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ON MULTIPLICATION OPERATORS ON THE BERGMAN SPACE: SIMILARITY, UNITARY EQUIVALENCE AND REDUCING SUBSPACES

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ABSTRACT. In this paper, we study similarity, unitary equivalence and reducing subspace problems of multiplication operators with symbols of finite Blaschke products on the Bergman space $L^2_a(\mathbb{D})$. By using Rudin's method, we establish a representation theorem of L^2_a-functions related to a given finite Blaschke product. As an immediate consequence, one sees that for two finite Blaschke products B_1, B_2, M_{B_1} is similar to M_{B_2} if and only if $\deg B_1 = \deg B_2$. By a different method, this similarity result also was independently obtained by Jiang and Li. Then we turn to the study of reducing subspaces of multiplication operators. It is shown that if B is a finite Blaschke product with $\deg B \leqslant 6$, then the number of minimal reducing subspaces of M_B is at most $\deg B$. The best previous known results were for the cases of $\deg B = 2, 3, 4$.

KEYWORDS: *Bergman space, finite Blaschke product, minimal reducing subspaces, unitary equivalence, similarity.*

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1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let dA denote the normalized area measure on $\mathbb{D}$. Denote by $L^2_a(\mathbb{D})$ the Bergman space, consisting of analytic functions on $\mathbb{D}$ which are squarely integrable with respect to the measure dA. For each bounded analytic function h on $\mathbb{D}$, we denote by M_h the multiplication operator on $L^2_a(\mathbb{D})$ with the symbol h.

Recall that in a Hilbert space H, a (closed) subspace M is called *invariant* for an operator T if $TM \subseteq M$. If in addition $T^*M \subseteq M$, then M is called a *reducing subspace* of T. And a nontrivial reducing subspace M is called *minimal* if the only reducing subspaces contained in M are M and 0. Below, for a Hilbert space of functions analytic on $\mathbb{D}$, by an invariant (respectively, reducing) subspace we mean a closed invariant (respectively, reducing) subspace of the coordinate operator M_z.

There are many motivations to study the reducing subspaces of multiplication operators on the Bergman space. One is from the equivalence of the Invariant Subspace Problem and the problem of the invariant subspace lattice of the Bergman space. Precisely, the Invariant Subspace Problem asks that, if T is a bounded linear operator on a separable Hilbert space H and $\dim H = \infty$, then must T have a proper invariant subspace? It is well known that this problem is equivalent to the following: if M and N are invariant subspaces of $L_a^2(\mathbb{D})$ such that $N \subseteqq M$ and $\dim M \ominus N = \infty$, then is there another invariant subspace L satisfying $N \subsetneqq L \subsetneqq M$? [5]. So it is interesting and important to get some deep information of the invariant space lattice in $L_a^2(\mathbb{D})$. However, the case of the Hardy space is quite different. By the Beurling Theorem [13], the invariant subspaces in the Hardy space are completely determined by inner functions.

Another motivation is consideration of the multiplication operators on the Hardy space $H^2(\mathbb{T})$. As pointed out in [11], for each inner function ϕ which is not a Mobius map, the reducing subspaces of M_ϕ are in one to one corresponding with the closed subspaces of $H^2(\mathbb{T}) \ominus \phi H^2(\mathbb{T})$ [7],[12]. For details, one may refer to [3], [4] and [16].

For the coordinate operator M_z, there is no nontrivial reducing subspace either on the Hardy space or on the Bergman space. But as for the multiplication operator M_{z^n} ($n \geqslant 2$), there are exactly n different minimal reducing subspaces on the Bergman space [18], [25], which is quite different from the case in the Hardy space. In [25], it is shown that for each Blaschke product B of degree 2, M_B has precisely two different minimal reducing subspaces. Zhu [25] conjectured that for a finite Blaschke product B of degree n, there are always n different minimal reducing subspaces. But it is proved that this conjecture fails when $n = 3$ [11]. And then Zhu's conjecture is modified as follows:

CONJECTURE 1.1 ([11]). Let B be a finite Blaschke product of degree n, then M_B has at most n minimal reducing subspaces.

In fact, it is shown in [11] and [23] that, if B is a Blaschke product of degree 3 or 4, then the number of minimal reducing subspaces of M_B is not larger than $\deg B$, the degree of B. In these papers, the minimal reducing subspaces are nicely described.

Now we turn to the question of similarity of multiplication operators on the Bergman space $L_a^2(\mathbb{D})$. Observe that M_{z^n} on $L_a^2(\mathbb{D})$ is similar to $M_z \otimes I_n$ on $L_a^2(\mathbb{D}) \otimes \mathbb{C}^n$. This result raises an interesting question which is related to the program mentioned above.

QUESTION 1.2 ([10]). Is M_B on $L_a^2(\mathbb{D})$ similar to $M_z \otimes I_n$ on $L_a^2(\mathbb{D}) \otimes \mathbb{C}^n$ (equivalently, similar to M_{z^n} on $L_a^2(\mathbb{D})$), where n is the degree of the finite Blaschke product B?

The answer to this question is yes, which was proved in [15]. Below, by using Rudin's method ([17], Chapter 7) we will establish a representation theorem of $L^2_a(\mathbb{D})$-functions, which is of independent interest, and from which one would have an immediate answer to the similarity problem.

THEOREM 1.3. *Suppose B is a finite Blaschke product of degree n, then there are bounded operators τ_i from $L^2_a(\mathbb{D})$ to $L^2_a(\mathbb{D})$ such that*

$$f(z) = \sum_{i=1}^{n} \tau_i f(B(z)) z^{i-1}, \quad z \in \mathbb{D}.$$

The representation is unique; that is, if there are $f_1, \ldots, f_n$ in $L^2_a(\mathbb{D})$ satisfying

$$f(z) = \sum_{i=1}^{n} f_i(B(z)) z^{i-1}, \quad z \in \mathbb{D},$$

then $f_i = \tau_i f$ for $i = 1, \ldots, n$.

Applying the above theorem immediately gives the following similarity theorem.

THEOREM 1.4 ([15]). *Given two finite Blaschke products B_1 and B_2, then M_{B_1} is similar to M_{B_2} on the Bergman space if and only if* $\deg B_1 = \deg B_2$.

Now let us turn back to the reducing subspace problem. Before continuing, let us have a look at the reducing subspaces from the von Neumann algebra's view. A basic argument shows that a reducing subspace of M_B is the range of an orthogonal projection commuting with M_B. For the Bergman space $L^2_a(\mathbb{D})$, we consider the commutant $\mathcal{C}_B$ of $\{M_B, M_B^*\}$, where B is a finite Blaschke product of degree n. Notice that $\mathcal{C}_B$ is a von Neumann algebra spanned by its projections. Therefore, characterizing minimal reducing subspaces is equivalent to characterizing minimal projections in $\mathcal{C}_B$. This means that Conjecture 1.1 says that $\mathcal{C}_B$ has at most n minimal projections. In fact, one has the following conclusion.

PROPOSITION 1.5. *Let B be a finite Blaschke product of degree n. Then the following are equivalent:*

(i) *Conjecture* 1.1 *holds.*
(ii) *$\mathcal{C}_B$ is abelian.*
(iii) *All minimal projections in $\mathcal{C}_B$ are orthogonal.*

Though it is hard to prove Conjecture 1.1, we have $\dim \mathcal{C}_B \leqslant n$, as will be see later in Section 4.

In this paper, B always denotes a finite Blaschke product, if there is no other explanation. For two reducing subspaces M and N of M_B, if there exists a unitary operator U from M onto N and U commutes with M_B, then M is called to be *unitarily equivalent* to N. If so, then we can extend U to $\widetilde{U}$ such that $\widetilde{U}|_M = U$ and $\widetilde{U}|_{M^\perp} = 0$. It follows that $\widetilde{U}$ commutes with both M_B and M_B^*. If we write p (respectively q) for the orthogonal projection from $L^2_a(\mathbb{D})$ onto M (respectively

N), then we get $p = \widetilde{U}^*\widetilde{U}$ and $q = \widetilde{U}\widetilde{U}^*$. That is, the two projections p and q are equivalent in $\mathcal{C}_B$. In this way, one sees that *the unitary equivalence between reducing subspaces is identified with the equivalence between projections in* $\mathcal{C}_B$. It is shown in [11] that Conjecture 1.1 is equivalent to that any two different minimal reducing subspaces are never unitarily equivalent, and hence equivalent to that two different minimal projections in $\mathcal{C}_B$ are never equivalent. Now write $\mathcal{C}_{B,j}$ $(1 \leqslant j \leqslant n)$ for the set consisting of all such minimal reducing subspaces M satisfying $\dim M \ominus BM = j$; and then Conjecture 1.1 is reduced to ask for each $1 \leqslant j \leqslant n$, whether any two minimal reducing subspaces in $\mathcal{C}_{B,j}$ are not unitarily equivalent. For $j = 1$, this is true, which is the following theorem.

