$AC(\sigma)$ OPERATORS

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ABSTRACT. In this paper we present a new extension of the theory of well-bounded operators to cover operators with complex spectrum. In previous work a new concept of the class of absolutely continuous functions on a non-empty compact subset σ of the plane, denoted $AC(\sigma)$, was introduced. An $AC(\sigma)$ operator is one which admits a functional calculus for this algebra of functions. The class of $AC(\sigma)$ operators includes all of the well-bounded operators and trigonometrically well-bounded operators, as well as all scalar-type spectral operators, but is strictly smaller than Berkson and Gillespie's class of AC operators. This paper develops the spectral properties of $AC(\sigma)$ operators and surveys some of the problems which remain in extending results from the theory of well-bounded operators.

KEYWORDS: *Functions of bounded variation, absolutely continuous functions, functional calculus, AC-operators, well-bounded operators.*

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1. INTRODUCTION

A Banach space operator is said to be well-bounded if it admits a functional calculus for $AC(J)$, the algebra of absolutely continuous functions on some compact interval $J \subseteq \mathbb{R}$. The motivation for the introduction of this class was to provide a theory which extended the spectral representation results which apply to self-adjoint operators to Banach space operators which may possess a conditionally rather than unconditionally convergent spectral expansion. Smart and Ringrose [27], [29], [30] showed that well-bounded operators always have an integral representation with respect to a family of projections known as a decomposition of the identity. The usefulness of this most general form of the theory is somewhat restricted however since the decomposition of the identity acts on the dual of the underlying Banach space and is in general not unique (see [18] for examples of this non-uniqueness).

In [8] a subclass of the well-bounded operators, the well-bounded operators of type (B), were introduced. The type (B) well-bounded operators, which

include those well-bounded operators acting on reflexive spaces, possess a theory of integration with respect to a family of projections which act on the original space. This family of projections, known as the spectral family, is uniquely determined by the operator. The integration theory provides an extension of the $AC(J)$ functional calculus to a $BV(J)$ functional calculus where $BV(J)$ is the algebra of functions of bounded variation on the interval J.

As is the case for a self-adjoint operator, the spectrum of a well-bounded operator must lie in the real line. The main obstacle to overcome if one wishes to extend the theory of well-bounded operators to cover operators whose spectrum may not lie in the real line, is that of obtaining a suitable concept of bounded variation for functions defined on a subset of the plane. Many such concepts exist in the literature. In [9], Berkson and Gillespie used a notion of variation ascribed to Hardy and Krause to define the AC operators. These are the operators which have an $AC_{HK}(J \times K)$ functional calculus where $AC_{HK}(J \times K)$ is the algebra of absolutely continuous functions in the sense of Hardy and Krause defined on a rectangle $J \times K \subset \mathbb{R}^2 \cong \mathbb{C}$. They showed ([9], Theorem 5) that an operator $T \in B(X)$ is an AC operator if and only if $T = R + iS$ where R and S are commuting well-bounded operators. In [7] it is shown that this splitting is not necessarily unique. Furthermore even if T is an AC operator on a Hilbert space H, it does not necessarily follow that αT is an AC operator for all $\alpha \in \mathbb{C}$. On the positive side, the AC operators include the trigonometrically well-bounded operators which have found important applications in harmonic analysis and differential equations (see [10] and [11]). An operator $T \in B(X)$ is said to be trigonometrically well-bounded if there exists a type (B) well-bounded operator $A \in B(X)$ such that $T = \exp(iA)$.

One of the problems in the theory well-bounded and AC operators is that the functional calculus of these operators is based on an algebra of functions whose domain is either an interval in the real axis or a rectangle in the plane. From an operator theory point of view, a much more natural domain is the spectrum, or at least a neighbourhood of the spectrum. Secondly, as we have already mentioned, the class of AC operators is not closed under multiplication by scalars. This is also undesirable, since if one has structural information about an operator T, this clearly gives similar information about αT. To overcome these problems, in [3] we defined $AC(\sigma)$, the Banach algebra of absolutely continuous functions whose domain is some compact set σ in the plane. In this paper we look at those operators which have an $AC(\sigma)$ functional calculus, which we call $AC(\sigma)$ operators.

Section 2 summarizes some of the main results from [3] concerning the function algebras $BV(\sigma)$ and $AC(\sigma)$. The question as to how one may patch together absolutely continuous functions defined on different domains is addressed in Section 3. These results will be needed in order to show that $AC(\sigma)$ operators are decomposable in the sense of [16].

In Section 4 we give some results which illustrate the extent of the class of $AC(\sigma)$ operators. In particular, we note that this class contains all scalar-type spectral operators, all well-bounded operators and all trigonometrically well-bounded operators.

In Section 5 we develop some of the main spectral properties of $AC(\sigma)$ operators. Here we show that the $AC(\sigma)$ operators form a proper subclass of the AC operators and hence such operators have a splitting into real and imaginary well-bounded parts. The natural conjecture that every $AC(\sigma)$ operator is in fact an $AC(\sigma(T))$ operator remains open. Resolving this question depends on being able to answer some difficult questions about the relationships between $AC(\sigma_1)$ and $AC(\sigma_2)$ for different compact sets σ_1 and σ_2. These issues are discussed in Section 6.

In Section 7 we examine the case where the $AC(\sigma)$ functional calculus for T is weakly compact. In this case one can construct a family of spectral projections associated with T which is rich enough to recover T via an integration process. This 'half-plane spectral family' is a generalization of the spectral family associated with a well-bounded operator of type (B). A full integration theory for this class of operators is, however, yet to be developed. In particular, it is not known whether one can always extend a weakly compact $AC(\sigma)$ functional calculus to a $BV(\sigma)$ functional calculus. The final section discusses some of the progress that has been obtained in pursuing such a theory, and lists some of the major obstacles that remain.

Throughout this paper let $\sigma \subset \mathbb{C}$ be compact and non-empty. For a Banach space X we shall denote the bounded linear operators on X by $B(X)$ and the bounded linear projections on X by $\text{Proj}(X)$. Given $T \in B(X)$ with the single valued extension property (see [19]) and $x \in X$ we denote the local spectrum of x (for T) by $\sigma_T(x)$. We shall write λ for the identity function $\lambda : \sigma \to \mathbb{C}, z \mapsto z$.

2. $BV(\sigma)$ AND $AC(\sigma)$

We shall briefly look at $BV(\sigma)$ and $AC(\sigma)$. In particular we look at how two dimensional variation is defined. More details may be found in [3].

To define two dimensional variation we first need to look at variation along curves. Let $\Gamma = C([0,1], \mathbb{C})$ be the set of curves in the plane. Let $\Gamma_L \subset \Gamma$ be the curves which are piecewise line segments. Let $S = \{z_i\}_{i=1}^n \subset \mathbb{C}$. We write $\Pi(S) \in \Gamma_L$ for the (uniform speed) curve consisting of line segments joining the vertices at $z_1, z_2, \ldots, z_n$ (in the given order). For $\gamma \in \Gamma$ we say that $\{s_i\}_{i=1}^n \subset \sigma$ is a *partition of γ over σ* if there exists a partition $\{t_i\}_{i=1}^n$ of $[0,1]$ such that $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ and such that $s_i = \gamma(t_i)$ for all i. We shall denote the partitions of γ over σ by $\Lambda(\gamma, \sigma)$. For $\gamma \in \Gamma$ and $S \in \Lambda(\gamma, \sigma)$ we denote by γ_S the curve $\Pi(S) \in \Gamma_L$.

The variation along $\gamma \in \Gamma$ for a function $f : \sigma \to \mathbb{C}$ is defined as

$$(2.1) \qquad \mathrm{cvar}(f, \gamma) = \sup_{\{s_i\}_{i=1}^n \in \Lambda(\gamma, \sigma)} \sum_{i=1}^{n-1} |f(s_{i+1}) - f(s_i)|.$$

To each curve $\gamma \in \Gamma$ we define a weight factor ρ. For $\gamma \in \Gamma$ and a line l we let $\mathrm{vf}(\gamma, l)$ denote the number of times that γ crosses l (for a precise definition of a crossing see Section 3.1 of [3]). Set $\mathrm{vf}(\gamma)$ to be the supremum of $\mathrm{vf}(\gamma, l)$ over all lines l. We set $\rho(\gamma) = \frac{1}{\mathrm{vf}(\gamma)}$. Here we take the convention that if $\mathrm{vf}(\gamma) = \infty$ then $\rho(\gamma) = 0$. We can extend the definition of ρ to include functions in $C[a, b]$ in the obvious way.

The two dimensional variation of a function $f : \sigma \to \mathbb{C}$ is defined to be

$$(2.2) \qquad \mathrm{var}(f, \sigma) = \sup_{\gamma \in \Gamma} \rho(\gamma) \, \mathrm{cvar}(f, \gamma).$$

The following properties of two dimensional variation which were shown in [3].

PROPOSITION 2.1. *Let $\sigma \subseteq \mathbb{C}$ be compact, and suppose that $f : \sigma \to \mathbb{C}$. Then*

$$\mathrm{var}(f, \sigma) = \sup_{\gamma \in \Gamma_{\mathrm{L}}} \rho(\gamma) \, \mathrm{cvar}(f, \gamma) = \sup \left\{ \rho(\gamma_S) \sum_{i=1}^{n-1} |f(s_{i+1}) - f(s_i)| : S = \{s_i\}_{i=1}^n \subseteq \sigma \right\}.$$

PROPOSITION 2.2. *Let $\sigma_1 \subset \sigma \subset \mathbb{C}$ both be compact. Let $f, g : \sigma \to \mathbb{C}, k \in \mathbb{C}$. Then*

(i) $\mathrm{var}(f + g, \sigma) \leqslant \mathrm{var}(f, \sigma) + \mathrm{var}(g, \sigma)$;
(ii) $\mathrm{var}(fg, \sigma) \leqslant \|f\|_\infty \mathrm{var}(g, \sigma) + \|g\|_\infty \mathrm{var}(f, \sigma)$;
(iii) $\mathrm{var}(kf, \sigma) = |k| \, \mathrm{var}(f, \sigma)$;
(iv) $\mathrm{var}(f, \sigma_1) \leqslant \mathrm{var}(f, \sigma)$.

