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COMMUTATOR IDEALS OF SUBALGEBRAS OF TOEPLITZ ALGEBRAS ON WEIGHTED BERGMAN SPACES

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ABSTRACT. For $\alpha > -1$, let A^2_α denote the corresponding weighted Bergman space of the unit ball. For any self-adjoint subset $G \subset L^\infty$, let $\mathfrak{T}(G)$ denote the C^*-subalgebra of $\mathfrak{B}(A^2_\alpha)$ generated by $\{T_f : f \in G\}$. Let $\mathfrak{C}\mathfrak{T}(G)$ denote the commutator ideal of $\mathfrak{T}(G)$. It was showed by D. Suárez (in 2004 for $n = 1$) and by the author (in 2006 for all $n \geqslant 1$) that $\mathfrak{C}\mathfrak{T}(L^\infty) = \mathfrak{T}(L^\infty)$ in the case $\alpha = 0$. In this paper we show that in the setting of weighted Bergman spaces, the identity $\mathfrak{C}\mathfrak{T}(G) = \mathfrak{T}(G)$ holds true for a class of subsets G including L^∞.

KEYWORDS: *Commutator ideals, Toeplitz operators, weighted Bergman spaces.*

MSC (2000): Primary 47B35; Secondary 47B47.

1. INTRODUCTION

For any integer $n \geqslant 1$, let $\mathbb{C}^n$ denote the Cartesian product of n copies of $\mathbb{C}$. For any $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$ and $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for the inner product and the associated Euclidean norm. Let $\mathbb{B}_n$ denote the open unit ball which consists of all $z \in \mathbb{C}^n$ with $|z| < 1$. Let $\mathbb{S}_n$ denote the unit sphere which consists of all $z \in \mathbb{C}^n$ with $|z| = 1$. We write $\overline{\mathbb{B}}_n$ for the closed unit ball which is $\mathbb{B}_n \cup \mathbb{S}_n$. Let $C(\mathbb{B}_n)$ (respectively, $C(\overline{\mathbb{B}}_n)$) denote the space of all functions that are continuous in the open unit ball (respectively, the closed unit ball).

Let ν denote the Lebesgue measure on $\mathbb{B}_n$ normalized so that $\nu(\mathbb{B}_n) = 1$. Fix a real number $\alpha > -1$. The weighted Lebesgue measure ν_α on $\mathbb{B}_n$ is defined by $\mathrm{d}\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \mathrm{d}\nu(z)$, where c_α is a normalizing constant so that $\nu_\alpha(\mathbb{B}_n) = 1$. A direct computation shows that $c_\alpha = \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$. For $1 \leqslant p \leqslant \infty$, let L^p_α denote the space $L^p(\mathbb{B}_n, \mathrm{d}\nu_\alpha)$. Note that L^∞_α is the same as $L^\infty = L^\infty(\mathbb{B}_n, \mathrm{d}\nu)$.

For any f in L^p_α, we denote by $\|f\|_{p,\alpha}$ the norm of f. So

$$\|f\|_{p,\alpha} = \Big[\int\limits_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\nu_\alpha\Big]^{1/p}.$$

The weighted Bergman space A^p_α consists of all holomorphic functions in $\mathbb{B}_n$ which are also in L^p_α. It is well-known that A^p_α is a closed subspace of L^p_α. In this paper we are only interested in the case $p = 2$. We denote the inner product in L^2_α by $\langle\cdot,\cdot\rangle_\alpha$. Since A^2_α is a closed subspace of the Hilbert space L^2_α, there is an orthogonal projection P_α from L^2_α onto A^2_α. For any $f \in L^2_\alpha$, we have

$$(P_\alpha f)(z) = \int\limits_{\mathbb{B}_n} \frac{f(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) = \langle f, K^\alpha_z\rangle_\alpha, \quad z \in \mathbb{B}_n,$$

where $K^\alpha_z(w) = \dfrac{1}{(1-\langle w,z\rangle)^{n+1+\alpha}}$. The functions K^α_z's are the reproducing kernels for A^2_α. For any $z \in \mathbb{B}_n$, let $k^\alpha_z(w) = \dfrac{K^\alpha_z(w)}{\|K^\alpha_z\|_{2,\alpha}}$. Then $\|k^\alpha_z\|_{2,\alpha} = 1$ and $\langle g, k^\alpha_z\rangle_\alpha = (1-|z|^2)^{(n+1+\alpha)/2} g(z)$ for $g \in A^2_\alpha$ and $z \in \mathbb{B}_n$.

For any multi-index $m = (m_1,\ldots,m_n)$ of non-negative integers, we write $|m| = m_1 + \cdots + m_n$ and $m! = m_1!\cdots m_n!$. For any $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$, we write $z^m = z_1^{m_1}\cdots z_n^{m_n}$. The standard orthonormal basis for A^2_α is $\{e_m : m \in \mathbb{N}^n\}$, where

$$e_m(z) = \Big[\frac{\Gamma(n+|m|+\alpha+1)}{m!\,\Gamma(n+\alpha+1)}\Big]^{1/2} z^m, \quad m \in \mathbb{N}^n, z \in \mathbb{B}_n.$$

For a more detailed discussion of A^2_α, see Chapter 2 in [7].

For any $f \in L^\infty$, the Toeplitz operator $T_f : A^2_\alpha \to A^2_\alpha$ is defined by the formula $T_f h = P_\alpha(fh)$ for $h \in A^2_\alpha$. We have, for $h \in A^2_\alpha$ and $z \in \mathbb{B}_n$:

$$(T_f h)(z) = \int\limits_{\mathbb{B}_n} \frac{f(w)h(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w). \tag{1.1}$$

We have $T^*_f = T_{\overline{f}}$ and since $\|P_\alpha\| = 1$, $\|T_f\| \leqslant \|f\|_\infty$ for all $f \in L^\infty$. There are examples of $f \in L^\infty$ so that $\|T_f\| < \|f\|_\infty$. Since T_f is an integral operator given by equation (1.1), we see that T_f is compact if f vanishes on the complement of a compact subset of $\mathbb{B}_n$.