THEOREM 1.6. *For two different minimal reducing subspaces M and N, if* $\dim M \ominus BM = \dim N \ominus BN = 1$, *then $M \perp N$, and they are not unitarily equivalent.*

As an application of Theorem 1.6, we give a quite different proof of the following result.

COROLLARY 1.7 ([25],[11],[23]). *Let B be a Blaschke product of degree n with $n \leqslant 4$, then M_B has at most n minimal reducing subspaces.*

Furthermore, a straightforward consequence of Theorem 1.6 is Theorem 27 in [11], which states that any minimal reducing subspace different from M_0 must be orthogonal to M_0, where M_0 is the distinguished subspace. Precisely, if $B(0) = 0$, then M_0 is the Bergman subspace spanned by $\{B'B^m : m \in \mathbb{Z}_+\}$ [19].

Our main theorem in this paper is as follows.

THEOREM 1.8. *Let B be a finite Blaschke product of degree $n = 5, 6$, then M_B has at most n minimal reducing subspaces.*

The paper is organized as follows.

In Section 2 by using Rudin's method ([17], Chapter 7) we establish a representation theorem of $L_a^2(\mathbb{D})$-functions, which is of independent interest. From this theorem, one easily sees that if B_1 and B_2 are two finite Blaschke products, then M_{B_1} is similar to M_{B_2} if and only if $\deg B_1 = \deg B_2$. This was independently obtained in [15], by a different method.

In Section 3 we study the reducing subspace problem. It is shown that for a given finite Blaschke product B, among all minimal reducing subspaces M of M_B satisfying $\dim M \ominus BM = 1$, any two of them are orthogonal. As an immediate consequence, this leads to a simpler proof of Corollary 1.7.

In Section 4 we give the proof of Theorem 1.8. Furthermore, it is shown $\dim \mathcal{C}_B \leqslant n$, where $n = \deg B$. Applying representation of finite dimensional C^*-algebras gives a shorter proof of Corollary 1.7.

2. SIMILARITY

The reducing subspace problem is of great interest in the study of Hilbert spaces of analytic functions. A lot of fruitful and deep results in this field have been got. We refer the reader to the references [4], [3], [7], [16], [18], [19], [11], [23], [22], [25]. As pointed out by R. Douglas [10], a related natural question is: Is M_B on $L^2_a(\mathbb{D})$ similar to $M_z \otimes I_n$ on $L^2_a(\mathbb{D}) \otimes \mathbb{C}^n$, where $n = \deg B$? The answer to this question is yes, which was proved in [15]. Below, by using Rudin's method ([17], Chapter 7) we will establish a representation theorem of $L^2_a(\mathbb{D})$-functions, which is of independent interest, and from which one would have an immediate answer to the similarity problem. We state it as follows.

THEOREM 2.1. *Suppose B is a finite Blaschke product of degree n, then there are bounded operators τ_i from $L^2_a(\mathbb{D})$ to $L^2_a(\mathbb{D})$ such that*

$$f(z) = \sum_{i=1}^{n} \tau_i f(B(z)) z^{i-1}, \quad f \in L^2_a(\mathbb{D}).$$

The representation is unique; that is, if there are $f_1, \dots, f_n$ in $L^2_a(\mathbb{D})$ satisfying

$$f(z) = \sum_{i=1}^{n} f_i(B(z)) z^{i-1}, \quad z \in \mathbb{D}, \tag{2.1}$$

then $f_i = \tau_i f$ for $i = 1, \dots, n$.

Proof. Before the proof, let us make an observation.

Since the derivative of a finite Blaschke product never vanishes on $\mathbb{T}$, there exists an $r \in (0,1)$ such that B' never vanishes on $\overline{A_r}$, where

$$A_r = \{z \in \mathbb{C} : \ r < |z| < 1\}.$$

It is easy to check that there is an r' $(r < r' < 1)$ satisfying

$$B^{-1}(\overline{A_{r'}}) \subseteq \overline{A_r}. \tag{2.2}$$

We first prove uniqueness. It suffices to show that if there are $f_1, \dots, f_n$ in $L^2_a(\mathbb{D})$ satisfying

$$\sum_{i=1}^{n} f_i(B(z)) z^{i-1} = 0, \quad z \in \mathbb{D}, \tag{2.3}$$

then $f_i \equiv 0$ for all i.

In fact, from the above observation we can pick an r_0 $(r' < r_0 < 1)$. By (2.2), for each $w \in r_0\mathbb{T}$,

$$B^{-1}(w) \subseteq B^{-1}(\overline{A_{r'}}) \subseteq \overline{A_r}.$$

Notice also that B' never vanishes on $\overline{A_r}$, so for each $w \in r_0\mathbb{T}$, $B^{-1}(w)$ consists of exactly n different points, denoted by $\beta_1(w), \dots, \beta_n(w)$. Thus by (2.3),

$$\sum_{i=1}^{n} f_i(w) \beta_j(w)^{i-1} = 0, \quad w \in r_0\mathbb{T} \text{ and } 1 \leqslant j \leqslant n.$$

Since for each $w \in r_0\mathbb{T}$, $(\beta_j(w)^{i-1})$ is just the Vandermonde matrix and hence the above equations force

$$f_1(w) = \cdots = f_n(w) = 0, \quad w \in r_0\mathbb{T}.$$

So $f_i \equiv 0$ for $1 \leqslant i \leqslant n$, as desired.

The proof for the existence. By a similar argument as above, for any $w \in \overline{A_{r'}}$, there exist n distinct points $\beta_j(w) \in \overline{A_r}$ satisfying

$$B(\beta_j(w)) = w, \quad j = 1, \dots, n. \tag{2.4}$$

Moreover, since $B'(\beta_j(w)) \neq 0$, there is an $\delta_w > 0$ such that B maps $O(\beta_j(w), \delta_w)$ biholomorphically onto some neighborhood of w. This gives the following fact.

FACT 2.2. For any $w \in \overline{A_{r'}}$, there exists an $\varepsilon_w > 0$, and n different functions $\beta_1, \dots, \beta_n$ which are holomorphic and injective on $O(w, \varepsilon_w)$ such that each of them satisfies (2.4), and $\beta_i(z) \neq \beta_j(z)$ for $z \in O(w, \varepsilon_w)$, $i \neq j$. Moreover, the derivative of each β_j^{-1} are bounded on $O(w, \varepsilon_w)$.

By Chapter 7 of [17], for each polynomial P, there are rational maps R_i in $A(\mathbb{D})$ such that

$$P(z) = \sum_{i=1}^{n} R_i(B(z)) z^{i-1}. \tag{2.5}$$

As done in the proof of Theorem 7.4.1 in [17], first we can define τ_i on polynomials and τ_i are well defined and linear. Precisely, from (2.5), $\tau_i P = R_i$. In what follows we will use the fact to prove that τ_i are bounded with respect to the Bergman norm.

Now rewrite (2.5) by

$$P(\beta_j(w)) = \sum_{i=1}^{n} R_i(w) \beta_j(w)^{i-1}, \quad 1 \leqslant j \leqslant n \text{ and } w \in \overline{A_{r'}}.$$

By Cramer's rule, we have for each $k (1 \leqslant k \leqslant n)$,

$$R_k(w) = \frac{\det V_k(P)(w)}{\det(\beta_j(w)^{i-1})}, \quad w \in \overline{A_{r'}}, \tag{2.6}$$

where $V_k(P)(w)$ denotes the matrix $(\beta_j(w)^{i-1})$ whose k-th column is replaced with $(P(\beta_1(w)), \dots, P(\beta_n(w)))^{\mathrm{T}}$. The above discussion shows that the Vandermonde determinant $\det(\beta_j(w)^{i-1})$ is continuous on $\overline{A_{r'}}$ and has no zero point. So by (2.6), there is a constant $C > 0$ such that

$$\int\limits_{A_{r'}} |R_k(w)|^2 \mathrm{d}A(w) \leqslant C \int\limits_{\overline{A_{r'}}} \sum_{j=1}^{n} |P(\beta_j(w))|^2 \mathrm{d}A(w). \tag{2.7}$$