For $f : \sigma \to \mathbb{C}$ set

$$(2.3) \qquad \|f\|_{BV(\sigma)} = \|f\|_\infty + \mathrm{var}(f, \sigma).$$

The functions of bounded variation with domain σ are defined to be

$$BV(\sigma) = \{f : \sigma \mapsto \mathbb{C} : \|f\|_{BV(\sigma)} < \infty\}.$$

To aid the reader we list here some of the main results from [3] and [4]. The affine invariance of these algebras (Theorem 2.5 and Proposition 2.8) is one of the main features of this theory and will be used regularly without comment.

PROPOSITION 2.3. *If $\sigma = [a, b]$ is an interval then the above definition of variation agrees with the usual definition of variation. Hence the above definition of $BV(\sigma)$ agrees with the usual definition of $BV[a, b]$ when $\sigma = [a, b]$.*

THEOREM 2.4. *Let $\sigma \subset \mathbb{C}$ be compact. Then $BV(\sigma)$ is a Banach algebra using the norm given in equation (2.3).*

THEOREM 2.5. *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. Then $BV(\sigma) \cong BV(\alpha\sigma + \beta)$.*

LEMMA 2.6. *Let $f : \sigma \to \mathbb{C}$ be a Lipschitz function with Lipschitz constant* $L(f) = \sup\limits_{z,w \in \sigma} \left| \frac{f(z) - f(w)}{z - w} \right|$. *Then* $\mathrm{var}(f, \sigma) \leqslant L(f)\,\mathrm{var}(\lambda, \sigma)$. *Hence* $f \in BV(\sigma)$.

We define $AC(\sigma)$ as being the subalgebra $BV(\sigma)$ generated by the functions 1, λ and $\overline{\lambda}$. (Note that λ and $\overline{\lambda}$ are always in $BV(\sigma)$.) We call functions in $AC(\sigma)$ the *absolutely continuous functions with respect to* σ. By Proposition 2.3 this coincides with the usual notion of absolute continuity if $\sigma = [a, b] \subset \mathbb{R}$ is an interval. In [3] the following properties of $AC(\sigma)$ are shown.

PROPOSITION 2.7. *Let $\sigma = [a, b]$ be a compact interval. Let $g \in BV(\sigma) \cap C(\sigma)$. Suppose that $\rho(g) > 0$. Then $\|f \circ g\|_{BV(\sigma)} \leqslant \frac{1}{\rho(g)} \|f\|_{BV(g(\sigma))}$ for all $f \in BV(g(\sigma))$.*

PROPOSITION 2.8. *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. Then $AC(\sigma) \cong AC(\alpha\sigma + \beta)$.*

PROPOSITION 2.9. *If $f \in AC(\sigma)$ and $f(z) \neq 0$ on σ then $\frac{1}{f} \in AC(\sigma)$. Indeed, if $M = \inf\limits_{z \in \sigma} |f(z)|$, then $\left\| \frac{1}{f} \right\|_{AC(\sigma)} \leqslant \frac{1}{M} + \frac{\mathrm{var}(f, \sigma)}{M^2}$.*

We shall also need some properties of $AC(\sigma)$ and $BV(\sigma)$ which were not included in [3].

PROPOSITION 2.10. *$BV(\sigma)$ is a lattice. If $f, g \in BV(\sigma)$, then*

$$\|f \vee g\|_{BV(\sigma)} \leqslant \|f\|_{BV(\sigma)} + \|g\|_{BV(\sigma)} \quad and \quad \|f \wedge g\|_{BV(\sigma)} \leqslant \|f\|_{BV(\sigma)} + \|g\|_{BV(\sigma)}.$$

Proof. Suppose that $\gamma \in \Gamma$ and that $\{s_i\}_{i=1}^{n} \in \Lambda(\gamma, \sigma)$. Note that for any a, a', b, b',

$$(2.4) \quad |(a \vee a') - (b \vee b')| \leqslant |(a \vee b) - (a' \vee b)| + |(a' \vee b) - (a' \vee b')| \leqslant |a - a'| + |b - b'|$$

and so

$$\sum_{i=1}^{n-1} |(f \vee g)(s_{i+1}) - (f \vee g)(s_i)| \leqslant \sum_{i=1}^{n-1} |f(s_{i+1}) - f(s_i)| + |g(s_{i+1}) - g(s_i)|.$$

Thus $\mathrm{cvar}(f \vee g, \gamma) \leqslant \mathrm{cvar}(f, \gamma) + \mathrm{cvar}(g, \gamma)$ and so

$$\begin{aligned}
\|f \vee g\|_{BV(\sigma)} &= \|f \vee g\|_{\infty} + \sup_{\gamma} \mathrm{cvar}(f \vee g, \gamma) \\
&\leqslant \|f\|_{\infty} + \|g\|_{\infty} + \sup_{\gamma} \{\mathrm{cvar}(f, \gamma) + \mathrm{cvar}(g, \gamma)\} \\
&\leqslant \|f\|_{\infty} + \sup_{\gamma} \mathrm{cvar}(f, \gamma) + \|g\|_{\infty} + \sup_{\gamma} \mathrm{cvar}(g, \gamma) \\
&= \|f\|_{BV(\sigma)} + \|g\|_{BV(\sigma)}.
\end{aligned}$$

The proof for $f \wedge g$ is almost identical. ∎

Note that $BV(\sigma)$ is not a *Banach* lattice, even in the case $\sigma = [0, 1]$.

The set $CTPP(\sigma)$ of functions on σ which are continuous and piecewise triangularly planar relative to σ was introduced in [3]. It is easy to see that $CTPP(\sigma)$ is a sublattice of $BV(\sigma)$.

COROLLARY 2.11. $AC(\sigma)$ is a sublattice of $BV(\sigma)$.

Proof. It suffices to show that if $f, g \in AC(\sigma)$, then $f \vee g \in AC(\sigma)$. Suppose then that $f, g \in AC(\sigma)$. Then there exist sequences $\{f_n\}, \{g_n\} \subseteq CTPP(\sigma)$ such that $f_n \to f$ and $g_n \to g$ in $BV(\sigma)$. As $CTPP(\sigma)$ is a lattice, $f_n \vee g_n \in CTPP(\sigma)$ for each n and, using (2.4), one can see that $(f_n \vee g_n) \to (f \vee g)$. This implies that $f \vee g$ lies in the closure of $CTPP(\sigma)$, namely $AC(\sigma)$. ∎

If one wishes to apply the results of local spectral theory, it is important that $AC(\sigma)$ forms an admissible algebra of functions in the sense of Colojoară and Foiaş [16]. The first step is to show that $AC(\sigma)$ admits partitions of unity.

LEMMA 2.12. *Let $\sigma \subset \mathbb{C}$ be compact. Then $AC(\sigma)$ is a normal algebra. That is, given any finite open cover $\{U_i\}_{i=1}^n$ of σ, there exist functions $\{f_i\}_{i=1}^n \subseteq AC(\sigma)$ such that:*

(i) $f_i(\sigma) \subset [0,1]$, *for all* $1 \leqslant i \leqslant n$;

(ii) $\operatorname{supp} f_i \subseteq U_i$ *for all* $1 \leqslant i \leqslant n$;

(iii) $\sum_{i=1}^n f_i = 1$ *on σ.*

Proof. This follows from the fact that $C^\infty(\sigma) \subseteq AC(\sigma)$ (see Proposition 4.7 of [3]). More precisely, let $\{U_i\}_{i=1}^n$ be a finite open cover of σ and let $U = \bigcup_{i=1}^n U_i$. Choose an open set V with $\sigma \subseteq V \subseteq \overline{V} \subseteq U$. Then there exist non-negative $f_1, \ldots, f_n \in C^\infty(V)$ such that $\sum_{i=1}^n f_i = 1$ on V (and hence on σ), and $\operatorname{supp} f_i \subseteq U_i$ for all $1 \leqslant i \leqslant n$ (see page 44 of [22]). ∎

For $f \in AC(\sigma)$ and $\xi \notin \operatorname{supp} f$, define

$$
f_{\bar{\xi}}(z) = \begin{cases} \frac{f(z)}{z - \bar{\xi}} & z \in \sigma \setminus \{\bar{\xi}\}, \\ 0 & z \in \sigma \cap \{\bar{\xi}\}. \end{cases}
$$

Recall that an algebra $\mathcal{A}$ of functions (defined on some subset of $\mathbb{C}$) is admissible if it contains the polynomials, is normal, and $f_{\bar{\xi}} \in \mathcal{A}$ for all $f \in \mathcal{A}$ and all $\xi \notin \operatorname{supp} f$.

PROPOSITION 2.13. *Let $\sigma \subset \mathbb{C}$ be compact. Then $AC(\sigma)$ is an admissible inverse-closed algebra.*

Proof. All that remains is to show that the last property hold in $AC(\sigma)$. Suppose then that $f \in AC(\sigma)$ and $\xi \notin \operatorname{supp} f$. Given that $\operatorname{supp} f$ is compact, there exists $h \in C^\infty(\mathbb{C})$ such that $h(z) = (z - \xi)^{-1}$ on $\operatorname{supp} f$ and $h(z) \equiv 0$ on some neighbourhood of ξ. Again using Proposition 4.7 in [3] we have that $h|\sigma \in AC(\sigma)$ and hence that $f_{\bar{\xi}} = fh \in AC(\sigma)$. ∎

3. PATCHING THEOREMS

The relationship between $\mathrm{var}(f,\sigma_1)$, $\mathrm{var}(f,\sigma_2)$ and $\mathrm{var}(f,\sigma_1 \cup \sigma_2)$ is in general rather complicated. The following theorem will allow us to patch together functions defined on different sets.

THEOREM 3.1. *Suppose that* $\sigma_1, \sigma_2 \subseteq \mathbb{C}$ *are nonempty compact sets which are disjoint except at their boundaries. Suppose that* $\sigma = \sigma_1 \cup \sigma_2$ *is convex. If* $f : \sigma \to \mathbb{C}$, *then*

$$\max\{\mathrm{var}(f,\sigma_1), \mathrm{var}(f,\sigma_2)\} \leqslant \mathrm{var}(f,\sigma) \leqslant \mathrm{var}(f,\sigma_1) + \mathrm{var}(f,\sigma_2)$$

and hence

$$\max\{\|f\|_{BV(\sigma_1)}, \|f\|_{BV(\sigma_2)}\} \leqslant \|f\|_{BV(\sigma)} \leqslant \|f\|_{BV(\sigma_1)} + \|f\|_{BV(\sigma_2)}.$$

Thus, if $f|\sigma_1 \in BV(\sigma_1)$ *and* $f|\sigma_2 \in BV(\sigma_2)$, *then* $f \in BV(\sigma)$.