Let $\mathfrak{B}(A^2_\alpha)$ denote the C^*-algebra of all bounded linear operators on A^2_α. Let $\mathcal{K}$ denote the ideal of $\mathfrak{B}(A^2_\alpha)$ that consists of all compact operators. For any subset G of L^∞, let $\mathfrak{T}(G)$ denote the Banach subalgebra of $\mathfrak{B}(A^2_\alpha)$ generated by all T_f with $f \in G$ and let $\mathfrak{CT}(G)$ denote the closed two-sided ideal of $\mathfrak{T}(G)$ generated by all commutators $[T_f, T_g] = T_f T_g - T_g T_f$ with $f, g \in G$. When G is self-adjoint (i.e. $G = G^* = \{\overline{f} : f \in G\}$), $\mathfrak{T}(G)$ is a C^*-algebra and so the quotient $\mathfrak{T}(G)/\mathfrak{CT}(G)$ is also a C^*-algebra. The algebra $\mathfrak{T} = \mathfrak{T}(L^\infty)$ is called the full Toeplitz algebra. Its commutator ideal is denoted by $\mathfrak{CT}$.

It is well-known that $\mathfrak{CT}(C(\overline{\mathbb{B}}_n)) = \mathcal{K}$. This was showed by L.A. Coburn [1] for the unweighted Bergman space ($\alpha = 0$) but his argument can be easily adapted to the weighted cases. In the unweighted case, D. Suárez [6] showed that $\mathfrak{CT} = \mathfrak{T}$ for $n = 1$. This result was generalized by the author to all $n \geqslant 1$ in [2]. The purpose of this paper is to show that the identity $\mathfrak{CT}(G) = \mathfrak{T}(G)$ holds true in the general setting of weighted Bergman spaces for a class of self-adjoint subsets G of L^∞. In the current situation, even though explicit computations used by Suárez cannot be used, many of his results play an important role. In particular, Proposition 2.3 is a version of Proposition 2.9 in [6] for Toeplitz operators on weighted Bergman spaces.

We next introduce a metric on $\mathbb{B}_n$ which we will mainly use in the paper. For $z \in \mathbb{B}_n$, let φ_z denote the Mobius automorphism of $\mathbb{B}_n$ that interchanges 0 and z. For any $w \in \mathbb{B}_n$, let $\rho(z,w) = |\varphi_z(w)|$. Then ρ is a metric on $\mathbb{B}_n$ (called the pseudo-hyperbolic metric) which is invariant under the action of the automorphism group Aut($\mathbb{B}_n$) of $\mathbb{B}_n$. For any $z, w, u \in \mathbb{B}_n$, we have

$$\rho(z,w) \geqslant \frac{\rho(z,u) - \rho(u,w)}{1 - \rho(z,u)\rho(u,w)}. \tag{1.2}$$

See Section 2 in [2] for more details about the metric ρ.

For any a in $\mathbb{B}_n$ and any $0 < r < 1$, let $E(a,r) = \{z \in \mathbb{B}_n : \rho(z,a) < r\}$, which is the ball of radius r centered at a with respect to the metric ρ.

A sequence $\{z_j : j \in J\} \subset \mathbb{B}_n$ (J is either a non-empty finite set or $J = \mathbb{N}$) is said to be separated if $r = \inf\{\rho(z_{j_1}, z_{j_2}) : j_1, j_2 \in J, j_1 \neq j_2\} > 0$. The number r is called the degree of separation of the sequence.

The following theorem is the main result of the paper.

THEOREM 1.1. *Suppose $W \subset \mathbb{B}_n$ is a measurable set and suppose there exist real numbers $r, R \in (0,1)$ and a separated sequence $\{z_j : j \in J\} \subset \mathbb{B}_n$ (J is either a non-empty finite set or $J = \mathbb{N}$) so that*

$$\bigcup_{j\in J} E(z_j, r) \subset W \subset \bigcup_{j\in J} E(z_j, R). \tag{1.3}$$

Let $G_1(W) = \{f \in C(\mathbb{B}_n) : f$ vanishes on $\mathbb{B}_n \backslash W\}$ and $G_2(W) = \{f \in L^\infty : f$ vanishes a.e. on $\mathbb{B}_n \backslash W\} = \chi_W L^\infty$. Let G be a self-adjoint linear subspace of $G_2(W)$ that contains $G_1(W)$. Then if J is finite, $\mathcal{K} = \mathfrak{CT}(G) = \mathfrak{T}(G)$. If $J = \mathbb{N}$, we have $\mathcal{K} \subsetneq \mathfrak{CT}(G) = \mathfrak{T}(G)$.

Applying Theorem 1.1 with $G = G_2(W)$, we conclude that for any measurable set $W \subset \mathbb{B}_n$ satisfying (1.3),

$$\mathfrak{CT}(\chi_W L^\infty) = \mathfrak{T}(\chi_W L^\infty). \tag{1.4}$$

Now by an argument using Zorn's Lemma, we see that for any $0 < R < 1$ there is a separated sequence $\{z_j : j \in \mathbb{N}\}$ of $\mathbb{B}_n$ so that $\mathbb{B}_n = \bigcup_{j\in\mathbb{N}} E(z_j, R)$. We then get $\mathfrak{C}\mathfrak{T} = \mathfrak{T}$. This identity in the case of unweighted Bergman space is our earlier result in [2].

Note that (1.4) holds true in many cases where W need not satisfy (1.3). In fact, if W is any measurable subset of $\mathbb{B}_n$ so that T_{χ_W} is a compact operator then $\mathfrak{T}(\chi_W L^\infty) \subset \mathcal{K}$, and if W has nonempty interior then $\mathcal{K} \subset \mathfrak{T}(\chi_W L^\infty)$ by Remark 2.9. It then follows that $\mathfrak{T}(\chi_W L^\infty) = \mathcal{K}$ and so $\mathcal{K} = \mathfrak{C}\mathfrak{T}(\chi_W L^\infty) = \mathfrak{T}(\chi_W L^\infty)$. Thus (1.3) is not a necessary condition for (1.4) to hold true. It would be interesting to know an explicit description of those W's that give (1.4) but at this time we do not know if there exists such a description.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries and basic results. In Section 3, we give a proof for Theorem 1.1.

2. BASIC RESULTS

In the first part of this section, we discuss how some results of Suárez can be generalized to the case of weighted Bergman space A^2_α. These results play an important role in the proof of Theorem 1.1. In the second part of the section, we show that a Toeplitz operator with radial symbol (see definition below) is a diagonal operator with respect to the standard orthonormal basis of A^2_α. A formula for the eigenvalues of the operator is given. We also show that the C^*-algebra $\mathfrak{T}(fC(\overline{\mathbb{B}}_n))$ with f a radial function is irreducible.

