \in \mathcal{A}, b \in \widehat{\mathcal{A}}$ . ■

Except for the appearance of the co-opposite comultiplication  $\widehat{\Delta}^{\text{cop}}$ , the previous proposition shows that  $\langle \cdot | \cdot \rangle$  is a suitable dual pairing map that corresponds to the pairing map on (finite-dimensional) Hopf algebras. In what follows, we will give an alternative description of this pairing map, using Haar weights.

**PROPOSITION 4.3.** *Let  $a \in \widehat{\lambda}(\widehat{\mathcal{T}})$  and  $b \in \lambda(\mathcal{T})$  be such that we can write  $\mathcal{F}(a) = (\omega \otimes \text{id})(W)$ , and  $\mathcal{F}^{-1}(b) = (\text{id} \otimes \theta)(W^*)$ . Assume further that  $\theta \in \widehat{M}_*^\sharp$  (as in Lemma 3.3). Then  $\mathcal{F}(a) \in \widehat{\mathcal{A}}, \mathcal{F}^{-1}(b) \in \mathcal{A}$ , and we have:*

$$\langle \mathcal{F}(a) | \mathcal{F}^{-1}(b) \rangle = \langle \widehat{\Lambda}(\mathcal{F}(a)), \Lambda(\mathcal{F}^{-1}(b)^*) \rangle.$$

*Proof.* It is obvious that  $\mathcal{F}(a) = (\omega \otimes \text{id})(W) \in \widehat{\mathcal{A}}$ . Whereas,  $\theta \in \widehat{M}_*^\sharp$  implies that  $\mathcal{F}^{-1}(b) = (\text{id} \otimes \theta)(W^*) = (\text{id} \otimes \theta^\sharp)(W^*)^* = (\text{id} \otimes \overline{\theta}^\sharp)(W) \in \mathcal{A}$ .

Meanwhile, by equation (2.4) and Proposition 3.5, we have:

$$\langle \widehat{\Lambda}(\mathcal{F}(a)), \Lambda(\mathcal{F}^{-1}(b)^*) \rangle = \omega(\mathcal{F}^{-1}(b)).$$

In addition, by writing  $\mathcal{F}^{-1}(b)^* = (\text{id} \otimes \theta^\sharp)(W^*)$ , we have

$$\begin{aligned} \langle \widehat{\Lambda}(\mathcal{F}(a)), \Lambda(\mathcal{F}^{-1}(b)^*) \rangle &= \overline{\langle \Lambda(\mathcal{F}^{-1}(b)^*), \widehat{\Lambda}(\mathcal{F}(a)) \rangle} = \overline{\langle \Lambda((\text{id} \otimes \theta^\sharp)(W^*)), \widehat{\Lambda}(\mathcal{F}(a)) \rangle} \\ &= \overline{\theta^\sharp(\mathcal{F}(a)^*)} = \overline{\theta^\sharp(\mathcal{F}(a))}, \end{aligned}$$

where we used equation (2.6) and Proposition 3.7. Combining these results, we obtain the following:

$$\begin{aligned} \langle \widehat{\Lambda}(\mathcal{F}(a)), \Lambda(\mathcal{F}^{-1}(b)^*) \rangle &= \omega(\mathcal{F}^{-1}(b)) = \overline{\theta^\sharp}(\mathcal{F}(a)) = \omega((\text{id} \otimes \overline{\theta^\sharp})(W)) = \overline{\theta^\sharp}((\omega \otimes \text{id})(W)) \\ &= (\omega \otimes \overline{\theta^\sharp})(W) = \langle (\omega \otimes \text{id})(W) \mid (\text{id} \otimes \overline{\theta^\sharp})(W) \rangle \\ &= \langle \mathcal{F}(a) \mid \mathcal{F}^{-1}(b) \rangle. \quad \blacksquare \end{aligned}$$

Observe in the proof above that we needed to work with elements in  $\widehat{\lambda}(\widehat{M}_*^\sharp)$  to ensure that taking involution is valid. As Lemma 3.3 shows, this is closely related to the antipode operation: in fact, it can be shown that  $\theta^\sharp = \overline{\theta} \circ \widehat{S}$ . Considering these, our preferred subspaces from now on will be:  $D := \widehat{\lambda}(\widehat{\mathcal{I}} \cap \widehat{M}_*^\sharp) \subseteq \mathcal{A}$ , and  $\widehat{D} := \lambda(\mathcal{I} \cap M_*^\sharp) \subseteq \widehat{\mathcal{A}}$ . Actually,  $D$  and  $\widehat{D}$  are dense subalgebras in  $M$  and  $\widehat{M}$ , respectively (See Lemma 3.3, and see also Proposition 2.6 of [7]). For elements in  $D$  and  $\widehat{D}$ , the following holds.

