EXTREME FLATNESS OF NORMED MODULES AND ARVESON–WITTSTOCK TYPE THEOREMS

A.YA. HELEMSKII

Dedicated to the memory of Graham R. Allan

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ABSTRACT. Let L be a separable infinite-dimensional Hilbert space $\mathcal{B} := \mathcal{B}(L)$. A contractive right $\mathcal{B}$-module X is called *semi-Ruan module*, if for every $u, v \in X$ and mutually orthogonal projections $P, Q \in \mathcal{B}$ we have $\|u \cdot P + v \cdot Q\| \leqslant (\|u \cdot P\|^2 + \|v \cdot Q\|^2)^{1/2}$. For an arbitrary Hilbert space H we consider the Hilbert tensor product $L \otimes H$ as a left $\mathcal{B}$-module with the outer multiplication $a \cdot \zeta := (a \otimes \mathbf{1}_H)\zeta; a \in \mathcal{B}, \zeta \in L \otimes H$. We prove that *for every isometric morphism $\alpha : Y \to Z$ of right semi-Ruan modules the operator $\alpha \otimes_{\mathcal{B}} \mathbf{1}_{L \otimes H}$ is also isometric.* As corollaries, we obtain several theorems on extensions of morphisms with the preservation of their norms and a new proof of the Arveson–Wittstock Extension Theorem in operator theory.

KEYWORDS: *Ruan bimodule, semi-Ruan module, extreme flatness, extreme injectivity, Arveson–Wittstock Theorem.*

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We show in this paper that a certain class of normed modules over the algebra of all bounded operators on a Hilbert space possesses a homological property which is a kind of a functional-analytic version of the standard algebraic property of flatness. We mean the preservation, under projective tensor multiplication of modules, of the property of a given morphism to be isometric. As an application, we obtain several extension theorems for different types of modules, called Arveson–Wittstock type theorems. These, in their turn, have, as a straight corollary, the "genuine" Arveson–Wittstock Theorem in its non-matricial presentation. We recall that the latter theorem plays the role of a "quantum" version of the classical Hahn–Banach theorem on the extension of bounded linear functionals. It was originally proved in [16], and a crucial preparatory step was done in [1]. As to the monographical presentation, see the textbooks [3], [12].

1. PRELIMINARIES

Throughout the paper we shall denote by $\mathcal{B}(E, F)$ the space of all bounded operators acting between normed spaces E and F, always equipped with the operator norm. We shall denote by $\mathcal{F}(E, F)$ the subspace of this space consisting of the finite-rank operators. As usual, we set $\mathcal{B}(E) := \mathcal{B}(E, E)$ and $\mathcal{F}(E) := \mathcal{F}(E, E)$.

The identity operator on E is denoted by $\mathbf{1}_E$.

The inner product in Hilbert spaces is denoted by $\langle \, \cdot \, , \, \cdot \, \rangle$. The complex-conjugate Hilbert space of a Hilbert space H is denoted by H^c.

In our future arguments we shall come across some tuples of isometric operators between Hilbert spaces, say H and K. Let $S_k; k = 1, \ldots, n$ be such a tuple, and suppose that the final projections $P_k := S_k S_k^*$ of these operators are pairwise orthogonal. We recall that in this situation we have the following equalities:

$$(1.1) \qquad S_k = P_k S_k, \quad S_k^* = S_k^* P_k \quad \text{and, as a corollary, } S_k^* S_l = 0 \text{ for } k \neq l.$$

Another class of operators we shall need is that of rank-one operators. For the same H and K as above, and for $\xi \in K$ and $\eta \in H$, we denote by $\xi \circ \eta$ the rank-one operator taking $\zeta \in H$ to $\langle \zeta, \eta \rangle \xi \in K$. We recall the equalities

$$(1.2) \quad (\xi \circ \eta)(\xi' \circ \eta') = \langle \xi', \eta \rangle \xi \circ \eta', \quad a(\xi \circ \eta) = (a\xi) \circ \eta \quad \text{and} \quad (\xi \circ \eta)a = \xi \circ (a^* \eta),$$

that are valid whenever their ingredients make sense.

As usual, the symbol "$\otimes$" denotes the algebraic tensor product of linear spaces and operators. Further, we use the symbol "$\overset{\cdot}{\otimes}$" for the Hilbert tensor product of Hilbert spaces as well as for the Hilbert tensor product of operators acting between these spaces. Finally, the symbol "$\otimes_{\mathrm{p}}$" denotes the *non-completed* projective tensor product of normed spaces.

Further, we choose a separable infinite-dimensional Hilbert space, denote it by L, and fix it throughout the whole paper. Sometimes in what follows this Hilbert space will be referred as the "canonical" one. (Our experience shows that, as a whole, it is more convenient to make an "abstract" choice, and not be tied to, say, l_2 or $L^2(\,\cdot\,)$.) For brevity, we denote the operator algebras $\mathcal{B}(L)$ and $\mathcal{F}(L)$ by $\mathcal{B}$ and $\mathcal{F}$, respectively.

Throughout the paper, *the terms left module, right module and bimodule (= two-sided module) always mean a unital module of the relevant type over the operator algebra* $\mathcal{B}$; we shall never consider other basic algebras. The respective outer (= module) multiplications will be denoted by a dot: "$\,\boldsymbol{\cdot}\,$". The words *(bi)module morphism* always mean a morphism of the $\mathcal{B}$-(bi)modules in question.

Let X be a left (respectively, right) module. A *left* (respectively, *right*) *support* of the element $u \in X$ is, by definition, each projection $P \in \mathcal{B}$ such that $P \cdot u = u$ (respectively, $u \cdot P = u$). If we have a bimodule, and P is both a left and right support of the element u, then we say that P is (just) a *support* of u.

Let X be a left module and simultaneously a normed space. We recall that X is called a *contractive* left module if we have $\|a \cdot u\| \leqslant \|a\| \|u\|$ for all $a \in \mathcal{B}$

and $u \in X$. Similarly, the condition $\|u \cdot a\| \leqslant \|a\| \|u\|$ leads to the notion of a contractive right module, and the two mentioned conditions together lead to the notion of a contractive bimodule.

If X and Y are two contractive left modules, we denote the space of all bounded (as operators) morphisms between X and Y by ${}_\mathcal{B}\mathbf{h}(X, Y)$. The relevant spaces for the cases of right modules and bimodules will be denoted by $\mathbf{h}_\mathcal{B}(X, Y)$ and ${}_\mathcal{B}\mathbf{h}_\mathcal{B}(X, Y)$, respectively. We equip these spaces with the operator norm, that is, we consider them as normed subspaces of $\mathcal{B}(X, Y)$.

Let X be a contractive left module. Then the complex conjugate normed space X^c becomes a contractive right module with the outer multiplication $x \cdot a$, defined as the former $a^* \cdot x$. Similarly, a right outer multiplication on X gives rise to a left one on X^c, defined by $a \cdot x := x \cdot a^*$. We call X^c, equipped with the relevant structure of the contractive right or left module, *the complex conjugate module of X*. Obviously, every bounded morphism $\varphi : X \to Y$ of contractive left (respectively, right) modules, being considered as a map from X^c into Y^c, becomes a morphism of right (respectively, left) modules with the same norm.