LEMMA 2.1. *Let* $0 < r < R < 1$ *and* $\{z_j\}_{j=1}^\infty \subset \mathbb{B}_n$ *such that* $E(z_{j_1}, r) \cap E(z_{j_2}, r) = \emptyset$ *if* $j_1 \neq j_2$. *Let*

$$\Phi(z,w) = \sum_{j=1}^\infty \chi_{E(z_j,r)}(z)\chi_{\mathbb{B}_n \setminus E(z_j,R)}(w)|1-\langle z,w\rangle|^{-n-1-\alpha}.$$

Then for any $0 < \beta < \alpha + 1$, *we have:*

(i) $\displaystyle\int_{\mathbb{B}_n} \Phi(z,w)(1-|w|^2)^{-\beta}\,\mathrm{d}\nu_\alpha(w) \leqslant C_1(1-|z|^2)^{-\beta}$ *for all* $z \in \mathbb{B}_n$;

(ii) $\displaystyle\int_{\mathbb{B}_n} \Phi(z,w)(1-|z|^2)^{-\beta}\,\mathrm{d}\nu_\alpha(z) \leqslant C_2(1-|w|^2)^{-\beta}$ *for all* $w \in \mathbb{B}_n$;

where $C_1 = C_1(n,\alpha,\beta), C_2 = C_2(n,\alpha,\beta,r,R)$ *and* $C_2 \to 0$ *as* $R \uparrow 1$.

Proof. For any $z, w \in \mathbb{B}_n$, $\Phi(z,w) \leqslant |1-\langle z,w\rangle|^{-n-1-\alpha}$, so

$$\int_{\mathbb{B}_n}\Phi(z,w)(1-|w|^2)^{-\beta}\mathrm{d}\nu_\alpha(w) \leqslant \int_{\mathbb{B}_n}\frac{(1-|w|^2)^{-\beta}}{|1-\langle z,w\rangle|^{n+1+\alpha}}\mathrm{d}\nu_\alpha(w) = c_\alpha\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-\beta}}{|1-\langle z,w\rangle|^{n+1+\alpha}}\mathrm{d}\nu(w).$$

Applying Theorem 1.12 in [7] with $t = \alpha - \beta > -1$ and $c = \beta > 0$, we get a constant $C_1 = C_1(n,\alpha,\beta)$ so that the inequality in (i) holds.

Now fix $w \in \mathbb{B}_n$. For any $z \in \mathbb{B}_n$, if $\Phi(z,w) \neq 0$ then there is an integer $j \geqslant 1$ so that $z \in E(z_j, r)$ and $w \notin E(z_j, R)$. Then

$$\rho(z,w) \geqslant \frac{\rho(w,z_j) - \rho(z,z_j)}{1 - \rho(w,z_j)\rho(z,z_j)} \geqslant \frac{R-r}{1-Rr}.$$

Let $\delta = \dfrac{R-r}{1-Rr}$. It is clear that for each $0 < r < 1$, $\delta \uparrow 1$ as $R \uparrow 1$. We have

$$\begin{aligned}
&\int_{\mathbb{B}_n} \Phi(z,w)(1-|z|^2)^{-\beta} \mathrm{d}\nu_\alpha(z) \\
&\quad \leqslant \int_{\mathbb{B}_n \setminus E(w,\delta)} \Phi(z,w)(1-|z|^2)^{-\beta} \mathrm{d}\nu_\alpha(z) \leqslant \int_{\mathbb{B}_n \setminus E(w,\delta)} \frac{(1-|z|^2)^{-\beta}}{|1-\langle z,w\rangle|^{n+1+\alpha}} \mathrm{d}\nu_\alpha(z) \\
&\quad = \int_{|\eta|>\delta} \frac{(1-|\varphi_w(\eta)|^2)^{-\beta}}{|1-\langle \varphi_w(\eta), w\rangle|^{n+1+\alpha}} \frac{(1-|w|^2)^{n+1+\alpha}}{|1-\langle \eta, w\rangle|^{2n+2+2\alpha}} \mathrm{d}\nu_\alpha(\eta) \\
&\qquad \text{(by the change-of-variable } z = \varphi_w(\eta) \text{ — see Proposition 1.13 in [7])} \\
&\quad = \int_{|\eta|>\delta} \frac{(1-|w|^2)^{-\beta}(1-|\eta|^2)^{-\beta}}{|1-\langle w,\eta\rangle|^{n+1+\alpha-2\beta}} \mathrm{d}\nu_\alpha(\eta) \\
&\quad = c_\alpha (1-|w|^2)^{-\beta} \int_{|\eta|>\delta} \frac{(1-|\eta|^2)^{\alpha-\beta}}{|1-\langle w,\eta\rangle|^{n+1+\alpha-2\beta}} \mathrm{d}\nu(\eta).
\end{aligned}$$

Here in the fourth equality, we have used the formulas

$$1 - |\varphi_w(\eta)|^2 = \frac{(1-|w|^2)(1-|\eta|^2)}{|1-\langle w,\eta\rangle|^2} \quad \text{and} \quad 1 - \langle \varphi_w(\eta), w\rangle = \frac{1-|w|^2}{1-\langle \eta, w\rangle}.$$

See Theorem 2.2.2 in [4] or Lemma 1.3 in [7].

For any $1 < \gamma < \infty$, Holder's inequality gives

$$\begin{aligned}
&\int_{|\eta|>\delta} \frac{(1-|\eta|^2)^{\alpha-\beta}}{|1-\langle w,\eta\rangle|^{n+1+\alpha-2\beta}} \mathrm{d}\nu(\eta) \\
&\quad \leqslant \Big(\int_{|\eta|>\delta} \frac{(1-|\eta|^2)^{(\alpha-\beta)\gamma}}{|1-\langle w,\eta\rangle|^{(n+1+\alpha-2\beta)\gamma}} \mathrm{d}\nu(\eta) \Big)^{1/\gamma} (1-\delta^{2n})^{1-1/\gamma} \\
&\quad \leqslant (1-\delta^{2n})^{1-1/\gamma} \Big(\int_{\mathbb{B}_n} \frac{(1-|\eta|^2)^{(\alpha-\beta)\gamma}}{|1-\langle w,\eta\rangle|^{(n+1+\alpha-2\beta)\gamma}} \mathrm{d}\nu(\eta) \Big)^{1/\gamma}.
\end{aligned}$$