Proof. The left-hand inequalities are obvious.

Note that given any points $z \in \sigma_1 \setminus \sigma_2$ and $w \in \sigma_2 \setminus \sigma_1$ there exists a point u on the line joining z and w with u in $\sigma_1 \cap \sigma_2$. To see this, let $\alpha(t) = (1-t)z + tw$ and let $t_0 = \inf\{t \in [0,1] : \alpha(t) \in \sigma_2\}$. By the convexity of σ, $\alpha(t) \in \sigma_1$ for all $0 \leqslant t < t_0$. The closedness of the subsets then implies that $u = \alpha(t_0) \in \sigma_1 \cap \sigma_2$.

Suppose then that $S = \{z_0, z_1, \ldots, z_n\} \subseteq \sigma$. For any j for which z_j and z_{j+1} lie in different subsets, using the above remark, expand S to add an extra vertex on the line joining z_j and z_{j+1} which lies in both σ_1 and σ_2. (Note that the addition of these extra vertices does not change the value of $\rho(\gamma_S)$ and can only increase the variation of f between the vertices.) Write the vertices of γ which lie in σ_1 as $S_1 = \{z_0^1, z_1^1, \ldots, z_{k_1}^1\}$ and those which lie in σ_2 as $S_2 = \{z_0^2, z_1^2, \ldots, z_{k_2}^2\}$, preserving the original ordering. Note that for every j, $\{z_j, z_{j+1}\}$ is a subset of at least one of the sets S_1 and S_2. Thus

$$\sum_{j=1}^{n} |f(z_j) - f(z_{j-1})| \leqslant \sum_{i=1}^{2} \sum_{j=1}^{k_i} |f(z_j^i) - f(z_{j-1}^i)|$$

where an empty sum is interpreted as having value 0. Recall that if $S' \subseteq S$ then $\rho(\gamma_{S'}) \geqslant \rho(\gamma_S)$. Thus

$$\rho(\gamma_S) \sum_{j=1}^{n} |f(z_j) - f(z_{j-1})| \leqslant \sum_{i=1}^{2} \rho(\gamma_{S_i}) \sum_{j=1}^{k_i} |f(z_j^i) - f(z_{j-1}^i)|$$

$$\leqslant \sum_{i=1}^{2} \rho(\gamma_{S_i}) \, \mathrm{cvar}(f, \sigma_i) \leqslant \sum_{i=1}^{2} \mathrm{var}(f, \sigma_i).$$

The results follows on taking a supremum over finite $S \subseteq \sigma$. ∎

Note that the convexity of σ is vital in Theorem 3.1. Without this condition it is easy to construct examples where $\mathrm{var}(f,\sigma_1) + \mathrm{var}(f,\sigma_2) = 0$ for a nonconstant function f.

Later, we will need to show that we can patch two absolutely continuous functions together. For notational simplicity, the following lemma is stated in terms of specific sets σ_1 and σ_2, but the affine invariance result (Proposition 2.8) implies that this immediately also applies to any two rectangles that meet along an edge.

LEMMA 3.2. *Suppose that* $\sigma_1 = [0,1] \times [0,1]$, *that* $\sigma_2 = [1,2] \times [0,1]$ *and that* $\sigma = \sigma_1 \cup \sigma_2$. *Suppose that* $f : \sigma \to \mathbb{C}$ *and that* $f_i = f|\sigma_i$ ($i = 1,2$). *If* $f_1 \in AC(\sigma_1)$ *and* $f_2 \in AC(\sigma_2)$, *then* $f \in AC(\sigma)$ *and*

$$\|f\|_{BV(\sigma)} \leqslant \|f_1\|_{BV(\sigma_1)} + \|f_2\|_{BV(\sigma_2)}.$$

Proof. By replacing f with the function $(x,y) \to f(x,y) - f(1,y)$ we may assume that $f|(\sigma_1 \cap \sigma_2) = 0$. (Note that $(x,y) \to f(1,y)$ is always in $AC(\sigma)$.)

Suppose first that $f_2 = 0$. Fix $\varepsilon > 0$. As $f_1 \in AC(\sigma_1)$ there exists $p \in CTPP(\sigma_1)$ with $\|f_1 - p\|_{BV(\sigma_1)} < \frac{\varepsilon}{4}$. By the definition of $CTPP(\sigma_1)$ there is a triangulation $\{A_i\}_{i=1}^n$ of σ_1 such that $p|A_i$ is planar (see Section 4 of [3]). Note that $b(y) = p(1,y)$ is a piecewise linear function on $[0,1]$ with $\|b\|_{BV[0,1]} = \|f_1 - p\|_{BV(\sigma_1 \cap \sigma_2)} < \frac{\varepsilon}{4}$. Extend p to σ_2 by setting $p(x,y) = b(y)$. Note that $p \in CTPP(\sigma)$ and by Proposition 4.4 in [3], $\|p|\sigma_2\|_{BV(\sigma_2)} < \frac{\varepsilon}{4}$. Thus, using Theorem 3.1,

$$\|f - p\|_{BV(\sigma)} \leqslant \|f - p\|_{BV(\sigma_1)} + \|f - p\|_{BV(\sigma_2)} < \frac{\varepsilon}{2}.$$

For arbitrary f_2, the same argument will produce a function $q \in CTPP(\sigma)$ which approximates to within $\frac{\varepsilon}{2}$ the function which is f_2 on σ_2 and zero on σ_1. Thus the piecewise planar function $p + q$ approximates f to within ε on σ. It follows that $f \in AC(\sigma)$. The norm estimate is given by Theorem 3.1. ∎

The conditions on σ_1 and σ_2 in Lemma 3.2 could be relaxed considerably. Since we will not need this greater generality in this paper, we have not attempted to determine the most general conditions on these sets for which the above proof works. It is worth noting that one does need *some* conditions on σ_1 and σ_2 or else the pasted function need not even be of bounded variation.

A major issue in much of this paper will be whether one can always extend an $AC(\sigma)$ function to a larger domain.

QUESTION 3.3. Suppose that $\sigma_1 \subseteq \sigma_2$ are nonempty compact sets. Does there exist $C = C(\sigma_1, \sigma_2)$ such that for every $f \in AC(\sigma_1)$ there exists $\widetilde{f} \in AC(\sigma_2)$ such that $\widetilde{f}|\sigma_1 = f$ and $\|\widetilde{f}\|_{BV(\sigma_2)} \leqslant C\|f\|_{BV(\sigma_1)}$?

The following special case will be needed in Section 5 to show that $AC(\sigma)$ operators are decomposable.

THEOREM 3.4. *Let* σ *denote the closed square* $[0,1] \times [0,1]$, *and let* $\partial\sigma$ *denote the boundary of* σ. *Suppose that* $b \in AC(\partial\sigma)$. *Then there exists* $f \in AC(\sigma)$ *such that* $f|\partial\sigma = b$ *and* $\|f\|_{BV(\sigma)} \leqslant 28\|b\|_{BV(\partial\sigma)}$.

Proof. Recall that by Proposition 4.4 in [3], if $h \in AC[0,1]$ is any absolutely continuous function of one variable, then its extension to the square, $\widehat{h}(x,y) = h(x)$, is in $AC(\sigma)$ with $\|\widehat{h}\| = \|h\|_{BV[0,1]}$.

Define $f_{\mathrm{s}} : \sigma \to \mathbb{C}$ by $f_{\mathrm{s}}(x,y) = (1-y)\, b(x,0)$. Since f_{s} is the product of AC functions of one variable, it is absolutely continuous on σ and

$$\|f_{\mathrm{s}}\|_{BV(\sigma)} \leqslant 2\|b(\cdot,0)\|_{BV[0,1]} \leqslant 2\|b\|_{BV(\partial\sigma)}.$$

Similarly, we define

$$f_{\mathrm{e}}(x,y) = (1-x)\, b(0,y), \quad f_{\mathrm{n}}(x,y) = y\, b(x,1), \quad f_{\mathrm{w}}(x,y) = x\, b(1,y).$$

Let $g = f_{\mathrm{s}} + f_{\mathrm{e}} + f_{\mathrm{n}} + f_{\mathrm{w}}$. Then $g \in AC(\sigma)$ and $\|g\|_{BV(\sigma)} \leqslant 8\|b\|_{BV(\partial\sigma)}$.

Let $\Delta_{\mathrm{l}} = \{(x,y) : 0 \leqslant y \leqslant x \leqslant 1\}$ and $\Delta_{\mathrm{u}} = \{(x,y) : 0 \leqslant x \leqslant y \leqslant 1\}$ denote the lower and upper closed triangles inside σ. Now let p_{l} be the affine function determined by the condition that it agrees with $b - g$ at the points $(0,0), (1,0)$ and $(1,1)$. Similarly, let p_{u} be the affine function which agrees with $b - g$ at the points $(0,0), (0,1)$ and $(1,1)$. Note that $p_{\mathrm{l}}(x,x) = p_{\mathrm{u}}(x,x)$ for all x. Let

$$p(x,y) = \begin{cases} p_{\mathrm{l}}(x,y) & (x,y) \in \Delta_{\mathrm{l}}, \\ p_{\mathrm{u}}(x,y) & (x,y) \in \Delta_{\mathrm{u}}. \end{cases}$$

Then $p \in CTPP(\sigma) \subseteq AC(\sigma)$. Now (using the facts about $AC(\sigma)$ functions which only vary in one direction)

$$\mathrm{var}(p, \Delta_{\mathrm{l}}) \leqslant \max\{|p(0,0) - p(1,0)|, |p(0,0) - p(1,1)|, |p(1,0) - p(1,1)|\}.$$

Note that

$$|p(0,0) - p(1,0)| \leqslant |b(0,0) - b(1,0)| + |g(0,0) - g(1,0)|$$
$$\leqslant \mathrm{var}(b, \partial\sigma) + \mathrm{var}(g, \sigma) \leqslant 9\|b\|_{BV(\partial\sigma)}.$$

This bound also holds for the other terms and hence $\|p\|_{BV(\Delta_{\mathrm{l}})} \leqslant 10\|b\|_{BV(\partial\sigma)}$. Applying the same argument in the upper triangle, and then using Theorem 3.1 gives that $\|p\|_{BV(\sigma)} \leqslant 20\|b\|_{BV(\partial\sigma)}$.