We recall several standard constructions. Let X be a left contractive module. Then its dual space X^* is a right contractive module with the outer multiplication defined by

$$[f \cdot a](x) := f(a \cdot x) \quad a \in A, x \in X, f \in X^* .$$

Similarly, the dual to a right contractive module becomes a left contractive module with the help of the equality $[a \cdot f](x) := f(x \cdot a)$, and the dual to a contractive bimodule becomes itself a contractive bimodule with the help of both of these equalities. If X and Y are two left contractive modules, then the normed space $\mathcal{B}(X, Y)$ is a contractive bimodule with outer multiplications defined by

$$[a \cdot \varphi](x) := a \cdot (\varphi(x)) \quad \text{and} \quad [\varphi \cdot a](x) := \varphi(a \cdot x) ;$$

here $\varphi \in \mathcal{B}(X, Y)$, etc. Finally, if X is a left and Y is a right contractive module, then the normed space $X \otimes_p Y$ is a contractive bimodule with the outer multiplications uniquely defined by

$$a \cdot (x \otimes y) := (a \cdot x) \otimes y \quad \text{and} \quad (x \otimes y) \cdot a := x \otimes (y \cdot a) .$$

We shall also need the notion of the module and bimodule tensor products, in their projective non-completed version. Suppose that either X is a right and Y is a left contractive modules, or both of X and Y are contractive bimodules. The normed spaces $X \otimes_\mathcal{B} Y$ in the first case, and $X \otimes_{\mathcal{B}-\mathcal{B}} Y$ in the second one, called respectively *the module and bimodule tensor product of X and Y*, are defined in terms of the universal property with respect to the class of balanced, bounded, bilinear operators from $X \times Y$ into normed spaces; cf., e.g., [4]. Namely, in the one-sided case a bounded bilinear operator $\mathcal{R} : X \times Y \to E$, where E is a normed space, is called *balanced* if $\mathcal{R}(x \cdot a, y) = \mathcal{R}(x, a \cdot y)$ for all $a \in \mathcal{B}$, $x \in X$, and $y \in Y$. In the two-sided case such a bilinear operator is called *balanced*, if, in addition to the indicated equalities, we also have $\mathcal{R}(a \cdot x, y) = \mathcal{R}(x, y \cdot a)$.

As to explicit constructions, the spaces $X \otimes_\mathcal{B} Y$ and $X \otimes_{\mathcal{B}-\mathcal{B}} Y$ can be realized as the normed quotient spaces of $X \otimes_p Y$, the projective tensor product of the underlying normed spaces of our (bi)modules. Namely,

$$X \otimes_\mathcal{B} Y = X \otimes_p Y/N_1,$$

where N_1 is the closure of

$$\text{span}\{x \cdot a \otimes y - x \otimes a \cdot y : a \in \mathcal{B}, x \in X, y \in Y\},$$

whereas

$$X \otimes_{\mathcal{B}-\mathcal{B}} Y = X \otimes_p Y/N_2,$$

where N_2 is the closure of

$$\text{span}\{a \cdot x \otimes y - x \otimes y \cdot a, x \cdot a \otimes y - x \otimes a \cdot y : a \in \mathcal{B}, x \in X, y \in Y\}.$$

Consequently, for the elementary tensors in $X \otimes_\mathcal{B} Y$ and $X \otimes_{\mathcal{B}-\mathcal{B}} Y$, that is, cosets $x \otimes_\mathcal{B} y := x \otimes y + N_1$ and $x \otimes_{\mathcal{B}-\mathcal{B}} y := x \otimes y + N_2$, we have the identities

(1.3) $x \cdot a \otimes_\mathcal{B} y = x \otimes_\mathcal{B} a \cdot y, \quad a \cdot x \otimes_{\mathcal{B}-\mathcal{B}} y = x \otimes_{\mathcal{B}-\mathcal{B}} y \cdot a,$

$$x \cdot a \otimes_{\mathcal{B}-\mathcal{B}} y = x \otimes_{\mathcal{B}-\mathcal{B}} a \cdot y.$$

Finally, the norm of an element u in $X \otimes_\mathcal{B} Y$ or in $X \otimes_{\mathcal{B}-\mathcal{B}} Y$ is equal to

(1.4) $\inf \left\{ \sum_{k=1}^{n} \|x_k\| \|y_k\| \right\},$

where the infimum is taken over all possible representations of u in the form $\sum_{k=1}^{n} x_k \otimes_\mathcal{B} y_k$ or, according to the case, $\sum_{k=1}^{n} x_k \otimes_{\mathcal{B}-\mathcal{B}} y_k$.

Note the following attractive property of module tensor products over $\mathcal{B}$.

PROPOSITION 1.1. *Let X be a right and Y a left module. Then every $u \in X \otimes_\mathcal{B} Y$ can be represented as a single elementary tensor. Moreover, if*

$$u = \sum_{k=1}^{n} x_k \otimes_\mathcal{B} y_k \quad x_k \in X, y_k \in Y,$$

and $S_k; k = 1, \ldots, n$ is an arbitrary family of isometric operators on L with pairwise orthogonal final projections $P_k := S_k S_k^*$, then such a representation can be taken as $u = x \otimes_\mathcal{B} y$, where $x := \sum_{k=1}^{n} x_k \cdot S_k^*$ and $y := \sum_{k=1}^{n} S_k \cdot y_k$.

Proof. By (1.1) and (1.3), we have

$$x \otimes_\mathcal{B} y = \sum_{k,l=1}^{n} x_k \cdot S_k^* \otimes_\mathcal{B} S_l \cdot y_l = \sum_{k,l=1}^{n} x_k \cdot S_k^* S_l \otimes_\mathcal{B} \cdot y_l = \sum_{k=1}^{n} x_k \otimes_\mathcal{B} y_k = u. \quad \blacksquare$$

Finally, we recall that the construction of the module tensor product has functorial properties. Namely, if $\alpha : X_1 \to X_2$ and $\beta : Y_1 \to Y_2$ are bounded morphisms of contractive right and left modules, respectively, then there exists a bounded operator $\alpha \otimes_\mathcal{B} \beta : X_1 \otimes_\mathcal{B} Y_1 \to X_2 \otimes_\mathcal{B} Y_2$, uniquely defined by the rule

$x \otimes_{\mathcal{B}} y \mapsto \alpha(x) \otimes_{\mathcal{B}} \beta(y)$. Moreover, we have $\|\alpha \otimes_{\mathcal{B}} \beta\| \leqslant \|\alpha\|\|\beta\|$. If we deal with contractive bimodules and their bimodule morphisms, then there exists a bounded operator $\alpha \otimes_{\mathcal{B}-\mathcal{B}} \beta : X_1 \otimes_{\mathcal{B}-\mathcal{B}} Y_1 \rightarrow X_2 \otimes_{\mathcal{B}-\mathcal{B}} Y_2$, uniquely defined by the rule $x \otimes_{\mathcal{B}-\mathcal{B}} y \mapsto \alpha(x) \otimes_{\mathcal{B}-\mathcal{B}} \beta(y)$, and we have $\|\alpha \otimes_{\mathcal{B}-\mathcal{B}} \beta\| \leqslant \|\alpha\|\|\beta\|$.