If $\alpha - \beta \geqslant 0$, choose any $\gamma = \gamma(n,\alpha,\beta) > 1$ so that $(n+1-\beta)\gamma - (n+1) < 0$ (since $n+1-\beta < n+1$, it is possible to choose such a γ). If $\alpha - \beta < 0$, choose any $\gamma = \gamma(n,\alpha,\beta) > 1$ so that $\gamma < \dfrac{1}{\beta-\alpha}$ and $(n+1-\beta)\gamma - (n+1) < 0$ (since $0 < \beta - \alpha < 1$ and $n+1-\beta < n+1$, it is possible to choose such a γ). With the above choice of γ, we have $t = (\alpha - \beta)\gamma > -1$ and $c = (n+1-\beta)\gamma - (n+1) < 0$. By Theorem 1.12 in [7], there is a constant $\widetilde{C}_2 = \widetilde{C}_2(n,\alpha,\beta)$ so that for all $w \in \mathbb{B}_n$,

$$\int\limits_{\mathbb{B}_n} \frac{(1-|\eta|^2)^{(\alpha-\beta)\gamma}}{|1-\langle w,\eta\rangle|^{(n+1+\alpha-2\beta)\gamma}}\, \mathrm{d}\nu(\eta) \leqslant \widetilde{C}_2.$$

We then have the inequality in (ii) with $C_2 = c_\alpha(1-\delta^{2n})^{1-1/\gamma}(\widetilde{C}_2)^{1/\gamma}$. Since $\delta \uparrow 1$ as $R \uparrow 1$, we see that $C_2 \to 0$ as $R \uparrow 1$. ∎

With the help of Lemma 2.1, the proof of the following lemma is the same as that of Lemma 2.6 in [6].

LEMMA 2.2. *Suppose that* $0 < r < R < 1$ *and* $\{w_j\}_{j=1}^{\infty} \subset \mathbb{B}_n$ *so that* $E(w_{j_1}, r) \cap E(w_{j_2}, r) = \emptyset$ *for all* $j_1 \neq j_2$. *Let* $a_j, A_j \in L^\infty$ *be functions of norm* $\leqslant 1$ *such that* $a_j(z) = 0$ *for almost every* z *in* $B_n \backslash E(w_j, r)$ *and* $A_j(z) = 0$ *for almost every* z *in* $E(w_j, R)$, $j = 1, 2, \ldots$. *Then the operator* $\sum\limits_{j\in\mathbb{N}} M_{a_j} P M_{A_j}$ *is bounded on* L^2, *with norm bounded by some constant* $k(R) \to 0$ *when* $R \to 1$.

The following proposition is a version of Proposition 2.9 in [6] for Toeplitz operators on the weighted Bergman space A^2_α. Since we have Lemma 2.2 and all the needed properties of the metric ρ, the proof is identical to that of Suárez.

PROPOSITION 2.3. *Suppose* $0 < r < 1$ *and* $\{w_j\}_{j=1}^{\infty} \subset \mathbb{B}_n$ *so that* $E(w_{j_1}, r) \cap E(w_{j_2}, r) = \emptyset$ *for all* $j_1 \neq j_2$. *Let* $c_j^1, \ldots, c_j^l, a_j, b_j, d_j^1, \ldots, d_j^k \in L^\infty$ *be functions of norm* $\leqslant 1$ *that vanish almost everywhere on* $B_n \backslash E(w_j, r)$, *for* $j = 1, 2, \ldots$. *Then the operator*

$$\sum_{j\in\mathbb{N}} T_{c_j^1} \cdots T_{c_j^l} (T_{a_j} T_{b_j} - T_{b_j} T_{a_j}) T_{d_j^1} \cdots T_{d_j^k}$$

belongs to the commutator ideal $\mathfrak{CT}(G)$, *where* G *is the closed subalgebra of* L^∞ *generated by the functions* $c_j^1, \ldots, c_j^l, a_j, b_j, d_j^1, \ldots, d_j^k$, $j = 1, 2, \ldots$.

For any multi-indices $m = (m_1, \ldots, m_n)$ and $k = (k_1, \ldots, k_n)$ in $\mathbb{N}^n$, we say that $m \preceq k$ (or $k \succeq m$) if $m_j \leqslant k_j$ for all $1 \leqslant j \leqslant n$. We write $m + k = (m_1 + k_1, \ldots, m_n + k_n)$ and $k - m = (k_1 - m_1, \ldots, k_n - m_n)$. It is clear that $k - m \in \mathbb{N}^n$ if $m \preceq k$. We have $|m + k| = |m| + |k|$ and $|k - m| = |k| - |m|$. Recall that the standard orthonormal basis for A^2_α is $\{e_m : m \in \mathbb{N}^n\}$, where

$$e_m(z) = \Big[\frac{\Gamma(n+|m|+\alpha+1)}{m!\,\Gamma(n+\alpha+1)}\Big]^{1/2} z^m, \quad m \in \mathbb{N}^n, z \in \mathbb{B}_n.$$

It is clear that for any multi-indices m and k, there is a positive constant $d_\alpha(m,k)$ so that $e_m e_k = d_\alpha(m,k) e_{k+m}$. Note that $d_\alpha(0,k) = d_\alpha(m,0) = 1$ for all m and k. Here 0 denotes $(0, \ldots, 0)$.

A function f on $\mathbb{B}_n$ is called a radial function if there is a function $\widetilde{f}$ defined on $[0,1)$ so that $f(z) = \widetilde{f}(|z|)$ for all $z \in \mathbb{B}_n$. For such an f and any real number

$s \geqslant 0$, put

$$\omega_\alpha(f,s) = \frac{\Gamma(n+s+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+s)} \int_0^1 r^{n+s-1}(1-r)^\alpha \widetilde{f}(r^{1/2})\mathrm{d}r.$$

If $f(z) = \chi_{E(0,\delta)}(z) = \chi_{[0,\delta)}(|z|)$ for some $0 < \delta < 1$ then for any $s \geqslant 0$,

$$\begin{aligned}\omega_\alpha(\chi_{E(0,\delta)}, s) &= \frac{\Gamma(n+s+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+s)} \int_0^1 r^{n+s-1}(1-r)^\alpha \chi_{[0,\delta)}(r^{1/2})\mathrm{d}r \\ &= \frac{\Gamma(n+s+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+s)} \int_0^{\delta^2} r^{n+s-1}(1-r)^\alpha \mathrm{d}r.\end{aligned}$$

Since $\min\{1,(1-\delta^2)^\alpha\} \leqslant (1-r)^\alpha \leqslant \max\{1,(1-\delta^2)^\alpha\}$ for all $0 \leqslant r \leqslant \delta^2$, we have

$$\min\{1,(1-\delta^2)^\alpha\} \leqslant \omega_\alpha(\chi_{E(0,\delta)}, s) \frac{\Gamma(\alpha+1)\Gamma(n+s)}{\Gamma(n+s+\alpha+1)} \frac{(n+s)}{\delta^{2(n+s)}} \leqslant \max\{1,(1-\delta^2)^\alpha\}.$$

This then implies that for any $0 < r < R < 1$, we have

$$\lim_{s\to\infty} \frac{\omega_\alpha(\chi_{E(0,r)}, s)}{\omega_\alpha(\chi_{E(0,R)}, s)} = 0. \tag{2.1}$$

