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GALOIS OBJECTS AND COCYCLE TWISTING FOR LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. In this article, we investigate the notion of a Galois object for a locally compact quantum group $\mathbb{G}$. Such an object consists of a von Neumann algebra N, together with an ergodic integrable action of $\mathbb{G}$ on N for which the crossed product is a type I factor. We show how to construct from this data a possibly different locally compact quantum group. By way of application, we prove the following statement: any twisting of a locally compact quantum group by a unitary 2-cocycle is again a locally compact quantum group.

KEYWORDS: *Locally compact quantum groups, Galois objects, 2-cocycles, projective representations.*

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INTRODUCTION

In commutative geometry, the importance of principal fiber bundles can hardly be overestimated. When passing to non-commutative geometry, they become even more intriguing: one can have interesting principal bundles over a point! In this article, we investigate this phenomenon in the framework of locally compact quantum groups.

In Hopf algebra theory, "non-commutative principal bundles" are known under the name "*faithfully flat Hopf–Galois extensions*". A *Hopf–Galois extension* consists of the following data: a Hopf algebra (H, Δ_H) (say over a field k), a unital k-algebra A, and a coaction $\alpha : A \to A \underset{k}{\otimes} H$. These have to satisfy the following property: with B the fixed point algebra of α, the map

$$G : A \underset{B}{\otimes} A \to A \underset{k}{\otimes} H : x \otimes y \to \alpha(x)(y \otimes 1), \tag{0.1}$$

called *the Galois map*, must be a bijection. Here the surjectivity corresponds geometrically to the freeness of the action, while the injectivity corresponds to the action being proper (actually, "to the action being Cartan" is the more accurate analogy). Saying that the Hopf Galois extension is *faithfully flat*, means that A is faithfully flat as a right B-module (in which case the injectivity of the map G comes for free). This corresponds to the local triviality of the bundle. Finally, if we want to have a fiber bundle over a point, we should ask that $B = k \cdot 1_A$ (the condition of being "faithfully flat" becoming obsolete). The couple (A, α) is then called a *(right) Galois object for* (H, Δ_H). See [20] for a nice overview of these concepts.

We now briefly indicate how the above definitions have to be adapted in the setting of locally compact quantum groups. We will work only in the von Neumann algebra framework. While this is certainly not sufficient to study "locally compact quantum principal fiber bundles", it turns out to be sufficient if one considers a bundle over a point (i.e., there is automatically a C^*-algebraic picture available). So let (M, Δ) be a von Neumann algebraic quantum group, which is to be interpreted as being $\mathcal{L}^\infty(\mathbb{G})$ for some locally compact quantum group $\mathbb{G}$ (see [14] and [27]). Let N be a von Neumann algebra, and $\alpha : N \to N \otimes M$ a right coaction. Denote by N^α the subalgebra of fixed points. The map α is called *integrable*, if the operator-valued weight $(\iota \otimes \varphi)\alpha$ from N to N^α is semi-finite, where φ denotes the left invariant nsf weight for (M, Δ). This is our non-commutative analogue of the action being proper. (In fact, one would also like to use the notion of integrability to define properness on the level of C^*-algebras, but the situation there is much more subtle, see e.g. [19].) When this condition is satisfied, one is able to construct an analogue of the Galois map (0.1) on the level of $\mathcal{L}^2$-spaces. It will automatically be isometric. When it is actually a unitary, then we call α a *Galois coaction*. Finally, when also $N^\alpha = \mathbb{C}$, i.e. when α is ergodic, we call (N, α) a Galois object. This turns out to be equivalent with the condition given in the abstract.

One reason which makes Galois objects so interesting, is that in general they carry with them not one, but two Hopf algebras: if (A, α) is a (right) Galois object for a Hopf algebra (H, Δ_H), then one can construct from this a *second* "reflected" Hopf algebra (L, Δ_L) and a (left) coaction γ of L on A, such that (A, γ) becomes a left Galois object, and such that γ and α commute. This turns out to be a (part of a) non-commutative generalization of the *Ehresmann construction*, where one lets a locally compact group act freely and properly on a locally compact space, and constructs from this a locally compact groupoid with an action on this same space, commuting with the group action (see e.g. [16], Example 1.1.5).

We show in this article that such a reflected quantum group also exists when dealing with Galois objects for locally compact quantum groups. While the new locally compact quantum group can be *constructed* more or less as on the algebraic level, there is one technical point which is much less straightforward to establish: namely, the construction gives a priori only a Hopf–von Neumann algebra, and one still has to see if there are invariant weights available. The existence of these weights is the main theorem of this paper (Theorem 5.10). In fact, we prefer an approach *dual* to the one in Hopf algebra theory, so we rather construct a von

Neumann algebraic quantum group $(\widehat{P}, \Delta_{\widehat{P}})$, whose dual then plays the role of (L, Δ_L).

An important corollary of our results, is that any cocycle twist of a locally compact quantum group is again a locally compact quantum group. I.e.: if Ω is a unitary 2-cocycle for a von Neumann algebraic quantum group $(\widehat{M}, \widehat{\Delta})$ (so $\Omega \in \widehat{M} \otimes \widehat{M}$ and $(\Omega \otimes 1)(\widehat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \widehat{\Delta})(\Omega)$), then one can show that the cocycle twisted convolution algebra $\widehat{M} \underset{\Omega}{\ltimes} \mathbb{C}$ together with its dual coaction constitutes a Galois object for (M, Δ), and the new von Neumann algebraic quantum group $(\widehat{P}, \Delta_{\widehat{P}})$ which we obtain is precisely $\widehat{M}$ itself with the new coproduct $\widehat{\Delta}_\Omega := \Omega\widehat{\Delta}(\cdot)\Omega^*$. We want to note that in [8], a special type of such cocycle deformations is discussed: a cocycle on a classical subobject, satisfying certain conditions, is lifted to the whole locally compact quantum group. In this case, more concrete formulas are available for describing the weights on the twisted locally compact quantum group.

As mentioned already, the theory of Galois objects is well-developed for Hopf algebras. It was also investigated for compact quantum groups in [2], which was in turn based on the work of Wassermann on ergodic actions of compact groups on von Neumann algebras ([29],[30]). We then investigated this notion for algebraic quantum groups in [3]. It can be shown that the $*$-Galois objects of [3] can be completed to analytic objects of the kind discussed in this paper (similar to the completion of $*$-algebraic quantum groups to locally compact quantum groups, as is done in [11]), although we have not included a detailed exposition of this fact in this paper.

The specific content of this paper is as follows: in the *first two sections*, we establish notation and preliminaries concerning operator valued weights. Our general references for this part are the first four chapters of [21] for the theory of non-commutative integration, Section 10 of [5] for some results about inclusions of von Neumann algebras.

In the third section, we treat the notion of a (right) *Galois coaction* for a von Neumann algebraic quantum group, as briefly explained above. This notion already appeared implicitly at various places in the literature, for it turns out to be equivalent with the following property: with α denoting the coaction of a locally compact quantum group (M, Δ) on a von Neumann algebra N, being Galois is the same as saying that $N \rtimes M$ can be represented *faithfully* on $\mathcal{L}^2(N)$ by a certain canonical map ρ. Our general references for this part are [14] and [27] for the theory of locally compact quantum groups in the von Neumann algebraic setting, and [22] for the theory of coactions for locally compact quantum groups (which are there just termed "actions").

In the fourth section, we study *Galois objects*, i.e. Galois coactions (N, α) for which α is *ergodic*. We show that a Galois object has as rich a structure as a locally compact quantum group: we can associate with N certain invariant weights, related by a modular element, and also a one-parameter scaling group. The references for this part are [14], [27] and [22].

In the fifth section, we use these results to construct a (possibly new) von Neumann algebraic quantum group $(\widehat{P}, \Delta_{\widehat{P}})$ from such a Galois object. The reference for this part is Section IX.3 of [21].

In the sixth section, we consider the special case of cocycle twisted locally compact quantum groups.

In the seventh section, we define the notion of a projective corepresentation for a locally compact quantum group, and we show the connection with coactions on type I-factors.

1. PRELIMINARIES AND NOTATION

The scalar product of a Hilbert space will be anti-linear in the second argument. If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, we denote by $B(\mathcal{H}, \mathcal{K})$ the Banach space of all bounded operators between $\mathcal{H}$ and $\mathcal{K}$, by $B(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$, and by $B_0(\mathcal{H})$ the algebra of all compact operators. If $\xi, \eta \in \mathcal{H}$, we write

$$\omega_{\xi,\eta} : B(\mathcal{H}) \to \mathbb{C} : x \to \langle x\xi, \eta \rangle.$$

If u is a unitary on $\mathcal{H}$, we will denote

$$\mathrm{Ad}(u) : B(\mathcal{H}) \to B(\mathcal{H}) : x \to uxu^*.$$

If $\mathcal{H}_1, \mathcal{H}_2$ are two Hilbert spaces, we will denote by Σ the flip map

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_1 : \xi \otimes \eta \to \eta \otimes \xi.$$

We will also frequently use leg numbering notation: if $\mathcal{H}_i$ are Hilbert spaces and

$$u : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3 \otimes \mathcal{H}_4$$

is an operator, we denote for example by u_{12} the operator

$$u \otimes 1 : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_5 \to \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5,$$

and by u_{13} the operator

$$\Sigma_{23} u_{12} \Sigma_{23} : \mathcal{H}_1 \otimes \mathcal{H}_5 \otimes \mathcal{H}_2 \to \mathcal{H}_3 \otimes \mathcal{H}_5 \otimes \mathcal{H}_4.$$

If u is already indexed, say $u = u_1$, then we write $u_{1,13}$ for u_{13}.

If N is a von Neumann algebra, we denote by N_* its predual. We denote by $\mathcal{L}^2(N)$ the universal Hilbert space for GNS-constructions. We denote the spatial tensor product of two von Neumann algebras by $\otimes$.

Let φ_N be a fixed normal semi-finite faithful (nsf) weight on N. We will then sometimes index the modular structure by N instead of φ_N (so the modular automorphism group for example is written as σ_t^N). We will then write the modular operator as ∇_N (since the symbol Δ will be used for the comultiplication of a quantum group). When we work with another weight ψ_N, we will then always use ψ_N as an index. We will always write $\mathcal{N}_{\varphi_N} = \{n \in N : \varphi_N(n^*n) < \infty\}$ for the space of square integrable elements for φ_N, we write $\mathcal{M}^+_{\varphi_N} = \{n \in N^+ : \varphi_N(n) < \infty\}$ for the space of positive integrable elements, and $\mathcal{M}_{\varphi_N} = \mathrm{span}\{\mathcal{M}^+_{\varphi_N}\} = \mathcal{N}^*_{\varphi_N}\mathcal{N}_{\varphi_N}$ for the space of integrable elements. The GNS map $\mathcal{N}_{\varphi_N} \to \mathcal{L}^2(N)$ for φ_N is denoted by Λ_N. We denote by $\mathcal{T}_{\varphi_N}$ the canonical Tomita algebra for φ_N (inside N):

$$\mathcal{T}_{\varphi_N} = \{x \in N : x \text{ analytic for } \sigma^N_t \text{ and } \sigma^N_z(x) \in \mathcal{N}_{\varphi_N} \text{ for all } z \in \mathbb{C}\}.$$

We then also call $\Lambda_N(\mathcal{T}_{\varphi_N})$ the Tomita algebra for φ_N (inside $\mathcal{L}^2(N)$).

The opposite weight of φ_N will be denoted by φ^{op}_N. We see it as a weight on the commutant $N' \subseteq B(\mathcal{L}^2(N))$. It has a natural GNS-construction in $\mathcal{L}^2(N)$: with J_N denoting the modular conjugation of φ_N, we have a GNS map

$$\Lambda^{\mathrm{op}}_N : \mathcal{N}_{\varphi^{\mathrm{op}}_N} \to \mathcal{L}^2(N) : J_N n^* J_N \to J_N \Lambda_N(n^*).$$

Sometimes however, we will also allow elements of N as input of Λ^{op}_N: then in fact we first identify N with the opposite von Neumann algebra N^{op} as a linear space (an operation we will write as $n \to n^{\mathrm{op}}$), and then we identify N^{op} with N' as a $*$-algebra by sending n^{op} to $J_N n^* J_N$. So for $n \in \mathcal{N}^*_{\varphi_N}$, we will also write $\Lambda^{\mathrm{op}}_N(n) = J_N\Lambda_N(n^*)$. This notation is consistent, since $J_N n^* J_N = n$ for elements in the center.

When N_1 and N_2 are two von Neumann algebras, and φ_{N_i} an nsf weight on N_i, we denote by $\varphi_{N_1} \otimes \varphi_{N_2}$ their tensor product (Definition 4.2 in [21]), which is an nsf weight on $N_1 \otimes N_2$. We denote its GNS-map with $\Lambda_{N_1} \otimes \Lambda_{N_2}$. One can show that

$$\varphi_{N_1} \otimes \varphi_{N_2} = \varphi_{N_1} \circ (\iota \otimes \varphi_{N_2}),$$

where $(\iota \otimes \varphi_{N_2})$ is an nsf operator valued weight from $N_1 \otimes N_2$ to N_1, defined as

$$\omega((\iota \otimes \varphi_2)(x)) := \varphi_2((\omega \otimes \iota)(x))$$

for $x \in (N_1 \otimes N_2)^+$ and $\omega \in (N_1 \otimes N_2)^+_*$. By symmetry, this gives us a Fubini theorem.