**COROLLARY 4.4.** *Let  $D \subseteq \mathcal{A}$  and  $\widehat{D} \subseteq \widehat{\mathcal{A}}$  be as defined above, and let  $a \in D$  and  $b \in \widehat{D}$ . Then we have the following description of the dual pairing, in terms of the inner product:*

$$\langle b \mid a \rangle = \langle \widehat{\Lambda}(b), \Lambda(a^*) \rangle.$$

*Proof.* By Fourier inversion theorem (see Theorem 3.8), we can write:  $a = \mathcal{F}^{-1}(\mathcal{F}(a))$  and  $b = \mathcal{F}(\mathcal{F}^{-1}(b))$ . So we are able to use the result of the previous proposition.  $\blacksquare$

Finally, we have the following alternative (up till now not appeared in the literature) description of the dual pairing, given in terms of the Haar weights:

**THEOREM 4.5.** *Let  $D := \widehat{\lambda}(\widehat{\mathcal{I}} \cap \widehat{M}_*^\sharp) \subseteq \mathcal{A}$  and  $\widehat{D} := \lambda(\mathcal{I} \cap M_*^\sharp) \subseteq \widehat{\mathcal{A}}$  be the dense subalgebras defined earlier. The dual pairing map  $\langle \cdot \mid \cdot \rangle : \widehat{\mathcal{A}} \times \mathcal{A} \rightarrow \mathbb{C}$  given in Proposition 4.2 takes the following form, if we restrict it to the level of the subspaces  $D$  and  $\widehat{D}$ :*

$$\langle b \mid a \rangle = \varphi(a\mathcal{F}^{-1}(b)) = \widehat{\varphi}(\mathcal{F}(a^*)^*b) = (\varphi \otimes \widehat{\varphi})[(a \otimes 1)W^*(1 \otimes b)].$$

Of course, it will satisfy the properties (i), (ii), (iii) of Proposition 4.2.

*Proof.* Recall that the Fourier inversion theorem is very much valid in spaces  $D$  and  $\widehat{D}$ . So for  $a \in D$  and  $b \in \widehat{D}$ , we can use the result of Proposition 4.3 and its corollary above that  $\langle b \mid a \rangle = \langle \widehat{\Lambda}(b), \Lambda(a^*) \rangle$ . Now note from Proposition 3.7 that  $\widehat{\Lambda}(b) = \Lambda(\mathcal{F}^{-1}(b))$  in  $\mathcal{H}$ . Thus we have:

$$\langle b \mid a \rangle = \langle \widehat{\Lambda}(b), \Lambda(a^*) \rangle = \langle \Lambda(\mathcal{F}^{-1}(b)), \Lambda(a^*) \rangle = \varphi(a\mathcal{F}^{-1}(b)).$$

Similarly, we have  $\langle b \mid a \rangle = \langle \widehat{\Lambda}(b), \Lambda(a^*) \rangle = \langle \widehat{\Lambda}(b), \widehat{\Lambda}(\mathcal{F}(a^*)) \rangle = \widehat{\varphi}(\mathcal{F}(a^*)^*b)$ . Since we know from Definition 3.6 that  $\mathcal{F}^{-1}(b) = (\text{id} \otimes \widehat{\varphi})(W^*(1 \otimes b))$ , and since Definition 3.4 implies that  $\mathcal{F}(a^*)^* = (\varphi \otimes \text{id})((a \otimes 1)W^*)$ , we conclude that:

$$\langle b \mid a \rangle = \varphi(a\mathcal{F}^{-1}(b)) = \widehat{\varphi}(\mathcal{F}(a^*)^*b) = (\varphi \otimes \widehat{\varphi})[(a \otimes 1)W^*(1 \otimes b)]. \quad \blacksquare$$

## 5. APPENDIX: CASE OF AN ORDINARY GROUP

In this Appendix, we will show how some of the results in the earlier sections are manifested in the case of an ordinary locally compact group. Obviously, the results here will be mostly familiar. On the other hand, this exercise will give us a clearer understanding of the picture, and will provide a further justification of the definitions we chose in the earlier sections.