LEMMA 2.4. *Let f be a bounded radial function on $\mathbb{B}_n$. Then for any multi-indices m and k we have*

$$P_\alpha(f e_m \bar{e}_k) = \begin{cases} d_\alpha(k, m-k)\omega_\alpha(f, |m|)e_{m-k} & \text{if } k \preceq m, \\ 0 & \text{otherwise.}\end{cases}$$

Proof. For any multi-index l, consider

$$\begin{aligned}\langle P_\alpha(f e_m \bar{e}_k), e_l\rangle_\alpha &= \langle f e_m, e_k e_l\rangle_\alpha = d_\alpha(k,l)\langle f e_m, e_{k+l}\rangle_\alpha \\ &= d_\alpha(k,l)c_\alpha \int_{\mathbb{B}_n} f(z)e_m(z)\bar{e}_{k+l}(z)(1-|z|^2)^\alpha \mathrm{d}\nu(z) \\ &= d_\alpha(k,l)c_\alpha \int_0^1 2nr^{2n-1}(1-r^2)^\alpha \widetilde{f}(r)\mathrm{d}r \int_{\mathbb{S}_n} e_m(r\zeta)\bar{e}_{k+l}(r\zeta)\mathrm{d}\sigma(\zeta) \\ &= d_\alpha(k,l)c_\alpha \int_0^1 2nr^{2n+|m|+|k|+|l|-1}(1-r^2)^\alpha \widetilde{f}(r)\mathrm{d}r \int_{\mathbb{S}_n} e_m(\zeta)\bar{e}_{k+l}(\zeta)\mathrm{d}\sigma(\zeta).\end{aligned}$$

Thus if $m \neq k+l$, $\langle P_\alpha(fe_m\bar{e}_k), e_l\rangle_\alpha = 0$. If $m = k+l$, we have

$$\begin{aligned}
&\langle P_\alpha(fe_m\bar{e}_k), e_l\rangle_\alpha\\
&= d_\alpha(k,l)c_\alpha \frac{\Gamma(n+|m|+\alpha+1)}{m!\,\Gamma(n+1+\alpha)} \frac{\Gamma(n)\,m!}{\Gamma(n+|m|)} \int_0^1 2nr^{2n+2|m|-1}(1-r^2)^\alpha \widetilde{f}(r)\mathrm{d}r\\
&\quad \text{(see Lemma 1.11 in [7])}\\
&= d_\alpha(k,l) \frac{\Gamma(n+|m|+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+|m|)} \int_0^1 2r^{2n+2|m|-1}(1-r^2)^\alpha \widetilde{f}(r)\mathrm{d}r\\
&= d_\alpha(k,l) \frac{\Gamma(n+|m|+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+|m|)} \int_0^1 r^{n+|m|-1}(1-r)^\alpha \widetilde{f}(r^{1/2})\mathrm{d}r = d_\alpha(k,l)\omega_\alpha(f,|m|).
\end{aligned}$$

Therefore,

$$P_\alpha(fe_m\bar{e}_k) = \begin{cases} d_\alpha(k,m-k)\omega_\alpha(f,|m|)e_{m-k} & \text{if } k \preceq m,\\ 0 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

REMARK 2.5. Lemma 2.4 implies that for a radial function $f \in L^\infty$, we have $T_f e_m = \omega_\alpha(f,|m|)e_m$ for all multi-indices m. Therefore, the Toeplitz operator T_f is diagonal with respect to the standard orthonormal basis. In fact, $T_f = \sum_{m\in\mathbb{N}^n} \omega_\alpha(f,|m|)e_m \otimes e_m$. Here $(e_m \otimes e_m)h = \langle h, e_m\rangle e_m$ for all $h \in A^2_\alpha$.

LEMMA 2.6. *Let $\{s_m : m \in \mathbb{N}^n\}$ be a bounded set of strictly positive real numbers. Let $0 < r < R < 1$. Then for any $\delta > 0$, there is a number $\lambda > 0$ depending on δ, r, R and $\left\{\frac{\omega_\alpha(\chi_{E(0,r)},|m|)}{s_m} : m \in \mathbb{N}^n\right\}$ so that*

$$T_{\chi_{E(0,r)}} \leqslant \lambda \sum_{m\in\mathbb{N}^n} s_m e_m \otimes e_m + \delta T_{\chi_{E(0,R)}}.$$

Proof. As in the remark preceding the lemma,

$$T_{\chi_{E(0,r)}} = \sum_{m\in\mathbb{N}^n} \omega_\alpha(\chi_{E(0,r)},|m|)e_m \otimes e_m, \quad \text{and} \quad T_{\chi_{E(0,R)}} = \sum_{m\in\mathbb{N}^n} \omega_\alpha(\chi_{E(0,R)},|m|)e_m \otimes e_m.$$

Let $\delta > 0$ be given. We need to find a $\lambda > 0$ so that

$$\omega_\alpha(\chi_{E(0,r)},|m|) \leqslant \lambda s_m + \delta\omega_\alpha(\chi_{E(0,R)},|m|)$$

for all multi-indices m. By (2.1), there is an integer $M \geqslant 1$ so that

$$\omega_\alpha(\chi_{E(0,r)},|m|) \leqslant \delta\omega_\alpha(\chi_{E(0,R)},|m|)$$

for all multi-indices m with $|m| \geqslant M$. Put

$$\lambda = \max\left\{\frac{\omega_\alpha(\chi_{E(0,r)},|m|)}{s_m} : |m| \leqslant M\right\}.$$

We then have the required inequality. $\blacksquare$

For any $1 \leqslant j \leqslant n$, let $\delta_j = (\delta_{1j}, \ldots, \delta_{nj})$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

PROPOSITION 2.7. *Let $\{p_s\}_{s=1}^{\infty}$ be a sequence of distinct positive integers so that $\sum\limits_{s=1}^{\infty} p_s^{-1} = \infty$. Let $f \in L^\infty$ be a radial function which is not identically zero in $\mathbb{B}_n$. Let G be a self-adjoint subset of L^∞ so that for any integer $s \geqslant 1$ and integer $1 \leqslant j \leqslant n$, there is a multi-index $k \in \mathbb{N}^n$ with $|k| = p_s$ and $f e_k \bar{e}_{k+\delta_j} \in G$ (hence, $\bar{f}\bar{e}_k e_{k+\delta_j} \in G$ as well). Then $\mathfrak{T}(G)$ is irreducible.*

Proof. Suppose Q is an operator on A^2_α so that $TQ = QT$ for all $T \in \mathfrak{T}(G)$. Let m be a multi-index in $\mathbb{N}^n$. Since f is not identically zero and $\sum\limits_{s=1}^{\infty} p_s^{-1}$ diverges, there is an integer $s \geqslant 1$ so that $\omega_\alpha(f, p_s + 1 + |m|) \neq 0$. This is a consequence of Muntz–Szasz's Theorem (Theorem 15.26 in [5]). Now let $1 \leqslant j \leqslant n$. By the assumption about G, there is a multi-index $k \in \mathbb{N}^n$ so that $|k| = p_s$ and $f e_k \bar{e}_{k+\delta_j} \in G$. We then have $T_{f e_k \bar{e}_{k+\delta_j}} Q = Q T_{f e_k \bar{e}_{k+\delta_j}}$.