2. RUAN BIMODULES AND SEMI-RUAN ONE-SIDED MODULES

DEFINITION 2.1 (cf. the definition of an "abstract operator space" in p. 20 of [3] or p. 180–181 of [12]). A contractive bimodule Y is called a *Ruan bimodule* if it satisfies the following condition (the *Ruan axiom*):

(R) for all $u, v \in X$ with orthogonal supports, we have

$$\|u + v\| = \max\{\|u\|, \|v\|\}.$$

EXAMPLE 2.2. Consider the Banach space $\mathcal{B}(L \overset{\cdot}{\otimes} H, L \overset{\cdot}{\otimes} K)$, where H and K are arbitrary Hilbert spaces (whatever Hilbert dimension, finite or infinite, they would have). It is easy to check that this space is a Ruan bimodule with respect to the outer multiplications, defined by $a \cdot \widetilde{b} := (a \overset{\cdot}{\otimes} \mathbf{1}_K)\widetilde{b}$ and $\widetilde{b} \cdot a := \widetilde{b}(a \overset{\cdot}{\otimes} \mathbf{1}_H); a \in \mathcal{B}, \widetilde{b} \in \mathcal{B}(L \overset{\cdot}{\otimes} H, L \overset{\cdot}{\otimes} K)$.

For Ruan bimodules, the axiom (R) can be strengthened.

PROPOSITION 2.3. *Let $u_1, \ldots, u_n$ be elements of a Ruan bimodule X with pairwise orthogonal left supports, say P_k, and pairwise orthogonal right supports, say $Q_k; k = 1, \ldots, n$. Then*

$$\|u_1 + \cdots + u_n\| = \max\{\|u_1\|, \ldots, \|u_n\|\}.$$

Proof. For brevity, set $u := \sum\limits_{k=1}^{n} u_k$, and take an arbitrary tuple $S_k; k = 1, \ldots, n$ of isometric operators on L with pairwise orthogonal final projections.

At first we compare the norms of the elements u and $v := \sum\limits_{k=1}^{n} S_k \cdot u_k \cdot S_k^*$. Set also $a := \sum\limits_{k=1}^{n} P_k S_k^*$ and $b := \sum\limits_{k=1}^{n} S_k Q_k$. Then the equalities (1.1) imply that

$$a \cdot v \cdot b = \sum_{k,l,m=1}^{n} P_k S_k^* \cdot (S_l \cdot u_l \cdot S_l^*) \cdot S_m Q_m$$

$$= \sum_{k,l,m=1}^{n} P_k \cdot (S_k^* S_l \cdot u_l \cdot S_l^* S_m) \cdot Q_m = \sum_{k=1}^{n} P_k \cdot u_k \cdot Q_m = u.$$

Therefore $\|u\| \leqslant \|a\|\|v\|\|b\|$. Further, the C^*-identity gives $\|a\| = \|aa^*\|^{1/2}$ and $\|b\| = \|b^*b\|^{1/2}$. Again using (1.1), we have

$$aa^* = \sum_{k,l=1}^{n} P_k S_k^* S_l P_l^* = \sum_{k=1}^{n} P_k,$$

and similarly $b^*b = \sum\limits_{k=1}^{n} Q_k$. Thus both of these operators are projections, and hence their norm is 1. Consequently, $\|u\| \leqslant \|v\|$.

Now observe that, by the same equalities (1.1), the final projection of S_k is a support of the element $S_k \cdot u_k \cdot S_k^* \in X; k = 1, \ldots, n$. Since these projections are pairwise orthogonal, it follows from (R) that

$$\|v\| = \max\{\|S_1 \cdot u_1 \cdot S_1^*\|, \ldots, \|S_n \cdot u_n \cdot S_n^*\|\}.$$

Since X is contractive, and $S_k^* S_k = \mathbf{1}_L$, we have, for every k, $\|S_k \cdot u_k \cdot S_k^*\| = \|u_k\|$. Therefore $\|v\| = \max\{\|u_1\|, \ldots, \|u_n\|\}$.

Thus $\|u\| \leqslant \max\{\|u_1\|, \ldots, \|u_n\|\}$. But we obviously have $u_k = P_k \cdot u \cdot Q_k$, and our bimodule is contractive. From this, we have the reverse inequality. ∎

We turn from bimodules to one-sided modules. As experience shows, the obvious version of the condition (R) for these modules is not very workable. The following, more "tolerant" definition happens to be more useful.

DEFINITION 2.4. A contractive left module X is a *left semi-Ruan module*, if it satisfies the following condition:

(lsR)   if $u, v \in X$ have orthogonal left supports, then

$$\|u + v\| \leqslant (\|u\|^2 + \|v\|^2)^{1/2}.$$

Similarly, with the obvious modifications, we introduce the notion of a *right semi-Ruan module*. The respective condition will be denoted by (rsR). (B. Magajna in Corollary 2.2 of [10], pursuing different aims, considers a certain class of left modules over arbitrary C^*-algebras. It is not hard to see that in the case when the algebra in question is $\mathcal{B}$, this class coincides with the class of Banach semi-Ruan modules. We are indebted to D. Blecher, who drew our attention to the paper of Magajna.)

Needless to say, we have a similar estimate for several summands. Namely, *if elements $u_1, \ldots, u_n$ of a one-sided semi-Ruan module have respective one-sided pairwise orthogonal supports, then $\|u_1 + \cdots + u_n\| \leqslant (\|u_1\|^2 + \cdots + \|u_n\|^2)^{1/2}$.*

Clearly, every sub-bimodule of a Ruan module is itself a Ruan module, and similar hereditary property holds for one-sided semi-Ruan modules. Note also the following obvious observation.

PROPOSITION 2.5. *The complex conjugate module (cf. the previous section) of a semi-Ruan module is itself a semi-Ruan module.*

Here is our most important pair of examples.

EXAMPLE 2.6. For an arbitrary Hilbert space H, the Hilbert space $L \,\dot{\otimes}\, H$ is obviously a left semi-Ruan module with respect to the outer multiplication

$$a \cdot \zeta := (a \,\dot{\otimes}\, \mathbf{1}_H)\zeta \quad a \in \mathcal{B}, \zeta \in L \,\dot{\otimes}\, H.$$

Its complex conjugate right semi-Ruan module is, of course, the Hilbert space $L^c \dot\otimes H^c$ with the outer multiplication $\zeta \cdot a := (a^* \dot\otimes \mathbf{1}_H)\zeta$. Note that this latter module is, by virtue of the Riesz representation theorem, nothing else than the dual to the left module $L \dot\otimes H$.

PROPOSITION 2.7 (cf. Proposition 2 of [8]). *Every Ruan bimodule, considered as a left or right module, is a respective one-sided semi-Ruan module.*

Proof. Let X be our bimodule, and let $u_1, u_2 \in X$ have, to be definite, pair-wise orthogonal left supports P_1 and P_2. Of course, we may suppose that $u_1, u_2 \neq 0$. Take isometric $S_1, S_2 \in \mathcal{B}$ with orthogonal final projections, and set

$$v := \frac{1}{\|u_1\|}u_1 \cdot S_1^* + \frac{1}{\|u_2\|}u_2 \cdot S_2^*, \quad P := P_1 + P_2, \quad \text{and} \quad b := \|u_1\|S_1 + \|u_2\|S_2.$$

Then the equalities (1.1) easily imply that

$$P \cdot v \cdot b = u_1 + u_2.$$

Therefore $\|u_1 + u_2\| \leqslant \|P\|\|v\|\|b\|$. But, of course, P is a projection, and the C^*-identity immediately gives

$$\|b\| = (\|u_1\|^2 + \|u_2\|^2)^{1/2}.$$

Finally, the summands in v obviously have orthogonal left and orthogonal right supports. Therefore, since X is contractive, Proposition 2.3 gives $\|v\| = 1$. The rest is clear. ∎

REMARK 2.8. However, a contractive bimodule, which is a left and a right semi-Ruan module, is not, generally speaking, a Ruan bimodule. One can take, as a counter-example, $L \dot\otimes L^c$ or $L \otimes_{\mathrm{p}} L^c$.