Let $f = g + p$. Clearly $f \in AC(\sigma)$ and $\|f\|_{BV(\sigma)} \leqslant 28\|b\|_{BV(\partial\sigma)}$. Note that $f_{\mathrm{e}}(x,0), f_{\mathrm{n}}(x,0), f_{\mathrm{w}}(x,0)$ and $p(x,0)$ are all affine functions of x, and hence $f(x,0) - b(x,0)$ is an affine function. But $f(0,0) = g(0,0) + b(0,0) - g(0,0) = b(0,0)$ and $f(1,0) = b(1,0)$ and so it follows that $f(x,0) = b(x,0)$ for all $x \in [0,1]$. Similar arguments hold for the remaining three sides and so $f|\partial\sigma = b$ as required. ∎

At the expense of lengthening the reasoning, one could reduce the constant 28 in the above theorem. It would be interesting to know the optimal constant; it seems unlikely that the above construction would provide this.

In building up AC functions in Section 6, we shall need to make use of the following straightforward extension lemma.

LEMMA 3.5. *Let σ denote the boundary of the square $[0,1] \times [0,1]$. Denote the four edges of the square as $\{\sigma_i\}_{i=1}^{4}$. Let J be a nonempty subset of $\{1,2,3,4\}$ and let $\sigma_J = \bigcup_{i \in J} \sigma_i$. Then given any $b \in AC(\sigma_J)$ there exists $\widehat{b} \in AC(\sigma)$ with $\widehat{b}|\sigma_J = b$ and $\|\widehat{b}\|_{BV(\sigma)} \leqslant 4\|b\|_{BV(\sigma_J)}$.*

Proof. Let T denote the circle passing through the 4 vertices of σ, and let π denote the map from σ to T defined by projecting along the rays coming out of the centre of σ. Consider a finite list of points $S = \{z_1, \ldots, z_n\} \subseteq \sigma$ with corresponding path $\gamma_S = \Pi(z_1, \ldots, z_n)$. Choose a line ℓ in $\mathbb{C}$ for which γ_S has $\mathrm{vf}(\gamma_S)$ entry points on ℓ. Note that you can always do this with ℓ passing through the interior of σ and hence ℓ is determined by two points $w_1, w_2 \in \sigma$. Let ℓ_π denote the line through $\pi(w_1)$ and $\pi(w_2)$. Since the projection π preserves which side of a line points lie on, $\gamma_{\pi(S)}$ has $\mathrm{vf}(\gamma_S)$ entry points on ℓ_π. Conversely, if $\gamma_{\pi(S)}$ has $\mathrm{vf}(\gamma_{\pi(S)})$ entry points on a line ℓ, then γ must have at least $\frac{\mathrm{vf}(\gamma_{\pi(S)})}{2}$ entry points on the inverse image of ℓ under π. (The factor of $\frac{1}{2}$ comes from the fact the inverse image of ℓ may lie along one of the edges of σ.) It follows then that

(3.1) $$\frac{1}{2}\rho(\gamma_S) \leqslant \rho(\gamma_{\pi(S)}) \leqslant \rho(\gamma_S).$$

Suppose then that $f \in BV(\sigma)$. Let $f_\pi : T \to \mathbb{C}$ be $f_\pi = f \circ \pi^{-1}$. From (3.1) it is clear that

$$\frac{1}{2}\mathrm{var}(f_\pi, T) \leqslant \mathrm{var}(f, \sigma) \leqslant \mathrm{var}(f_\pi, T)$$

and so $f_\pi \in BV(T)$. The same estimate holds when comparing the variation of $f \in BV(\sigma_J)$ and that of f_π on the corresponding subset T_J of T. But, by Corollary 5.6 in [4], $BV(T)$ is 2-isomorphic to the subset of $BV[0,1]$ consisting of functions which agree at the endpoints. In this final space, one can extend an AC function from a finite collection of subintervals K to the whole of $[0,1]$ by linear interpolation, without increasing the norm. Note that absolute continuity is preserved by the isomorphisms between these function spaces. The factor 4 comes from collecting together the norms along the following composition of maps

$$
\begin{array}{ccc}
AC(\sigma_J) & & AC(\sigma) \\
{\scriptstyle 2}\Big\downarrow{\scriptstyle \pi} & & {\scriptstyle 1}\Big\uparrow{\scriptstyle \pi^{-1}} \\
AC(T_J) & & AC(T) \\
{\scriptstyle 2}\Big\downarrow & & {\scriptstyle 1}\Big\uparrow \\
AC(K) & \xrightarrow[\text{extend}]{1} & AC[0,1]. \quad \blacksquare
\end{array}
$$

Note that if σ_J consists of either one side, or else 2 contiguous sides, then one may extend b to all of σ without increasing of norm using Proposition 4.4 of

[3]. We do not know whether this is true if, for example, σ_J consists of 2 opposite sides of the square.

4. $AC(\sigma)$ OPERATORS: DEFINITIONS AND EXAMPLES

DEFINITION 4.1. Suppose that $\sigma \subseteq \mathbb{C}$ is a nonempty compact set and that T is a bounded operator on a Banach space X. We say that T is an $AC(\sigma)$ operator if T admits a bounded $AC(\sigma)$ functional calculus. That is, T is an $AC(\sigma)$ operator if there exists a bounded unital Banach algebra homomorphism $\psi : AC(\sigma) \to B(X)$ for which $\psi(\lambda) = T$.

Where there seems little room for confusion we shall often say that T is an $AC(\sigma)$ operator where one should more properly say that T is an $AC(\sigma)$ operator *for some σ*.

Before proceeding to give some of the general properties of $AC(\sigma)$ operators, it is appropriate to give the reader some idea of how this class is related to other standard classes of operators which arise in spectral theory.

EXAMPLE 4.2. Let H be a Hilbert space and let $T \in B(H)$ be normal. Then T has a $C(\sigma(T))$ functional calculus ψ. Then $\psi|AC(\sigma(T))$ is a linear homomorphism from $AC(\sigma(T))$ into $B(X)$. Furthermore $\|\psi(f)\| \leqslant \|\psi\|\|f\|_\infty \leqslant \|\psi\|\|f\|_{BV(\sigma(T))}$ for all $f \in AC(\sigma)$ and so $\psi|AC(\sigma(T))$ is continuous from $AC(\sigma(T))$ into $B(H)$. Hence T is an $AC(\sigma(T))$ operator. Indeed, by the same argument any scalar type spectral operator (or even scalar-type prespectral operator) T on a Banach space X is also an $AC(\sigma(T))$ operator. (See [18] for the definitions of these latter classes of operators.)

The operators in the previous example are associated with spectral expansions which are of an unconditional nature. The motivation for the present theory is of course to cover operators such as well-bounded operators, which admit less constrained types of spectral expansion.

LEMMA 4.3. *Let $T \in B(X)$ be an $AC(\sigma)$ operator. Suppose that $\sigma \subset \sigma'$ where $\sigma' \subset \mathbb{C}$ is compact. Then T is an $AC(\sigma')$ operator.*

Proof. Let ψ be a $AC(\sigma)$ functional calculus for T. Define $\psi_{\sigma'} : AC(\sigma') \to B(X) : f \mapsto \psi(f|\sigma)$. Then $\psi_{\sigma'}$ is a unital linear homomorphism. Furthermore $\psi_{\sigma'}(\lambda) = \psi(\lambda|\sigma) = T$. Finally we note from the inequality $\|f|\sigma\|_{BV(\sigma)} \leqslant \|f\|_{BV(\sigma')}$ that $\psi_{\sigma'}$ is continuous. Hence $\psi_{\sigma'}$ is an $AC(\sigma')$ functional calculus for T. ∎

The following result was announced in Section 2 of [3].

PROPOSITION 4.4. *Let $T \in B(X)$. The following are equivalent:*
  (i) *T is well-bounded.*
 (ii) *T is an $AC(\sigma)$ operator for some $\sigma \subset \mathbb{R}$.*
(iii) *$\sigma(T) \subset \mathbb{R}$ and T is an $AC(\sigma(T))$ operator.*

Proof. Trivially (iii) implies (ii). Lemma 4.3 shows that (ii) implies (i). Say T is well-bounded with functional calculus $\psi : AC(J) \to B(X)$ for some interval J. In [3] we define a linear isometry $\iota : AC(\sigma(T)) \to AC(J)$. Define $\psi_{\sigma(T)} : AC(\sigma(T)) \to B(X) : f \mapsto \psi(\iota(f))$. We show that $\psi_{\sigma(T)}$ is an $AC(\sigma(T))$ functional calculus for T which will complete the proof. Clearly $\psi_{\sigma(T)}$ is linear and continuous. Furthermore, since $\iota(\lambda|\sigma(T)) = \lambda$, we have that $\psi_{\sigma(T)}(\lambda) = T$. To see that $\psi_{\sigma(T)}$ is a homomorphism we note that if $f, g \in AC(\sigma(T))$ then $(\iota(fg) - \iota(f)\iota(g))(\sigma(T)) = \{0\}$. Theorem 4.4.4 of [1] says we can find a sequence $\{h_n\}_{n=1}^{\infty} \subset AC(J)$ such that $\lim_n \|h_n - (\iota(fg) - \iota(f)\iota(g))\|_{BV(J)} = 0$ and such that for each n, h_n is zero on a neighbourhood of $\sigma(T)$. This last condition, by Proposition 3.1.12 of [15], implies that $\psi(h_n) = 0$ for all n. Hence $\psi(\iota(fg) - \iota(f)\iota(g)) = \lim_n \psi(h_n) = 0$, which shows that $\psi_{\sigma(T)}$ is a homomorphism as claimed. ∎

As a result of the last proposition we prefer to use the term "real $AC(\sigma)$ operator" rather than the term well-bounded operator. As well as being less descriptive, the term well-bounded operator also suffers from the fact that it is used for quite a different concept in the local theory of Banach spaces (see [24] for example). We shall however stick with the traditional term for the remainder of this paper.

The next theorem shows that some important classes of AC operators are also $AC(\sigma)$ operators.