Let $\varphi = Q1 \in A^2_\alpha$. From Lemma 2.4, $T_{f e_k \bar{e}_{k+\delta_j}} 1 = P_\alpha(f e_k \bar{e}_{k+\delta_j}) = 0$, so $T_{f e_k \bar{e}_{k+\delta_j}} \varphi = T_{f e_k \bar{e}_{k+\delta_j}} Q1 = Q T_{f e_k \bar{e}_{k+\delta_j}} 1 = 0$. Using this together with the identities

$$
\begin{aligned}
(2.2) \qquad P_\alpha(\bar{f}\bar{e}_k e_{k+\delta_j} e_m) &= d_\alpha(k+\delta_j, m) P_\alpha(\bar{f}\bar{e}_k e_{k+\delta_j+m}) \\
&= d_\alpha(k+\delta_j, m) d_\alpha(k, \delta_j + m) \omega_\alpha(\bar{f}, |k| + 1 + |m|) e_{\delta_j+m} \\
&= d_\alpha(k+\delta_j, m) d_\alpha(k, \delta_j + m) \omega_\alpha(\bar{f}, p_s + 1 + |m|) e_{\delta_j+m}
\end{aligned}
$$

we get

$$
\begin{aligned}
0 &= \langle T_{f e_k \bar{e}_{k+\delta_j}} \varphi,\ e_m \rangle_\alpha = \langle \varphi,\ \bar{f}\bar{e}_k e_{k+\delta_j} e_m \rangle_\alpha = \langle \varphi,\ P_\alpha(\bar{f}\bar{e}_k e_{k+\delta_j} e_m) \rangle_\alpha \\
&= d_\alpha(k+\delta_j, m) d_\alpha(k, \delta_j + m) \omega_\alpha(f, p_s + 1 + |m|) \langle \varphi,\ e_{\delta_j+m} \rangle_\alpha.
\end{aligned}
$$

From this we conclude that $\langle \varphi,\ e_{\delta_j+m} \rangle_\alpha = 0$. Since this is true for all multi-indices m and any $1 \leqslant j \leqslant n$, it follows that φ is a constant function. So there is a complex number c so that $\varphi(z) = c$ for all $z \in \mathbb{B}_n$.

We next show that $Qe_m = ce_m$ for all multi-indices $m \in \mathbb{N}^n$ by induction on $|m|$. Since $Q1 = c$ and $1 = e_{(0,\ldots,0)}$, we have $Qe_m = ce_m$ for $|m| = 0$. Now suppose this holds true for all multi-indices m with $|m| = l \geqslant 0$. Let m' be a multi-index with $|m'| = l + 1$. Then there is a multi-index m with $|m| = l$ and an integer $1 \leqslant j \leqslant n$ so that $m' = m + \delta_j$. Choose $s \geqslant 1$ so that $\omega_\alpha(\bar{f}, p_s + 1 + |m|) \neq 0$. Then there is a multi-index k so that $|k| = p_s$ and $\bar{f}\bar{e}_k e_{k+\delta_j} \in G$. Since $QT_{\bar{f}\bar{e}_k e_{k+\delta_j}} = T_{\bar{f}\bar{e}_k e_{k+\delta_j}} Q$ and $Qe_m = ce_m$, we have

$$
(2.3) \qquad QP_\alpha(\bar{f}\bar{e}_k e_{k+\delta_j} e_m) = QT_{\bar{f}\bar{e}_k e_{k+\delta_j}} e_m = T_{\bar{f}\bar{e}_k e_{k+\delta_j}} Qe_m = cP_\alpha(\bar{f}\bar{e}_k e_{k+\delta_j} e_m).
$$

By (2.2) and the fact that $d_\alpha(k+\delta_j, m)d_\alpha(k, \delta_j+m)\omega_\alpha(\overline{f}, p_s+1+|m|) \neq 0$, equation (2.3) implies $Qe_{m'} = ce_{m'}$.

Thus, $Q = cI$, where I is the identity operator on A^2_α. Since the commutant of $\mathfrak{T}(G)$ consists of only multiples of the identity operator, we conclude that $\mathfrak{T}(G)$ is irreducible. ∎

REMARK 2.8. Let $G_f = fC(\overline{\mathbb{B}}_n) = \{fg : g \in C(\overline{\mathbb{B}}_n)\}$ where $f \in L^\infty$ is a radial function which is not identically zero in $\mathbb{B}_n$. Proposition 2.7 then implies that $\mathfrak{T}(G_f)$ is irreducible. Since $\mathfrak{T}(G_f)$ clearly contains a nonzero compact operator (for example, T_{fg} for an appropriate $g \in C_c(\mathbb{B}_n)$), we conclude that $\mathcal{K} \subset \mathfrak{T}(G_f)$. If it happens that T_f is compact then since $T_{fg} - T_fT_g$ is compact for any $g \in C(\overline{\mathbb{B}}_n)$ (see [1]), we see that T_{fg} is compact for all $g \in C(\overline{\mathbb{B}}_n)$. Hence in this case, $\mathfrak{T}(G_f) = \mathcal{K}$.

For each $z \in \mathbb{B}_n$, the formula $U_z(f) = (f \circ \varphi_z)k^\alpha_z$, $f \in L^2_\alpha$ defines a bounded operator on L^2_α. It is easy to verify that U_z is a self-adjoint unitary operator with A^2_α as a reducing subspace and $U_zT_fU_z = T_{f\circ\varphi_z}$ on A^2_α for all $z \in \mathbb{B}_n$ and all $f \in L^\infty$. See, for example, Lemmas 7 and 8 in [3].