Since all neighborhoods $O(w,\varepsilon_w)$ of $w \in \overline{A_{r'}}$ consist of an open cover of $\overline{A_{r'}}$, then we can pick finite of them

$$O(w_1,\varepsilon_1),\ldots,O(w_N,\varepsilon_N),$$

whose union contains $\overline{A_{r'}}$. Then by (2.7),

$$\begin{aligned}\int_{A_{r'}} |R_k(w)|^2 \mathrm{d}A(w) &\leqslant C \int_{\bigcup_{l=1}^N O(w_l,\varepsilon_l)\cap\overline{\mathbb{D}}} \sum_{j=1}^n |P(\beta_j(w))|^2 \mathrm{d}A(w)\\ &\leqslant C \sum_{l=1}^N \sum_{j=1}^n \int_{\beta_j(O(w_l,\varepsilon_l)\cap\overline{\mathbb{D}})} |P(z)|^2 |(\beta_j^{-1})'(z)|^2 \mathrm{d}A(z)\\ &\leqslant CN \int_{\mathbb{D}} nM|P(z)|^2 \mathrm{d}A(z) = CnNM \int_{\mathbb{D}} |P(z)|^2 \mathrm{d}A(z),\end{aligned}$$

where

$$M = \sup\,\{|(\beta_j^{-1})'(z)|^2 : z \in O(w_l,\varepsilon_l), 1 \leqslant j \leqslant n, 1 \leqslant l \leqslant N\} < \infty.$$

On the other hand, there is a numerical constant C' satisfying

$$\int_{\mathbb{D}} |f(w)|^2 \mathrm{d}A(w) \leqslant C' \int_{A_{r'}} |f(w)|^2 \mathrm{d}A(w), \quad f \in L_a^2(\mathbb{D}),$$

and hence $\int_{\mathbb{D}} |R_k(w)|^2 \mathrm{d}A(w) \leqslant C' \int_{A_{r'}} |R_k(w)|^2 \mathrm{d}A(w)$. Combining this inequality with the the above arguments shows that

$$\|\tau_k P\|^2 = \|R_k\|^2 \leqslant K\|P\|^2, \tag{2.8}$$

where $K = nMNCC' < \infty$ depends only on B.

Since every Bergman function f is a limit of polynomials $\{p_m\}$ in the Bergman norm. And by boundedness of τ_k $(1 \leqslant k \leqslant n)$, see (2.8), $\{\tau_k p_m\}_m$ has uniformly bounded Bergman norm, and hence $\{\tau_k p_m\}_m$ is a normal family. Then it follows that there is a subsequence $\{p_{m_l}\}$ such that $\{\tau_k p_{m_l}\}_l$ converges to some holomorphic function f_k (in the Bergman space) uniformly on compact subsets of $\mathbb{D}$. And it is easy to check such f_k satisfy

$$f(z) = \sum_{k=1}^n f_k(B(z)) z^{k-1}, \quad z \in \mathbb{D}.$$

Then by uniqueness of the representation, such f_k are independent of the choices of $\{p_m\}$ and its subsequence. Put $\tau_k f = f_k$ and the proof is complete. ∎

REMARK 2.3. The above theorem remains true if the Bergman space is replaced with the weighted Bergman spaces and Hardy spaces $H^p(\mathbb{D})(p \geqslant 1)$, and the proof is similar.

Then we have the following corollary.

COROLLARY 2.4 ([15]). *Let B be a finite Blaschke product of degree n, then there is a bounded invertible operator S from $L_a^2(\mathbb{D})$ to $L_a^2(\mathbb{D}) \otimes \mathbb{C}^n$ satisfying*

$$SM_B = (M_z \otimes I_n)S.$$

Consequently, for two finite Blaschke product B_1 and B_2, M_{B_1} is similar to M_{B_2} on the Bergman space if and only if $\deg B_1 = \deg B_2$.

Proof. Assume the operators $\tau_1, \ldots, \tau_n$ are defined as in Theorem 2.1 and set $Sf = (\tau_1 f, \ldots, \tau_n f)$ for each $f \in L_a^2(\mathbb{D})$. By Theorem 2.1, it is easy to check that S is invertible.

Moreover, we have $SM_B = (M_z \otimes I_n)S$. In fact, for each $f \in L_a^2(\mathbb{D})$,

$$M_B f = B \sum_{i=1}^{n} \tau_i f(B) z^{i-1} = \sum_{i=1}^{n} (z\tau_i f)(B) z^{i-1},$$

and hence by uniqueness,

$$SM_B f = (z\tau_1 f, \ldots, z\tau_n f) = (M_z \otimes I_n)Sf.$$

That is, $SM_B = (M_z \otimes I_n)S$.

As a direct result, we see that for two finite Blaschke products B_1 and M_{B_1} is similar to M_{B_2} on the Bergman space if and only if B_1 and B_2 have the same degree. ∎

3. REDUCING SUBSPACE PROBLEM

In this section, we will study the reducing subspace problem. Some lemmas will be established to prove our main theorem in this section, Theorem 3.1. As an application, we will give a short proof of Corollary 1.7 [11], [23]. And at the end of this section, we will give a proof of Proposition 1.5.

First we state our main theorem in this section as follows.

THEOREM 3.1. *For two different minimal reducing subspaces M and N, if one of the following holds:*

(i) $\dim M \ominus BM = \dim N \ominus BN = 1$,

(ii) $\dim M \ominus BM \neq \dim N \ominus BN$,

then $M \perp N$, and they are not unitarily equivalent.

It is well known that there is a unique distinguished minimal reducing subspace M_0 satisfying $\dim M_0 \ominus BM_0 = 1$ [14]. Precisely, if $B(0) = 0$, then M_0 is the Bergman subspace spanned by $\{B'B^m : m \in \mathbb{Z}_+\}$ [19]. So Theorem 3.1 implies that any minimal reducing subspace different from M_0 must be orthogonal to M_0, also see Theorem 27 of [11].

Furthermore, as an application of Theorem 3.1, we will give a quite different and simpler proof for Corollary 1.7. Also in the end of Section 4, we will use representation of finite dimensional C^*-algebras to give a shorter proof of Corollary 1.7. We restate it here.

COROLLARY 3.2 ([25], [11], [23]). *Let B be a finite Blaschke product of degree n with $n \leqslant 4$, then M_B has at most n minimal reducing subspaces.*

Proof. Once Theorem 3.1 has been proved, then we can give the proof of Corollary 3.2 as follows. It suffices to consider the following three cases: $\deg B = 2$, $\deg B = 3$ and $\deg B = 4$.

Case I. $\deg B = 2$. Let M_0 be the distinguished minimal reducing subspace. Clearly $M_0^\perp$ is a reducing subspace, and since $\dim M_0^\perp \ominus BM_0^\perp = 1$, $M_0^\perp$ is minimal. But as mentioned above, all minimal reducing subspaces other than M_0 are contained in $M_0^\perp$. This implies that there is no other minimal reducing subspace other than M_0 and $M_0^\perp$.

Case II. $\deg B = 3$. Notice that all minimal reducing subspaces other than M_0 are contained in $M_0^\perp$, and $\dim M_0^\perp \ominus BM_0^\perp = 2$, so either $M_0^\perp$ is minimal or all minimal reducing subspaces N satisfy $\dim N \ominus BN = 1$. It suffices to consider the latter case. Now assume all minimal reducing subspaces N satisfy $\dim N \ominus BN = 1$, then by Theorem 3.1, any two different minimal reducing subspaces are orthogonal to each other, and hence there exist no more than 3 minimal reducing subspaces.

Case III. $\deg B = 4$. A similar argument as in Case II shows that if all minimal reducing subspaces N satisfy $\dim N \ominus BN = 1$, then the number of minimal reducing subspaces is not larger than 4; and if there is a minimal reducing subspace M_1 satisfying $\dim M_1 \ominus BM_1 = 3$, then M_B has no minimal reducing subspace other than M_0 and M_1. So it remains to deal with the case when there is a minimal reducing subspace M_1 satisfying $\dim M_1 \ominus BM_1 = 2$.

Now set $M_2 = L_a^2(\mathbb{D}) \ominus (M_0 \oplus M_1)$. Then M_2 is reducing. Moreover, since $\dim M_2 \ominus BM_2 = 1$, M_2 is minimal. We claim that there is no other minimal reducing subspace expect M_0, M_1 and M_2. To this end, assume conversely that there is another minimal reducing space N ($N \neq M_i$, $i = 0,1,2$). Then either $\dim N \ominus BN = 2$ or $\dim N \ominus BN = 1$. If $\dim N \ominus BN = 2$, then by Theorem 3.1, N is orthogonal to M_0 and M_2. Therefore $N \subseteq M_1$, and by minimality, $N = M_1$. This is a contradiction. If $\dim N \ominus BN = 1$, then by Theorem 3.1, $N \perp M_i$ ($i = 0,1,2$). So $N \perp L_a^2(\mathbb{D})$, which is impossible. The proof of Corollary 3.2 is complete. ∎

Before proving Theorem 3.1, let us state a theorem which comes from Theorem 3 in and Theorem 31 [11]. But we include a similar but abbreviated proof.