THEOREM 4.5. *Let $A \in B(X)$ be well-bounded with functional calculus $\psi : AC(J) \to B(X)$ for some interval J. Let $f \in AC(J)$ be such that $\rho(f) > 0$. Then $\psi(f)$ is an $AC(f(J))$ operator.*

Proof. Define $\psi_f : AC(f(J)) \to B(X) : g \mapsto \psi(g \circ f)$. Then ψ_f is a unital linear homomorphism and $\psi_f(\lambda) = \psi(f)$. By Proposition 2.7, ψ_f is continuous. ∎

COROLLARY 4.6. *Let $A \in B(X)$ be well-bounded and p be a polynomial of one variable. Then $p(A)$ is an $AC(p(\sigma(A)))$ operator.*

COROLLARY 4.7. *Let $A \in B(X)$ be a well-bounded operator. Then $\exp(iA)$ is an $AC(i \exp(\sigma(A)))$ operator.*

We noted earlier that the trigonometrically well-bounded operators are the operators which can be expressed in the form $\exp(iA)$ where $A \in B(X)$ is a well-bounded operator of type (B). (Indeed one can also insist that $\sigma(A) \subset [0, 2\pi]$.) As usual, we denote the unit circle in $\mathbb{C}$ by $\mathbb{T}$.

COROLLARY 4.8 (Theorem 6.2 in [4]). *If $T \in B(X)$ is trigonometrically well-bounded then T is an $AC(\mathbb{T})$ operator. Indeed, if X is reflexive, then T is trigonometrically well-bounded operator if and only if it is an $AC(\mathbb{T})$ operator.*

We end this section with a more concrete example.

EXAMPLE 4.9. Suppose that $1 < p < \infty$ and that X is the usual Hardy space $H^p(\mathbb{D})$ of analytic functions on the unit disk. Consider the unbounded operator $Af(z) = zf'(z)$, $f \in H^p(\mathbb{D})$ (with natural domain $\{f : Af \in H^p(\mathbb{D})\}$). This operator arises, for example, as the analytic generator of a semigroup of composition operators, $T_t f(z) = f(e^{-t}z)$; see [26], which includes a summary of many of the spectral properties of A. The spectrum of A is $\sigma(A) = \mathbb{N} = \{0, 1, 2, \dots\}$ with the corresponding spectral projections $P_k(\sum a_n z^n) = a_k z^k$ ($k \in \mathbb{N}$) giving just the usual Fourier components. Suppose then that $\mu \notin \sigma(A)$. The resolvent operator $R(\mu, A) = (\mu I - A)^{-1}$ is a compact operator with spectrum $\sigma(R(\mu, A)) = \left\{\frac{1}{\mu-k}\right\}_{k=0}^{\infty} \cup \{0\}$. Using Theorem 3.3 in [14] it follows easily from the properties of Fourier series that if $x \in \mathbb{R} \setminus \mathbb{N}$, then $R(x, A)$ is well-bounded. If we fix such an x and take $\mu \notin \mathbb{R}$, then $R(\mu, A) = f(R(x, A))$ where $f(t) = \frac{t}{1+(\mu-x)t}$ is a Möbius transformation. If J is any compact interval containing $\sigma(R(x, A))$ then $\rho(f(J)) = \frac{1}{2}$. Thus $R(\mu, A)$ is an $AC(f(J))$ operator. Thus, all the resolvents of A are compact $AC(\sigma)$ operators (for some σ). Note that none of the resolvents is scalar-type spectral unless $p = 2$.

5. PROPERTIES OF $AC(\sigma)$ OPERATORS

All $AC(\sigma)$ operators belong to the larger class of decomposable operators (in the sense of [16]). This will follow immediately from the requirement that the functional calculus map $\psi : AC(\sigma) \to B(X)$ be what Colojoară and Foiaş term an $AC(\sigma)$-spectral function. Recall that by Proposition 2.13, $AC(\sigma)$ is an admissible algebra.

Suppose that $f \in AC(\sigma)$. Let $\Omega_f \subseteq \mathbb{C}$ be the open set $\mathbb{C} \setminus \text{supp} f$. By Proposition 2.13, $\Phi_f(\xi) = f_\xi$ is a well-defined map from Ω_f to $AC(\sigma)$.

Following Section 3.1 of [16], the functional calculus map $\psi : AC(\sigma) \to B(X)$ is an $AC(\sigma)$-spectral function if, for all $f \in AC(\sigma)$, the map $\psi \circ \Phi_f : \Omega_f \to B(X)$ is analytic on Ω_f.

Since ψ is linear, it suffices to show that the map Φ_f is differentiable at each point $\xi_0 \in \Omega_f$. To establish this we shall need a technical lemma.

As in [5], let $|x + iy|_\infty = \max(|x|, |y|)$. For $\xi_0 \in \mathbb{C}$ and $\delta > 0$ let

$$B_\infty(\xi_0, \delta) = \{z \in \mathbb{C} : |\xi_0 - z|_\infty < \delta\}.$$

LEMMA 5.1. *Suppose that $f \in AC(\sigma)$, $\xi_0 \in \Omega_f$ and that $\delta > 0$ is chosen so that $B_\infty(\xi_0, 3\delta) \subseteq \Omega_f$. Then there exists a constant $C(\delta, \sigma)$ such that for all $\xi \in B_\infty(\xi_0, \delta)$, there exists $r_\xi \in AC(\sigma)$ which satisfies:*
  (i) $r_\xi(z) = \frac{1}{\xi-z}$ *for all $z \in \sigma \setminus B_\infty(\xi_0, 2\delta)$, and*
  (ii) $\|r_\xi\|_{AC(\sigma)} \leqslant C(\delta, \sigma)$.

Proof. Suppose first that $\xi_0 \in \sigma$. (The case where $\xi_0 \notin \sigma$ is similar, but with slightly different norm bounds. The details are left to the reader.)

Let σ_0 denote the smallest closed square (with sides parallel to the axes) containing σ and $B_\infty(\xi_0, 3\delta)$. Let σ_1 denote the $\overline{B}_\infty(\xi_0, 2\delta)$ and let σ_2 denote $\sigma_0 \setminus B_\infty(\xi_0, 2\delta)$.

Suppose that $\xi \in B_\infty(\xi_0, \delta)$. The function $z \mapsto \xi - z$ is absolutely continuous on σ_2 with variation equal to $d = d(\sigma, \delta)$, the length of the diagonal of σ_0. Since $|\xi - z| \geqslant \delta$ on σ_2, Lemma 2.9 implies that $r_\xi : \sigma_2 \to \mathbb{C}, z \mapsto (\xi - z)^{-1}$ is in $AC(\sigma_2)$ with

$$\|r_\xi\|_{AC(\sigma_2)} \leqslant \frac{1}{\delta} + \frac{d}{\delta^2}.$$

Clearly $\partial \sigma_1 \subseteq \sigma_2$ so by Lemmas 3.9 and 4.5 in [3], $r_\xi|\partial\sigma_1 \in AC(\partial\sigma_1)$. Using Theorem 3.4, we can extend r_ξ to σ_1 so that $r_\xi|\sigma_1 \in AC(\sigma_1)$ and $\|r_\xi\|_{AC(\sigma_1)} \leqslant 28\|r_\xi\|_{AC(\partial\sigma_1)} \leqslant 28\|r_\xi\|_{AC(\sigma_2)}$.

By splitting σ_0 into 9 smaller rectangles and then using Lemma 3.2 repeatedly, one can deduce that $r_\xi \in AC(\sigma_0)$, and that one has a bound on $\|r_\xi\|_{AC(\sigma_0)}$ which depends only on σ and δ. Taking the restriction of this function to the original domain σ completes the construction. ∎

PROPOSITION 5.2. *The functional calculus map ϕ for an $AC(\sigma)$ operator $T \in B(X)$ is an $AC(\sigma)$-spectral function.*

Proof. Fix $f \in AC(\sigma)$, $\xi_0 \in \Omega_f$ and $\delta > 0$ so that $B_\infty(\xi_0, 3\delta) \subseteq \Omega_f$. Using Lemma 5.1, choose a family of functions r_ξ for $\xi \in B_\infty(\xi_0, \delta)$. Note that $\Phi_f(\xi) = r_\xi f \in AC(\sigma)$. Thus

$$\frac{\Phi_f(\xi) - \Phi_f(\xi_0)}{\xi - \xi_0} = \frac{(r_\xi - r_{\xi_0})f}{\xi - \xi_0} = -r_\xi r_{\xi_0} f = -r_{\xi_0}^2 f + r_{\xi_0}(r_{\xi_0} - r_\xi)f$$

$$= -r_{\xi_0}^2 f + r_{\xi_0}(\xi - \xi_0)r_\xi r_{\xi_0} f \to -r_{\xi_0}^2 f$$

as $\xi \to \xi_0$, by the uniform bound on the norms of the functions r_ξ. Composing Φ_f with the linear map ψ preserves differentiability so $\psi \circ \Phi_f : \Omega_f \to B(X)$ is analytic. ∎

PROPOSITION 5.3. *Let $T \in B(X)$ be an $AC(\sigma)$ operator. Then*

(i) $\sigma(T) \subseteq \sigma$.

(ii) *T is decomposable.*

This follows from Proposition 5.2 using Theorems 3.1.6 and 3.1.16 in [16].

In general it is easy to pass between spectral properties of an operator T and those of affine translations of T. One of the main motivations for developing this theory was to provide a suitably broad class of operators which is closed under such transformations. From Theorem 2.8 we get the following.

THEOREM 5.4. *Let $T \in B(X)$ be an $AC(\sigma)$ operator. Let $\alpha, \beta \in \mathbb{C}$. Then $\alpha T + \beta I$ is an $AC(\alpha\sigma + \beta)$ operator.*

Proof. Let $\theta : AC(\sigma) \to AC(\alpha\sigma + \beta)$ be the isomorphism of Theorem 2.8. Let ψ be the $AC(\sigma)$ functional calculus for T. Then it is routine to check that the map $\psi_{\alpha,\beta} : AC(\alpha\sigma + \beta) \to B(X) : f \mapsto \psi(\theta^{-1}(f))$ is an $AC(\alpha\sigma + \beta)$ functional calculus for $\alpha T + \beta I$. ∎

THEOREM 5.5. *Let $T \in B(X)$ be an $AC(\sigma)$ operator. Then $T = R + \mathrm{i}S$ where R, S are commuting well-bounded operators. Further, $\sigma(R) = \mathrm{Re}(\sigma(T))$ and $\sigma(S) = \mathrm{Im}(\sigma(T))$.*

Proof. Let ψ be an $AC(\sigma)$ functional calculus for T. Proposition 5.4 in [3] shows that the map $u : AC(\mathrm{Re}(\sigma)) \to AC(\sigma)$ defined by $u(f)(z) = f(\mathrm{Re}(z))$ is a norm-decreasing linear homomorphism. Then the map $\psi_{\mathrm{Re}(\sigma)} : AC(\mathrm{Re}(\sigma)) \to B(X) : f \mapsto \psi(u(f))$ is a continuous linear unital homomorphism. Hence $R := \psi_{\mathrm{Re}(\sigma)}(\lambda|\mathrm{Re}(\sigma)) = \psi(\mathrm{Re}(\lambda))$ is well-bounded. Similarly $S := \psi(\mathrm{Im}(\lambda))$ is well-bounded. Then $T = \psi(\lambda) = \psi(\mathrm{Re}(\lambda) + \mathrm{i}\,\mathrm{Im}(\lambda)) = R + \mathrm{i}S$. Finally we note that R and S commute since $AC(\sigma)$ is a commutative algebra and ψ is a homomorphism.