REMARK 2.9. Suppose $f = h \circ \varphi_z$ for some radial function $h \in L^\infty$ which is not identically zero and $z \in \mathbb{B}_n$. Let $G_f = fC(\overline{\mathbb{B}}_n)$ and $G_h = hC(\overline{\mathbb{B}}_n)$ then $G_f = \{g \circ \varphi_z : g \in G_h\}$. Since $T_{g\circ\varphi_z} = U_zT_gU_z$ for any $g \in G_h$, we see that $\mathfrak{T}(G_f) = U_z\mathfrak{T}(G_h)U_z$. Therefore, all the statements in Remark 2.8 remain true for this f.

3. PROOF OF THE MAIN THEOREM

We begin this section by the following lemma which implies that for all $\varepsilon \in (0,1)$, there is a continuous function g_ε supported in the set $\{z \in \mathbb{B}_n : |z| < \varepsilon\}$ so that $[T_{g_\varepsilon}, T_{\overline{g}_\varepsilon}]$ is a diagonal operator (with respect to the standard orthonormal basis of A^2_α) which has only nonzero eigenvalues.

LEMMA 3.1. *Let $\widetilde{g} : [0,\infty) \longrightarrow [0,\infty)$ be a measurable function with $\|\widetilde{g}\|_\infty > 0$ and $\widetilde{g}(t) = 0$ for all $t \geqslant 1$. For any $0 < \varepsilon < 1$ and any multi-index $m_0 \in \mathbb{N}^n$ with $|m_0| \geqslant 1$, let $g_\varepsilon(z) = \widetilde{g}(|z|\varepsilon^{-1})$ and $f_{\varepsilon,m_0}(z) = e_{m_0}(z)g_\varepsilon(z)$, for all $z \in \mathbb{B}_n$. Then the operator $T_{\varepsilon,m_0} = [T_{f_{\varepsilon,m_0}}, T_{\overline{f}_{\varepsilon,m_0}}]$ is diagonal with respect to the standard orthonormal basis. Furthermore, for each fixed m_0, for all but at most countably many ε, $T_{\varepsilon,m_0}e_m \neq 0$ for all $m \in \mathbb{N}^n$.*

Proof. For any multi-index m, from Lemma 2.4 we have

$$\begin{aligned} T_{f_{\varepsilon,m_0}}e_m &= P_\alpha(f_{\varepsilon,m_0}e_m) = P_\alpha(e_{m_0}g_\varepsilon e_m) = d_\alpha(m, m_0)P_\alpha(g_\varepsilon e_{m_0+m}) \\ &= d_\alpha(m, m_0)\omega_\alpha(g_\varepsilon, |m_0+m|)e_{m_0+m} = d_\alpha(m, m_0)\omega_\alpha(g_\varepsilon, |m_0|+|m|)e_{m_0+m}, \end{aligned}$$

and

$$T_{\overline{f}_{\varepsilon,m_0}} e_m = P_\alpha(\overline{g}_\varepsilon \overline{e}_{m_0} e_m) = P_\alpha(g_\varepsilon \overline{e}_{m_0} e_m) = \begin{cases} d_\alpha(m_0, m-m_0)\omega_\alpha(g_\varepsilon, |m|)e_{m-m_0} & \text{if } m_0 \preceq m, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} (T_{\overline{f}_{\varepsilon,m_0}} T_{f_{\varepsilon,m_0}})e_m &= T_{\overline{f}_{\varepsilon,m_0}}(d_\alpha(m, m_0)\omega_\alpha(g_\varepsilon, |m_0| + |m|)e_{m_0+m}) \\ &= d_\alpha^2(m_0, m)\omega_\alpha^2(g_\varepsilon, |m_0| + |m|)e_m, \\ (T_{f_{\varepsilon,m_0}} T_{\overline{f}_{\varepsilon,m_0}})e_m &= \begin{cases} d_\alpha^2(m_0, m-m_0)\omega_\alpha^2(g_\varepsilon, |m|)e_m & \text{if } m_0 \preceq m, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It then follows that $T_{\varepsilon,m_0} = [T_{f_{\varepsilon,m_0}}, T_{\overline{f}_{\varepsilon,m_0}}]$ is a diagonal operator with respect to the standard orthonormal basis. We have $T_{\varepsilon,m_0} = \sum_{m\in\mathbb{N}^n} s_m(\varepsilon, m_0)e_m \otimes e_m$, where $s_m(\varepsilon, m_0) = d_\alpha^2(m_0, m-m_0)\omega_\alpha^2(g_\varepsilon, |m|) - d_\alpha^2(m_0, m)\omega_\alpha^2(g_\varepsilon, |m_0| + |m|)$ if $m_0 \preceq m$ and $s_m(\varepsilon, m_0) = -d_\alpha^2(m_0, m)\omega_\alpha^2(g_\varepsilon, |m_0| + |m|)$ if $m_0 \not\preceq m$.

For any real number $s \geqslant 0$,

$$\begin{aligned} \omega_\alpha(g_\varepsilon, s) &= \frac{\Gamma(n+s+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+s)} \int_0^1 r^{n+s-1}(1-r)^\alpha \widetilde{g}(r^{1/2}\varepsilon^{-1})\,\mathrm{d}r \\ &= \frac{\Gamma(n+s+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+s)} \varepsilon^{2n+2s} \int_0^1 u^{n+s-1}(1-\varepsilon^2 u)^\alpha \widetilde{g}(u^{1/2})\,\mathrm{d}u \\ &\qquad \text{(by the change-of-variable } u = \varepsilon^{-2}r). \end{aligned}$$

Now suppose $|m_0| \geqslant 1$. By assumption about $\widetilde{g}$, we have $s_m(\varepsilon, m_0) \neq 0$ for all $0 < \varepsilon < 1$ if $m \not\succeq m_0$. Now let $m \succeq m_0$. Then $s_m(\varepsilon, m_0)$ can be written as

$$s_m(\varepsilon, m_0) = A \int_0^1 (\varepsilon^{2|m_0|}u^{|m_0|} - B)u^{n+|m|-1}(1-\varepsilon^2 u)^\alpha \widetilde{g}(u^{1/2})\,\mathrm{d}u,$$

where A and B are nonzero numbers with B independent of ε.

Since the function

$$\theta(\tau) = \int_0^1 (\tau^{2|m_0|}u^{|m_0|} - B)u^{n+|m|-1}(1-\tau^2 u)^\alpha \widetilde{g}(u^{1/2})\,\mathrm{d}u$$

is holomorphic in the open unit disc $|\tau| < 1$ and $\theta(0) \neq 0$, the set $\{\tau : |\tau| < 1 \text{ and } \theta(\tau) = 0\}$ is at most countable. So the set $\{\varepsilon : 0 < \varepsilon < 1 \text{ and } s_m(\varepsilon, m_0) = 0\}$ is at most countable for each multi-index m. This implies that $\{\varepsilon : 0 < \varepsilon < 1 \text{ and } s_m(\varepsilon, m_0) = 0 \text{ for some } m \in \mathbb{N}^n\}$ is at most countable. The conclusion of the lemma then follows. ∎

We are now in the position of proving Theorem 1.1.