We recall the definition of the Connes–Sauvageot tensor product.

If $\mathcal{H}$ is a left N-module, by which we mean a Hilbert space carrying a unital normal representation π_{l} of N, and φ_N is a nsf weight on N, a vector $\xi \in \mathcal{H}$ is called *right bounded* with respect to φ_N if the map

$$\Lambda_N(\mathcal{N}_{\varphi_N}) \to \mathcal{H} : \Lambda_N(x) \to \pi_{\mathrm{l}}(x)\xi$$

is bounded, in which case we denote its closure by $R^{\pi_{\mathrm{l}},\varphi_N}(\xi)$ (or R_ξ if π_{l} and φ_N are fixed). We denote by ${}_{\varphi_N}\mathcal{H}$ the space of right bounded vectors for π_{l}. Similarly, if $\mathcal{H}$ is a right N-module, by which we mean a Hilbert space carrying a unital normal anti-representation π_{r} of N, a vector $\xi \in \mathcal{H}$ is called *left bounded* with respect to φ_N if the map

$$\Lambda^{\mathrm{op}}_N(\mathcal{N}^*_{\varphi_N}) \to \mathcal{H} : J_N\Lambda_N(x^*) \to \pi_{\mathrm{r}}(x)\xi$$

is bounded, in which case we denote its closure by $L^{\pi_{\mathrm{r}},\varphi_N}(\xi)$ (or L_ξ if π_{r} and φ_N are fixed). We denote by $\mathcal{H}_{\varphi_N}$ the space of left bounded vectors for π_{r}. Remark that when we regard $\mathcal{H}$ as a left N^{op}-module in the natural way, then the right bounded vectors with respect to φ_N^{op} are exactly the left bounded vectors with respect to φ_N.

If $(\mathcal{H}^{\mathrm{r}}, \pi_{\mathrm{r}})$ is a faithful right N-module, $(\mathcal{H}^{\mathrm{l}}, \pi_{\mathrm{l}})$ a faithful left N-module, and φ_N a nsf weight on N, we denote by $\mathcal{H}^{\mathrm{r}} \underset{\varphi_N}{{}_{\pi_{\mathrm{r}}}\otimes_{\pi_{\mathrm{l}}}} \mathcal{H}^{\mathrm{l}}$ (or simply $\mathcal{H}^{\mathrm{r}} \underset{\varphi_N}{\otimes} \mathcal{H}^{\mathrm{l}}$ when $\pi_{\mathrm{l}}, \pi_{\mathrm{r}}$ are clear) their Connes–Sauvageot tensor product with respect to $\pi_{\mathrm{l}}, \pi_{\mathrm{r}}$ and φ_N. It is the Hilbert space closure of the algebraic tensor product of $\mathcal{H}^{\mathrm{r}}_{\varphi_N}$ and $\mathcal{H}^{\mathrm{l}}$ with respect to the scalar product

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \pi_{\mathrm{l}}(L_{\eta_1}^* L_{\xi_1})\xi_2, \eta_2 \rangle,$$

modulo vectors of norm zero. In fact, we could as well start with the algebraic tensor product of $\mathcal{H}^{\mathrm{r}}_{\varphi_N}$ and ${}_{\varphi_N}\mathcal{H}^{\mathrm{l}}$, since the image of this tensor product in the previous Hilbert space will be dense. On elementary tensors of the last space, we can give a different form of the scalar product, namely

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \pi_{\mathrm{r}}(R_{\eta_2}^* R_{\xi_2})\xi_1, \eta_1 \rangle.$$

The image of such an elementary tensor in $\mathcal{H}^{\mathrm{r}} \underset{\varphi_N}{\otimes} \mathcal{H}^{\mathrm{l}}$ will then be denoted by the same symbol, with $\otimes$ replaced by $\underset{\varphi_N}{{}_{\pi_{\mathrm{r}}}\otimes_{\pi_{\mathrm{l}}}}$ or simply $\underset{\varphi_N}{\otimes}$.

Note that these spaces carry faithful normal left representations π_{r}' and π_{l}' of respectively $\pi_{\mathrm{r}}(N)'$ and $\pi_{\mathrm{l}}(N)'$, determined by

$$\pi_{\mathrm{r}}'(n_1)\pi_{\mathrm{l}}'(n_2)(\xi_1 \underset{\varphi_N}{\otimes} \xi_2) = (n_1\xi_1) \underset{\varphi_N}{\otimes} (n_2\xi_2),$$

where $n_1 \in \pi_{\mathrm{r}}(N)', n_2 \in \pi_{\mathrm{l}}(N)', \xi_1 \in \mathcal{H}^{\mathrm{r}}_{\varphi_N}, \xi_2 \in {}_{\varphi_N}\mathcal{H}^{\mathrm{l}}$. If $N_1 \subseteq B(\mathcal{H}^{\mathrm{r}})$ is a von Neumann algebra containing $\pi_{\mathrm{r}}(N)$, and $N_2 \subseteq B(\mathcal{H}^{\mathrm{l}})$ is a von Neumann algebra containing $\pi_{\mathrm{l}}(N)$, the von Neumann algebra

$$N_1 \underset{N}{{}_{\pi_{\mathrm{r}}}*_{\pi_{\mathrm{l}}}} N_2 := (\pi_{\mathrm{r}}'(N_1') \cup \pi_{\mathrm{l}}'(N_2'))'$$

is called the *fiber product* of N_1 and N_2. As an abstract von Neumann algebra, it only depends on N, N_1, N_2 and the maps $\pi_{\mathrm{r}} : N \to N_1$ and $\pi_{\mathrm{l}} : N \to N_2$. For further properties of the fiber product, see [7].

We will also need the notion of *intertwiners* and a *linking algebra*. Suppose we are given two right N-modules $(\mathcal{H}_2, \pi_{\mathrm{r},2})$ and $(\mathcal{H}_1, \pi_{\mathrm{r},1})$. Denote

$$Q_{ij} = \{x \in B(\mathcal{H}_j, \mathcal{H}_i) : x\pi_{\mathrm{r},j}(n) = \pi_{\mathrm{r},i}(n)x \text{ for all } n \in N\}.$$

We call Q_{12} the space of *intertwiners* between the right N-modules $(\mathcal{H}_2, \pi_{\mathrm{r},2})$ and $(\mathcal{H}_1, \pi_{\mathrm{r},1})$. In fact, it is a self-dual Q_{11}-Q_{22}-Hilbert W^*-bimodule (see [18]). The *linking algebra* between $(\mathcal{H}_2, \pi_{\mathrm{r},2})$ and $(\mathcal{H}_1, \pi_{\mathrm{r},1})$ is the von Neumann algebra

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix},$$

acting on $\binom{\mathcal{H}_1}{\mathcal{H}_2} = \mathcal{H}_1 \oplus \mathcal{H}_2$ in the obvious way. It is the commutant of the direct sum right representation $\pi_{\mathrm{r},1} \oplus \pi_{\mathrm{r},2}$. Most of the time, we will identify the Q_{ij} as subspaces of Q, indexing the units of the Q_{ii} then to emphasize that we consider them as projections in Q. If θ_1 is a weight on Q_{11} and θ_2 a weight on Q_{22}, the *balanced weight* $\theta_1 \oplus \theta_2$ is the weight $Q^+ \to [0, +\infty] : \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \to \theta_1(x) + \theta_2(w)$.

We now briefly recall the definition of a locally compact quantum group, mainly to fix notation.

Let M be a von Neumann algebra, Δ a faithful normal unital $*$-homomorphism $M \to M \otimes M$ which satisfies coassociativity:

$$(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta,$$

where ι denotes the identity map. Then the pair (M, Δ) is called a *Hopf–von Neumann algebra*. A Hoph–von Neumann algebra is called *coinvolutive* if there exists an anti-multiplicative $*$-involution $R : M \to M$ such that

$$\Delta \circ R = (R \otimes R) \circ \Delta^{\mathrm{op}},$$

where $\Delta^{\mathrm{op}} = \mathrm{Ad}(\Sigma) \circ \Delta$. A Hopf–von Neumann algebra is called a *von Neumann algebraic quantum group* if there exist nsf weights φ and ψ on M such that

$$(\iota \otimes \varphi) \circ \Delta = \varphi, \quad (\psi \otimes \iota) \circ \Delta = \psi.$$

These identities should be interpreted as follows: for any $\omega \in M_*^+$, the weight $\psi \circ (\iota \otimes \omega)\Delta$ should equal the weight $\omega(1)\psi$, and similarly for φ. (These are in fact the *strong forms* of invariance, and they follow from weaker ones (see Proposition 3.1 of [14]).) A von Neumann algebraic quantum group will automatically be a coinvolutive Hopf–von Neumann algebra for a canonical map R. As we mentioned already in the introduction, the terminology "locally compact quantum group" is a formal one used to guide intuition: one sometimes writes (M, Δ) as $(\mathcal{L}^\infty(\mathbb{G}), \Delta)$ and refers to the symbol $\mathbb{G}$ as the *locally compact quantum group associated to* (M, Δ).

We refer to [14] and [27] for further definitions and formulas. We shall also use notations as in those papers. Specifically, we denote by φ a (fixed) left invariant nsf weight, by S the antipode, by τ_t the one-parameter scaling group and by R the unitary antipode (so that $S = R \circ \tau_{-\mathrm{i}/2}$). We scale the right invariant weight ψ such that $\psi = \varphi \circ R$. We establish the GNS-constructions for φ in the standard form $\mathcal{L}^2(M)$, writing just Λ for the GNS-map associated with φ. We follow the convention of [14] by taking a GNS-construction Λ_δ for ψ in $\mathcal{L}^2(N)$ by defining $\Lambda_\delta(x) := \Lambda(x\delta^{1/2})$ for $x \in M$ a left multiplier of the square root of the modular element δ *such that* $x\delta^{1/2} \in \mathcal{N}_\varphi$, and then closing Λ_δ. If ν is the scaling constant of (M, Δ), then Λ_δ and Λ_ψ are related by $\Lambda_\delta = \nu^{\mathrm{i}/8}\Lambda_\psi$. The modular one-parameter group for φ is denoted simply by σ_t, and its corresponding modular operator by ∇. The modular one-parameter group for ψ is denoted by $\sigma'_t = \mathrm{Ad}(\delta^{it}) \circ \sigma_t$, its modular operator by $\nabla\!\!\!/$. The canonical (self-dual) one-parameter group of unitaries implementing the scaling group will be denoted by P^{it}.

We denote the dual von Neumann algebraic quantum group by $(\widehat{M}, \widehat{\Delta})$, and also all its other structures are denoted as for (M, Δ), but with a $\widehat{\ }$ on top. By W and $\widehat{W} = \Sigma W^* \Sigma$ we denote the left regular corepresentation of respectively (M, Δ) and $(\widehat{M}, \widehat{\Delta})$. We write V and $\widehat{V}$ for the right regular corepresentation of respectively (M, Δ) and $(\widehat{M}, \widehat{\Delta})$. We will also from time to time work with the commutant von Neumann algebraic quantum groups (M', Δ') and $(\widehat{M}', \widehat{\Delta}')$. For the relationship between all these quantum groups, we refer again to [14].

For most of the paper, we will work with a fixed von Neumann algebraic quantum group (M, Δ). When (P, Δ_P) is the von Neumann algebraic realization of another locally compact quantum group, we will use the same notations but with a subscript P. Note that we also use the symbol P for the scaling operator, since this is standard notation, but in any case, there should not arise any occasion where a von Neumann algebra could get mixed up with an operator!

2. FURTHER PRELIMINARIES ON OPERATOR VALUED WEIGHTS

We collect in this preliminary section some results about operator valued weights. While they are well-known to specialists, we have chosen to present them here in considerable detail, as we do not know a convenient reference for the specific results we need.

Let $N_0 \subseteq N$ be a unital inclusion of von Neumann algebras, and T a normal semi-finite faithful operator valued weight from N^+ to the positive extended cone $(N_0)^{+,\mathrm{ext}}$ of N_0. Let μ be a fixed nsf weight on N_0, and denote by φ_N the nsf weight $\mu \circ T$. Denote the semi-cyclic representation associated with φ_N by $(\mathcal{L}^2(N), \Lambda_N, \pi_{\mathrm{l}})$, realized in the standard form. Most of the time, we will write n instead of $\pi_{\mathrm{l}}(n)$ for $n \in N$. Denote by π_{r} (or π_{r}^N for emphasis) the anti-representation $n \to J_N n^* J_N$ of N, where J_N is the modular conjugation. Denote $N_2 = \pi_{\mathrm{r}}(N_0)'$, then

$$N_0 \subseteq N \subseteq N_2$$

is called the basic construction. We will also use π_{l} for the natural representation of N_2 on $\mathcal{L}^2(N)$, and θ_{r} for the natural anti-representation

$$\theta_{\mathrm{r}} : N_2 \to B(\mathcal{L}^2(N)) : \theta(x) = J_N x^* J_N.$$

Note that this will of course *not* turn $\mathcal{L}^2(N)$ into a N_2-N_2-bimodule in general.