THEOREM 3.3 ([11]). *Let M and N be two minimal reducing subspaces of M_B. If M and N are not orthogonal, then M is unitarily equivalent to N.*

Proof. Let P denote the orthogonal projection from $L_a^2(\mathbb{D})$ onto N and put $p = P|_M$.

Claim. The operator $p : M \to N$ is injective and $\overline{\text{Range } p} = N$. Moreover, p commutes with both M_B and M_B^*.

To this end, we first prove that p commutes with both M_B and M_B^*. In fact, we have for each $f \in M$,

$$Bf = pBf + (1-p)Bf, \quad \text{and} \quad Bf = Bpf + B(1-p)f.$$

Comparing the above two identities, and noting that N and $N^\perp$ are both reducing, we get $pBf = Bpf, f \in M$. That is, p commutes with M_B. Similarly, p commutes with M_B^*.

Next we will show that p is injective. To reach a contradiction, assume there is an $f_0 \in M$ ($f_0 \neq 0$) such that $pf_0 = 0$. That is, $f_0 \perp N$. Since N is reducing, it follows that the reducing subspace $[f_0]$ generated by f_0 is orthogonal to N. Since M is minimal, $M = [f_0]$, and hence $M \perp N$, which is a contradiction to our assumption.

Since p commutes with both M_B and M_B^*, it follows that $\overline{\text{Range } p}$ is a reducing subspace contained in N. Notice also that $\overline{\text{Range } p} \neq 0$ since p is injective, so by the minimality of N, $\overline{\text{Range } p} = N$, completing the proof of the claim.

Now let $p = u|p|$ be the polar decomposition of p, where $|p| = \sqrt{p^*p}$ and u is a partial isometry from M to N satisfying

$$u : |p|f \mapsto pf, \quad f \in M. \tag{3.1}$$

Below, we will see that u is a unitary operator from M onto N. In fact, by the claim p commutes with both M_B and M_B^*, and hence $|p|$ commutes with both M_B and M_B^*. This means that $\overline{|p|M}$ is a reducing subspace contained in M, and by minimality of M, $\overline{|p|M} = M$. Therefore by (3.1), u is an isometry and thus has closed range. On the other hand, (3.1) gives Range $p \subsetneqq$ Range u. Noting $\overline{\text{Range } p} = N$, we have Range $u = N$, which implies that u is a unitary operator from M onto N. Then it is easy to check that u commutes with both M_B and M_B^*, completing the proof. ∎

Before continuing, let us make an observation. It is well known that the reducing subspaces of M_B are the same as those of $M_{\varphi_a(B)}$ $(a \in \mathbb{D})$, where

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D}.$$

It follows from a simple calculation that, if a is not in the finite set

$$\{z : \text{ there exists } z' \text{ such that } B'(z') = 0 \text{ and } B(z') = B(z)\},$$

then $\varphi_a(B)$ has only simple zeros. Thus, *when studying the reducing subspaces of M_B, we can reduce to the case that B has only simple zeros.*

Below, we always assume B, B_1 and B_2 are finite Blaschke products which have only simple zeros, if there is no other explanation.

Next we will establish a proposition, which plays an important role in the proof of Theorem 3.1. Given two Blaschke products B_1 and B_2 with $\deg B_1 = \deg B_2 = n$, let $\beta_1, \dots, \beta_n$ (respectively, $\gamma_1, \dots, \gamma_n$) be n branches of B_1^{-1} (respectively, B_2^{-1}). Take Δ to be some connected domain such that all β_i and γ_i are (single valued) analytic on some neighborhood of $\overline{\Delta}$. In particular, since B_1 and B_2 have only simple zeros, we can choose Δ to be an open disk containing 0. Then we have the following proposition, whose special form is shown in the proof of Lemma 1 in [21].

PROPOSITION 3.4. *If M_i is a closed subspace of $L_a^2(\mathbb{D})$ which is invariant under M_{B_i} $(i = 1, 2)$, and if $U : M_1 \to M_2$ is a unitary operator such that $UM_{B_1} = M_{B_2}U$, then there exists an $n \times n$ numerical unitary matrix W such that*

$$W \begin{pmatrix} f(\beta_1(w))\beta_1'(w) \\ \vdots \\ f(\beta_n(w))\beta_n'(w) \end{pmatrix} = \begin{pmatrix} g(\gamma_1(w))\gamma_1'(w) \\ \vdots \\ g(\gamma_n(w))\gamma_n'(w) \end{pmatrix}, \quad w \in \Delta,$$

where $f \in M_1$ and $g = Uf$.

To prove the above proposition, we need the following, which is of independent interest.

LEMMA 3.5. *Let H be a Hilbert space and suppose e_λ^k, f_μ^k $(1 \leqslant k \leqslant n$ and $\lambda, \mu \in \Lambda)$ are vectors in H satisfying*

$$\sum_{k=1}^{n} e_\lambda^k \otimes e_\mu^k = \sum_{k=1}^{n} f_\lambda^k \otimes f_\mu^k, \quad \lambda, \mu \in \Lambda,$$

then there is an $n \times n$ numerical unitary matrix W such that

$$W \begin{pmatrix} e_\lambda^1 \\ \vdots \\ e_\lambda^n \end{pmatrix} = \begin{pmatrix} f_\lambda^1 \\ \vdots \\ f_\lambda^n \end{pmatrix}, \quad \lambda \in \Lambda.$$

In the case that Λ is a singlet, Lemma 3.5 is known, see Proposition 5.1 of [2] or Proposition A.1 of [1].

Proof. For each $\lambda \in \Lambda$, set

$$\begin{aligned} A_\lambda : \mathbb{C}^n &\to H \\ (c_1, \dots, c_n) &\mapsto \sum_{k=1}^{n} c_k e_\lambda^k, \end{aligned}$$

and

$$\begin{array}{rcl} B_\lambda : \mathbb{C}^n & \to & H \\ (c_1, \ldots, c_n) & \mapsto & \sum_{k=1}^{n} c_k f_\lambda^k. \end{array}$$

It follows from simple computations that

$$\begin{array}{rcl} A_\lambda^* : H & \to & \mathbb{C}^n \\ h & \mapsto & (\langle h, e_\lambda^1 \rangle, \ldots, \langle h, e_\lambda^n \rangle). \end{array}$$

Moreover, for any $\lambda, \mu \in \Lambda$, we have

$$A_\mu A_\lambda^* h = \sum_{k=1}^{n} \langle h, e_\lambda^k \rangle e_\mu^k = \sum_{k=1}^{n} (e_\mu^k \otimes e_\lambda^k) h = \sum_{k=1}^{n} (f_\mu^k \otimes f_\lambda^k) h = B_\mu B_\lambda^* h, \quad h \in H.$$

So $A_\mu A_\lambda^* = B_\mu B_\lambda^*$, and then it is easy to check that

$$\sum_{i=1}^{l} c_i A_{\lambda_i}^* h \mapsto \sum_{i=1}^{l} c_i B_{\lambda_i}^* h, \quad h \in H \text{ and } \lambda_i \in \Lambda$$

is a well defined isometry from some subspace of $\mathbb{C}^n$ to another. This isometry can be extended to a unitary map $V^* : \mathbb{C}^n \to \mathbb{C}^n$. By the definition of V^*, we have

$$V^* A_\lambda^* = B_\lambda^*, \quad \lambda \in \Lambda,$$

and hence

$$A_\lambda V = B_\lambda. \tag{3.2}$$

Let (3.2) act on $(0, \ldots, 1, 0, \ldots, 0)$ where 1 is at the k-th coordinate, and denote $\mathbf{v}_k$ the k-th column of V, then

$$\mathbf{v}_k^{\mathrm{T}} \begin{pmatrix} e_\lambda^1 \\ \vdots \\ e_\lambda^n \end{pmatrix} = f_\lambda^k, \quad \lambda \in \Lambda,$$

where $\mathbf{v}_k^{\mathrm{T}}$ is the transpose of $\mathbf{v}_k$. Let W be the transpose V^{T} of V and the above identities imply that

$$W \begin{pmatrix} e_\lambda^1 \\ \vdots \\ e_\lambda^n \end{pmatrix} = \begin{pmatrix} f_\lambda^1 \\ \vdots \\ f_\lambda^n \end{pmatrix}, \quad \lambda \in \Lambda. \quad \blacksquare$$