Proof of Theorem 1.1. First of all, Remark 2.8 shows that $\mathcal{K} \subset \mathfrak{T}(G)$. Second, by Lemma 3.1, there is an $\varepsilon > 0$ and a continuous function η_ε supported in $E(0,r)$ so that $[T_{\eta_\varepsilon}, T_{\overline{\eta}_\varepsilon}]$ is a diagonal operator (with respect to the standard orthonormal basis) with only nonzero eigenvalues.

Fix an $\widetilde{R}$ so that $R < \widetilde{R} < 1$. Let $\delta > 0$ be given. Applying Lemma 2.6, we get a number $\lambda > 0$ so that

$$T_{\chi_{E(0,R)}} \leqslant \lambda [T_{\eta_\varepsilon}, T_{\overline{\eta}_\varepsilon}]^2 + \delta T_{\chi_{E(0,\widetilde{R})}}.$$

For each $j \in J$, multiplying U_{z_j} to both sides of the above inequality and using the identities $U_{z_j} T_f U_{z_j} = T_{f \circ \varphi_{z_j}}$, $\varphi_{z_j}(E(0,R)) = E(z_j, R)$ and $\varphi_{z_j}(E(0,\widetilde{R})) = E(z_j, \widetilde{R})$, we get

$$T_{\chi_{E(z_j,R)}} \leqslant \lambda U_{z_j} [T_{\eta_\varepsilon}, T_{\overline{\eta}_\varepsilon}]^2 U_{z_j} + \delta T_{\chi_{E(z_j,\widetilde{R})}} = \lambda [T_{\eta_\varepsilon \circ \varphi_{z_j}}, T_{\overline{\eta}_\varepsilon \circ \varphi_{z_j}}]^2 + \delta T_{\chi_{E(z_j,\widetilde{R})}}.$$

It then follows that

$$T_{\chi_W} \leqslant \sum_{j \in J} T_{\chi_{E(z_j,R)}} \leqslant \lambda \sum_{j \in J} [T_{\eta_\varepsilon \circ \varphi_{z_j}}, T_{\overline{\eta}_\varepsilon \circ \varphi_{z_j}}]^2 + \delta \sum_{j \in J} T_{\chi_{E(z_j,\widetilde{R})}}. \tag{3.1}$$

Note that all the sums above are taken in the strong operator topology of $\mathfrak{B}(L_a^2)$.

Now since the set $\{z_j : j \in J\}$ is separated, by Lemma 2.3 in [2] we can decompose $J = J_1 \cup \cdots \cup J_M$ where $E(z_{j_1}, \widetilde{R}) \cap E(z_{j_2}, \widetilde{R}) = \emptyset$ for all $j_1 \neq j_2$ in J_s, $1 \leqslant s \leqslant M$. Then

$$\sum_{j \in J} T_{\chi_{E(z_j,\widetilde{R})}} = \sum_{s=1}^{M} \sum_{j \in J_s} T_{\chi_{E(z_j,\widetilde{R})}} \leqslant \sum_{s=1}^{M} T_{\chi_{\mathbb{B}_n}} = M.$$

On the other hand, Proposition 2.3 implies that the operators

$$A_s = \sum_{j \in J_s} [T_{\eta_\varepsilon \circ \varphi_{z_j}}, T_{\overline{\eta}_\varepsilon \circ \varphi_{z_j}}]^2, \quad \text{for } 1 \leqslant s \leqslant M,$$

belong to $\mathfrak{CT}(G)$. Thus, $A = \sum\limits_{j \in J} [T_{\eta_\varepsilon \circ \varphi_{z_j}}, T_{\overline{\eta}_\varepsilon \circ \varphi_{z_j}}]^2 = \sum\limits_{s=1}^{M} A_s$ also belongs to $\mathfrak{CT}(G)$. Now inequality (3.1) becomes $T_{\chi_W} \leqslant \lambda A + \delta M$.

Let π denote the canonical quotient map from $\mathfrak{T}(G)$ onto the quotient algebra $\mathfrak{T}(G)/\mathfrak{CT}(G)$. Since $\pi(A) = 0$, we then have $0 \leqslant \pi(T_{\chi_W}) \leqslant \delta M$ for arbitrary $\delta > 0$. This implies $\pi(T_{\chi_W}) = 0$. For any function $f \in G$ with $f \geqslant 0$, we have $0 \leqslant f \leqslant \|f\|_\infty \chi_W$. This gives $0 \leqslant T_f \leqslant \|f\|_\infty T_{\chi_W}$ and hence, $\pi(T_f) = 0$. For an arbitrary function $f \in G$, since f can be written as a linear combination of positive functions in G, we conclude that $\pi(T_f) = 0$. So $\mathfrak{CT}(G) = \mathfrak{T}(G)$.

Now if J is finite then for any f in G, f vanishes a.e. on the complement of a compact subset of $\mathbb{B}_n$, so T_f is compact. This implies that $\mathfrak{T}(G) \subset \mathcal{K}$. Hence $\mathcal{K} = \mathfrak{CT}(G) = \mathfrak{T}(G)$.

Finally, suppose $J = \mathbb{N}$. We will show that T_{χ_W} is not compact and this implies that $\mathcal{K} \subsetneq \mathfrak{T}(G)$. Since the sequence $\{z_j\}_{j=1}^{\infty}$ is separated, we have $|z_j| \to 1$

as $j \to \infty$. Hence $k_{z_j} \to 0$ weakly as $j \to \infty$. But for each j, since $k_{z_j} = U_{z_j}1$, we have

$$\langle T_{\chi_W} k_{z_j}, k_{z_j} \rangle \geqslant \langle T_{\chi_{E(z_j,r)}} k_{z_j}, k_{z_j} \rangle = \langle U_{z_j} T_{\chi_{E(0,r)}} U_{z_j} U_{z_j} 1, U_{z_j} 1 \rangle = \langle T_{\chi_{E(0,r)}} 1, 1 \rangle.$$

So $\|T_{\chi_W} k_{z_j}\| \geqslant \langle T_{\chi_{E(0,r)}} 1, 1 \rangle$ for all $j \geqslant 1$. Hence T_{χ_W} is not compact. Thus in this case $\mathcal{K} \subsetneq \mathfrak{C}\mathfrak{T}(G) = \mathfrak{T}(G)$. ∎

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