Note also, that the l_2-sum of a family of one-sided semi-Ruan modules is also a semi-Ruan module of the same type.

PROPOSITION 2.9. *Let X be a right semi-Ruan module, Y a left semi-Ruan module, and $u \in X \otimes_{\mathcal{B}} Y$. Then*

$$\|u\| = \inf\{\|x\|\|y\|\},$$

where the infimum is taken over all possible representations of u in the form $u = x \otimes_{\mathcal{B}} y; x \in X, y \in Y$. (Such representations exist by Proposition 1.1).

Proof. Denote the indicated infimum by $\|u\|'$. It follows from (1.4) that $\|u\| \leqslant \|u\|'$. Our task is to establish the reverse inequality.

Take an arbitrary representation of u in the form $\sum_{k=1}^{n} x_k \otimes_{\mathcal{B}} y_k$. Obviously, without loss of generality we may suppose that $\|x_k\| = \|y_k\|; k = 1, \ldots, n$. Let $S_k, P_k; k = 1, \ldots, n,\ x$ and y be as in Proposition 1.1. The formulae (1.1) imply

that P_k is the right support of $x_k \cdot S_k^*$ and the left support of $S_k \cdot y_k; k = 1, \ldots, n$. Therefore the conditions (rsR) and (lsR) imply for our contractive modules that

$$
\|u\|' \leqslant \|x\|\|y\| \leqslant \Big(\sum_{k=1}^{n} \|x_k \cdot S_k^*\|^2 \Big)^{1/2} \Big(\sum_{k=1}^{n} \|S_k \cdot y_k\|^2 \Big)^{1/2}
$$
$$
= \Big(\sum_{k=1}^{n} \|x_k\|^2 \Big)^{1/2} \Big(\sum_{k=1}^{n} \|y_k\|^2 \Big)^{1/2} = \sum_{k=1}^{n} \|x_k\|^2 = \sum_{k=1}^{n} \|x_k\|\|y_k\|.
$$

Taking all possible representations of u as sums of elementary tensors and using (1.4), we obtain $\|u\|' \leqslant \|u\|$. ∎

Let X be a contractive bimodule, and let Y be a contractive left module. We consider the space $X \otimes_{\mathcal{B}} Y$, where X is considered as a right contractive module. Recall that in this situation $X \otimes_{\mathcal{B}} Y$ has the structure of a contractive left module with the outer multiplications uniquely defined by $a \cdot (x \otimes_{\mathcal{B}} y) := (a \cdot x) \otimes_{\mathcal{B}} y$. Similarly, if X is a contractive module and Y is a contractive bimodule, then the space $X \otimes_{\mathcal{B}} Y$, where now Y is considered as a contractive left module, is a contractive right module with the outer multiplications uniquely defined by $(x \otimes_{\mathcal{B}} y) \cdot a := x \otimes_{\mathcal{B}} (y \cdot a)$.

PROPOSITION 2.10. *Let X be a Ruan bimodule, and let Y be a left semi-Ruan module. Then $X \otimes_{\mathcal{B}} Y$ is a left semi-Ruan module.*

Let X be a right semi-Ruan module, and let Y be a Ruan bimodule. Then $X \otimes_{\mathcal{B}} Y$ is a right semi-Ruan module.

Proof. Since the arguments concerning both assertions are strictly parallel, we restrict ourselves to the first one. Since X is contractive as a left module, the equality (1.4) obviously implies that $X \otimes_{\mathcal{B}} Y$ is also contractive as a left module. So we concentrate on the condition (lsR).

Let $u_1, u_2 \in X \otimes_{\mathcal{B}} Y$ have orthogonal left supports, say Q_1 and Q_2. By virtue of Proposition 1.1, we may suppose that $u_k = x_k \otimes_{\mathcal{B}} y_k : k = 1, 2$. Obviously, without loss of generality we may also suppose that $\|x_k\| = 1$ and $x_k := Q_k \cdot x_k; k = 1, 2$.

Take, for our u_k, the operators S_k and $P_k; k = 1, 2$ as in the just-mentioned proposition. Then we have

$$
u_1 + u_2 = (x_1 \cdot S_1^* + x_2 \cdot S_2^*) \otimes_{\mathcal{B}} (S_1 \cdot y_1 + S_2 \cdot y_2) \,.
$$

Further, the elements $x_k \cdot S_k^*; k = 1, 2$ have orthogonal left supports Q_k and orthogonal right supports P_k, respectively. Therefore, since X is a contractive bimodule, Proposition 2.3 implies that

$$
\|x_1 \cdot S_1^* + x_2 \cdot S_2^*\| = \max\{\|x_1 \cdot S_1^*\|, \|x_2 \cdot S_2^*\|\} = \max\{\|x_1\|, \|x_2\|\} = 1 \,.
$$

Consequently, $\|u_1 + u_2\| \leqslant \|S_1 \cdot y_1 + S_2 \cdot y_2\|$. But the elements $S_k \cdot y_k; k = 1, 2$ have orthogonal left supports P_k, and Y is contractive and satisfies (lsR). Thus

$$\|u_1 + u_2\| \leqslant (\|S_1 \cdot y_1\|^2 + \|S_2 \cdot y_2\|^2)^{1/2}$$
$$= (\|y_1\|^2 + \|y_2\|^2)^{1/2} = ((\|x_1\|\|y_1\|)^2 + (\|x_2\|\|y_2\|)^2)^{1/2}.$$

It remains to take all possible representations of u_1 and u_2 as elementary tensors, and to apply Proposition 2.9.   ∎

3. EXTREMELY FLAT AND EXTREMELY INJECTIVE (BI)MODULES

We give the following definition in the spirit of the well-known definitions of flat and of strictly flat Banach module ([4], Chapter VII, Section 1; [5], Chapter VII, Section 1.3).

DEFINITION 3.1. A contractive left module X is *extremely flat with respect to semi-Ruan modules* or, for short, *ESR-flat*, if, for every isometric morphism $\alpha : Y \to Z$ of right semi-Ruan modules, the operator $\alpha \otimes_{\mathcal{B}} \mathbf{1}_X : Y \otimes_{\mathcal{B}} X \to Z \otimes_{\mathcal{B}} X$ (see the end of Section 1) is also isometric.

We define similarly the "right-hand" version of this notion.

Finally, a contractive bimodule X is *extremely flat with respect to Ruan bimodules* or, for short, *ER-flat*, if, for every isometric morphism $\alpha : Y \to Z$ of Ruan bimodules, the operator $\alpha \otimes_{\mathcal{B}-\mathcal{B}} \mathbf{1}_X : Y \otimes_{\mathcal{B}-\mathcal{B}} X \to Z \otimes_{\mathcal{B}-\mathcal{B}} X$ is also isometric.