Proof of Proposition 3.4. Consider M_1 as a reproducing function space on $\mathbb{D}$, and let K_λ be the reproducing kernel of M_1 at $\lambda \in \mathbb{D}$, and put $L_\lambda = U K_\lambda$ (L_λ is not necessarily the reproducing kernel of M_2). Since $U M_{B_1} = M_{B_2} U$, it is easy to see that for any polynomials P and Q,

$$\langle P(B_1) K_\lambda, Q(B_1) K_\mu \rangle = \langle P(B_2) L_\lambda, Q(B_2) L_\mu \rangle, \quad \lambda, \mu \in \mathbb{D}.$$

That is,

$$\int_{\mathbb{D}} ((P\overline{Q}) \circ B_1(w) K_\lambda(w) \overline{K_\mu}(w) - (P\overline{Q}) \circ B_2(w) L_\lambda(w) \overline{L_\mu}(w)) \mathrm{d}A(w) = 0. \tag{3.3}$$

Now set

$$\mathcal{E} = \operatorname{span}\{P\overline{Q} : P, Q \text{ are polynomials}\}.$$

By the Weierstrass's Theorem, any continuous function on $\overline{\mathbb{D}}$ can be uniformly approximated by functions in $\mathcal{E}$. It follows from (3.3) that

$$\int_{\mathbb{D}} (f(B_1(w)) K_\lambda(w) \overline{K_\mu}(w) - f(B_2(w)) L_\lambda(w) \overline{L_\mu}(w)) \mathrm{d}A(w) = 0, \quad f \in C(\overline{\mathbb{D}}). \tag{3.4}$$

By the Dominated Convergence Theorem, the identity (3.4) holds for any function f in $L^\infty(\mathbb{D})$. In particular, for any $f \in L^\infty(\Delta)$, (3.4) gives that

$$\int_{B_1^{-1}(\Delta)} f(B_1(w)) K_\lambda(w) \overline{K_\mu}(w) \mathrm{d}A(w) = \int_{B_2^{-1}(\Delta)} f(B_2(w)) L_\lambda(w) \overline{L_\mu}(w) \mathrm{d}A(w),$$

and hence by our assumptions of Δ,

$$\int_\Delta f(z) \sum_{k=1}^n (K_\lambda \overline{K_\mu}) \circ \beta_k(z) |\beta_k'(z)|^2 \mathrm{d}A(z) = \int_\Delta f(z) \sum_{k=1}^n (L_\lambda \overline{L_\mu}) \circ \gamma_k(z) |\gamma_k'(z)|^2 \mathrm{d}A(z).$$

So

$$\sum_{k=1}^n (K_\lambda \overline{K_\mu}) \circ \beta_k(z) |\beta_k'(z)|^2 = \sum_{k=1}^n (L_\lambda \overline{L_\mu}) \circ \gamma_k(z) |\gamma_k'(z)|^2, \quad z \in \Delta. \tag{3.5}$$

Next we will apply Lemma 3.5. Let H be the Bergman space over Δ and $\Lambda = \mathbb{D}$. Set

$$e_\lambda^k = K_\lambda(\beta_k(z)) \beta_k'(z), f_\mu^k = L_\mu(\gamma_k(z)) \gamma_k'(z), \quad 1 \leqslant k \leqslant n,\ \lambda, \mu \in \mathbb{D}.$$

By (3.5), the Berezin transforms of $\sum_{k=1}^n e_\lambda^k \otimes e_\mu^k$ and $\sum_{k=1}^n f_\lambda^k \otimes f_\mu^k$ are equal, and hence

$$\sum_{k=1}^n e_\lambda^k \otimes e_\mu^k = \sum_{k=1}^n f_\lambda^k \otimes f_\mu^k, \quad \lambda, \mu \in \mathbb{D}.$$

Applying Lemma 3.5, we have

$$W \begin{pmatrix} K_\lambda(\beta_1(w)) \beta_1'(w) \\ \vdots \\ K_\lambda(\beta_n(w)) \beta_n'(w) \end{pmatrix} = \begin{pmatrix} L_\lambda(\gamma_1(w)) \gamma_1'(w) \\ \vdots \\ L_\lambda(\gamma_n(w)) \gamma_n'(w) \end{pmatrix}, \quad w, \lambda \in \Delta,$$

where W is an $n \times n$ unitary numerical matrix. This immediately leads to our conclusion. ∎

REMARK 3.6. In Proposition 3.4, we will pay special attention to the case that $B_1 = B_2$ and M_1 and M_2 are reducing for M_{B_1}. In this case, $\beta_i = \gamma_i$ for all i. Notice also that if $f \in M_1 \ominus B_1 M_1$, then $g = Uf \in M_2 \ominus B_1 M_2$.

If the assumption that B_1 and B_2 have only simple zeros is dropped, then Proposition 3.4 still holds, but Δ will be replaced with some open disk Δ' not containing 0. The proof is completely the same.

Before continuing, we need some notations. As mentioned above, B is a Blaschke product with only simple zeros, and the set Δ (above Proposition 3.4) is fixed to be a disk containing 0. Recall that there are n branches of B^{-1}; $\beta_1, \dots, \beta_n$ which are (single valued) analytic on some neighborhood of $\overline{\Delta}$. For each reducing subspace N of M_B, put

$$\mathcal{L}_{N,\Delta} = \operatorname{span}\left\{ \begin{pmatrix} h(\beta_1(w))\beta_1'(w) \\ \vdots \\ h(\beta_n(w))\beta_n'(w) \end{pmatrix} : h \in N, w \in \Delta \right\} \subseteq \mathbb{C}^n.$$

Now let us make an observation. Let M and M' be two orthogonal reducing subspaces of M_B, and take any f in M and g in M'. In the proof of Proposition 3.4, replace K_λ with f and K_μ with g, then we get

$$\langle P(B)f, Q(B)g \rangle = 0.$$

As done in the proof, at last we would have

$$\sum_{k=1}^{n} f(\beta_k)\beta_k' \otimes g(\beta_k)\beta_k' = 0,$$

from which we see $\mathcal{L}_{M,\Delta} \perp \mathcal{L}_{M',\Delta}$. So we conclude that *if M and M' are two orthogonal reducing subspaces, then $\mathcal{L}_{M,\Delta} \perp \mathcal{L}_{M',\Delta}$.* Furthermore, we have the following lemma.

LEMMA 3.7. *For each reducing subspace M, we have*

$$\dim \mathcal{L}_{M \ominus BM,\Delta} = \dim \mathcal{L}_{M,\Delta} = \dim M \ominus BM.$$

Proof. Denote M by M_1 and put $M_2 = M^\perp$, and then $\mathcal{L}_{M_1,\Delta} \perp \mathcal{L}_{M_2,\Delta}$. Therefore $\dim \mathcal{L}_{M_1,\Delta} + \dim \mathcal{L}_{M_2,\Delta} \leqslant n$, and hence

$$\dim \mathcal{L}_{M_1 \ominus BM_1,\Delta} + \dim \mathcal{L}_{M_2 \ominus BM_2,\Delta} \leqslant n.$$

Notice also that $\dim M_1 \ominus BM_1 + \dim M_2 \ominus BM_2 = n$, so it suffices to show that

$$\dim \mathcal{L}_{M_i \ominus BM_i,\Delta} \geqslant \dim M_i \ominus BM_i \quad (i = 1, 2).$$

To this end, consider M_1 and set $r = \dim M_1 \ominus BM_1$. Pick r linearly independent functions $h_1, \dots, h_r$ in $M_1 \ominus BM_1$, and we will show that there is a $w \in \Delta$

such that the matrix

$$\begin{pmatrix} h_1(\beta_1(w))\beta_1'(w) & \cdots & h_r(\beta_1(w))\beta_1'(w) \\ \vdots & \ddots & \vdots \\ h_1(\beta_n(w))\beta_n'(w) & \cdots & h_r(\beta_n(w))\beta_n'(w) \end{pmatrix}$$

has rank r; equivalently, the matrix

$$\begin{pmatrix} h_1(\beta_1(w)) & \cdots & h_r(\beta_1(w)) \\ \vdots & \ddots & \vdots \\ h_1(\beta_n(w)) & \cdots & h_r(\beta_n(w)) \end{pmatrix}$$

has rank r. And we denote the above matrix by $H(w)$ for simplicity.