From now on, let  $G$  be a locally compact group, with a fixed left Haar measure,  $dx$ . Let  $\mathcal{H}$  denote the Hilbert space  $L^2(G)$ . Let us work with the two subalgebras,  $M$  and  $\widehat{M}$ , of  $\mathcal{B}(\mathcal{H})$ , as follows.

First consider the commutative von Neumann algebra  $M = L^\infty(G)$ , where  $a \in L^\infty(G)$  is viewed as the multiplication operator  $\pi_a$  on  $\mathcal{H} = L^2(G)$ , given by  $\pi_a \xi(x) = a(x)\xi(x)$ . Next consider the group von Neumann algebra  $\widehat{M} = \mathcal{L}(G)$ , given by the left regular representation. That is, for  $b \in C_c(G)$ , let  $L_b \in \mathcal{B}(\mathcal{H})$  be such that  $L_b \xi(x) = \int b(z)\xi(z^{-1}x) dz$ . We take  $\mathcal{L}(G)$  to be the  $W^*$ -closure of  $L(C_c(G))$ . These are well-known von Neumann algebras, and it is also rather well-known that we can give (mutually dual) quantum group structures on them. We briefly review the results below.

Let  $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) = \mathcal{B}(L^2(G \times G))$  be defined by  $W\xi(x, y) = \xi(x, x^{-1}y)$ . It is easy to show that it is a multiplicative unitary operator. We also have

$$M = \overline{\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^w,$$

and the comultiplication on  $M$  is given by  $\Delta c = W^*(1 \otimes c)W$ , for  $c \in M$ . In effect, this will give us  $(\Delta a)(x, y) = a(xy)$ , for  $a \in L^\infty(G)$ . The antipode map  $S : a \rightarrow S(a)$  is such that  $S(a)(x) = a(x^{-1})$ , while the left Haar weight is just  $\varphi(a) = \int a(x) dx$ . In this way,  $(M, \Delta)$  becomes a (commutative) von Neumann algebraic quantum group.

Meanwhile, we can show without difficulty that

$$\widehat{M} = \overline{\{(\omega \otimes \text{id})(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^w,$$

and the comultiplication on  $\widehat{M}$  is given by  $\widehat{\Delta}d = \Sigma W(d \otimes 1)W^*\Sigma$ , for  $d \in \widehat{M}$ . For  $b \in C_c(G)$ , this reads  $(L \otimes L)_{\widehat{\Delta}b} \xi(x, y) = \int b(t)\xi(t^{-1}x, t^{-1}y) dt$ . The antipode map is  $\widehat{S} : b \rightarrow \widehat{S}(b)$  such that  $\widehat{S}(b)(x) = \delta(x^{-1})b(x^{-1})$ , where  $\delta$  is the modular function. And, the left Haar weight is given by  $\widehat{\varphi}(b) = b(1)$ , where  $1 = 1_G$  is the group identity element. In this way, we have now a (co-commutative) von Neumann algebraic quantum group  $(\widehat{M}, \widehat{\Delta})$ .

As in Proposition 4.2, a dual pairing map can be considered at the level of certain dense subalgebras of  $\widehat{M}$  and  $M$ . For convenience, consider  $\pi(C_c(G)) \subseteq M$  and  $L(C_c(G)) \subseteq \widehat{M}$ . The dual pairing defined by the multiplicative unitary operator  $W$ , as given in equation (4.1), becomes

$$(5.1) \quad \langle L_b | \pi_a \rangle = \int a(x)b(x) dx,$$

for  $\pi_a \in \pi(C_c(G))$  and  $b \in L(C_c(G))$ . We will skip the proof here (see Proposition 5.4 instead), though it is really not very difficult, using the operator  $W$  defined above.

Now that we are pretty much through with the brief review, let us turn our attention to our aim of interpreting the results from earlier sections in this current setting of  $L^\infty(G)$  and  $\mathcal{L}(G)$ .