REMARK 3.2. The word "extremely" is chosen because isometric operators or morphisms are exactly the so-called extreme monomorphisms in some principal categories of spaces or (bi)modules in functional analysis (cf., e.g., [2] and Chapter 0, Section 5 of [7].

As simplest examples, the module $\mathcal{B}$ is ESR-flat as a left and as a right contractive module, whereas the bimodule $\mathcal{B} \otimes_{\mathrm{p}} \mathcal{B}$ is an ER-flat contractive bimodule. Of course, this is because tensoring by $\mathcal{B}$ in the one-sided case and by $\mathcal{B} \otimes_{\mathrm{p}} \mathcal{B}$ in the two-sided case does not change a given space. In addition, one can easily show that $\mathcal{B} \otimes_{\mathrm{p}} l_1$ and $(\mathcal{B} \otimes_{\mathrm{p}} \mathcal{B}) \otimes_{\mathrm{p}} l_1$ are ESR-flat as a one-sided module and ER-flat as a two-sided module, respectively. Note, that in these examples, tensoring by the respective (bi)module preserve the isometry of morphisms of all given contractive modules, and not only (semi-)Ruan modules. The properties of the latter modules will be seen to be indispensable when, very soon, we proceed to other examples, more important for our aims.

We emphasize that the given definition does not require that our extremely flat (bi)module is itself a (semi-)Ruan (bi)module. However, in our principal examples that will be the case.

Let us show that several standard constructions preserve the property of extreme flatness.

PROPOSITION 3.3. *If a left or right contractive module is* ESR-*flat, then the same is true for its complex conjugate module.*

Proof. To be definite, consider a left ESR-module X. Our task is to prove that, for every isometric morphism of left semi-Ruan modules $\alpha : Y \to Z$, the operator $\mathbf{1}_{X^c} \otimes_{\mathcal{B}} \alpha : X^c \otimes_{\mathcal{B}} Y \to X^c \otimes_{\mathcal{B}} Z$ is isometric.

Consider $\mathbf{1}_{X^c} \otimes_{\mathcal{B}} \alpha$ as acting between the respective complex conjugate normed spaces $(X^c \otimes_{\mathcal{B}} Y)^c$ and $(X^c \otimes_{\mathcal{B}} Z)^c$. It is obvious that the first space coincides with $Y^c \otimes_{\mathcal{B}} X$ up to an isometric isomorphism, uniquely defined by taking $x \otimes y$ to $y \otimes x$, and similarly that the second space coincides with $Z^c \otimes_{\mathcal{B}} X$. Moreover, under such an identification the operator $\mathbf{1}_{X^c} \otimes_{\mathcal{B}} \alpha$ transforms to $\alpha \otimes_{\mathcal{B}} \mathbf{1}_X : Y^c \otimes_{\mathcal{B}} X \to Z^c \otimes_{\mathcal{B}} X$, where α, now being considered as a map between Y^c and Z^c, is, of course, an isometric morphism of the respective complex conjugate right modules. But the latter are, by Proposition 2.5, semi-Ruan modules.

The rest is clear. ∎

PROPOSITION 3.4. *Let X be a left and Y a right* ESR-*flat contractive module. Suppose that at least one of them is a semi-Ruan module. Then the bimodule $X \otimes_{\mathrm{p}} Y$ (cf. Section 1) is* ER-*flat.*

Proof. To be definite, suppose that Y is a semi-Ruan module. Let $\alpha : Z_1 \to Z_2$ be an isometric morphism of Ruan bimodules. Our task is to show that the operator $\mathbf{1}_{X \otimes_{\mathrm{p}} Y} \otimes_{\mathcal{B}-\mathcal{B}} \alpha : (X \otimes_{\mathrm{p}} Y) \otimes_{\mathcal{B}-\mathcal{B}} Z_1 \to (X \otimes_{\mathrm{p}} Y) \otimes_{\mathcal{B}-\mathcal{B}} Z_2$ is also isometric.

It is known (and easy to verify) that the latter operator is weakly isometrically equivalent to the operator $(\mathbf{1}_Y \otimes_{\mathcal{B}} \alpha) \otimes_{\mathcal{B}} \mathbf{1}_X : (Y \otimes_{\mathcal{B}} Z_1) \otimes_{\mathcal{B}} X \to (Y \otimes_{\mathcal{B}} Z_2) \otimes_{\mathcal{B}} X$. Recall (cf., e.g., [7]) that this means that there exists a commutative diagram

$$
\begin{array}{ccc}
(X \otimes_{\mathrm{p}} Y) \otimes_{\mathcal{B}-\mathcal{B}} Z_1 & \xrightarrow{\ \mathbf{1}_{X \otimes_{\mathrm{p}} Y} \otimes_{\mathcal{B}-\mathcal{B}} \alpha\ } & (X \otimes_{\mathrm{p}} Y) \otimes_{\mathcal{B}-\mathcal{B}} Z_2 \ , \\
\downarrow & & \downarrow \\
(Y \otimes_{\mathcal{B}} Z_1) \otimes_{\mathcal{B}} X & \xrightarrow{\ (\mathbf{1}_Y \otimes_{\mathcal{B}} \alpha) \otimes_{\mathcal{B}} \mathbf{1}_X\ } & (Y \otimes_{\mathcal{B}} Z_2) \otimes_{\mathcal{B}} X
\end{array}
$$

where the vertical arrows depict isometric isomorphisms of normed spaces. In the case that we are considering, these isomorphisms, a kind of "complicated associativity", are uniquely defined by taking an elementary tensor $(x \otimes y) \otimes_{\mathcal{B}-\mathcal{B}} z$ to $(y \otimes_{\mathcal{B}} z) \otimes_{\mathcal{B}} x$; here $x \in X, y \in Y$, and z belongs to Z_1 or Z_2.

We see that it is sufficient to show that the operator $(\mathbf{1}_Y \otimes_{\mathcal{B}} \alpha) \otimes_{\mathcal{B}} \mathbf{1}_X$ is isometric. But Y is ESR-flat, and, by Proposition 2.7, α is a morphism of left semi-Ruan modules. Therefore the operator $\mathbf{1}_Y \otimes_{\mathcal{B}} \alpha : Y \otimes_{\mathcal{B}} Z_1 \to Y \otimes_{\mathcal{B}} Z_2$ is isometric. However, this operator is, of course, a morphism of right modules; moreover, by Proposition 2.9, it is a morphism of semi-Ruan modules. It remains to recall that X is also extremely flat. ∎

The property of extreme flatness which we introduced is intimately con-nected with the question of the extension of bounded morphisms, descending from the classical Hahn–Banach Theorem.

DEFINITION 3.5. A contractive left module X is *extremely injective with re-spect to semi-Ruan modules* or, for short, *ESR-injective*, if, for every isometric mor-phism $\alpha : Y \to Z$ of left semi-Ruan modules and an arbitrary bounded morphism of left modules $\Phi : Y \to X$, there exists a bounded morphism of left modules $\Psi : Z \to X$ such that the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \alpha\ } & Z \\
{\scriptstyle \Phi}\downarrow & \swarrow{\scriptstyle \Psi} & \\
X & &
\end{array}
$$

is commutative and $\|\Phi\| = \|\Psi\|$. In other words, every bounded morphism of left modules from Y into X can be extended, after the identification of Y with a submodule of Z, to a morphism from Z to X with the same norm.