In fact, the matrix $H(w)$ has rank r when $w = 0$. To see this, assume conversely that $H(0)$ has rank less than r. Then the columns of $H(0)$ span a subspace in $\mathbb{C}^n$ with dimension less than r, and hence there is a nonzero vector $\mathbf{c} = (c_1, \dots, c_r)$ in $\mathbb{C}^r$ satisfying

$$\sum_{i=1}^{r} c_i h_i(\beta_j(0)) = 0 \quad (1 \leqslant j \leqslant n).$$

That is,

$$\Big\langle \sum_{i=1}^{r} c_i h_i, K_{\beta_j(0)} \Big\rangle = 0 \quad (1 \leqslant j \leqslant n). \tag{3.6}$$

By our assumption, B has only simple zeros $\{\beta_j(0)\}_{j=1}^{n}$, and thus the set $\{K_{\beta_j(0)} : 1 \leqslant j \leqslant n\}$ spans $L_a^2(\mathbb{D}) \ominus BL_a^2(\mathbb{D})$. So (3.6) gives that $\sum\limits_{i=1}^{r} c_i h_i \in BL_a^2(\mathbb{D})$. On the other hand,

$$\sum_{i=1}^{r} c_i h_i \in M_1 \ominus BM_1 \subseteq L_a^2(\mathbb{D}) \ominus BL_a^2(\mathbb{D}),$$

and hence $\sum\limits_{i=1}^{r} c_i h_i = 0$, which is a contradiction to the linear independence of $h_1, \dots, h_r$. So

$$\dim \mathcal{L}_{M_1 \ominus BM_1, \Delta} \geqslant \dim M_1 \ominus BM_1.$$

A similar argument shows that $\dim \mathcal{L}_{M_2 \ominus BM_2, \Delta} \geqslant \dim M_2 \ominus BM_2$, as desired. ∎

To prove Theorem 3.1, we also need the following lemma.

LEMMA 3.8. *Let B be a finite Blaschke product (B need not have only simple zeros) and M is a reducing subspace of M_B. Then for any Mobius map ϕ,*

$$\dim M \ominus BM = \dim M \ominus \phi(B)M.$$

Proof. For each $a \in \mathbb{D}$, consider the map $\varphi_a(B)$. Since $M_{\varphi_a(B)}$ is a Fredholm operator on the Bergman space and M is a reducing subspace of $M_{\varphi_a(B)}$, then it is easy to check that

$$M_{\varphi_a(B)}|_M : M \to M$$

is also a Fredholm operator. Moreover, since $a \to M_{\varphi_a(B)}|_M$ is a continuous map from $\mathbb{D}$ to bounded operators on M, then the Index $M_{\varphi_a(B)}|_M$ is a continuous integer-valued function in a. Then from

$$\operatorname{Index} M_{\varphi_a(B)}|_M = -\dim M \ominus \varphi_a(B)M,$$

we get

$$\dim M \ominus BM = \dim M \ominus \varphi_a(B)M, \ a \in \mathbb{D}.$$

This immediately leads to our conclusion. ∎

Now we are prepared to prove Theorem 3.1.

Proof of Theorem 3.1. (i) As mentioned above, for a finite Blaschke product B, there is always a Mobius map ϕ such that $\phi(B)$ has only simple zeros. Notice also that M_B and $M_{\phi(B)}$ have the same (minimal) reducing subspaces. Then by Lemma 3.8, without loss of generality, we can assume that B is a finite Blaschke product with only simple zeros, and let M and N be two distinct minimal reducing subspaces of M_B.

Now suppose $\dim M \ominus BM = \dim N \ominus BN = 1$ and assume conversely that M is unitarily equivalent to N; that is, there exists a unitary operator $U : M \to N$ such that $UM_B = M_B U$. Then by Proposition 3.4 and Remark 3.6, there exists an open disk Δ and an $n \times n$ numerical unitary matrix W such that

$$W \begin{pmatrix} f(\beta_1(w))\beta_1'(w) \\ \vdots \\ f(\beta_n(w))\beta_n'(w) \end{pmatrix} = \begin{pmatrix} Uf(\beta_1(w))\beta_1'(w) \\ \vdots \\ Uf(\beta_n(w))\beta_n'(w) \end{pmatrix}, \quad w \in \Delta, \tag{3.7}$$

where $f \in M$.

Now take $f = f_0$, a nonzero function in M and let $g_0 = Uf_0 \in N$. Since by Lemma 3.7 $\dim \mathcal{L}_{M,\Delta} = \dim \mathcal{L}_{N,\Delta} = 1$, then there exist two nonzero vectors $\mathbf{c}$ and $\mathbf{d}$ in $\mathbb{C}^n$ such that $\mathbf{c}$ spans $\mathcal{L}_{M,\Delta}$ and $\mathbf{d}$ spans $\mathcal{L}_{N,\Delta}$. Moreover, $\mathbf{c}$ and $\mathbf{d}$ can be chosen carefully such that

$$\begin{pmatrix} f_0(\beta_1(w))\beta_1'(w) \\ \vdots \\ f_0(\beta_n(w))\beta_n'(w) \end{pmatrix} = f_0(\beta_1(w))\beta_1'(w)\mathbf{c}, \quad w \in \Delta,$$

$$\begin{pmatrix} g_0(\beta_1(w))\beta_1'(w) \\ \vdots \\ g_0(\beta_n(w))\beta_n'(w) \end{pmatrix} = g_0(\beta_1(w))\beta_1'(w)\mathbf{d}, \quad w \in \Delta.$$

Then by (3.7), we have on Δ

$$f_0(\beta_1)\beta_1' W\mathbf{c} = g_0(\beta_1)\beta_1'\mathbf{d},$$

and hence $f_0 = cg_0$ for a constant $c \neq 0$. So the reducing subspace $[f_0]$ generated by f_0 equals $[g_0]$. Since M is minimal, $M = [f_0]$ and similarly $N = [g_0]$, and hence

$M = N$, which is a contradiction. So M is not unitarily equivalent to N. Applying Theorem 3.3 yields $M \perp N$, as desired.

(ii) If $\dim M \ominus BM \neq \dim N \ominus BN$, then obviously M is not unitarily equivalent to N. Therefore by Theorem 3.3, $M \perp N$. The proof of Theorem 3.1 is complete. ∎

Finally, we will apply Theorem 3.3 to give a proof of Proposition 1.4. We restate it here.

PROPOSITION 3.9. *Let B be a finite Blaschke product of degree n. Then the following are equivalent:*

(i) *Conjecture* 1.1 *holds; that is, M_B has at most n minimal reducing subspaces.*

(ii) *$\mathcal{C}_B$ is abelian.*

(iii) *All minimal projections in $\mathcal{C}_B$ are orthogonal.*

Proof. (iii) $\Rightarrow$ (ii) follows from the fact that $\mathcal{C}_B$ is a von Neumann algebra spanned by its minimal projections.

(ii) $\Rightarrow$ (iii) follows from a simple fact: if $\mathcal{A}$ is an abelian von Neumann algebra and $\mathcal{A}$ has a minimal projection, then all minimal projections in $\mathcal{A}$ are orthogonal.

(iii) $\Rightarrow$ (i) Notice that every minimal reducing subspace is exactly the range of a minimal projection in $\mathcal{C}_B$, so (iii) implies all minimal reducing subspaces are orthogonal. Thus there are only finite minimal reducing subspaces, denoted by M_i $(0 \leqslant i \leqslant t)$, whose direct sum is the Bergman space. Notice that

$$L^2_a(\mathbb{D}) \ominus BL^2_a(\mathbb{D}) = \bigoplus_{i=0}^{t} M_i \ominus BM_i,$$

and hence $t \leqslant n$, where n is degree of B, as desired. One can also see [11].

(i) $\Rightarrow$ (iii) follows from Theorem 31 in [11]. Here we include a similar but shorter proof.

Suppose conversely that (iii) does not hold, and then there exist two minimal reducing subspaces M and N which are not orthogonal. By Theorem 3.3, M is unitarily equivalent to N; that is, there is a unitary operator U from M onto N commuting with M_B. Now for each $0 < a < 1$, set

$$M_a = \{f + aUf : f \in M\}.$$

It is easy to see that each M_a is a reducing subspace and is minimal since M is minimal. Moreover, it is not difficult to check that if $0 < a < a' < 1$, then $M_a \neq M_{a'}$. (Or see the proof of Theorem 31 in [11]). This is a contradiction to (i). ∎

4. IN THE CASE OF $\deg B = 5, 6$

In this section, we will give the proof of Theorem 1.8, that is restated here.