Consider $x \in \mathcal{N}_T = \{n \in N : T(n^*n) \text{ is bounded}\}$. Then $xn \in \mathcal{N}_{\varphi_N}$ when $n \in \mathcal{N}_\mu$, and $\Lambda_\mu(n) \to \Lambda_N(xn)$ extends from $\Lambda_\mu(\mathcal{N}_\mu)$ to a bounded operator

$$\Lambda_T(x) : \mathcal{L}^2(N_0) \to \mathcal{L}^2(N),$$

following the notations of Theorem 10.6 of [5]. Its adjoint is then determined by $\Lambda_T(x)^*\Lambda_N(y) = \Lambda_\mu(T(x^*y))$ for $y \in \mathcal{N}_{\varphi_N} \cap \mathcal{N}_T$. The operators of the form $\Lambda_T(x)\Lambda_T(y)^*$, with $x, y \in \mathcal{N}_T$, will generate a σ-weakly-dense sub-$*$-algebra of N_2, and if we denote by T_2 the canonical operator valued weight from N_2 onto N associated with T, then we have, as follows from Theorem 10.7 of [5],

$$\Lambda_T(x)\Lambda_T(y)^* \in \mathcal{M}_{T_2} := \mathcal{N}_{T_2}^*\mathcal{N}_{T_2}$$

with

$$T_2(\Lambda_T(x)\Lambda_T(y)^*) = xy^*.$$

Consider now $\mathcal{L}^2(N)$ as an N_2-N_0-bimodule, and denote by $\overline{\mathcal{L}^2(N)}$ the conjugate bimodule. Then it is well-known that there is a unitary N_2-N_2-bimodule map

$$\mathcal{L}^2(N) \underset{\mu}{\otimes} \overline{\mathcal{L}^2(N)} \to \mathcal{L}^2(N_2) : \Lambda_N(x) \underset{\mu}{\otimes} \overline{\Lambda_N^{\mathrm{op}}(y)} \to \Lambda_{\varphi_2}(\Lambda_T(x)\Lambda_T(y)^*)$$

for $x, y \in \mathcal{N}_{\varphi_N} \cap \mathcal{N}_{\varphi_N}^* \cap \mathcal{N}_T$, where $\varphi_2 = \varphi_N \circ T_2$. As said, since we will need some more information about this statement, of which we know no appropriate reference in the literature, we will give a proof of it.

We first prove a lemma about interchanging the analytic continuation of a modular one-parameter group with an operator valued weight.

LEMMA 2.1. *Let Q be the linking algebra between the right N_0-modules $\mathcal{L}^2(N)$ and $\mathcal{L}^2(N_0)$, and consider the balanced weight $\varphi_2 \oplus \mu$ on Q. Let $x \in N$ be such that x is analytic for σ_t^N and $\sigma_z^N(x) \in \mathcal{N}_T$ for all $z \in \mathbb{C}$. Then $\Lambda_T(x)$ is analytic for σ_t^Q, with $\sigma_z^Q(\Lambda_T(x)) = \Lambda_T(\sigma_z^N(x))$ for all $z \in \mathbb{C}$.*

Recall that the notion of a linking algebra between two right von Neumann modules, and the notion of balanced weight, were given in the section on preliminaries, of which we also use the notation.

Proof. First remark that $\Lambda_T(x) \in Q_{12}$ by Lemma 10.6.(i) of [5]. Choose $y \in \mathcal{N}_\mu$ and $u, v \in \mathcal{N}_{\varphi_N}$ with v in the Tomita algebra $\mathcal{T}_{\varphi_N} \subseteq N$ for φ_N. Denote

$$f(z) = \langle \Lambda_T(\sigma_z^N(x))\Lambda_\mu(y), J_N\sigma_{\mathrm{i}/2}^N(v)J_N\Lambda_N(u)\rangle, \quad \text{for } z \in \mathbb{C}.$$

Then

$$f(z) = \langle J_N\sigma_{\mathrm{i}/2}^N(v)^*J_N\Lambda_N(\sigma_z^N(x)y), \Lambda_N(u)\rangle = \langle \sigma_z^N(x)\Lambda_N(yv), \Lambda_N(u)\rangle,$$

and so f is analytic. Moreover, if $z = r + \mathrm{i}s$ with $r, s \in \mathbb{R}$, then since $\sigma_t^\mu = (\sigma_t^N)_{:N_0}$,

$$\begin{aligned} |f(z)| &= |\langle \sigma_{\mathrm{i}s}^N(x)\nabla_N^{-\mathrm{i}r}\Lambda_N(yv), \nabla_N^{-\mathrm{i}r}\Lambda_N(u)\rangle| \\ &= |\langle \Lambda_N(\sigma_{\mathrm{i}s}^N(x)\sigma_{-r}^\mu(y)), J_N\sigma_{\mathrm{i}/2}^N(\sigma_{-r}^N(v))J_N\Lambda_N(\sigma_{-r}^N(u))\rangle| \\ &= |\langle \Lambda_T(\sigma_{\mathrm{i}s}^N(x))\nabla_\mu^{-\mathrm{i}r}\Lambda_\mu(y), \nabla_N^{-\mathrm{i}r}J_N\sigma_{\mathrm{i}/2}^N(v)J_N\Lambda_N(u)\rangle|, \end{aligned}$$

and so we can conclude, by the Phragmén–Lindelöf principle, that the modulus of f is bounded on every horizontal strip by $M_x\|\omega\|$, where

$$\omega = \omega_{\Lambda_\mu(y),J_N\sigma^N_{i/2}(v)J_N\Lambda_N(u)} \in B(\mathcal{L}^2(N_0),\mathcal{L}^2(N))_*,$$

and M_x is a number depending only on x and the chosen strip. The same is of course true for linear combinations of such ω, and since these span a dense subspace of $B(\mathcal{L}^2(N_0),\mathcal{L}^2(N))_*$, we get that $z \to \Lambda_T(\sigma_z^N(x))$ is bounded on compact sets. But then this function is analytic (for example by condition A.1.(iii) in the appendix of [21]). Since σ_t^Q is implemented by $\nabla_N^{it}\oplus\nabla_\mu^{it}$ and since we have that $\nabla_N^{it}\Lambda_T(x)\nabla_\mu^{-it} = \Lambda_T(\sigma_t^N(x))$, the result follows. ∎

We can now provide a convenient Tomita algebra for φ_2. Let $\mathcal{T}_{\varphi_N}\subseteq N$ be the Tomita algebra for φ_N, and denote

$$\mathcal{T}_{\varphi_N,T} = \{x\in\mathcal{T}_{\varphi_N}\cap\mathcal{N}_T\cap\mathcal{N}_T^* : \sigma_z^N(x)\in\mathcal{N}_T\cap\mathcal{N}_T^* \text{ for all } z\in\mathbb{C}\}.$$

(This space is called the Tomita algebra for φ_N and T in Proposition 2.2.1 of [4].) Denote the linear span of $\{\Lambda_T(x)\Lambda_T(y)^* : x,y\in\mathcal{T}_{\varphi_N,T}\}$ by $\mathfrak{A}_2$, and further denote by $(\mathcal{L}^2(N_2),\Lambda_{N_2},\pi_1^{N_2})$ the natural semi-cyclic representation for φ_2.

PROPOSITION 2.2. *We have $\mathfrak{A}_2\subseteq\mathcal{D}(\Lambda_{N_2})$, and $\mathfrak{A}_2$ is a Tomita algebra for (N_2,φ_2).*

By the second statement, we mean that $\Lambda_{N_2}(\mathfrak{A}_2)$ is a sub-Tomita algebra of the natural Tomita algebra $\Lambda_{N_2}(\mathcal{T}_{\varphi_2})$ for φ_2, closed in $\mathcal{L}^2(N_2)$, which still has N_2 as its left von Neumann algebra, and also with the corresponding weight on N_2 coinciding with φ_2.

Proof. For $x,y\in\mathcal{T}_{\varphi_N,T}$, we know that $\Lambda_T(x)\Lambda_T(y)^*\in\mathcal{M}_{T_2}$, with

$$T_2(\Lambda_T(x)\Lambda_T(y)^*) = xy^*.$$

Since $x,y\in\mathcal{T}_{\varphi_N}$, also $xy^*\in\mathcal{M}_{\varphi_N}$. Hence $\mathfrak{A}_2\subseteq\mathcal{M}_{\varphi_2}$, and so certainly $\mathfrak{A}_2\subseteq\mathcal{D}(\Lambda_{N_2})$.

It is clear that $\mathfrak{A}_2$ is closed under the $*$-involution. Now choose $x,y,u,v\in\mathcal{T}_{\varphi_N,T}$. Then

$$(\Lambda_T(u)\Lambda_T(v)^*)(\Lambda_T(x)\Lambda_T(y)^*) = \Lambda_T(uT(v^*x))\Lambda_T(y)^*.$$

We want to show that $uT(v^*x)\in\mathcal{T}_{\varphi_N,T}$. It is clear that

$$uT(v^*x)\in\mathcal{N}_{\varphi_N}^*\cap\mathcal{N}_T\cap\mathcal{N}_T^*.$$

By the previous lemma, we have, using notation as there, that $\Lambda_T(v)$ and $\Lambda_T(x)$ are analytic for σ_t^Q, with

$$\sigma_z^Q(\Lambda_T(a)) = \Lambda_T(\sigma_z^N(a)) \quad \text{for all } z\in\mathbb{C}, a\in\{u,v\}.$$

But then also $\Lambda_T(v)^*\Lambda_T(x) = T(v^*x)$ analytic for σ_t^Q, with

$$\sigma_z^Q(T(v^*x)) = T(\sigma_{\bar{z}}^N(v)^*\sigma_z^N(x)) \quad \text{for all } z\in\mathbb{C}.$$

Since σ_t^Q restricts to σ_t^μ on N_0, and also σ_t^N restricts to σ_t^μ on N_0, we get that $uT(v^*x)$ is analytic for σ_t^N, with

$$\sigma_z^N(uT(v^*x)) = \sigma_z^N(u)T(\sigma_{\bar{z}}^N(v)^*\sigma_z^N(x)) \quad \text{for all } z \in \mathbb{C}.$$

Since $\mathcal{T}_{\varphi_N,T}$ is invariant under all σ_z^N with $z \in \mathbb{C}$, we get that

$$\sigma_z^N(uT(v^*x)) \in \mathcal{N}_{\varphi_N}^* \cap \mathcal{N}_T \cap \mathcal{N}_T^* \quad \text{for all } z \in \mathbb{C}.$$

Hence $uT(v^*x) \in \mathcal{T}_{\varphi_N,T}$, and thus

$$(\Lambda_T(u)\Lambda_T(v)^*)(\Lambda_T(x)\Lambda_T(y)^*) \in \mathfrak{A}_2.$$

We have shown so far that $\Lambda_{N_2}(\mathfrak{A}_2) \subseteq \Lambda_{N_2}(\mathcal{N}_{\varphi_2} \cap \mathcal{N}_{\varphi_2}^*)$ is a sub-left Hilbert algebra. But by the previous lemma, $\mathfrak{A}_2$ consists of analytic elements for σ_t^Q, which restricts to $\sigma_t^{N_2}$ on N_2. So in fact $\Lambda_{N_2}(\mathfrak{A}_2)$ is a sub-Tomita algebra of $\Lambda_{N_2}(\mathcal{T}_{\varphi_2})$.

Now we show that $\mathfrak{A}_2$ is σ-weakly dense in N_2. For this, it is enough to show that $\Lambda_T(\mathcal{T}_{\varphi_N,T})$ is strongly dense in Q_{12}. Note that $\Lambda_T(\mathcal{T}_{\varphi_N,T})$ is closed under right multiplication with elements from $\mathcal{T}_\mu \subseteq N_0$, which are σ-weakly dense in N_0. Then by a similar argument as in the proof of Theorem 10.6.(ii) of [5], it is sufficient to prove that if $z \in Q_{12}$ and $z^*\Lambda_T(x) = 0$ for all $x \in \mathcal{T}_{\varphi_N,T}$, then $z = 0$. So suppose z satisfies this condition. Choose $y \in \mathcal{N}_\mu$ analytic for σ_t^μ. Then

$$\pi_{\mathrm{r}}^{N_0}(\sigma_{\mathrm{i}/2}^\mu(y))z^*\Lambda_N(x) = z^*\pi_{\mathrm{r}}^N(\sigma_{\mathrm{i}/2}^N(y))\Lambda_N(x) = z^*\Lambda_N(xy) = z^*\Lambda_T(x)\Lambda_\mu(y) = 0.$$

Letting $\pi_{\mathrm{r}}^{N_0}(\sigma_{\mathrm{i}/2}^\mu(y))$ tend to 1, we see that z^* vanishes on $\Lambda_N(\mathcal{T}_{\varphi_N,T})$. Now choose $x \in \mathcal{M}_{\varphi_N} \cap \mathcal{M}_T$. Then

$$x_n = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-nt^2}\sigma_t^N(x)\mathrm{d}t$$

is in $\mathcal{T}_{\varphi_N,T}$ by Lemma 10.12 of [5], and $\Lambda_N(x_n)$ converges to $\Lambda_N(x)$. Hence z^* vanishes also on $\Lambda_N(\mathcal{M}_{\varphi_N} \cap \mathcal{M}_T)$. Since $\mathcal{N}_{\varphi_N} \cap \mathcal{N}_T$ is weakly dense in N and $\Lambda_N(\mathcal{N}_{\varphi_N} \cap \mathcal{N}_T)$ is normdense in $\mathcal{L}^2(N)$, we get that $z^* = 0$, and the density claim follows.