The identification of $\sigma(R)$ and $\sigma(S)$ follows immediately from the spectral mapping theorem (see Theorem 3.2.1 in [16]). ∎

Splittings which arise from an $AC(\sigma)$ functional calculus we call *functional calculus splittings*.

COROLLARY 5.6. *The $AC(\sigma)$ operators are a proper subset of the AC operators of Berkson and Gillespie.*

Proof. We note that not all AC operators are $AC(\sigma)$ operators. Example 4.1 of [7] shows that the class of AC operators is not closed under multiplication by scalars even on Hilbert spaces. ∎

Not all splittings into commuting real and imaginary well-bounded parts arise from an $AC(\sigma)$ functional calculus. This was shown in the next example which first appeared in [7].

EXAMPLE 5.7. Let $X = L^\infty[0,1] \oplus L^1[0,1]$. Define $A \in B(X)$ by $A(f,g) = (\lambda f, \lambda g)$. It is not difficult to see that A is well-bounded and that $\sigma(A) = [0,1]$. Let $T = (1+\mathrm{i})A = A + \mathrm{i}A$. By Theorem 5.4, T is an $AC(\sigma(T))$ operator where $\sigma(T)$ is the line segment from 0 to $1 + \mathrm{i}$.

The operator T has an infinite number splittings. Define $Q \in B(X)$ by $Q(f,g) = (0,f)$. In [7] it is shown that $A + \alpha Q$ is well-bounded for any $\alpha \in \mathbb{C}$. But then $T = A + \mathrm{i}A = A + Q + \mathrm{i}(A + \mathrm{i}Q)$.

The second splitting cannot come from an $AC(\sigma)$ functional calculus. Say T has an $AC(\sigma)$ functional calculus ψ. Since $\sigma(T)$ is a line segment we can use similar reasoning as to that in Proposition 4.4 to conclude that if $f \in AC(\sigma)$ is such that $f(\sigma(T)) = \{0\}$ then $\psi(f) = 0$. Hence if $g|\sigma(T) = h|\sigma(T)$ then $\psi(g) = \psi(h)$. In particular since $\mathrm{Re}(\lambda)|\sigma(T) = \mathrm{Im}(\lambda)|\sigma(T)$ we can only have $AC(\sigma)$ functional calculus splittings of the form $T = R + \mathrm{i}R$.

We do not know if it is possible to have several splittings each arising from an $AC(\sigma)$ functional calculus. The following tells us to what extent we can expect splittings to be unique.

PROPOSITION 5.8. *Let $T \in B(X)$ be an $AC(\sigma)$ operator. Suppose that $T = R_1 + iS_1 = R_2 + iS_2$ where R_1, S_1 and R_2, S_2 are pairs of commuting well-bounded operators. Then R_1 and R_2 are quasinilpotent equivalent in the sense of [15] (as are S_1 and S_2). Suppose that $\{R_1, S_1, R_2, S_2\}$ is a commuting set. Then $(R_1 - R_2)^2 = (S_1 - S_2)^2 = 0$. Furthermore suppose that $\{R_1, S_1, R_2, S_2\}$ are all type (B) well-bounded operators. Then $R_1 = R_2$ and $S_1 = S_2$.*

This is Theorem 3.2.6 of [16] and Theorem 3.7 of [7].

6. THE SUPPORT OF THE FUNCTIONAL CALCULUS

Suppose that $\psi : AC(\sigma) \to B(X)$ is the functional calculus map for an $AC(\sigma)$ operator T. The support of ψ is defined as the smallest closed set $F \subseteq \mathbb{C}$ such that if $\operatorname{supp} f \cap F = \varnothing$, then $\psi(f) = 0$. It follows from Theorem 5.2 and Theorem 3.1.6 of [16] that the support of ψ is $\sigma(T)$.

It is natural therefore to ask whether such an operator T must admit an $AC(\sigma(T))$ functional calculus. By Proposition 4.4, this is certainly the case if T is well-bounded (that is, if $\sigma(T) \subseteq \mathbb{R}$), but the general case remains open.

We shall now give a partial answer to this question, and show that one may always at least shrink σ down to be a compact set not much bigger than $\sigma(T)$.

DEFINITION 6.1. A set $G \subseteq \mathbb{C}$ is said to be *gridlike* if it is a closed polygon with sides parallel to the axes.

Note that we do not require that a gridlike set be convex, or even simply connected.

PROPOSITION 6.2. *Suppose that V is a gridlike set, that σ is compact and that $V \subseteq \sigma$. Let $I_V = \{f \in AC(\sigma) : f \equiv 0 \text{ on } V\}$. Then $AC(\sigma)/I_V \cong AC(V)$ as Banach algebras.*

Proof. Define $\Theta : AC(\sigma)/I_V \to AC(V)$ by $\Theta([f]) = f|V$. Then clearly

$$\Theta([f]) = \Theta([g]) \iff f|V \equiv g|V \iff f - g \in I_V$$

and so Θ is well-defined and one-to-one. It is also easy to see that Θ is an algebra homomorphism. Since

$$\|\Theta([f])\| = \|f|V\|_{BV(V)} = \inf_{g \in I_V} \|f + g|V\|_{BV(V)} \leqslant \inf_{g \in I_V} \|f + g\|_{BV(\sigma)} = \|[f]\|_{AC(\sigma)/I_V}$$

the map Θ is bounded.

The hard part of the proof is to show that Θ is onto. That is, given $f \in AC(V)$, there exists $F \in AC(\sigma)$ so that $F|V = f$.

Choose then a square $J \times K$ containing σ. Extending the edges of V produces a grid on $J \times K$, determining N closed subrectangles $\{\sigma_k\}_{k=1}^N$.

Suppose now that $f \in AC(V)$. Our aim is to define $\widehat{f} \in AC(J \times K)$ with $\widehat{f}|V = f$ and $\|\widehat{f}\|_{BV(J \times K)} \leqslant C\|f\|_{BV(V)}$.

Fix an ordering of the rectangles $\{\sigma_k\}$ so that

  (i) there exists k_0 such that $\sigma_k \subseteq V$ if and only if $k \leqslant k_0$, and
  (ii) for all ℓ, σ_1 intersects $\bigcup_{k<\ell} \sigma_k$ on at least one edge of σ_1.

Let E_0 denote the union of the edges of the rectangles σ_k for $k \leqslant k_0$ and let b be the restriction of f to E_0. Note that b is absolutely continuous on E_0 and if e is any edge of any rectangle σ_k ($k \leqslant k_0$), then $b|e \in AC(e)$ with $\|b|e\|_{BV(e)} \leqslant \|b\|_{BV(E_0)} \leqslant \|f\|_{BV(V)}$. Now apply Lemma 3.5 to recursively extend b to the set E of all edges of rectangles σ_k, $1 \leqslant k \leqslant N$, so that $b \in AC(E)$ and $\|b\|_{BV(E)} \leqslant C_N\|f\|_{BV(V)}$.

For $1 \leqslant k \leqslant k_0$, let $f_k = f|\sigma_k$, so that $f_k \in AC(\sigma_k)$ and $\|f_k\|_{BV(\sigma_k)} \leqslant \|f\|_{BV(V)}$. Suppose alternatively that $k_0 < k \leqslant N$. By Theorem 3.4 we can find $f_k \in AC(\sigma_k)$ with $f_k|\partial\sigma_k = b|\partial\sigma_k$ and $\|f_k\|_{BV(\sigma_k)} \leqslant 28\|b|\partial\sigma_k\|_{BV(\partial\sigma_k)} \leqslant 28C_N\|f\|_{BV(V)}$.

Define $\widehat{f} : J \times K \to \mathbb{C}$ such that $\widehat{f}|\sigma_k = f_k$. That $\widehat{f}$ is in $AC(J \times K)$ with $\|\widehat{f}\|_{BV(J \times K)} \leqslant 28C_N N\|f\|_{BV(V)}$ follows from Lemma 3.2 (first patching together all the rectangles in each row, and then all the rows together). We can now let $F = \widehat{f}|\sigma$.

It follows then that Θ is onto and hence is a Banach algebra isomorphism. ∎

THEOREM 6.3. *Let $T \in B(X)$ be an $AC(\sigma)$ operator for some $\sigma \subset \mathbb{C}$. Let U be an open neighbourhood of $\sigma(T)$. Then T is an $AC(\overline{U})$ operator.*

Proof. Suppose that T, σ and U are as stated. Choose a square $J \times K$ containing $U \cup \sigma$. By Lemma 4.3, T admits an $AC(J \times K)$ functional calculus ψ.

Consider an equispaced grid on $J \times K$, determining n^2 subsquares $\{\sigma_k\}_{k=1}^{n^2}$. Let $V = V(n)$ be the union of all those σ_k which intersect $\sigma(T)$. For n large enough

$$\sigma(T) \subseteq \text{int}(V) \subseteq V \subseteq U.$$

For the rest of the proof, fix such an n.

As in Proposition 6.2, let $I_V = \{f \in AC(J \times K) : f|V \equiv 0\}$, so that $AC(J \times K)/I_V \cong AC(V)$ via the isomorphism Θ. Note that $I_V \subseteq \ker(\psi)$ since if $f \in I_V$, then $\text{supp} f \cap \sigma(T) = \emptyset$. Thus the map $\widetilde{\psi} : AC(J \times K)/I_V \to B(X)$,

$$\widetilde{\psi}([f]) = \psi(f)$$

is a well-defined algebra homomorphism with $\|\widetilde{\psi}\| \leqslant \|\psi\|$.