THEOREM 4.1. *Let B be a Blaschke product of degree n with $n = 5$ or 6, then M_B has at most n minimal reducing subspaces.*

Before proving Theorem 4.1, let us recall some preliminary facts. As in Section 2 in [23], let $B = \frac{P}{Q}$ be a Blaschke product of order n, where P and Q are two polynomials of degree n. Let

$$f(w,z) = P(w)Q(z) - P(z)Q(w),$$

and the Riemann surface S_B equals the locus S_f of solutions of the equation $f(w,z) = 0$ in $\mathbb{D}^2$. For details of this Riemann surface, see [23]. Set

$$F = \{z : \text{there exists } z' \text{ such that } B'(z') = 0 \text{ and } B(z') = B(z)\}$$

and it is known that F is a finite set contained in $\mathbb{D}$ (for example, see Bochner's theorem [24]). Suppose P is a polygon drawn through all points in F and a fixed point on the unit circle such that $\mathbb{D} - P$ is simply connected. By Theorem 12.3 in [6], the n distinct roots $w = \rho_k(z)$ $(1 \leqslant k \leqslant n)$ of the equation

$$f(z,w) = 0 \text{ (or equivalently, } B(w) - B(z) = 0)$$

are holomorphic functions in $\mathbb{D} - P$. Then we have the following.

LEMMA 4.2. *Suppose $B = z^2\varphi_\alpha\varphi_\beta\varphi_\gamma$ with $\alpha, \beta, \gamma \in \mathbb{D}$, then all ρ_k can be extended analytically to the unit disk if and only if $\alpha = \beta = \gamma = 0$.*

Proof. If $\alpha = \beta = \gamma = 0$, then clearly ρ_k has the form $\omega^k z$, where $\omega = \mathrm{e}^{\mathrm{i}2\pi/5}$.

Now assume $B = z^2\varphi_\alpha\varphi_\beta\varphi_\gamma$ and all ρ_k can be extended analytically to the unit disk. We will show $\alpha = \beta = \gamma = 0$. Since each ρ_k satisfies

$$B(\rho_k(z)) - B(z) = 0, \tag{4.1}$$

then $|\rho_k(z)| \to 1$ $(|z| \to 1)$ and hence all ρ_k are Blaschke products, of degree one. One of them is the identity map. There are two cases.

Case I. There is a k such that $\rho_k(z) = \bar{c}z$, where $|c| = 1$ and $c \neq 1$. Then from (4.1), we see

$$B(\bar{c}z) = B(z). \tag{4.2}$$

First we show that either $\alpha = \beta = \gamma = 0$ or $\alpha\beta\gamma \neq 0$. If not, then there are essentially two cases to discuss. If $\alpha \neq 0$ and $\beta = \gamma = 0$, then by (4.2), $c\alpha = \alpha$. So $\alpha = 0$, which is a contradiction. If $\alpha\beta \neq 0$ and $\gamma = 0$, then by comparing the zeros of two sides of (4.2) we get $c = -1$ and $\beta = -\alpha$. Then $B = z^3\varphi_{\alpha^2}(z^2)$ and $B(\bar{c}z) \neq B(z)$, which is a contradiction.

Next we will exclude the case $\alpha\beta\gamma \neq 0$ to finish the proof. Again by (4.2), $(c\alpha, c\beta, c\gamma)$ is a permutation of (α, β, γ). Without loss of generality, $\beta = ca$ and then $(c^2\alpha, c\gamma)$ is a permutation of (α, γ). Since $c \neq 1$, $c\gamma \neq \gamma$, so $c^2\alpha = \gamma$ and

$c\gamma = \alpha$. Therefore $c^3 = 1$. By a simple computation, we get $B(\bar{c}z) = \bar{c}^2 B(z) \neq B(z)$, which is a contradiction.

Case II. There is a k such that $\rho_k(z)$ is a Mobius map other than the identity map and $\rho_k(0) \neq 0$. Write $\rho_k = \phi$. By $B \circ \phi = B$, z^2 is a factor of $B \circ \phi$. So we can assume that $\alpha = \beta = \phi(0)$. Now $\alpha \neq 0$ is a zero of B with multiplicity $\geqslant 2$, and by $B \circ \phi = B$, α is also a zero of $B \circ \phi$ with multiplicity $\geqslant 2$. This implies $\phi(\alpha) = 0$. So there exists a constant ξ ($|\xi| = 1$) satisfying $\phi = \xi \varphi_\alpha$, and by $\alpha = \phi(0)$, we get $\phi = \varphi_\alpha$. By (4.1),

$$z^2 \varphi_\alpha^2 \varphi_\gamma \circ \varphi_\alpha = z^2 \varphi_\alpha^2 \varphi_\gamma.$$

So $\varphi_\gamma \circ \varphi_\alpha = \varphi_\gamma$, which is impossible. ∎

To prove Theorem 4.1, we also need the following lemma, which comes from Lemma 2 of [20]. However, Sun's proof reads difficult. We include a different proof for completeness.

LEMMA 4.3 ([20]). *Suppose B be a finite Blaschke product of degree n, and S is a unitary operator which commutes with M_B, then there exist constants c_k $(1 \leqslant k \leqslant n)$ satisfying $\sum_{k=1}^{n} |c_k|^2 = 1$ and*

$$Sh(w) = \sum_{k=1}^{n} c_k \rho_k'(w) h \circ \rho_k(w), \quad h \in L_a^2(\mathbb{D}),\, w \in \mathbb{D}. \tag{4.3}$$

REMARK 4.4. Let us make clear what (4.3) means. First, both sides of (4.3) are analytic in $\mathbb{D} - P$, so (4.3) holds for $w \in \mathbb{D} - P$; notice that the left side Sh of (4.3) is analytic in $\mathbb{D}$, so the right side of (4.3) can be extend analytically to $\mathbb{D}$. Therefore (4.3) holds on $\mathbb{D}$.

Proof. First we will prove Lemma 4.3 in the case that B has only simple zeros. Let $\beta_1, \ldots, \beta_n$ be n branches of B^{-1} and they are locally analytic. And regard $\rho_k(z)$ $(1 \leqslant k \leqslant n)$ as n different branches of $B^{-1} \circ B$; clearly, locally $(\rho_1(z), \ldots, \rho_n(z))$ is a permutation of $(\beta_1 \circ B, \ldots, \beta_n \circ B)$.

Fix β_1 and without loss of generality, we may assume that the set Δ in Proposition 3.4 is such that $\beta_1(\Delta) \subsetneqq \mathbb{D} - P$ (otherwise we may replace Δ with some open disk Δ' and 0 is not necessarily in Δ'). Also we can assume

$$\rho_i|_{\beta_1(\Delta)} = \beta_i \circ B|_{\beta_1(\Delta)}.$$

Then we have

$$\rho_i \circ \beta_1|_\Delta = \beta_i|_\Delta, \tag{4.4}$$

and hence

$$\rho_i' \circ \beta_1 \beta_1'|_\Delta = \beta_i'|_\Delta. \tag{4.5}$$

Since S commutes with M_B, by Proposition 3.4 there exists an $n \times n$ numerical unitary matrix W such that

$$W \begin{pmatrix} f(\beta_1(w))\beta_1'(w) \\ \vdots \\ f(\beta_n(w))\beta_n'(w) \end{pmatrix} = \begin{pmatrix} Sf(\beta_1(w))\beta_1'(w) \\ \vdots \\ Sf(\beta_n(w))\beta_n'(w) \end{pmatrix}, \quad f \in L_a^2(\mathbb{D}) \text{ and } w \in \Delta.$$

So there are n constants $c_1, \ldots, c_n$ satisfying $\sum\limits_{i=1}^{n} |c_i|^2 = 1$ and

$$Sf(\beta_1(w))\beta_1'(w) = \sum_{i=1}^{n} c_i f(\beta_i(w))\beta_i'(w), \quad f \in L_a^2(\mathbb{D}) \text{ and } w \in \Delta.$$

Therefore by (4.4) and (4.5), we have for each $f \in L_a^2(\mathbb{D})$

$$Sf(\beta_1(w))\beta_1'(w) = \sum_{i=1}^{n} c_i f(\rho_i \circ \beta_1(w))\rho_i' \circ \beta_1(w)\beta_1'(w), \quad w \in \Delta,$$

and thus

$$Sf(\beta_1(w)) = \sum_{i=1}^{n} c_i f(\rho_i \circ \beta_1(w))\rho_i' \circ \beta_1(w), \quad w \in \Delta.$$

That is, (4.3) holds on $\beta_1(\Delta)$. Moreover, since the two sides of (4.3) are analytic on the connected set $\mathbb{D} - P$, then (4.3) holds on $\mathbb{D} - P$. And by Remark 4.4, we have (4.3) on $\mathbb{D}$. Then the proof is complete in the case that B has only simple zeros.