We define the "right" version of this notion in the obvious symmetric way,.

Finally, by replacing words "left module" by "bimodule" and also "semi-Ruan" by "Ruan", we obtain the definition of a bimodule, *extremely injective with respect to Ruan bimodules* or, for short, of an *ER-injective bimodule*.

PROPOSITION 3.6. (i) *Let X be a contractive left or right normed module. Then it is* ESR-*flat if and only if its dual right or, respectively, left module X^* is* ESR-*injective.*

(ii) *Let X be a contractive bimodule. Then it is* ER-*flat if and only if its dual bimodule X^* is* ER-*injective.*

Proof. Since the argument is parallel in all three cases, we shall restrict our-selves to the case of a given left module.

It is obvious that the assertion that X^* is ESR-injective is equivalent to the following statement: for every isometric morphism $\alpha : Y \to Z$ of right semi-Ruan modules, the operator $\alpha_* : \mathbf{h}_{\mathcal{B}}(Z, X^*) \to \mathbf{h}_{\mathcal{B}}(Y, X^*) : \beta \mapsto \beta\alpha$, otherwise, the relevant restriction operator, is strictly co-isometric. (The latter property means that our operator maps the closed unit ball in the domain space onto the closed unit ball in the range space). According to the law of the adjoint associativity, also called the exponential law (see, e.g., Chapter III, Section 3.8 of [2] or Chapter VI, Section 3.2 of [4]), the normed space $\mathbf{h}_{\mathcal{B}}(Y, X^*)$ coincides with the space $(Y \otimes_{\mathcal{B}} X)^*$ up to the isometric isomorphism, taking a morphism $\varphi : Y \to X^*$ to the functional $f : Y \otimes_{\mathcal{B}} X \to \mathbb{C}$, well-defined by $f(y \otimes_{\mathcal{B}} x) = [\varphi(y)](x)$. Similarly, $\mathbf{h}_{\mathcal{B}}(Y, X^*)$ is identified with $(Z \otimes_{\mathcal{B}} X)^*$. Moreover, one can easily check that we

have a commutative diagram

$$
\begin{array}{ccc}
\mathbf{h}_{\mathcal{B}}(Y, X^*) & \xrightarrow{\ \alpha_*\ } & \mathbf{h}_{\mathcal{B}}(Y, X^*) \\
\downarrow & & \downarrow \\
(Z \otimes_{\mathcal{B}} X)^* & \xrightarrow{\ \alpha^\bullet\ } & (Y \otimes_{\mathcal{B}} X)^*
\end{array}
$$

where the vertical arrows depict indicated isometric isomorphisms of normed spaces, and $\alpha^\bullet$ is the operator which is adjoint to $\alpha \otimes_{\mathcal{B}} \mathbf{1}_X : Y \otimes_{\mathcal{B}} X \to Z \otimes_{\mathcal{B}} X$. Consequently, the operators α_* and $\alpha^\bullet$ are simultaneously strictly co-isometric or not. But, as an obvious corollary (in fact, an equivalent formulation) of the Hahn–Banach theorem, an adjoint operator is strictly co-isometric if and only if the original operator is isometric. The rest is clear. ∎

As a byproduct, we have the following result.

PROPOSITION 3.7. *Suppose that X is a contractive left, right or two-sided module, and X_0 is a dense submodule of the respective type. Then X is* ESR- *(or, according to the sense,* ER-*) flat if and only if the same is true of X_0.*

Proof. Indeed, the dual (bi)modules of X and X_0 coincide, and hence they are simultaneously extremely injective or not. Then the previous proposition works. ∎

In what follows, an assertion that (bi)modules of this or that class are ESR- (or ER-) injective, will be referred as a "theorem of the Arveson–Wittstock type". This is because assertions of that type have their origin in the "genuine" Arveson–Wittstock theorem of quantum functional analysis (= operator space theory). As to Proposition 3.6, it suggests a certain way to establish such theorems, reducing questions about extreme injectivity to those about extreme flatness.

4. EXTREME FLATNESS OF CERTAIN MODULES

Choose, in addition to our canonical Hilbert space L, an arbitrary Hilbert space H. In this section we shall prove the extreme flatness of some (bi)modules connected with this space.

At first, take the algebraic tensor product $L \otimes H$ as a subspace of $L \overset{\cdot}{\otimes} H$ with the induced norm. It is obviously a submodule with respect to the outer multiplication in the latter space, considered in the Example 2.

In what follows, the symbol $\mathcal{S}(\,\cdot\,,\,\cdot\,)$ denotes the space of Schmidt operators between two Hilbert spaces, equipped with the Schmidt norm $\|a\|_{\mathcal{S}} := \mathrm{tr}(a^*a)^{1/2}$, whereas $\mathcal{F}_{\mathcal{S}}(\,\cdot\,,\,\cdot\,)$ denotes its dense normed subspace consisting of finite-rank operators (that is $\mathcal{F}(\,\cdot\,,\,\cdot\,)$, considered with the Schmidt norm). We recall that $L \overset{\cdot}{\otimes} H$, as a normed space, can be identified with the space $\mathcal{S}(H^c, L)$ by

means of the isometric isomorphism uniquely defined by taking the elementary tensor $\xi \otimes \eta$ to the rank-one operator $\xi \circ \eta$ (cf., e.g., Chapter 3, Section 4.3 of [7]). Clearly, this isometric isomorphism identifies $L \otimes H$ with $\mathcal{F}_{\mathcal{S}}(H^{c}, L)$.

Note that the space $\mathcal{S}(H^{c}, L)$ is a left contractive module with respect to the usual operator composition: for $a \in \mathcal{B}$ and $b \in \mathcal{S}(H^{c}, L)$, we set $a \cdot b := ab$. Now, returning to the mentioned isometric isomorphism, we see that it actually provides the identification of $L \overset{\cdot}{\otimes} H$ with $\mathcal{S}(H^{c}, L)$ and of $L \otimes H$ with $\mathcal{F}_{\mathcal{S}}(H^{c}, L)$ as left contractive modules. This can be immediately checked on elementary tensors.

From now on we denote the left module $\mathcal{F}_{\mathcal{S}}(H^{c}, L)$ briefly by $\mathcal{X}$. Take an arbitrary right semi-Ruan module Y. For a time, the main object of our study will be the normed space $Y \otimes_{\mathcal{B}} \mathcal{X}$.

Let $c : L \to H^{c}$ be a bounded operator. Consider the bilinear operator

$$\mathcal{T}_{c}^{Y} : Y \times \mathcal{X} \to Y : (y, b) \mapsto y \cdot (bc) .$$

Of course, $\mathcal{T}_{c}^{Y}$ is bounded, and $\|\mathcal{T}_{c}^{Y}\| \leqslant \|c\|$. Furthermore, one can immediately check that this bilinear operator is balanced. Therefore (see Section 1), it gives rise to the bounded operator from $Y \otimes_{\mathcal{B}} \mathcal{X}$ into Y, uniquely defined by

$$y \underset{\mathcal{B}}{\otimes} b \mapsto y \cdot (bc) \quad y \in Y, b \in \mathcal{X}$$

and having norm $\leqslant \|c\|$. Denote this operator by T_{c}^{Y}.