**PROPOSITION 5.1.** *Let  $M=L^\infty(G)$  and  $\widehat{M}=\mathcal{L}(G)$  be the mutually dual quantum groups as above, and consider the subalgebras  $\pi(C_c(G)) \subseteq M$  and  $L(C_c(G)) \subseteq \widehat{M}$ . Then the Fourier transform and the inverse Fourier transform of Section 3 reads as follows:*

- (i) For  $a \in C_c(G)$ , we have  $\pi_a \in M$  and  $\mathcal{F}(\pi_a) = L_a \in \widehat{M}$ .
- (ii) For  $b \in C_c(G)$ , we have  $L_b \in \widehat{M}$  and  $\mathcal{F}^{-1}(L_b) = \pi_b \in M$ .

The Fourier inversion theorem is obvious.

*Proof.* For  $a \in C_c(G)$  and any  $\xi \in \mathcal{H}$ , we have, by Definition 3.4 that:

$$\mathcal{F}(\pi_a)\xi(y) = (\varphi \otimes \text{id})(W(\pi_a \otimes 1))\xi(y).$$

Remembering the definitions of  $W$  and  $\varphi$  given above, it becomes:

$$\mathcal{F}(\pi_a)\xi(y) = \int a(x)\xi(x^{-1}y) dx = L_a\xi(y).$$

From this, we obtain  $\mathcal{F}(\pi_a) = L_a$ .

Meanwhile, for  $b \in C_c(G)$  and any  $\xi \in \mathcal{H}$ , we have, by Definition 3.6 that:

$$\mathcal{F}^{-1}(L_b)\xi(x) = (\text{id} \otimes \widehat{\varphi})(W^*(1 \otimes b)).$$

Since  $W^*\xi(x, y) = \xi(x, xy)$  and since  $\widehat{\varphi}(b) = b(1)$ , this becomes:

$$\mathcal{F}^{-1}(L_b)\xi(x) = b(x \cdot 1)\xi(x) = b(x)\xi(x) = \pi_b\xi(x).$$

We thus obtain  $\mathcal{F}^{-1}(L_b) = \pi_b$ . ■

What this proposition shows is that under the Fourier transform, not much seems to be happening at the level of functions in  $C_c(G)$ , and in turn, also at the level of the Hilbert space  $L^2(G)$  (see Proposition 5.2 below). The reason for this “trivialization” phenomenon is due to the way the general theory of locally compact quantum groups works with the same Hilbert space (in this case  $\mathcal{H} = L^2(G)$ ) for both  $M$  and  $\widehat{M}$ . In the abelian locally compact group case, this would have been the same as saying that  $L^2(G) = L^2(\widehat{G})$ , making the Fourier transform rather “hidden”, thereby hindering its development so far.

In the abelian case (for instance, when  $G = \mathbb{R}$ ), the proper approach in abstract harmonic analysis is to use the Fourier transform to obtain the spatial isomorphism,  $F : L^2(G) \cong L^2(\widehat{G})$ , not an identification. In that setting, the above result can be given a further interpretation: Namely, for  $a \in C_c(G)$ , we have  $\mathcal{F}(\pi_a) = L_a \cong \pi_{\widehat{a}}$ , where  $\widehat{a}$  is the usual Fourier transform of  $a \in C_c(G)$  while the last equivalence is given by the spatial isomorphism between  $L^2(G)$  and  $L^2(\widehat{G})$ . Or,  $\mathcal{F}(\pi_a) = F^{-1}\pi_{\widehat{a}}F$ . (This is relatively easy to verify.) This remark certainly



gives a justification of our map  $\mathcal{F}$  being a suitable generalization of the ordinary Fourier transform.

The next proposition concerns the Plancherel formula, as in Proposition 3.9.

PROPOSITION 5.2. *As before, let  $a \in C_c(G)$  and  $b \in C_c(G)$ . Then:*

- (i)  $\widehat{\varphi}(\mathcal{F}(\pi_a)^* \mathcal{F}(\pi_a)) = \varphi(\pi_a^* \pi_a) = \|a\|_2^2$ .
- (ii)  $\varphi(\mathcal{F}^{-1}(L_b)^* \mathcal{F}^{-1}(L_b)) = \widehat{\varphi}(L_b^* L_b) = \|b\|_2^2$ .