Now let $\mathcal{G}$ be the closure of $\Lambda_{N_2}(\mathfrak{A}_2)$. Then for $x \in N_2$, we get that $\pi_{\mathrm{l}}^{N_2}(x)$ will restrict to an operator $\pi_{\mathrm{l}}^{\mathfrak{A}_2}(x) : \mathcal{G} \to \mathcal{G}$, since $\mathfrak{A}_2$ is dense in N_2. Then the left von Neumann algebra associated with $\Lambda_{N_2}(\mathfrak{A}_2)$ is $\pi_{\mathrm{l}}^{\mathfrak{A}_2}(N_2)$. If we denote by $\varphi_{1(1/2)}$ the weight on $\pi_{\mathrm{l}}^{\mathfrak{A}_2}(N_2)$ associated with $\Lambda_{N_2}(\mathfrak{A}_2)$, then it is clear that φ_2, the weight $(\varphi_{1(1/2)} \circ \pi_{\mathrm{l}}^{\mathfrak{A}_2})$ and $\mathfrak{A}_2$ satisfy the conditions of Proposition VIII.3.15 of [21], hence $\varphi_2 = \varphi_{1(1/2)} \circ \pi_{\mathrm{l}}^{\mathfrak{A}_2}$, which finishes the proof. ∎

REMARK 2.3. It also follows easily from Lemma 10.12 of [5] that $\mathcal{T}_{\varphi_N,T}$ itself is σ-weakly dense in N.

Let $\mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N)$ denote the Connes–Sauvageot tensor product, with its natural N_2-N_2-bimodule structure. Denote by $\mathcal{K}$ the natural image of the algebraic tensor product $\Lambda_N(\mathcal{T}_{\varphi_N,T}) \odot \Lambda_N(\mathcal{T}_{\varphi_N,T})$ inside $\mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N)$.

THEOREM 2.4. *The space $\mathcal{K}$ is dense in $\mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N)$, and the map*

$$\mathcal{K} \to \mathcal{L}^2(N_2) : \Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y) \to \Lambda_{N_2}(\Lambda_T(x)\Lambda_T(y^*)^*)$$

extends to a unitary equivalence of N_2-N_2-bimodules.

Proof. First note that the expression on the left is well-defined by Theorem 10.6.(v) of [5], and then by definition, we have for $x, y, z, w \in \mathcal{T}_{\varphi_N,T}$ that

$$\begin{aligned}\langle \Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y), \Lambda_N(z) \underset{\mu}{\otimes} \Lambda_N(w)\rangle &= \langle (\Lambda_T(z)^*\Lambda_T(x))\Lambda_N(y), \Lambda_N(w)\rangle \\ &= \varphi_N(w^*T(z^*x)y) \\ &= \langle \Lambda_{N_2}(\Lambda_T(x)\Lambda_T(y^*)^*), \Lambda_{N_2}(\Lambda_T(z)\Lambda_T(w^*)^*)\rangle,\end{aligned}$$

so that the given map extends to a well-defined partial isometry. Since $\Lambda_N(\mathcal{T}_{\varphi_N,T})$ is dense in $\mathcal{L}^2(N)$ (which was proven in the course of the previous proposition), we have that $\mathcal{K}$ is dense in $\mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N)$. Since also $\Lambda_{N_2}(\mathfrak{A}_2)$ is dense in $\mathcal{L}^2(N_2)$, the extension is in fact a unitary.

The fact that it is a bimodule map follows from a straightforward computation (since we only have to check the bimodule property for operators in $\mathfrak{A}_2$ and vectors in $\mathcal{K}$ and $\Lambda_{N_2}(\mathfrak{A}_2)$). ∎

REMARK 2.5. If we identify $\mathcal{L}^2(N)$ with $\overline{\mathcal{L}^2(N)}$ as an N_0-N_2-bimodule by the unitary $\Lambda_N(y) \to \overline{\Lambda_N^{\mathrm{op}}(y^*)}$, we get the isomorphism $\mathcal{L}^2(N) \underset{\mu}{\otimes} \overline{\mathcal{L}^2(N)} \to \mathcal{L}^2(N_2)$ mentioned before. In some sense, this is a more natural unitary, but in our specific setting, the former one is easier to work with.

In the following, we will hence identify $\mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N)$ and $\mathcal{L}^2(N_2)$ in this manner.

LEMMA 2.6. *Let x, y be elements of $\mathcal{T}_{\varphi_N,T}$, and let p be an element of $\mathcal{N}_{\varphi_2}$. Then*

$$\langle \Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y), \Lambda_{N_2}(p)\rangle = \langle \Lambda_N(x), p\Lambda_N(\sigma^N_{-\mathrm{i}}(y^*))\rangle.$$

Conversely, if $p \in N_2$ and $\xi \in \mathcal{L}^2(N_2)$ are such that

$$\langle \Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y), \xi\rangle = \langle \Lambda_N(x), p\Lambda_N(\sigma^N_{-\mathrm{i}}(y^*))\rangle$$

for all $x, y \in \mathcal{T}_{\varphi_N,T}$, then $p \in \mathcal{N}_{\varphi_2}$ and $\Lambda_{N_2}(p) = \xi$.

Proof. Suppose $p = \Lambda_T(z)\Lambda_T(w^*)^*$ for some $z, w \in \mathcal{T}_{\varphi_N,T}$. Then since $w^*T(z^*x) \in \mathcal{N}_{\varphi_N} \cap \mathcal{N}^*_{\varphi_N}$, we have

$$\begin{aligned}\langle \Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y), \Lambda_{N_2}(p)\rangle &= \langle \Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y), \Lambda_N(z) \underset{\mu}{\otimes} \Lambda_N(w)\rangle \\ &= \varphi_N(w^*T(z^*x)y) = \varphi_N(\sigma^N_i(y)w^*T(z^*x)) \\ &= \langle \Lambda_N(w^*T(z^*x)), \Lambda_N(\sigma^N_{-\mathrm{i}}(y^*))\rangle \\ &= \langle \Lambda_T(w^*)\Lambda_T(z)^*\Lambda_N(x), \Lambda_N(\sigma^N_{-\mathrm{i}}(y^*))\rangle \\ &= \langle \Lambda_N(x), p\Lambda_N(\sigma^N_{-\mathrm{i}}(y^*))\rangle.\end{aligned}$$

As $\mathfrak{A}_2$, being a left Hilbert algebra for φ_2, is a strong-norm core for Λ_{N_2}, the result holds true for any $p \in \mathcal{N}_{\varphi_2}$.

Now we prove the converse statement. So let $p \in N_2$ and $\xi \in \mathcal{L}^2(N_2)$ be such that

$$\langle \Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y), \xi\rangle = \langle \Lambda_N(x), p\Lambda_N(\sigma^N_{-\mathrm{i}}(y^*))\rangle$$

for all $x, y \in \mathcal{T}_{\varphi_N,T}$. Then, since $\mathfrak{A}_2^{\mathrm{op}}$ is a strong-norm core for $\Lambda^{\mathrm{op}}_{N_2}$, it is enough to prove that $p\Lambda^{\mathrm{op}}_{N_2}(a) = \pi_{\mathrm{r}}^{N_2}(a)\xi$ for all $a \in \mathfrak{A}_2$. Now if $a = \Lambda_T(x)\Lambda_T(y^*)^*$, then

$$\Lambda^{\mathrm{op}}_{N_2}(a) = J_{N_2}\Lambda_{N_2}(a^*) = \Lambda_N(\sigma^N_{-\mathrm{i}/2}(x)) \underset{\mu}{\otimes} \Lambda_N(\sigma^N_{-\mathrm{i}/2}(y)).$$

So if also $b \in \mathfrak{A}_2$ with $b = \Lambda_T(z)\Lambda_T(w^*)^*$, $w, z \in \mathcal{T}_{\varphi_N,T}$, then

$$\langle \Lambda_{N_2}(b), p\Lambda^{\mathrm{op}}_{N_2}(a)\rangle = \langle \Lambda_N(z), p\Lambda_T(\sigma^N_{-\mathrm{i}/2}(x))\Lambda_T(\sigma^N_{-\mathrm{i}/2}(y)^*)^*\Lambda_N(\sigma^N_{-\mathrm{i}}(w^*))\rangle$$

by the first part of the lemma. On the other hand, we have

$$\begin{aligned}\langle \Lambda_{N_2}(b), \pi_{\mathrm{r}}^{N_2}(a)\xi\rangle &= \langle \pi_{\mathrm{r}}^{N_2}(a)^*\Lambda_{N_2}(b), \xi\rangle = \langle \Lambda_{N_2}(b\sigma^{N_2}_{\mathrm{i}/2}(a)^*), \xi\rangle \\ &= \langle \Lambda_{N_2}(\Lambda_T(z)\Lambda_T(w^*)^*\Lambda_T(\sigma^N_{\mathrm{i}/2}(y)^*)\Lambda_T(\sigma^N_{\mathrm{i}/2}(x))^*), \xi\rangle \\ &= \langle \Lambda_{N_2}(\Lambda_T(z)\Lambda_T(\sigma^N_{\mathrm{i}/2}(x)T(\sigma^N_{\mathrm{i}/2}(y)w^*))^*), \xi\rangle \\ &= \langle \Lambda_N(z), p\Lambda_N(\sigma^N_{-\mathrm{i}/2}(x)T(\sigma^N_{-\mathrm{i}/2}(y)\sigma^N_{-\mathrm{i}}(w^*)))\rangle,\end{aligned}$$

which equals our earlier expression, hence proving $p\Lambda^{\mathrm{op}}_{N_2}(a) = \pi_{\mathrm{r}}^{N_2}(a)\xi$ for all $a \in \mathfrak{A}_2$. ∎

We prove two further results which naturally belong in this section, but of which only the second one will be further used in the present paper.

LEMMA 2.7. *Let $N_0 \subseteq N$ be a unital inclusion of von Neumann algebras, let $T : N \to N_0$ an* nsf *operator valued weight, μ an* nsf *weight on N_0, and let φ_N be the composed* nsf *weight $\mu \circ T$. Suppose $x \in N$ and $z \in B(\mathcal{L}^2(N_0), \mathcal{L}^2(N))$ are such, that for any $y \in \mathcal{N}_\mu$, we have $xy \in \mathcal{N}_{\varphi_N}$ and $\Lambda_N(xy) = z\Lambda_\mu(y)$. Then $x \in \mathcal{N}_T$ with $\Lambda_T(x) = z$.*

Proof. Choose $y, w \in \mathcal{N}_\mu$ with w in the Tomita algebra of μ. Then

$$\pi_{\mathrm{r}}(w)z\Lambda_\mu(y)=\pi_{\mathrm{r}}(w)\Lambda_N(xy)=\Lambda_N(xy\sigma^N_{-\mathrm{i}/2}(w))=z\Lambda_\mu(y\sigma^\mu_{-\mathrm{i}/2}(w))=z\pi_{\mathrm{r}}^{N_0}(w)\Lambda_\mu(y),$$

so that z is a right N_0-module map. It follows that $z^*z \in N_0$.