PROPOSITION 4.1. *Let $u \in Y \otimes_{\mathcal{B}} \mathcal{X}$ be represented as an elementary tensor $y \otimes_{\mathcal{B}} b$ (cf. Proposition 1.1). Further, let $P \in \mathcal{F}$ be the projection on $\mathrm{Im}(b)$. Then $u = y \cdot P \otimes_{\mathcal{B}} b$, and there exists an operator $c \in \mathcal{F}(L, H^{c})$ such that $T_{c}^{Y}(u) = y \cdot P$.*

Proof. Since, of course, we have $Pb = b$, formulae (1.3) give the first of the desired equalities. Further, it is clear from the fact that $\dim(\mathrm{Im}(b)) < \infty$ that there exists $c \in \mathcal{F}(L, H^{c})$ such that $bc = P$.

The second desired equality follows immediately.   ∎

REMARK 4.2. From this, as a first application, one can easily obtain that our normed tensor product $Y \otimes_{\mathcal{B}} \mathcal{X}$ coincides with the *algebraic* tensor product of Y and $\mathcal{X}$ over $\mathcal{B}$. In other words, the subspace

$$N_{1} := \mathrm{span}\{x \cdot a \otimes y - x \otimes a \cdot y\}$$

is closed in $Y \otimes_{\mathrm{p}} \mathcal{X}$ (cf. Section 1), and thus the quotient semi-norm on $(Y \otimes_{\mathrm{p}} \mathcal{X})/N_{1}$ is actually a norm. But we do not need this observation.

Now let $\alpha : Y \to Z$ be an arbitrary bounded morphism of contractive right semi-Ruan modules. Then, by virtue of the functorial properties of the module tensor product (see Section 1), the operator $\alpha \otimes_{\mathcal{B}} \mathbf{1}_{\mathcal{X}} : Y \otimes_{\mathcal{B}} \mathcal{X} \to Z \otimes_{\mathcal{B}} \mathcal{X}$ appears.

Note that for every $c \in \mathcal{B}(L, H^c)$ we have the commutative diagram

(4.1)
$$
\begin{array}{ccc}
Y \otimes_\mathcal{B} \mathcal{X} & \xrightarrow{\ T_c^Y\ } & Y \\
{\scriptstyle \alpha \otimes_\mathcal{B} \mathbf{1}_\mathcal{X}} \downarrow & & \downarrow {\scriptstyle \alpha} \\
Z \otimes_\mathcal{B} \mathcal{X} & \xrightarrow{\ T_c^Z\ } & Z\,.
\end{array}
$$

This can be immediately verified on elementary tensors in $Y \otimes_\mathcal{B} \mathcal{X}$.

PROPOSITION 4.3. *If α is an injective map, then the same is true of $\alpha \otimes_\mathcal{B} \mathbf{1}_\mathcal{X}$.*

Proof. Suppose that, for $u \in Y \otimes_\mathcal{B} \mathcal{X}$, we have $\alpha \otimes_\mathcal{B} \mathbf{1}_\mathcal{X}(u) = 0$. Take y, P and c as in Proposition 4.1. Then the commutative diagram above gives $y \cdot P = T_c^Y(u) = 0$. But this, of course, means that $u = 0$. ∎

At last, we are ready to prove our main theorem.

THEOREM 4.4. *Let H be an arbitrary Hilbert space. Then the left contractive modules $L \otimes H$ and $L \overset{\cdot}{\otimes} H$ are ESR-flat.*

Proof. Taking into account Proposition 3.7, it is sufficient to show that the module $\mathcal{X} := \mathcal{F}_\mathcal{S}(H^c, L)$, that is, as we remember, $L \otimes H$ in disguise, have the desired property.

Let $\alpha : Y \to Z$ be an isometric morphism of left modules. Consequently (cf. Section 1), $\alpha \otimes_\mathcal{B} \mathbf{1}_\mathcal{X}$ is a contractive operator. Therefore our task is to prove that, for every $v \in Y \otimes_\mathcal{B} \mathcal{X}$ and $u := (\alpha \otimes_\mathcal{B} \mathbf{1}_\mathcal{X})(v)$, we have $\|v\| \leqslant \|u\|$.

Take the representation of u as $z \otimes_\mathcal{B} b$, as provided by Proposition 1.1 (with Z in the role of Y). After this, take the respective P and c, indicated in Proposition 4.1. Then the commutative diagram (4.1) gives

$$
z \cdot P = T_c^Z(u) = T_c^Z(\alpha \otimes_\mathcal{B} \mathbf{1}_\mathcal{X})(v) = \alpha(y)\,,
$$

where $y := T_c^Y(v) \in Y$. From this we have that $(\alpha \otimes_\mathcal{B} \mathbf{1}_\mathcal{X})(y \otimes_\mathcal{B} b) = u$, and, because of Proposition 4.3, $v = y \otimes_\mathcal{B} b$. Now, remembering that α is an isometric operator, we obtain the estimate

$$
\|v\| \leqslant \|y\| \|b\| = \|z \cdot P\| \|b\| \leqslant \|z\| \|b\|.
$$

Further, $L \otimes H$ is a semi-Ruan module, and hence the same is true of its "alter ego" $\mathcal{X}$. It remains to take the infimum of numbers $\|z\| \|b\|$ over all possible representations of u as elementary tensors in the previous estimate, and then to apply Proposition 2.9. ∎

REMARK 4.5. As a matter of fact, every semi-Ruan module is ESR-flat. This was shown by the referee of our paper in his report. The argument, which is much more lengthy and sophisticated than the proof of Theorem 4.4, is suggested by some results of Lambert [9].

As an immediate corollary of Theorem 4.4, we have the following theorem.

THEOREM 4.6. *Let H and K be arbitrary Hilbert spaces. Then:*

(i) *the right contractive modules $L^c \otimes H$ and $L^c \dot{\otimes} H$ are* ESR-*flat;*

(ii) *the contractive bimodules* $(L \otimes H) \otimes_p (L^c \otimes K)$, $(L \dot{\otimes} H) \otimes_p (L^c \dot{\otimes} K)$, *and their completion* $(L \dot{\otimes} H) \overset{p}{\otimes} (L^c \dot{\otimes} K)$ *are* ER-*flat.*

Proof. (i) This follows from the previous theorem and Proposition 3.3, being applied to $L \otimes H^c$ and $L \dot{\otimes} H^c$.

(ii) This follows from the previous theorem, combined with the assertion (i), Proposition 3.4 and also, in the case of the third indicated bimodule, with Proposition 3.7. ∎

REMARK 4.7. We do not know whether the indicated (bi)modules are extremely flat in the "absolute" sense. By this, in the case, say, of left modules, we mean the following property of a given X: the operator $\alpha \otimes_B \mathbf{1}_X$ is isometric whenever α is an isometric morphism between arbitrary (and not only semi-Ruan) right normed modules. It is somehow doubtful that the answer is "yes". Anyhow, if we consider the similarly defined "absolute extreme" version of projectivity, then the module $L \dot{\otimes} H$ certainly does not possess this stronger property provided $\dim H = \infty$. As it was shown in [6], such a module is not projective even in the usual sense of Banach homology.