*Proof.* We just need to use the result of Proposition 3.9. Meanwhile, remember that for  $a \in L^\infty(G)$ , the involution is given by  $a^*(x) = \overline{a(x)}$ . Whereas, for  $L_b \in \mathcal{L}(G)$ , recall  $(L_b)^* = L_{b^*}$ , where  $b^*(x) = \delta(x^{-1})\overline{b(x^{-1})}$ . ■

The convolution product (see Definition 3.10 and Proposition 3.11) will also take the familiar form. The result below confirms that the formula given in Proposition 3.11 is a natural generalization of the familiar classical convolution product. But of course, we could have obtained the same result using Definition 3.10 together with Proposition 5.1.

PROPOSITION 5.3. (i) *For  $a, c \in C_c(G) \subseteq L^\infty(G)$ , their convolution product reads  $(a * c)(y) = \int a(x)c(x^{-1}y) dx$ .*

(ii) *For  $L_b, L_d \in L(C_c(G)) \subseteq \mathcal{L}(G)$ , their convolution product is  $(b * d)(x) = b(x)d(x)$ .*

*Proof.* We will use the results from Section 3 and the definitions from the earlier part of this Appendix. Note by the way that in our special case,  $S^{-1} = S$  and also  $\widehat{S}^{-1} = \widehat{S}$ . So for  $a, c \in C_c(G) \subseteq L^\infty(G)$ , we have:

$$(a * c)(y) = (\varphi \otimes \text{id})([(S^{-1} \otimes \text{id})(\Delta c)](a \otimes 1)) = \int c(x^{-1}y)a(x) dx.$$

And, for  $L_b, L_d \in L(C_c(G)) \subseteq \mathcal{L}(G)$  and any  $\xi \in L^2(G)$ , observe that we have the following, where  $bd(t) = b(t)d(t)$ :

$$\begin{aligned} L_{b*d}\xi(y) &= (\widehat{\varphi} \otimes \text{id})([( \widehat{S}^{-1} \otimes \text{id})(L \otimes L)_{\Delta d}](L_b \otimes 1))\xi(y) \\ &= \int d(t)b(t \cdot 1)\xi(t^{-1}y) dt = \int b(t)d(t)\xi(t^{-1}y) dt = L_{bd}\xi(y). \quad \blacksquare \end{aligned}$$

Finally, let us observe how the new description of the dual pairing given in Theorem 4.5 plays out in our special case.

PROPOSITION 5.4. *For  $\pi_a \in \pi(C_c(G)) \subseteq L^\infty(G)$  and  $L_b \in L(C_c(G)) \subseteq \mathcal{L}(G)$ , the dual pairing can be given by the formula in Theorem 4.5:*

$$\langle L_b | \pi_a \rangle = (\varphi \otimes \widehat{\varphi})[(a \otimes 1)W^*(1 \otimes b)] = \varphi(\pi_a \mathcal{F}^{-1}(L_b)) = \widehat{\varphi}(\mathcal{F}(\pi_a^*)^* L_b) = \int a(x)b(x) dx,$$

which agrees with the equation (5.1).

*Proof.* By Proposition 5.1, we know that  $\mathcal{F}^{-1}(L_b) = \pi_b$ . Therefore, we have:

$$\langle L_b | \pi_a \rangle = \varphi(\pi_a \pi_b) = \varphi(\pi_{ab}) = \int a(x)b(x) dx.$$

Alternatively, with  $\mathcal{F}(\pi_a^*) = L_{\bar{a}}$ , we arrive at the same result by

$$\langle L_b | \pi_a \rangle = \widehat{\varphi}((L_{\bar{a}})^* L_b) = \int \delta(x^{-1}) a(x^{-1}) b(x^{-1} \cdot 1) dx = \int a(x) b(x) dx. \quad \blacksquare$$

As the proof of this last proposition shows, the characterization of the dual pairing we obtained in Section 3, via the Haar weight, inner product and the Fourier transform (see Corollary of Proposition 4.3 and Theorem 4.5), can be quite useful: compared to the approach based only on the definition, the new characterization may often provide us a simpler approach to results involving the dual pairing. This aspect may turn out to be useful in the future, in more complicated cases (for instance, see [5]).

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