Now by Lemma 4.7 of [21], there exists a closed positive (possibly unbounded) operator A, such that $\Lambda_\mu(y) \in \mathcal{D}(A)$ and

$$\omega_{\Lambda_\mu(y),\Lambda_\mu(y)}(T(x^*x)) = \langle A\Lambda_\mu(y), A\Lambda_\mu(y)\rangle.$$

Also, since for any element $u \in N_0^{+,\mathrm{ext}}$, one can find a sequence $u_n \in N_0^+$ such that $u_n \nearrow u$ pointwise on $(N_0)_*^+$ (see the proof of Proposition 4.17(ii) in [21]), we get that $\omega_{\Lambda_\mu(y),\Lambda_\mu(y)}(T(x^*x)) = \mu(y^*T(x^*x)y)$, using Corollary 4.9 of [21] (which allows us to extend weights to the extended positive cone). Using the bimodularity of T, we get

$$\begin{aligned}\langle A\Lambda_\mu(y), A\Lambda_\mu(y)\rangle &= \omega_{\Lambda_\mu(y),\Lambda_\mu(y)}(T(x^*x)) = \mu(y^*T(x^*x)y) = \mu(T(y^*x^*xy)) \\ &= \langle \Lambda_N(xy), \Lambda_N(xy)\rangle = \langle z\Lambda_\mu(y), z\Lambda_\mu(y)\rangle,\end{aligned}$$

from which we conclude that A is bounded. Hence $T(x^*x)$ is bounded, and then of course $\Lambda_T(x) = z$ follows. ∎

LEMMA 2.8. *Let* $\begin{matrix} N_{10} & \subseteq & N_{11} \\ \cup\!| & & \cup\!| \\ N_{00} & \subseteq & N_{01}\end{matrix}$ *be unital normal inclusions of von Neumann algebras. Denote, for $i \in \{0,1\}$, by Q_i the linking algebra between the right N_{i0}-modules $\mathcal{L}^2(N_{i0})$ and $\mathcal{L}^2(N_{i1})$. Suppose T_1 is an* nsf *operator valued weight $N_{11}^+ \to N_{10}^{+,\mathrm{ext}}$ whose restriction T_0 to N_{01}^+ is an* nsf *operator valued weight $N_{01}^+ \to N_{00}^{+,\mathrm{ext}}$. Then there is a natural normal embedding of Q_0 into Q_1, determined by $\Lambda_{T_0}(x) \to \Lambda_{T_1}(x)$ for $x \in \mathcal{N}_{T_0}$.*

REMARK 2.9. The inclusion will in general *not* be unital. Consider for example the case where $N_{11} = M_2(\mathbb{C})$ and all other algebras equal to $\mathbb{C}$.

Proof. By assumption, if $x, y \in \mathcal{N}_{T_0}$, then $x, y \in \mathcal{N}_{T_1}$, and $T_0(x^*y) = T_1(x^*y)$. Denote by $\widetilde{\mathcal{Q}}_1$ the $*$-algebra generated by the $\Lambda_{T_1}(x)$, $x \in \mathcal{N}_{T_0}$, and by $\widetilde{Q}_1$ its σ-weak closure. Denote by $\mathcal{Q}_0$ the $*$-algebra generated by the $\Lambda_{T_0}(x)$, $x \in \mathcal{N}_{T_0}$. We want to show that Q_0 and $\widetilde{Q}_1$ are isomorphic in the indicated way.

Now for $a_i, b_i \in \mathcal{N}_{T_0}$, it is easy to check that

$$\sum_i \Lambda_{T_1}(a_i)\Lambda_{T_1}(b_i)^* = 0 \quad \text{if and only if} \quad \sum_i \Lambda_{T_0}(a_i)\Lambda_{T_0}(b_i)^* = 0,$$

so we already have an isomorphism F at the level of $\mathcal{Q}_0$ and $\widetilde{\mathcal{Q}}_1$. Denote by e_0 the unit of N_{00}, seen as a projection in Q_0, and denote by e_1 the unit of N_{00} as a projection in $\widetilde{Q}_1$. Suppose that x_i is a bounded net in $\mathcal{Q}_0$ which converges to 0 in the σ-weak topology. Then for any $a, b \in \mathcal{Q}_0$, we have that $e_0ax_ibe_0$ converges to 0 σ-weakly. Applying F, we get that $e_1F(a)F(x_i)F(b)e_1$ converges σ-weakly to 0, and then also $ce_1F(a)F(x_i)F(b)e_1d$, for any $c, d \in \widetilde{Q}_1$. Since $\widetilde{Q}_1e_1\widetilde{Q}_1$ is σ-weakly dense in $\widetilde{Q}_1$, we get that $F(x_i)$ converges σ-weakly to 0. Since the same argument applies to F^{-1}, we see that F extends to a $*$-isomorphism between Q_0 and $\widetilde{Q}_1$, and we are done. ∎

REMARK 2.10. We could also have used the results from [18] concerning self-dual Hilbert W^*-modules to prove this lemma.

3. GALOIS COACTIONS

Let (M, Δ) be the von Neumann algebraic realization of a locally compact quantum group. Let N be a von Neumann algebra equipped with a right coaction α of (M, Δ), by which we mean a faithful normal unital $*$-homomorphism

$$\alpha : N \to N \otimes M$$

such that the coaction property is satisfied:

$$(\iota \otimes \Delta)\alpha = (\alpha \otimes \iota)\alpha.$$

Denote by N^α the von Neumann algebra of coinvariants:

$$N^\alpha = \{x \in N : \alpha(x) = x \otimes 1\}.$$

In this paper, we will only work with integrable coactions, so the normal faithful operator valued weight

$$T = (\iota \otimes \varphi)\alpha : N^+ \to (N^\alpha)^{+,\mathrm{ext}},$$

where φ is the left invariant weight for (M, Δ), is assumed to be semi-finite ([22], Proposition 1.3 and Definition 1.4). Let μ be a fixed nsf weight on N^α, and denote by φ_N the nsf weight $\mu \circ T$. It will be δ-invariant (see Definition III.1 of [6] and Definition 2.3 of [22]). With the exception that N_0 is now written N^α, we will use notation as in the previous section.

Recall from Theorem 5.3 of [22] that the integrability of α is equivalent with the existence of a canonical map

$$\rho : N \rtimes M \to B(\mathcal{L}^2(N)),$$

which we will explicitly write down later on. Here $N \rtimes M$ denotes the crossed product of N with respect to the coaction α, i.e.

$$N \rtimes M = (\alpha(N) \cup (1 \otimes \widehat{M}'))'' \subseteq B(\mathcal{L}^2(N) \otimes \mathcal{L}^2(M)).$$

We can also consider the map

$$\mathcal{K} \to \mathcal{L}^2(N) \otimes \mathcal{L}^2(M) : \Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y) \to (\Lambda_N \otimes \Lambda)(\alpha(x)(y \otimes 1))$$

for $x, y \in \mathcal{T}_{\varphi_N,T}$, where $\mathcal{K}$ was introduced just before Theorem 2.4, and $\mathcal{T}_{\varphi_N,T}$ just before Proposition 2.2. Then this is easily seen to be a well-defined isometry. Denote its extension by

$$G : \mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N) \to \mathcal{L}(N) \otimes \mathcal{L}^2(M).$$

The main goal of this section is to prove

THEOREM 3.1. *The map ρ is faithful if and only if G is a unitary.*

The result will follow from the following set of lemmas and propositions, which conclude with Lemma 3.6.

Consider the dual weight $\varphi_{N\rtimes M}$ of φ_N on $N \rtimes M$ ([22], Definition 3.1). Then there is a natural semi-cyclic representation $(\mathcal{L}^2(N) \otimes \mathcal{L}^2(M), \Lambda_{N\rtimes M}, \pi_1^{N\rtimes M})$ for $\varphi_{N\rtimes M}$, determined by

$$\Lambda_{N\rtimes M}((1 \otimes m)\alpha(x)) = \Lambda_N(x) \otimes \widehat{\Lambda}^{\mathrm{op}}(m)$$

for $x \in \mathcal{N}_{\varphi_N}$ and $m \in \mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}}$. Most of the time, we will suppress the symbol $\pi_1^{N\rtimes M}$. Note that we use here the results of [22], adapted to the setting of right coactions.

Denote by $U \in B(\mathcal{L}^2(N)) \otimes M$ the *unitary implementation* of α (i.e., the unitary implementation for α^{op} in the sense of [22], with its legs interchanged). By Proposition 4.3 and Theorem 4.4 of [22], it can be defined as

$$U = J_{N\rtimes M}(J_N \otimes \widehat{J}),$$

with $J_{N\rtimes M}$ the modular conjugation of the dual weight $\varphi_{N\rtimes M}$, as well as by the formula

$$(\iota \otimes \omega_{\xi,\eta})(U)\Lambda_N(z) = \Lambda_N((\iota \otimes \omega_{\delta^{-1/2}\xi,\eta})\alpha(z)), \tag{3.1}$$

where $\xi, \eta \in \mathcal{L}^2(M)$ with $\xi \in \mathcal{D}(\delta^{-1/2})$, $z \in \mathcal{N}_{\varphi_N}$. The surjective normal $*$-homomorphism ρ from $N \rtimes M$ to N_2 mentioned before is then given on the generators of $N \rtimes M$ by

$$\begin{cases} \rho(\alpha(x)) = \pi_1(x) & \text{for } x \in N, \\ \rho(1 \otimes (\iota \otimes \omega)(V)) = (\iota \otimes \omega)(U) & \text{for } \omega \in M_*, \end{cases}$$

where we recall that π_1 is just the standard representation for N, that V is the right regular multiplicative unitary for (M, Δ), and that N_2 is the von Neumann algebra in the basic construction $N^\alpha \subseteq N \subseteq N_2$.

In the following proposition, we also use the associated basic construction for the weight T, i.e. $N^\alpha \overset{T}{\subseteq} N \overset{T_2}{\subseteq} N_2$ denotes the basic construction obtained from $N^\alpha \overset{T}{\subseteq} N$, as explained in the beginning of the previous section. We also denote again $\varphi_2 = \varphi_N \circ T_2$.

PROPOSITION 3.2. *If $m \in \mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}}$ and $z \in \mathcal{N}_{\varphi_N}$, then $\rho((1 \otimes m)\alpha(z)) \in \mathcal{N}_{\varphi_2}$ and*

$$G^*(\Lambda_N(z) \otimes \widehat{\Lambda}^{\mathrm{op}}(m)) = \Lambda_{N_2}(\rho((1 \otimes m)\alpha(z))).$$

Proof. Choose $m \in \mathcal{N}_{\widehat{\varphi}^{\rm op}}$ of the form $(\iota \otimes \omega)(V)$, with ω such that the functional $x \to \overline{\omega(S(x)^*)}$ on $\mathcal{D}(S)$ extends to a normal functional ω^* on M, and such that moreover the functional $x \to \omega^*(x\delta^{-1/2})$ on the set of left multipliers of $\delta^{-1/2}$ in M extends to a normal functional ω^*_δ on M. Then, since $(\iota \otimes \omega)(V)^* = (\iota \otimes \omega^*)(V)$, we have, for $x, y \in \mathcal{T}_{\varphi_N,T}$ and $z \in \mathcal{N}_{\varphi_N}$, that

$$\begin{aligned}\langle \Lambda_N(x), \rho(1\otimes m)z\Lambda_N(\sigma^N_{-\mathrm{i}}(y^*))\rangle &= \langle \Lambda_N((\iota\otimes\omega^*_\delta)(\alpha(x))), \Lambda_N(z\sigma^N_{-\mathrm{i}}(y^*))\rangle \\ &= \varphi_N(\sigma^N_{\mathrm{i}}(y)z^*(\iota\otimes\omega^*_\delta)(\alpha(x))) \\ &= \varphi_N(z^*(\iota\otimes\omega^*_\delta)(\alpha(x))y).\end{aligned}$$

But for $a \in \mathcal{N}_\varphi$, we have $\langle \Lambda(a), \widehat{\Lambda}^{\rm op}(m)\rangle = \omega^*_\delta(a)$, so the final expression equals

$$\langle G(\Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y)), \Lambda_N(z)\otimes\widehat{\Lambda}^{\rm op}(m)\rangle.$$

Since such m form a strong-norm core for $\widehat{\Lambda}^{\rm op}$ (by standard smoothing arguments), we have

$$\langle \Lambda_N(x), \rho(1\otimes m)z\Lambda_N(\sigma^N_{-\mathrm{i}}(y^*))\rangle = \langle G(\Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y)), \Lambda_N(z)\otimes\widehat{\Lambda}^{\rm op}(m)\rangle,$$

for all $m \in \mathcal{N}_{\widehat{\varphi}^{\rm op}}$. By Lemma 2.6, we then get $\rho((1\otimes m)\alpha(z)) \in \mathcal{N}_{\varphi_2}$ and

$$\Lambda_{N_2}(\rho((1\otimes m)\alpha(z))) = G^*(\Lambda_N(z)\otimes\widehat{\Lambda}^{\rm op}(m))$$

for all $m \in \mathcal{N}_{\widehat{\varphi}^{\rm op}}$ and $z \in \mathcal{N}_{\varphi_N}$. ∎

LEMMA 3.3. *The map G is a left $N \rtimes M$-module morphism.*

Proof. Denoting again by $\pi_1^{N_2}$ the natural representation of N_2 on $\mathcal{L}^2(N)$, it is easy to see that $G\pi_1^{N_2}(x) = \alpha(x)G$ for all $x \in \mathcal{T}_{\varphi_N,T}$, hence this is true for all $x \in N$. Further, if $m \in \widehat{M}'$, $n \in \mathcal{N}_{\widehat{\varphi}^{\rm op}}$ and $z \in \mathcal{N}_{\varphi_N}$, then $\rho((1\otimes mn)\alpha(z)) \in \mathcal{N}_{\varphi_2}$ by the previous lemma, and we have

$$\pi_1^{N_2}(\rho(1\otimes m))G^*(\Lambda_N(z)\otimes\widehat{\Lambda}^{\rm op}(n)) = \Lambda_{N_2}(\rho((1\otimes mn)\alpha(z))) = G^*(\Lambda_N(z)\otimes\widehat{\Lambda}^{\rm op}(mn)),$$

hence $G\pi_1^{N_2}(\rho(1\otimes m)) = (1\otimes m)G$ for all $m \in \widehat{M}'$. Since $N \rtimes M$ is generated by $1 \otimes \widehat{M}'$ and $\alpha(N)$, the lemma is proven. ∎

REMARK 3.4. This implies that $\pi_1^{N_2}(\rho(x)) = G^*xG$ for $x \in N \rtimes M$, as G is an isometry.