We may therefore define $\widehat{\psi} : AC(\overline{U}) \to B(X)$ by $\widehat{\psi}(f) = \widetilde{\psi}([\Theta^{-1}(f|V)])$. Note that $\widehat{\psi}$ is a bounded algebra homomorphism and that, since $\Theta([\lambda]) = \lambda|V$, $\widehat{\psi}(\lambda) = \psi(\lambda) = T$. Thus $\widehat{\psi}$ is an $AC(\overline{U})$ functional calculus for T. ∎

COROLLARY 6.4. *Let $T \in B(X)$ be an $AC(\sigma_0)$ operator for some compact set σ_0. Then*

$$\sigma(T) = \bigcap \{\sigma \; : \; T \text{ has an } AC(\sigma) \text{ functional calculus}\}.$$

The proof of Theorem 6.3 depends on two vital facts. The first is that the map Θ is an isomorphism. The second is that $I_V \subseteq \ker(\psi)$. To show that every $AC(\sigma)$ operator is an $AC(\sigma(T))$ operator, it would suffice to show that:

(i) the restriction map $AC(\sigma) \to AC(\sigma(T))$, $f \mapsto f|\sigma(T)$ is onto. This is basically equivalent to answering Question 3.3.

(ii) given any $f \in AC(\sigma)$ with $f|\sigma(T) \equiv 0$, there exists a sequence $\{f_n\} \subseteq AC(\sigma)$ with $\|f - f_n\|_{BV(\sigma)} \to 0$ and $\operatorname{supp} f_n \cap \sigma(T) = \varnothing$ for all n.

Proving (i) and (ii) when $\sigma(T)$ is a complicated compact set would appear to require new ways of estimating the two-dimensional variation used in our definitions.

If $T \in B(X)$ is an $AC(\sigma(T))$ operator then T has spectral theorems similar to those for normal operators. Recall from [19] the definition of the local spectrum $\sigma_T(x)$ of $x \in X$ for an operator $T \in B(X)$ with the single-valued extension property. From [23] if $T \in B(X)$ is an $AC(\sigma)$ operator (and hence decomposable) then those $x \in X$ such that $\sigma_T(x) = \sigma(T)$ are second countable in X.

THEOREM 6.5. *Suppose that $T \in B(X)$ is an $AC(\sigma(T))$ operator with functional calculus $\psi : AC(\sigma(T)) \to B(X)$. Then ψ is injective. Hence we can identify $AC(\sigma(T))$ with a subalgebra of $B(X)$. Furthermore suppose that $x \in X$ is such that $\sigma_T(x) = \sigma(T)$. Then the map $AC(\sigma(T)) \to X : f \mapsto \psi(f)x$ is injective, and so we can identify $AC(\sigma(T))$ with a subspace of X.*

Proof. Let $x \in X$ be such that $\sigma_T(x) = \sigma(T)$. To prove the theorem it suffices to show that if $f \in AC(\sigma(T))$ and $f \neq 0$ then $\psi(f)x \neq 0$. Let $\lambda_0 \in \sigma(T)$ be such that $f(\lambda_0) \neq 0$. Since f is continuous we can find an open neighbourhood V of λ_0 such that $0 \notin f(V)$. We can choose $g \in AC(\sigma(T))$ such that $(fg)(V) = \{1\}$. If we show $\psi(fg)x \neq 0$ this will imply, since ψ is a homomorphism, that $\psi(f)x \neq 0$. Hence we can assume that $f(V) = \{1\}$. Let U be an open set such that $\{U, V\}$ is an open cover of $\sigma(T)$ and such that $\lambda_0 \notin U$. By Lemma 5.2.3 of [1] we can find non-zero $x_u, x_V \in X$ such that $x = x_u + x_V$ and where $\sigma_T(x_u) \subset U$ and $\sigma_T(x_V) \subset V$. Since $\sigma_T(x) \subset \sigma_T(x_u) \cup \sigma_T(x_V)$ we have that $\lambda_0 \in \sigma_T(x_V)$ and $\lambda_0 \notin \sigma_T(x_u)$. Assume that $\psi(f)x = 0$. Then $0 = \psi(f)(x_u + x_V) = \psi(f)x_u + x_V$ since f is one on V. It follows that $\sigma_T(x_V) = \sigma_T(-\psi(f)x_u) = \sigma_T(\psi(-f)x_u) \subset \sigma_T(x_u)$. Then we have the contradiction that $\lambda_0 \in \sigma_T(x_V) \subset \sigma_T(x_u) \not\ni \lambda_0$. Hence $\psi(f)x \neq 0$. ∎

Since every $AC(\sigma)$ operator is also an AC operator, the results of [17] give a representation theorem for compact $AC(\sigma)$ operators. Specifically, if $T \in B(X)$ is a compact $AC(\sigma)$ operator with nonzero eigenvalues $\{\mu_j\}$ and corresponding

Riesz projections $\{P_j\}$, then

(6.1) $$T = \sum_j \mu_j P_j$$

where the sum converges in norm under a particular specified ordering of the eigenvalues. Given a sequence of real numbers $\{\mu_j\}$ and disjoint projections $\{P_j\} \subseteq B(X)$, necessary and sufficient conditions are known which ensure that the operator defined via (6.1) is well-bounded (Theorem 3.3 in [14]). At present an analogous result for compact $AC(\sigma)$ operators in unknown. These questions are pursued more fully in [5] where, for example, various sufficient conditions for (6.1) to define a compact $AC(\sigma)$ operator are given.

7. SPECTRAL RESOLUTIONS

The theory of well-bounded operators is at its most powerful if one adds the additional assumption that the functional calculus map for T is 'weakly compact'. That is, for all $x \in X$, the map $\psi_x : AC(\sigma(T)) \to X$, $f \mapsto \psi(f)x$ is weakly compact. In this case T admits an integral representation with respect to a spectral family of projections $\{E(\mu)\}_{\mu \in \mathbb{R}}$. The integration theory for spectral families allows one to define

$$f(T) = \widehat{\psi}(f) = \int_{\sigma(T)}^{\oplus} f(\mu)\, dE(\mu)$$

for all $f \in BV(\sigma)$ giving an extended functional calculus map. (This integral is more usually written as $\int_J^{\oplus} f(\mu)\, dE(\mu)$, where J is some compact interval containing $\sigma(T)$. We have written it in the above form to stress that the value of the integral only depends on the values of f on $\sigma(T)$.) If ψ is not weakly compact, then there may be no spectral resolution consisting of projections on X. A suitable family of projections on X^*, known as a decomposition of the identity, does always exist, but the theory here is much less satisfactory.

Obviously extending this theory to cover general $AC(\sigma)$ operators with a weakly compact functional calculus is highly desirable. At present a full analogue of the well-bounded theory has not been found, but we are able to show that each such operator does admit a nice spectral resolution from which the operator may be recovered. The following definition extends the definition for well-bounded operators.

DEFINITION 7.1. Let $T \in B(X)$ be an $AC(\sigma)$ operator with functional calculus map ψ. Then T is said to be of type (B) if for all $x \in X$, the map $\psi_x : AC(\sigma(T)) \to X$, $f \mapsto \psi(f)x$ is weakly compact.

Obviously every $AC(\sigma)$ operator on a reflexive Banach space is of type (B), as is every scalar-type spectral operator on a general Banach space (see [21]). The

weak compactness of the functional calculus removes one of the potential complications with studying $AC(\sigma)$ operators.

LEMMA 7.2. *Let $T \in B(X)$ have a weakly compact $AC(\sigma)$ functional calculus. Then it has a unique splitting $T = R + iS$ where R and S are commuting type* (B) *well-bounded operators.*

Proof. Recall if we set $R = \psi(\text{Re}(\lambda))$ and $S = \psi(\text{Im}(\lambda))$ then R and S are commuting well-bounded operators. The $AC(\text{Re}(\sigma(T)))$ functional calculus for R is given by $f \mapsto \psi(u(f))$ and so is clearly weakly compact. Hence R is type (B). Similarly S is type (B). Uniqueness follows from Proposition 5.8. ∎

If T is a well-bounded operator of type (B) with spectral family $\{E(\mu)\}_{\mu \in \mathbb{R}}$, then, for each μ, $E(\mu)$ is the spectral projections for the interval $(-\infty, \mu]$. The natural analogue of this in the $AC(\sigma)$ operator setting is to index the spectral resolution by half-planes. Modelling the plane as $\mathbb{R}^2$, each closed half-plane is specified by a unit vector $\theta \in \mathbb{T}$ and a real number μ:

$$H(\theta, \mu) = \{z \in \mathbb{R}^2 : z \cdot \theta \leqslant \mu\}.$$

Let $\mathcal{H}$ denote the set of all half-planes in $\mathbb{R}^2$. The following provisional definition contains the minimal conditions one would require of a spectral resolution for an $AC(\sigma)$ operator.

DEFINITION 7.3. Let X be a Banach space. A half-plane spectral family on X is a family of projections $\{E(H)\}_{H \in \mathcal{H}}$ satisfying:
   (i) $E(H_1) E(H_2) = E(H_2) E(H_1)$ for all $H_1, H_2 \in \mathcal{H}$;
   (ii) there exists K such that $\|E(H)\| \leqslant K$ for all $H \in \mathcal{H}$;
   (iii) for all $\theta \in \mathbb{T}$, $\{E(H(\theta, \mu))\}_{\mu \in \mathbb{R}}$ forms a spectral family of projections;
   (iv) for all $\theta \in \mathbb{T}$, if $\mu_1 < \mu_2$, then $E(H(\theta, \mu_1)) E(H(-\theta, -\mu_2)) = 0$.
The radius of $\{E(H)\}$ is the (possibly infinite) value

$$r(\{E(H)\}) = \inf\{r : \text{for all } \theta, E(H(\theta, \mu)) = I \text{ for all } \mu > r\}.$$

Suppose that $\sigma \subset \mathbb{R}^2$ is a nonempty compact set. Given any unit direction vector θ, let $\sigma_\theta = \{z \cdot \theta : z \in \sigma\} \subseteq \mathbb{R}$. Define the subalgebra of all $AC(\sigma)$ functions which only depend on the component of the argument in the direction θ,

$$AC_\theta(\sigma) = \{f \in AC(\sigma) : \text{there exists } u \in AC(\sigma_\theta) \text{ such that } f(z) = u(z \cdot \theta)\}.$$

By Proposition 3.9 and Lemma 3.10 of [3], there is a norm 1 isomorphism $U_\theta : AC(\sigma_\theta) \to AC_\theta(\sigma)$.