Now we came to several Arveson–Wittstock type theorems. Here again we need the law of the adjoint associativity (= exponential law), now in a slightly different version. Namely, suppose that X is a left and Y is a right contractive module. In such a context, accordingly to what was said in Section 1, Y^* becomes a left contractive module, and $\mathcal{B}(X, Y^*)$, $X \otimes_p Y$, and $(X \otimes_p Y)^*$ become contractive bimodules. Then $\mathcal{B}(X, Y^*)$ coincides with $(X \otimes_p Y)^*$ up to the isometric bimodule isomorphism which takes an operator $\varphi : X \to Y^*$ to the functional $f : X \otimes_B Y \to \mathbb{C}$, well-defined by the formula $f(y \otimes_B x) = [\varphi(y)](x)$.

THEOREM 4.8. *Let H and K be arbitrary Hilbert spaces. Then the left contractive module $L \dot{\otimes} H$ and the right contractive module $L^c \dot{\otimes} H$ are* ESR-*injective, whereas the contractive bimodule $\mathcal{B}(L \dot{\otimes} H, L \dot{\otimes} K)$ (see Example* 2.2*) is* ER-*injective.*

Proof. To begin with, it is obvious that, up to an isometric isomorphism of modules of the relevant type, $L \dot{\otimes} H = (L^c \dot{\otimes} H^c)^*$ and $L^c \dot{\otimes} H = (L \dot{\otimes} H^c)^*$.

Furthermore, by virtue of the Riesz representation theorem, the normed space $\mathcal{B}(L \dot{\otimes} H, L \dot{\otimes} K)$, that is $\mathcal{B}(L \dot{\otimes} H, (L^c \dot{\otimes} K^c)^c)$, can be identified with the normed space $\mathcal{B}(L \dot{\otimes} H, (L^c \dot{\otimes} K^c)^*)$. Recalling that the latter space is also a contractive bimodule (of the type $\mathcal{B}(X, Y^*)$; cf. above), we immediately see that actually we have an identification of contractive bimodules. Finally, the bimodule

$\mathcal{B}(L \overset{\cdot}{\otimes} H, (L^c \overset{\cdot}{\otimes} K^c)^*)$ coincides, by the above mentioned law of the adjoint associativity, with the module $[(L \overset{\cdot}{\otimes} H) \otimes_p (L^c \overset{\cdot}{\otimes} K^c)]^*$.

Thus all we have to do in all three cases is to combine Theorems 4.4 and 4.6 with Proposition 3.6. ∎

5. THE ARVESON–WITTSTOCK THEOREM

In the concluding part of the paper we recall the Arveson–Wittstock theorem and show that it follows from Theorem 4.8. Being, so to say, in the air, it must be well known that this theorem can be easily deduced from the extension theorems for morphisms of bimodules. Nevertheless, for the completeness of the picture, we shall present some details.

In what follows, we use the principal definitions of quantum functional analysis (= operator space theory) in the frame-work of the non-coordinate approach. The main ideas of such an approach can be essentially found in the book of Pisier [13] and in the unpublished notes of Barry Johnson. The detailed definitions, in somewhat different form, are given in [8]; these are the amplification of a linear space and of a linear operator, a quantum space (= abstract operator space) a concrete quantum space (= concrete operator space), and, above all, a completely bounded operator and its completely bounded norm $\| \cdot \|_{cb}$.

THEOREM 5.1 (Arveson–Wittstock Theorem). *Let E be a quantum subspace of a quantum space G, and let H be an arbitrary Hilbert space. Then every completely bounded operator φ from E into the concrete quantum space $\mathcal{B}(H)$ can be extended to a completely bounded operator $\psi : G \to \mathcal{B}(H)$ such that $\|\psi\|_{cb} = \|\varphi\|_{cb}$.*

Proof. Since $\mathcal{B}(H)$ is concrete, its amplification $\mathcal{F} \otimes \mathcal{B}(H)$ is identified with a sub-bimodule of $\mathcal{B}(L \overset{\cdot}{\otimes} H)$. Let Φ be a coextension of the amplification $\varphi_\infty := \mathbf{1}_{\mathcal{F}} \otimes \varphi$ of φ to a morphism into $\mathcal{B}(L \overset{\cdot}{\otimes} H)$. Then Theorem 4.8, being considered for $Y := \mathcal{F} \otimes E$, $Z := \mathcal{F} \otimes G$, and $K := H$, provides an extension Ψ of Φ with the same norm.

Observe that the image of Ψ lies in $\mathcal{F} \otimes \mathcal{B}(H)$. Indeed, $\mathcal{F} \otimes Z = \mathrm{span}\{(\xi \circ \eta)z; \xi, \eta \in L, z \in Z\}$, and, by the equalities (1.2), we have $\xi \circ \eta = (\xi \circ e)p(e \circ \eta)$ for every $e \in L$; $\|e\| = 1$ and $p := e \circ e$. Therefore, taking into account the fact that Ψ is a morphism of $\mathcal{B}$-bimodules, it is sufficient to show that $\Psi(pz) = p \cdot \Psi(pz) \cdot p$ belongs to $\mathcal{F} \otimes \mathcal{B}(H)$. But it is indeed the case, since it is well known that, for every $\tilde{a} \in \mathcal{B}(L \overset{\cdot}{\otimes} H)$ we have $p \cdot \tilde{a} \cdot p = p \overset{\cdot}{\otimes} T$ for $T \in \mathcal{B}(H)$, well defined by $e \otimes T\xi = (p \overset{\cdot}{\otimes} \mathbf{1}_H)[\tilde{a}(e \otimes \xi)]; \xi \in H$.

Thus Ψ has a well-defined corestriction to $\mathcal{F} \otimes \mathcal{B}(H)$. This corestriction, being a bimodule morphism, obviously has the form $\psi_\infty := \mathbf{1}_{\mathcal{F}} \otimes \psi$ for some

operator $\psi : G \to \mathcal{B}(H)$. Further, $\|\psi\|_{\mathrm{cb}} = \|\psi_\infty\| = \|\Psi\| = \|\Phi\| = \|\varphi\|_{\mathrm{cb}}$. Finally, ψ_∞ is an extension of φ_∞, and this obviously implies that ψ is an extension of φ. ∎

REMARK 5.2. We should like to emphasize that we have deduced from Theorem 4.8 the non-coordinate version of the original Arveson–Wittstock Theorem, concerning just linear completely bounded operators. What we did not touch, is the later and more general form of the Arveson–Wittstock Theorem, dealing with completely bounded morphisms of bimodules over two arbitrary unital C^*-algebras. Different proofs of such a theorem, formulated in various degrees of generality, can be found in the papers of Wittstock ([17], Theorem 3.1), Suen [15], Muhly and Na ([11], Theorem 3.4), Pop ([14], Theorem 2.5). Note that it could be shown that the respective $\mathcal{B}$-bimodule version of the Arveson–Wittstock Theorem and our Theorem 4.8 are equivalent. This is because, as it was observed by the referee, there exist isometric functors from the categories of Ruan and semi-Ruan modules into the categories of operator $\mathcal{B}$-bimodules and operator $\mathcal{B}$-modules, respectively.

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A.YA. HELEMSKII, FACULTY OF MECHANICS AND MATHEMATICS, MOSCOW STATE UNIVERSITY, MOSCOW 119992, RUSSIA
E-mail address: helemskii@rambler.ru