LEMMA 3.5. *The following commutation relations hold:*

(i) $\nabla^{\mathrm{i}t}_{N\rtimes M}G = G\nabla^{\mathrm{i}t}_{N_2}$;

(ii) $J_{N\rtimes M}G = GJ_{N_2}$.

Here $\nabla_{N\rtimes M}$ denotes the modular operator for $\varphi_{N\rtimes M}$.

Proof. By the earlier identification of $\mathcal{L}^2(N_2)$ with $\mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N)$, it's easy to see that

$$\nabla_{N_2}^{\mathrm{i}t}(\Lambda_N(x) \underset{\mu}{\otimes} \Lambda_N(y)) = \Lambda_N(\sigma_t^N(x)) \underset{\mu}{\otimes} \Lambda_N(\sigma_t^N(y))$$

for $x, y \in \mathcal{T}_{\varphi_N,T}$, so for the first commutation relation, we must show that for all $x, y \in \mathcal{T}_{\varphi_N,T}$, we have

$$\nabla_{N \rtimes M}^{\mathrm{i}t}((\Lambda_N \otimes \Lambda)(\alpha(x)(y \otimes 1))) = (\Lambda_N \otimes \Lambda)(\alpha(\sigma_t^N(x))(\sigma_t^N(y) \otimes 1)).$$

Define the one-parameter group κ_t on M by $\kappa_t(a) = \delta^{-\mathrm{i}t}\tau_{-t}(a)\delta^{\mathrm{i}t}$ for $a \in M$. As in the proof of Proposition 4.3 in [22], one can show that

$$\nabla_{N \rtimes M}^{\mathrm{i}t} = \nabla_N^{\mathrm{i}t} \otimes q^{\mathrm{i}t},$$

where $q^{\mathrm{i}t}\Lambda(a) = \Lambda(\kappa_t(a))$ for $a \in \mathcal{N}_\varphi$. Since $\sigma_t^{\varphi_{N \rtimes M}} \circ \alpha = \alpha \circ \sigma_t^N$ by Proposition 3.7.2 of [22], we have for $x, y \in \mathcal{T}_{\varphi_N,T}$ and $\xi \in \mathcal{L}^2(M)$ that

$$\nabla_{N \rtimes M}^{\mathrm{i}t}(\alpha(x)(\Lambda_N(y) \otimes \xi)) = \alpha(\sigma_t^N(x))(\Lambda_N(\sigma_t^N(y)) \otimes q^{\mathrm{i}t}\xi).$$

Now let $a \in \mathcal{N}_\varphi$ be analytic for σ_t. Since σ_t commutes with κ_t, we have that $\kappa_t(a)$ is then also analytic for σ_t, with $\sigma_z(\kappa_t(a)) = \kappa_t(\sigma_z(a))$ for $t \in \mathbb{R}, z \in \mathbb{C}$. Hence for such a, and $x, y \in \mathcal{T}_{\varphi_N,T}$, we get

$$\begin{aligned}
\nabla_{N \rtimes M}^{\mathrm{i}t}&(1 \otimes J\sigma_{\mathrm{i}/2}(a)^* J)((\Lambda_N \otimes \Lambda)(\alpha(x)(y \otimes 1))) \\
&= \nabla_{N \rtimes M}^{\mathrm{i}t}(\Lambda_N \otimes \Lambda)(\alpha(x)(y \otimes a)) \\
&= (\Lambda_N \otimes \Lambda)(\alpha(\sigma_t^N(x))(\sigma_t^N(y) \otimes \kappa_t(a))) \\
&= (1 \otimes J\kappa_t(\sigma_{\mathrm{i}/2}(a))^* J)(\Lambda_N \otimes \Lambda)(\alpha(\sigma_t^N(x))(\sigma_t^N(y) \otimes 1)),
\end{aligned}$$

and letting $\sigma_{\mathrm{i}/2}(a)$ tend to 1, we see that

$$\nabla_{N \rtimes M}^{\mathrm{i}t}((\Lambda_N \otimes \Lambda)(\alpha(x)(y \otimes 1))) = (\Lambda_N \otimes \Lambda)(\alpha(\sigma_t^N(x))(\sigma_t^N(y) \otimes 1)),$$

which proves the first commutation relation.

It follows that $G^*\nabla_{N \rtimes M}^{1/2}$ equals the restriction of $\nabla_{N_2}^{1/2}G^*$ to $\mathcal{D}(\nabla_{N \rtimes M}^{1/2})$. Denote $t_{N \rtimes M} = J_{N \rtimes M}\nabla_{N \rtimes M}^{1/2}$ and $t_{N_2} = J_{N_2}\nabla_{N_2}^{1/2}$. Then

$$t_{N_2}G^* = J_{N_2}G^*\nabla_{N \rtimes M}^{1/2} \quad \text{on} \quad \mathcal{D}(\nabla_{N \rtimes M}^{1/2}).$$

So to prove that $G^*J_{N \rtimes M} = J_{N_2}G^*$, we only have to find a subset

$$K \subseteq \mathcal{D}(\nabla_{N \rtimes M}^{1/2}) = \mathcal{D}(t_{N \rtimes M})$$

whose image under $\nabla_{N \rtimes M}^{1/2}$ (or $t_{N \rtimes M}$) is dense in $\mathcal{L}^2(N \rtimes M)$, and on which $t_{N_2}G^*$ and $G^*t_{N \rtimes M}$ agree. But take

$$K = \mathrm{span}\{\alpha(x)\Lambda_{N \rtimes M}((1 \otimes m)\alpha(y)) : x, y \in \mathcal{T}_{\varphi_N,T}, m \in \mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}} \cap \mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}}^*\}.$$

Then clearly $K \subseteq \mathcal{D}(t_{N \rtimes M})$ and $t_{N \rtimes M}(K) = K$, since

$$t_{N \rtimes M}(\alpha(x)\Lambda_{N \rtimes M}((1 \otimes m)\alpha(y))) = \alpha(y^*)\Lambda_{N \rtimes M}((1 \otimes m^*)\alpha(x^*)).$$

Furthermore, if $x, y \in \mathcal{T}_{\varphi_N, T}$ and $m \in \mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}} \cap \mathcal{N}^*_{\widehat{\varphi}^{\mathrm{op}}}$, we get from Proposition 3.2 and Lemma 3.3 that $\rho(\alpha(x)(1 \otimes m)(\alpha(y)))$ and $\rho(\alpha(y^*)(1 \otimes m^*)(\alpha(x^*)))$ are both in $\mathcal{D}(\Lambda_{N_2})$, and that $G^*\alpha(x)\Lambda_{N \rtimes M}((1 \otimes m)\alpha(y)) \in \mathcal{D}(t_{N_2})$, with

$$\begin{aligned} t_{N_2} G^* \alpha(x) \Lambda_{N \rtimes M}((1 \otimes m)\alpha(y)) &= t_{N_2} \Lambda_{N_2}(\rho(\alpha(x)(1 \otimes m)\alpha(y))) \\ &= \Lambda_{N_2}(\rho(\alpha(y^*)(1 \otimes m^*)\alpha(x^*))) \\ &= G^* \alpha(y^*) \Lambda_{N \rtimes M}((1 \otimes m^*)\alpha(x^*)) \\ &= G^* t_{N \rtimes M} \alpha(x) \Lambda_{N \rtimes M}((1 \otimes m)\alpha(y)). \end{aligned}$$

Since K is dense in $\mathcal{L}^2(N \rtimes M)$, the second commutation relation is proven. ∎

Denote by p the central projection in $N \rtimes M$ such that $\ker(\rho) = (1 - p)(N \rtimes M)$. Denote by ρ_p the restriction of $\rho : N \rtimes M \to N_2$ to $p(N \rtimes M)$, and by $\widetilde{\varphi}_2$ the nsf weight $\varphi_{N \rtimes M} \circ \rho_p^{-1}$ on N_2.

LEMMA 3.6. *The projection GG^* equals p.*

Proof. By Lemma 3.3, G is a left $N \rtimes M$-module morphism, hence $GG^* \in (N \rtimes M)'$, and $GG^* \leqslant p$ since $G^*pG = \rho(p) = 1$. By the previous lemma, GG^* commutes with $J_{N \rtimes M}$, hence GG^* is in the center $\mathcal{Z}(N \rtimes M)$. Since $\rho(GG^*) = G^*(GG^*)G = 1$, we must have $GG^* = p$. ∎

As mentioned, Theorem 3.1 follows immediately from this, since G is unitary if and only if $p = 1$ if and only if ρ is faithful.

PROPOSITION 3.7. *The weight $\widetilde{\varphi}_2$ equals φ_2.*

Proof. If $m \in \mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}}$ and $z \in \mathcal{N}_{\varphi_N}$, then $\rho((1 \otimes m)\alpha(z)) \in \mathcal{N}_{\widetilde{\varphi}_2}$, and we can make a GNS-map $\Lambda_{\widetilde{\varphi}_2}$ for $\widetilde{\varphi}_2$ in $p(\mathcal{L}^2(N) \otimes \mathcal{L}^2(M))$ by

$$\Lambda_{\widetilde{\varphi}_2}(\rho((1 \otimes m)\alpha(z))) = p(\Lambda_{N \rtimes M}((1 \otimes m)\alpha(z))) = p(\Lambda_N(z) \otimes \widehat{\Lambda}^{\mathrm{op}}(m)),$$

since by the results of [22], the linear span of the $(1 \otimes m)\alpha(z)$ forms a σ-strong$*$-norm core for $\Lambda_{N \rtimes M}$. By Proposition 3.2 and the previous lemma,

$$\Lambda_{\widetilde{\varphi}_2}(\rho((1 \otimes m)\alpha(z))) = G(\Lambda_{\varphi_2}(\rho((1 \otimes m)\alpha(z)))).$$

Since G is a left $N \rtimes M$-module map, we obtain that also $(\mathcal{L}^2(N_2), G^* \circ \Lambda_{\widetilde{\varphi}_2}, \pi_1^{N_2})$ is a GNS-construction for $\widetilde{\varphi}_2$, and that $(G^* \circ \Lambda_{\widetilde{\varphi}_2}) \subseteq \Lambda_{\varphi_2}$.

By the first commutation relation of Lemma 3.5, it also follows that the modular operators for the GNS-constructions Λ_{φ_2} and $G^* \circ \Lambda_{\widetilde{\varphi}_2}$ are the same. Hence $\varphi_2 = \widetilde{\varphi}_2$ by Proposition VIII.3.16 of [21]. ∎

REMARK 3.8. This implies that T_2 equals $T_{N \rtimes M} \circ \rho_p^{-1}$ with $T_{N \rtimes M}$ the canonical operator valued weight $N \rtimes M \to N$, by Theorem IX.4.18 of [21]. Note that this result was obtained in Proposition 5.7 of [22] under the hypothesis that ρ was faithful.

It follows from Proposition 3.7 that G^* coincides with the map

$$Z : \mathcal{L}^2(N \rtimes M) \to \mathcal{L}^2(N_2) : \Lambda_{N \rtimes M}(z) \to \Lambda_{\widetilde{\varphi}_2}(\rho(z)), \quad z \in \mathcal{N}_{\varphi_{N \rtimes M}},$$

cf. the proof of Theorem 5.3 in [22]. So we can summarize our results by saying that the following square of $N \rtimes M$-bimodules and bimodule morphisms commutes:

$$\begin{tikzcd} \mathcal{L}^2(N_2) \arrow[r, "Z^*"] \arrow[d, "\cong"] & \mathcal{L}^2(N \rtimes M) \arrow[d, "\cong"] \\ \mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N) \arrow[r, "G"'] & \mathcal{L}^2(N) \otimes \mathcal{L}^2(M) \end{tikzcd} \tag{3.2}$$

DEFINITION 3.9. Let α be an integrable coaction of (M, Δ) on a von Neumann algebra N. We call the associated map

$$\rho : N \rtimes M \to N_2$$

the *Galois homomorphism* for α. We call the operator

$$\widetilde{G} = \Sigma G : \mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N) \to \mathcal{L}^2(M) \otimes \mathcal{L}^2(N)$$

the *Galois map* or the *Galois isometry* for (N, α). We call the coaction α *Galois* if the Galois homomorphism is bijective, or equivalently, if the Galois isometry is a unitary (in which case we call it of course the *Galois unitary*).

REMARK 3.10. The reason for putting a flip map in front of G, is to make it right N-linear in such a way that this is just right N-linearity on the second factors of the domain and range, so that "the second leg" of $\widetilde{G}$ is in N. See the section on Galois objects for more information.