Let $T \in B(X)$ be an $AC(\sigma)$ operator of type (B), with functional calculus map ψ. The algebra homomorphism $\psi_\theta : AC(\sigma_\theta) \to B(X)$, $u \mapsto \psi(U_\theta u)$ is clearly bounded and weakly compact. It follows then from the spectral theorem for well-bounded operators of type (B) (see, for example, [12]) that there exists a spectral

family $\{E(H(\theta,\mu))\}_{\mu\in\mathbb{R}}$, with $\|E(H(\theta,\mu))\| \leqslant 2\|\psi\|$ for all μ. We have thus constructed a uniformly bounded family of projections $\{E(H)\}_{H\in\mathcal{H}}$. To show that this family is a half-plane spectral family it only remains to verify (i) and (ii).

Suppose then that $E_1 = E(\theta_1,\mu_1)$ and $E_2 = E(\theta_2,\mu_2)$. For $\mu \in \mathbb{R}$ and $\delta > 0$, let $g_{\mu,\delta} : \mathbb{R} \to \mathbb{R}$ be the function which is 1 on $(-\infty,\mu]$, is 0 on $[\mu+\delta,\infty)$ and which is linear on $[\mu,\mu+\delta]$. Let $h_\delta = U_{\theta_1}(g_{\mu_1,\delta})$ and $k_\delta = U_{\theta_2}(g_{\mu_2,\delta})$. The proof of the spectral theorem for well-bounded operators shows that $E_1 = \lim_{\delta\to 0^+} \psi(h_\delta)$ and $E_2 = \lim_{\delta\to 0^+} \psi(k_\delta)$, where the limits are taken in the weak operator topology in $B(X)$. Thus, if $x \in X$ and $x^* \in X^*$,

$$\langle E_1 E_2 x, x^* \rangle = \lim_{\delta\to 0^+} \langle \psi(h_\delta)E_2 x, x^* \rangle = \lim_{\delta\to 0^+} \langle E_2 x, \psi(h_\delta)^* x^* \rangle$$

$$= \lim_{\delta\to 0^+} \left(\lim_{\beta\to 0^+} \langle \psi(h_\delta)\psi(k_\beta)x, x^* \rangle \right) = \lim_{\delta\to 0^+} \left(\lim_{\beta\to 0^+} \langle \psi(k_\beta)\psi(h_\delta)x, x^* \rangle \right)$$

$$= \lim_{\delta\to 0^+} \langle \psi(h_\delta)x, E_2^* x^* \rangle = \langle E_2 E_1 x, x^* \rangle.$$

Verifying (iv) is similar. Fix $\theta \in \mathbb{T}$ and $\mu_1 < \mu_2$. Let $E_1 = E(\theta,\mu_1)$ and $E_2 = E(-\theta,-\mu_2)$. Let $h_\delta = U_\theta(g_{\mu_1,\delta})$ and $k_\delta = U_{-\theta}(g_{-\mu_2,\delta})$ so that $E_1 = \lim_{\delta\to 0^+} \psi(h_\delta)$ and $E_2 = \lim_{\delta\to 0^+} \psi(k_\delta)$. The result follows by noting that for δ small enough, $h_\delta k_\delta = 0$.

We have shown then that $\{E(H)\}_{H\in\mathcal{H}}$ is a half-plane spectral family.

For notational convenience, we shall identify the direction vector $\theta \in \mathbb{R}^2$ with the corresponding complex number on the unit circle. Thus, for example, we identify $(0,1)$ with i.

For $\theta \in \mathbb{T}$, the spectral family $\{E(\theta,\mu)\}_{\mu\in\mathbb{R}}$ defines a well-bounded operator of type (B)

$$(7.1) \qquad T_\theta = \int_{\sigma_\theta} \mu\, dE(\theta,\mu).$$

Clearly the map $\lambda_\theta = z \cdot \theta$ lies in $AC_\theta(\sigma) \subseteq AC(\sigma)$ and the construction of the spectral family ensures that $\psi(\lambda_\theta) = T_\theta$. Since $\lambda = \theta\lambda_\theta + i\theta\lambda_{i\theta}$ we have that

$$(7.2) \qquad T = \theta T_\theta + i\theta\, T_{i\theta}.$$

In particular, using Theorem 5.5 and Theorem 5.8 we have that T has the unique splitting into real and imaginary parts

$$(7.3) \qquad T = T_1 + i\, T_i.$$

One consequence of these identities is that T may be recovered from the half-plane spectral family produced by the above construction.

Note that Theorem 5.5 and the fact that $\theta^{-1}T = T_\theta + iT_{i\theta}$ imply that $\sigma(T_\theta) = \text{Re}(\sigma(\theta^{-1}T))$. Thus, if $r(\cdot)$ denotes the spectral radius, then $r(T_\theta) \leqslant r(\theta^{-1}T) = r(T)$.

Since there exists $\theta \in \mathbb{T}$ for which $r(T_\theta) = r(T)$, we have the following result.

PROPOSITION 7.4. *With T and $\{E(H)\}$ as above, $r(\{E(H)\}) = r(T)$.*

Note that if we define $f_\theta \in AC(\sigma)$ by $f_\theta(z) = z \cdot \theta$, then $T_\theta = \psi(f_\theta) = f_\theta(T)$. In particular, if $\omega = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then $f_\omega = \frac{f_1 + f_i}{\sqrt{2}}$, and hence

$$T_\omega = \psi(f_\omega) = \frac{T_1 + T_i}{\sqrt{2}}.$$

This proves the following proposition. Note that in general the sum of two commuting well-bounded operators need not be well-bounded.

PROPOSITION 7.5. *Let T be an $AC(\sigma)$ operator of type* (B), *with unique splitting $T = R + iS$. Then $R + S$ is also well-bounded.*

QUESTION 7.6. Suppose that R and S are commuting well-bounded operators whose sum is well-bounded. Is $R + iS$ an $AC(\sigma)$ operator?

It is clear that given any half-plane spectral family $\{E(H)\}_{H \in \mathcal{H}}$ with finite radius, equation (7.3) defines $T \in B(X)$ which is an AC operator in the sense of Berkson and Gillespie. It is not clear however, that T need be an $AC(\sigma)$ operator. In particular, if we define T_θ via equation (7.1), then it is not known whether the identity (7.2) holds.

QUESTION 7.7. Is there a one-to-one correspondence between $AC(\sigma)$ operators of type (B) and half-plane spectral families with finite radius? If not, can one refine Definition 7.3 so that such a correspondence exists?

8. EXTENDING THE FUNCTIONAL CALCULUS

Given a $AC(\sigma)$ operators of type (B) its associated half-plane spectral family (as constructed above), it is natural to ask whether one can develop an integration theory which would enable the functional calculus to be extended to a larger algebra than $AC(\sigma)$.

The spectral family associated to a well-bounded operator T of type (B) allows one to associate a bounded projection with any set of the form $\bigcup_{j=1}^{n} \sigma(T) \cap I_j$, where $I_1, \ldots, I_n$ are disjoint intervals of $\mathbb{R}$. Let $\sigma = \sigma(T) \subset \mathbb{R}$ and let $\mathcal{N}(\sigma(T))$ denote the algebra of all such sets. It is easy to check the following.

THEOREM 8.1. *Let $T \in B(X)$ be a type* (B) *well-bounded operator with functional calculus ψ. Then there is a unique map $E : \mathcal{N}(\sigma(T)) \to \mathrm{Proj}(X)$ satisfying the following:*
  (i) $E(\varnothing) = 0$, $E(\sigma(T)) = I$;
  (ii) $E(A \cap B) = E(A)E(B) = E(B)E(A)$ for all $A, B \in \mathcal{N}(\sigma(T))$;
  (iii) $E(A \cup B) = E(A) + E(B) - E(A \cap B)$ for all $A, B \in \mathcal{N}(\sigma(T))$;
  (iv) $\|E(A)\|_{BV(\sigma)} \leqslant \|\psi\| \|\chi_A\|_{BV(J)}$ for all $A \in \mathcal{N}(\sigma(T))$;

(v) *if $S \in B(X)$ is such that $TS = ST$ then $E(A)S = SE(A)$ for all $A \in \mathcal{N}(\sigma(T))$;*

(vi) Range$(E(A)) = \{x \in X : \sigma_T(x) \subseteq A\}$.

For general $AC(\sigma)$ operators, the natural algebra of sets is that generated by the closed half-planes. This algebra has been studied in various guises, particularly in the setting of computational geometry. The sets that can be obtained by starting with closed half-planes and applying a finite number of unions, intersections and set complements are sometimes known as Nef polygons. The set of Nef polygons in the plane, $\mathcal{N}$, clearly contains all polygons, lines and points in the plane. For more information about Nef polygons, or more generally their n-dimensional analogues, Nef polyhedra, we refer the reader to [6], [20] or [25].

Let σ be a nonempty compact subset of $\mathbb{C}$. Define

$$\mathcal{N}(\sigma) = \{A \,:\, A = \sigma \cap P \text{ for some } P \in \mathcal{N}\}.$$

It is clear that given an $AC(\sigma)$ operator of type (B), one may use the half-plane spectral family constructed in Section 7 to associate a projection $E(A) \in B(X)$ with each set $A \in \mathcal{N}(\sigma)$. The major obstacle in developing a suitable integration theory in this setting is in providing an analogue of condition (iv) in Theorem 8.1.

Note that if $A \in \mathcal{N}(\sigma)$, then $\chi_A \in BV(\sigma)$. Rather than forming $E(A)$ by a finite combination of algebra operations, one might try to define $E(A)$ directly as we did when A was a half-plane. That is, one may try to write

$$E(A) = \text{WOT-}\lim_{\alpha} \psi(h_\alpha)$$

where $\{h_\alpha\}$ is a suitable uniformly bounded net of functions in $AC(\sigma)$ which approximates χ_A pointwise. It is shown in [1] that if A is a closed polygon then this may be done but only under the bound $\|h_\alpha\| \leqslant V_A$. Here V_A is a constant depending on A. This allows one to prove a weaker version of Theorem 8.1, with condition (iv) replaced by $\|E(A)\| \leqslant V_A \|\psi\|$. It remains an open question as whether one can do this with $V_A \leqslant 2\|\chi_A\|$. However, if A is a closed convex polygon contained in the interior of σ, then this is possible.

QUESTION 8.2. Does every $AC(\sigma)$ operator of type (B) admit a $BV(\sigma)$ functional calculus?

It might be noted in this regard that all the examples of $AC(\sigma)$ operators of type (B) given in Section 4 do admit such a functional calculus extension.

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