In general, by Remark 3.6 we still have Proposition 3.4, but then Δ will be replaced with some other disk. Then the same argument as above shows that (4.3) holds. ∎

We have an immediate corollary.

COROLLARY 4.5. *Let $\mathcal{C}_B$ be the commutant of $\{M_B, M_B^*\}$, then* $\dim \mathcal{C}_B \leqslant n$.

Proof. Notice that Lemma 4.3 implies all unitary operators in $\mathcal{C}_B$ span a subspace of dimension $\leqslant n$. Since $\mathcal{C}_B$ is a von Neumann algebra containing the identity, and any von Neumann algebra is the finite linear span of its unitary operators [8], then we get $\dim \mathcal{C}_B \leqslant n$, as desired. ∎

Now we comes to the proof of Theorem 4.1.

Proof of Theorem 4.1. Let B be a finite Blaschke product and there are two cases under consideration: $\deg B = 5$ and $\deg B = 6$.

Case I. $\deg B = 5$. Recall that for any Blaschke product B of degree 5, there always exists an a and c in $\mathbb{D}$ such that $\varphi_a \circ B \circ \varphi_c = z^2 \varphi_\alpha \varphi_\beta \varphi_\gamma$ with $\alpha, \beta, \gamma \in \mathbb{D}$. So without loss of generality, assume $B = z^2 \varphi_\alpha \varphi_\beta \varphi_\gamma$ as above.

Observe that there is always an orthogonal decomposition of $L_a^2(\mathbb{D})$:

$$L_a^2(\mathbb{D}) = \bigoplus_{i=0}^{t} M_i, \tag{4.6}$$

where each M_i $(0 \leqslant i \leqslant t)$ is a minimal reducing subspace. Here and below M_0 always denotes the distinguished reducing subspace. Since (4.6) gives

$$L_a^2(\mathbb{D}) \ominus BL_a^2(\mathbb{D}) = \bigoplus_{i=0}^{t} M_i \ominus BM_i, \tag{4.7}$$

then we get

$$\sum_{i=0}^{t} \dim M_i \ominus BM_i = \deg B = 5. \tag{4.8}$$

The next discussion is based on (4.7) and (4.8). In fact, noting $\dim M_0 \ominus BM_0 = 1$, it suffices to consider the following cases:

(i) $t = 4$ and $\dim M_i \ominus BM_i = 1$ $(1 \leqslant i \leqslant 4)$;
(ii) $t = 3$, $\dim M_i \ominus BM_i = 1$ $(1 \leqslant i \leqslant 2)$ and $\dim M_3 \ominus BM_3 = 2$;
(iii) $t = 1$ and $\dim M_1 \ominus BM_1 = 4$;
(iv) $t = 2$, $\dim M_1 \ominus BM_1 = 1$ and $\dim M_2 \ominus BM_2 = 3$;
(v) $t = 2$ and $\dim M_1 \ominus BM_1 = \dim M_2 \ominus BM_2 = 2$.

Cases (i)–(iv) can be done by similar arguments as in the proof of Corollary 3.2. For example, let us deal with case (ii) and suppose conversely that there is some other minimal reducing subspace other than M_i $(0 \leqslant i \leqslant 3)$, say N. Since $\dim M_i \ominus BM_i = 1$ $(0 \leqslant i \leqslant 2)$, then by Theorem 3.1, N is orthogonal to M_i $(0 \leqslant i \leqslant 2)$. Therefore $N \subsetneqq M_3$, and hence by minimality, $N = M_3$. This is a contradiction. Case (i), (iii) and (iv) can be done similarly. The difficulty lies in case (v) and we will discuss this case in details. Below, we will consider case (v), and show that there is no minimal reducing subspace other than M_0, M_1 and M_2.

To this end, assume conversely that M is a minimal reducing subspace other than M_0, M_1 and M_2. Then M is orthogonal to M_0, and hence $M \subsetneqq M_1 \oplus M_2$. A simple application of Theorem 3.3 shows that M_1 and M_2 are unitarily equivalent. That is, there is a unitary U from M_1 onto M_2 which commutes with M_B. Now extend U to $\widetilde{U}$ such that $\widetilde{U}|_{M_1} = U$ and $\widetilde{U}|_{M_1^\perp} = 0$. Let P_j denote the orthogonal projection from $L_a^2(\mathbb{D})$ onto M_j $(j = 0, 1, 2)$, and it is easy to check that

$$P_0, P_1, P_2, \widetilde{U} \text{ and } \widetilde{U}^*$$

are linearly independent. Moreover, for any pair $(c_1, c_2) \in \mathbb{T}^2$,

$$P_0 + \sum_{i=1}^{2} c_i P_i \quad \text{and} \quad P_0 + c_1\widetilde{U} + c_2\widetilde{U}^*$$

are unitary operators which commute with M_B. And it is easy to see that all such unitary operators span a subspace of dimension 5, then by Lemma 4.3, it is not difficult to show that for each k $(1 \leqslant k \leqslant n)$,

$$h \to \rho_k'(w) h \circ \rho_k(w)$$

is a well defined map from $L_a^2(\mathbb{D})$ to $L_a^2(\mathbb{D})$. Precisely, by Remark 4.4, the function $\rho_k'(w) h \circ \rho_k(w)$ that is analytic in $\mathbb{D} - P$ can be analytically extended to $\mathbb{D}$ for each $h \in L_a^2(\mathbb{D})$. In particular, $\rho_k'(w)$ is analytic in $\mathbb{D}$ and hence ρ_k can be extend to an analytic function over $\mathbb{D}$. Then by Lemma 4.2, we get $B(z) = -z^5$. In this case, it is well known that M_B has precisely 5 minimal reducing subspaces and each one of them, say N, satisfies $\dim N \ominus BN = 1$, which is a contradiction.

Case II. $\deg B = 6$. Similarly, we can assume $B = z^2 \varphi_\alpha \varphi_\beta \varphi_\gamma \varphi_\delta$ with $\alpha, \beta, \gamma, \delta \in \mathbb{D}$. And by (4.7) and similar arguments as in the proof of Case I, it suffices to consider this case:

$$t = 3, \quad \dim M_1 \ominus BM_1 = 1 \quad \text{and} \quad \dim M_i \ominus BM_i = 2 \ (i = 2, 3).$$

And by careful verifications, one can established a similar version of Lemma 4.2, which will derive a contradiction if we assume there is some other minimal reducing subspace different from M_i $(0 \leqslant i \leqslant 3)$. The proof is just like that of case (v). Thus Theorem 4.1 also holds in this case. The proof of Theorem 4.1 is complete. ∎

To end this section, we will apply Corollary 4.5 to give another proof of Corollary 3.2. And by Proposition 3.9 (=Proposition 1.4), we restate it as the following form.

COROLLARY 4.6 ([25], [11], [23]). *Let B be a finite Blaschke product of degree n with $1 \leqslant n \leqslant 4$, then $\mathcal{C}_B$ is abelian, and hence M_B has at most n minimal reducing subspaces.*

Proof. As mentioned above, $\mathcal{C}_B$ is a von Neumann algebra, and by Corollary 4.5, $\dim \mathcal{C}_B \leqslant \deg B = n$. Notice Theorem III.1.2 in [9] states that *any finite dimensional von Neumann algebra is $*$-isomorphic to the direct sum of full matrix algebras*

$$\bigoplus_{k=1}^{r} M_{n_k}(\mathbb{C}).$$

So we assume $\mathcal{C}_B$ is $*$-isomorphic to

$$\bigoplus_{k=1}^{r} M_{n_k}(\mathbb{C}). \tag{4.9}$$

If $1 \leqslant n \leqslant 3$, then $\dim \mathcal{C}_B \leqslant n \leqslant 3$. Since $\dim M_j(\mathbb{C}) = j^2$, clearly all n_k in (4.9) equal one. Thus $\mathcal{C}_B$ is abelian.

If $n = 4$, then $\dim \mathcal{C}_B \leqslant 4$. To reach a contradiction, assume that $\mathcal{C}_B$ is not abelian. By $\dim \mathcal{C}_B \leqslant 4$, it is easy to see that $\mathcal{C}_B \cong M_2(\mathbb{C})$, whose center is trivial.

But by Theorem 3.1, the orthogonal projection P_0 onto the distinguished subspace M_0 is orthogonal to any other minimal projection in $\mathcal{C}_B$. And since all minimal projections span $\mathcal{C}_B$, P_0 belongs to the center of $\mathcal{C}_B$. This is a contradiction. The proof is complete. ∎

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