Note that the notion of a Galois coaction already appeared, as far as we know, nameless at various places in the literature. The property of G being surjective is the motivation for the terminology, as the bijectivity of the above map $N \underset{N^\alpha}{\otimes} N \to N \otimes M$ (in the algebraic context of Hopf algebras) is precisely the condition to have a Galois coaction of a Hopf algebra. Also note that as these are the non-commutative generalizations of principal fiber bundles, we could call the space pertaining to a Galois coaction a *measured quantum principal fiber bundle* (with (M, Δ) as the principal fiber), an object which is quite trivial in the commutative setting! (There probably is no need to account for the "local triviality", as the functor $\mathcal{L}^2(N) \underset{\mu}{\otimes} -$ is automatically an equivalence between the categories of respectively left N^α and left N_2-modules (so the "faithful flatness" condition in the algebraic setup is automatically fulfilled).) Further note that the above square was essentially constructed in the setting of algebraic quantum groups in [28].

We give a further characterization of Galois coactions in the following corollary. Given an integrable coaction α of (M, Δ) on N, write $N_{00} = N^\alpha \otimes \mathbb{C}$, $N_{01} =$

$\alpha(N)$, $N_{10} = N \otimes \mathbb{C}$ and $N_{11} = N \otimes M$. Write T_1 for the operator valued weight $(\iota \otimes \varphi)$ from $(N \otimes M)^+$ to $(N \otimes \mathbb{C})^{+,\mathrm{ext}}$. Then T_1 restricts to the canonical operator valued weight $T_0 = T \circ \alpha^{-1}$ from $\alpha(N)^+$ to $(N^\alpha \otimes \mathbb{C})^{+,\mathrm{ext}}$, so we are in the situation of Lemma 2.8. Then also denote again by Q_0 and Q_1 the corresponding linking algebras. We regard $\mathcal{L}^2(N)$ and $\mathcal{L}^2(N) \otimes \mathcal{L}^2(M) \cong \mathcal{L}^2(N \rtimes M)$ as right $N \rtimes M$-modules in the natural way, using the Galois homomorphism for the first one.

COROLLARY 3.11. *The following statements are equivalent:*
(i) *the coaction α is Galois;*
(ii) *the inclusion $Q_0 \subseteq Q_1$ is unital;*
(iii) *the image of $(Q_0)_{12}$ in $(Q_1)_{12}$ is exactly the space of $N \rtimes M$-intertwiners.*

Proof. We will write $\widetilde{Q}$ for the linking algebra between the right $N \rtimes M$-modules $\mathcal{L}^2(N)$ and $\mathcal{L}^2(N \rtimes M)$. Denote explicitly the inclusion $Q_0 \subseteq Q_1$ by F. We first show that $F(Q_0) \subseteq \widetilde{Q}$. Take $x \in \mathcal{N}_T$. Then $\alpha(x) \in \mathcal{N}_{T_1}$, and it is easily seen that $\Lambda_{T_1}(\alpha(x))\Lambda_N(y) = (\Lambda_N \otimes \Lambda)(\alpha(x)(y \otimes 1))$ for $y \in \mathcal{N}_{\varphi_N}$. Hence $\Lambda_{T_1}(\alpha(x)) = G \circ l_x$, where we denote by l_x the map $\mathcal{L}^2(N) \to \mathcal{L}^2(N) \underset{\mu}{\otimes} \mathcal{L}^2(N)$ which sends $\xi \in \mathcal{L}^2(N)$ to $\Lambda_N(x) \underset{\mu}{\otimes} \xi$. But l_x is a right $N \rtimes M$-intertwiner, and we know that G is a right $N \rtimes M$-intertwiner by the diagram (3.2). Hence $\Lambda_{T_1}(\alpha(x)) \in \widetilde{Q}_{12}$, and then it follows that $F(Q_0) \subseteq \widetilde{Q}$.

Next, we show that $\rho \circ F_{11} \circ \widetilde{F} = \iota$, where F_{11} denotes the restriction of F to $(Q_0)_{11}$, and $\widetilde{F}$ is the isomorphism $N_2 \to (Q_0)_{11}$ determined by $\Lambda_T(x)\Lambda_T(y)^* \to \Lambda_{T_0}(\alpha(x))\Lambda_{T_0}(\alpha(y))^*$ for $x, y \in \mathcal{N}_T$. Namely: for $x, y \in \mathcal{N}_T$, we have

$$\begin{aligned}(\rho \circ F_{11})(\Lambda_{T_0}(\alpha(x))\Lambda_{T_0}(\alpha(y))^*) &= (\pi_1^{N_2})^{-1}(G^*(Gl_x l_y^* G^*)G) = (\pi_1^{N_2})^{-1}(l_x l_y^*)\\ &= \Lambda_T(x)\Lambda_T(y)^*,\end{aligned}$$

again by using the diagram (3.2).

By these observations, the equivalence of the first and third statement is immediate. Since we have also shown that in fact $F_{11} \circ \widetilde{F} = G(\pi_1^{N_2}(\cdot))G^*$, the equivalence of the first and second statement follows. ∎

We now present some natural examples of Galois coactions.

First, every dual coaction is Galois. More generally, recall that a coaction is called *semi-dual* if there exists a unitary $v \in B(\mathcal{L}^2(M)) \otimes N$ with $(\iota \otimes \alpha)(v) = \widehat{W}_{13}v_{12}$. Then such a semidual coaction is Galois, by Proposition 5.12 of [22].

Also, whenever α is an integrable coaction with $N \rtimes M$ a factor, then α is Galois since the Galois homomorphism ρ is necessarily faithful. A special case concerns the integrable *outer* coactions, i.e. the coactions for which

$$N \rtimes M \cap \alpha(N)' = \mathbb{C}1.$$

Next, suppose (M_1, Δ_1) and (M, Δ) are von Neumann algebraic quantum groups, with $(\widehat{M}_1, \widehat{\Delta}_1)$ a von Neumann algebraic quantum subgroup of $(\widehat{M}, \widehat{\Delta})$.

We mean by this that $\widehat{M}_1$ is a unital sub-von Neumann algebra of $\widehat{M}$, such that the restriction of $\widehat{\Delta}$ to $\widehat{M}_1$ coincides with $\widehat{\Delta}_1$ ([26], Definition 2.9).

Associated to (M_1, Δ_1) and (M, Δ), there is a canonical right coaction

$$\Gamma_{\mathrm{r}} : M \to M \otimes M_1$$

by right translation, and likewise a left coaction Γ_{l} by left translation. See for example the first paragraphs of Section 4 in [26] for the left setting.

PROPOSITION 3.12. *If (M_1, Δ_1) and (M, Δ) are von Neumann algebraic quantum groups, with $(\widehat{M}_1, \widehat{\Delta}_1)$ a von Neumann algebraic quantum subgroup of $(\widehat{M}, \widehat{\Delta})$, the associated coaction Γ_{r} is Galois. Conversely, if (M, Δ) and (M_1, Δ_1) are von Neumann algebraic quantum groups for which there is a right Galois coaction Γ_{r} of (M_1, Δ_1) on M such that $(\iota \otimes \Gamma_{\mathrm{r}})\Delta = (\Delta \otimes \iota)\Gamma_{\mathrm{r}}$, then $(\widehat{M}_1, \widehat{\Delta}_1)$ can be identified with a von Neumann algebraic quantum subgroup of $(\widehat{M}, \widehat{\Delta})$ in such a way that Γ_{r} is precisely the coaction by right translations.*

Proof. First suppose that $(\widehat{M}_1, \widehat{\Delta}_1)$ is a von Neumann algebraic quantum subgroup of $(\widehat{M}, \widehat{\Delta})$. Then we can also embed $\widehat{M}_1'$ into $\widehat{M}'$ by a normal map F which respects the comultiplications. Denote $V_\Gamma = (F \otimes \iota)(V_1)$, where $V_1 \in \widehat{M}_1' \otimes M_1$ is the right regular corepresentation of (M_1, Δ_1). The aforementioned coaction Γ_{r} is then explicitely given as $\Gamma_{\mathrm{r}}(x) = V_\Gamma(x \otimes 1)V_\Gamma^*$ for $x \in M$. We can make the following sequence of isomorphisms:

$$\begin{aligned} M \rtimes M_1 &= (\Gamma_{\mathrm{r}}(M) \cup (1 \otimes \widehat{M}_1'))'' \cong ((M \otimes 1) \cup V_\Gamma^*(1 \otimes \widehat{M}_1')V_\Gamma)'' \\ &= ((M \otimes 1) \cup (F \otimes \iota)(\widehat{\Delta}_1'(\widehat{M}_1')))'' \cong ((M \otimes 1) \cup \widehat{\Delta}'(F(\widehat{M}_1')))'' \cong (M \cup F(\widehat{M}_1')), \end{aligned}$$

where we have used that V_1 is also the left regular corepresentation for $(\widehat{M}_1', \widehat{\Delta}_1')$. Since it's easy to see that the resulting isomorphism satisfies the requirements for the Galois homomorphism (using that V_Γ is actually the corepresentation implementing Γ_{r}), the coaction is Galois.

Now suppose that we have a Galois coaction Γ_{r} such that $(\iota \otimes \Gamma_{\mathrm{r}})\Delta = (\Delta \otimes \iota)\Gamma_{\mathrm{r}}$. Denote by $(\widehat{A}_u', \widehat{\Delta}_u')$ the universal C^*-algebraic quantum group associated with $(\widehat{M}', \widehat{\Delta}')$, and similarly for $(\widehat{M}_1', \widehat{\Delta}_1')$ (cf. [12]). By the results in Section 12 of [12] (in the setting of right coactions), we get that there is a canonical non-degenerate $*$-homomorphism $F_u : \widehat{A}_{1,u}' \to M(\widehat{A}_u')$ which intertwines the comultiplications. Since Γ_{r} is Galois, we also have a faithful normal homomorphism $F : \widehat{M}_1' \to B(\mathcal{L}^2(M))$. Denote by U the corepresentation associated with Γ_{r}. By the results of [12], it is an element of $\widehat{M}' \otimes M_1$. Denote by π_u and $\pi_{1,u}$ respectively the canonical homomorphisms $M(\widehat{A}_u') \to \widehat{M}'$ and $M(\widehat{A}_{1,u}') \to \widehat{M}_1'$ from the multiplier C^*-algebras to the von Neumann algebras. Identify M_* and $(M_1)_*$ with their images in respectively $\widehat{A}_u'$ and $\widehat{A}_{1,u}'$ (noting that the dual of $(\widehat{M}', \widehat{\Delta}')$ is $(M, \Delta^{\mathrm{op}})$). Then we can deduce again from [12] that for $\omega \in (M_1)_*$, we have $\pi_u(F_u(\omega)) = (\iota \otimes \omega)(U)$. Since the π_u also commute with the comultiplications, we deduce that $F(\widehat{M}_1') \subseteq \widehat{M}'$, and that this embedding respects the comultiplications. Hence $(\widehat{M}_1', \widehat{\Delta}_1')$ is a von Neumann algebraic quantum subgroup of $(\widehat{M}', \widehat{\Delta}')$ (and then of course also $(\widehat{M}_1, \widehat{\Delta}_1)$ is a von Neumann algebraic quantum subgroup of $(\widehat{M}, \widehat{\Delta})$).

Finally, we should show that Γ_{r} is just the coaction naturally associated with this von Neumann algebraic quantum subgroup. But this is clear, as Γ_{r} is implemented by the corepresentation U, which equals V_Γ since also $\pi_u(F_u(\omega)) = (\iota \otimes \omega)(V_\Gamma)$. ∎

4. GALOIS OBJECTS

We will now treat in detail the case of ergodic Galois coactions, i.e. $N^\alpha = \mathbb{C}$.

DEFINITION 4.1. If N is a von Neumann algebra, (M, Δ) a von Neumann algebraic quantum group and α an *ergodic* Galois coaction of (M, Δ) on N, we call (N, α) a (right) *Galois object* for (M, Δ).

In this case, the constructions of the previous sections greatly simplify. First of all, $T = (\iota \otimes \varphi)\alpha$ itself will already be an nsf weight on N (identifying $\mathbb{C}$ with $\mathbb{C} \cdot 1_N$), so we denote it by φ_N. Then $\mathcal{N}_T = \mathcal{N}_{\varphi_N}$. There is a slight ambiguity of notation then, as $\Lambda_N(x)$ denotes either an element of $\mathcal{H}$ or a linear operator $\mathbb{C} \to \mathcal{H}$, but this ambiguity disappears if we identify the Hilbert spaces $B(\mathbb{C}, \mathcal{H})$ and $\mathcal{H}$ by sending x to $x \cdot 1$. Next, $N \rtimes M \overset{\rho}{\cong} N_2$ becomes the whole of $B(\mathcal{L}^2(N))$, and $\varphi_2 = \mathrm{Tr}(\cdot \nabla_N)$. Further, $\mathcal{L}^2(N_2)$ will be identified with $\mathcal{L}^2(N) \otimes \mathcal{L}^2(N)$ by the map

$$\Lambda_{N_2}(\Lambda_N(x)\Lambda_N(y^*)^*) \to \Lambda_N(x) \otimes \Lambda_N(y) \quad \text{for } x, y \in \mathcal{N}_{\varphi_N} \cap \mathcal{N}_{\varphi_N}^*.$$

For $x \in B(\mathcal{L}^2(N))$, we have $\pi_1^{N_2}(x) = x \otimes 1$, $\pi_{\mathrm{r}}^{N_2}(x) = 1 \otimes \pi_{\mathrm{r}}(x)$ (where $\pi_{\mathrm{r}}(x) = J_N x^* J_N$), $\nabla_{N_2}^{it} = \nabla_N^{it} \otimes \nabla_N^{it}$ and $J_{N_2} = \Sigma(J_N \otimes J_N)$. In the following, we will now also use the symbol $\widehat{\pi}_1$ to denote the left representation of $\widehat{M}'$ on $\mathcal{L}^2(N)$ (so $\widehat{\pi}_1(m) = \rho(1 \otimes m)$ for $m \in \widehat{M}'$), and we will write $\widehat{\theta}_{\mathrm{r}}(m)$ for $\theta_{\mathrm{r}}(\rho(1 \otimes m)) = J_N \widehat{\pi}_1(m)^* J_N$ when $m \in \widehat{M}'$. In fact, it's not difficult to see that for any integrable coaction, we have then $\widehat{\theta}_{\mathrm{r}}(m) = \widehat{\pi}_1(\widehat{R}'(m))$: just use that $(J_N \otimes \widehat{J})U(J_N \otimes \widehat{J}) = U^*$ and $(J \otimes \widehat{J})V(J \otimes \widehat{J}) = V^*$.

The aim of this section is to show that there is much extra structure on a Galois object (N, α), closely resembling the one of (M, Δ) itself. In particular, we are able to show that there exists an nsf invariant weight on N. To find it, we will search a 1-cocycle to deform φ_N.

Let (N, α) be a fixed Galois object, with Galois unitary $\widetilde{G} : \mathcal{L}^2(N) \otimes \mathcal{L}^2(N) \to \mathcal{L}^2(M) \otimes \mathcal{L}^2(N)$. To begin with, we will write down some commutation relations. In the following, $\alpha^{\mathrm{op}}(x) = \Sigma\alpha(x)\Sigma$ for $x \in N$, and $\widehat{\Pi}_{\mathrm{r}}(m)$ for $m \in \widehat{M}$ will denote the operator $\widehat{J}m^*\widehat{J}$ on $\mathcal{L}^2(M)$.

LEMMA 4.2. *For all $x \in N$ and $m \in \widehat{M}'$, we have:*
(i) $\widetilde{G}(x \otimes 1) = \alpha^{\mathrm{op}}(x)\widetilde{G}$;
(ii) $\widetilde{G}(\widehat{\pi}_{\mathrm{l}}(m) \otimes 1) = (m \otimes 1)\widetilde{G}$;
(iii) $\widetilde{G}(1 \otimes \pi_{\mathrm{r}}(x)) = (1 \otimes \pi_{\mathrm{r}}(x))\widetilde{G}$;
(iv) $\widetilde{G}(1 \otimes \widehat{\theta}_{\mathrm{r}}(m)) = (\widehat{\Pi}_{\mathrm{r}} \otimes \widehat{\theta}_{\mathrm{r}})((\widehat{\Delta}')^{\mathrm{op}}(m))\widetilde{G}$.

Proof. These equalities follow directly from the fact that G is a $N \rtimes M$-bimodule map. For the fourth one, we remark that the right representation $\pi_{\mathrm{r}}^{N \rtimes M}$ of $N \rtimes M$ on $\mathcal{L}^2(N) \otimes \mathcal{L}^2(M)$ is given by

$$\pi_{\mathrm{r}}^{N \rtimes M}(\alpha(x)) = \pi_{\mathrm{r}}(x) \otimes 1, \quad x \in N,$$
$$\pi_{\mathrm{r}}^{N \rtimes M}(1 \otimes m) = U(1 \otimes \widehat{\Pi}_{\mathrm{r}}(m))U^*, \quad m \in \widehat{M}',$$

a fact which is easy to recover using that $U = J_{N \rtimes M}(J_N \otimes \widehat{J})$. Now use that also $U = (\widehat{\pi}_{\mathrm{l}} \otimes \iota)(V)$, that V is the left multiplicative unitary for $(\widehat{M}', \widehat{\Delta}')$, and that $V(J \otimes \widehat{J}) = (J \otimes \widehat{J})V^*$. ∎

Note that $\mathcal{L}^2(N)$ is a natural right $\widehat{M}$-module, by an anti-representation

$$\widehat{\pi}_{\mathrm{r}} : m \to \widehat{\pi}_{\mathrm{l}}(\widehat{J}m^*\widehat{J}), \quad m \in \widehat{M}.$$

Denote by $\widehat{Q}$ the linking von Neumann algebra between the right $\widehat{M}$-modules $\mathcal{L}^2(M)$ and $\mathcal{L}^2(N)$. We will write

$$\widehat{Q} = \begin{pmatrix} \widehat{Q}_{11} & \widehat{Q}_{12} \\ \widehat{Q}_{21} & \widehat{Q}_{22} \end{pmatrix} = \begin{pmatrix} \widehat{P} & \widehat{N} \\ \widehat{O} & \widehat{M} \end{pmatrix}.$$

COROLLARY 4.3. (i) *$\widetilde{G} \in \widehat{O} \otimes N$.*
(ii) *$\widetilde{G}_{12}U_{13} = V_{13}\widetilde{G}_{12}$.*

Proof. The first statement follows by the second and third commutation relation in the previous lemma. Since for $\omega \in M_*$, we have $(\iota \otimes \omega)(U) = \widehat{\pi}_{\mathrm{l}}((\iota \otimes \omega)(V))$, the second statement also follows from the second commutation relation of the previous lemma. ∎

The following is just a restatement of Lemma 3.5.

LEMMA 4.4. *The map $\widetilde{G}$ satisfies the identity $\widetilde{G}(J_N \otimes J_N)\Sigma = \Sigma U \Sigma(\widehat{J} \otimes J_N)\widetilde{G}$.*

Now we prove a kind of pentagon equation:

PROPOSITION 4.5. *We have:*

$$\widehat{W}_{12}\widetilde{G}_{13}\widetilde{G}_{23} = \widetilde{G}_{23}\widetilde{G}_{12}.$$

Proof. For $x \in \mathcal{N}_{\varphi_N}$ and $\omega \in B(\mathcal{L}^2(N))_*$, we have $(\omega \otimes \iota)(\alpha(x)) \in \mathcal{N}_\varphi$, and

$$(\iota \otimes \omega)(\widetilde{G})\Lambda_N(x) = \Lambda((\omega \otimes \iota)(\alpha(x))).$$

This follows by first considering ω of the form $\omega_{\Lambda_N(y),\Lambda_N(z)}$ with $y, z \in \mathcal{N}_{\varphi_N}$, and then using the closedness of the map Λ to conclude that it holds in general. Now for $x \in \mathcal{N}_{\varphi_N}$, $\omega \in M_*$ and $\omega' \in N_*$, we have, using $\widehat{W} = \Sigma W^* \Sigma$,

$$\begin{aligned}(\iota \otimes \omega)(\widehat{W})(\iota \otimes \omega')(\widetilde{G})\Lambda_N(x) &= \Lambda((\omega' \otimes \omega \otimes \iota)((\iota \otimes \Delta)(\alpha(x))))\\ &= \Lambda((\omega' \otimes \omega \otimes \iota)((\alpha \otimes \iota)(\alpha(x))))\\ &= \Lambda((((\omega \otimes \omega') \circ \alpha^{\mathrm{op}}) \otimes \iota)(\alpha(x)))\\ &= (\iota \otimes ((\omega \otimes \omega') \circ \alpha^{\mathrm{op}}))(\widetilde{G})\Lambda_N(x),\end{aligned}$$

from which we conclude $\widehat{W}_{12}\widetilde{G}_{13} = (\iota \otimes \alpha^{\mathrm{op}})(\widetilde{G})$. Since $(\iota \otimes \alpha^{\mathrm{op}})(\widetilde{G}) = \widetilde{G}_{23}\widetilde{G}_{12}\widetilde{G}_{23}^*$, the result follows. ∎

REMARK 4.6. (i) Note that if N and M have separable preduals, then, choosing a unitary $u : \mathcal{L}^2(M) \to \mathcal{L}^2(N)$, the unitary $v = \widetilde{G}(u \otimes 1)$ in $B(\mathcal{L}^2(M)) \otimes N$ will satisfy $(\iota \otimes \alpha)(v) = \widehat{W}_{13}v_{12}$. So in this case there is a one-to-one correspondence between Galois objects and ergodic semi-dual coactions.

(ii) Note that for the trivial right Galois object (M, Δ) for (M, Δ), the map $\widetilde{G}$ is exactly $\widehat{W}$, while the map U becomes the right regular representation V.

We have the following density results:

PROPOSITION 4.7. (i) *The following space L is σ-weakly dense in N:*

$$L = \{(\omega \otimes \iota)(\widetilde{G}) : \omega \in B(\mathcal{L}^2(N), \mathcal{L}^2(M))_*\}.$$

(ii) *The space $K = \{(\iota \otimes \omega)(\widetilde{G}) : \omega \in B(\mathcal{L}^2(N))_*\}$ is σ-weakly dense in $\widehat{O}$.*

Proof. By the pentagon equation, the linear span of the $(\omega \otimes \iota)(\widetilde{G})$ will be an algebra. Further, for any $x \in \mathcal{N}_{\varphi_N}$ and $m \in \mathcal{N}_\varphi$, we have $(1 \otimes m^*)\alpha(x) \in \mathcal{M}_{(\iota\otimes\varphi)}$ and $(\omega_{\Lambda_N(x),\Lambda(m)} \otimes \iota)(\widetilde{G}) = (\iota \otimes \varphi)((1 \otimes m^*)\alpha(x))$. From this, we can conclude that the σ-weak closure of L also is the σ-weak closure of the span of $\{(\iota \otimes \omega)(\alpha(x)) : \omega \in M_*, x \in N\}$, so that this σ-weak closure will be a unital sub-von Neumann algebra of N (see also the proof of Proposition 1.21 of [27]). Now suppose $\omega \in N_*$ is orthogonal to L. By the bi-duality theorem (see [6], and also Theorem 2.6 of [22]), we have that $(\alpha(N) \cup (1 \otimes B(\mathcal{L}^2(M))))''$ equals $N \otimes B(\mathcal{L}^2(M))$. So for any $x \in N \otimes B(\mathcal{L}^2(N))$ and $\omega' \in B(\mathcal{L}^2(N))_*$, $(\iota \otimes \omega')(x)$ can be σ-weakly approximated by elements of the form $(\iota \otimes \omega')(x_n)$ with x_n in the algebra generated by $\alpha(N)$ and $1 \otimes B(\mathcal{H})$, and any such element can in turn be approximated by an element in the algebra generated by elements of the form $(\iota \otimes \omega'')(\alpha(x_{nm}))$, $\omega'' \in B(\mathcal{L}^2(M))_*$ and $x_{nm} \in N$, by using an orthogonal basis argument. It follows that ω vanishes on the whole of N, and hence L is σ-weakly dense in N.

For the second statement, note that, by the pentagon equation, K is closed under left multiplication with elements of the form $(\iota \otimes \omega)(\widehat{W})$ for $\omega \in M_*$. Hence, as in the proof of Proposition 2.2, it is enough to show that if $z \in \widehat{N}$ satisfies $K \cdot z = 0$, then $z = 0$. But take $x, y \in \mathcal{T}_{\varphi_N,T}$ (which is now just the Tomita algebra $\mathcal{T}_{\varphi_N}$ for φ_N), and $m \in \mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}}$. Then $(\iota \otimes \omega_{\Lambda_N(x),\Lambda_N(y)})(\widetilde{G}^*)\widehat{\Lambda}^{\mathrm{op}}(m) = \widehat{\pi}_1(m)x\Lambda_N(\sigma_{-\mathrm{i}}(y^*))$ by Lemma 2.6 and Proposition 3.2. Hence $K^* \cdot \mathcal{L}^2(M)$ is dense in $\mathcal{L}^2(N)$, and necessarily $z = 0$. ∎

PROPOSITION 4.8. *For any $m \in M'$, the operator $\widetilde{G}^*(m \otimes 1)\widetilde{G}$ lies in $N' \otimes N$.*

Proof. Clearly, the second leg lies in N. Since $\widetilde{G}(y \otimes 1)\widetilde{G}^* = \alpha^{\mathrm{op}}(y)$ for $y \in N$, the first leg of $\widetilde{G}^*(m \otimes 1)\widetilde{G}$ must be inside N'. ∎

Recall from the proof of Lemma 3.5 that $\nabla^{\mathrm{i}t}_{N \rtimes M} = \nabla^{\mathrm{i}t}_N \otimes q^{\mathrm{i}t}$, where we can also write $q^{\mathrm{i}t} = \delta^{-\mathrm{i}t}\widehat{\nabla}^{-\mathrm{i}t}$. Then $\kappa_t = q^{\mathrm{i}t}xq^{-\mathrm{i}t}$ defines a one-parameter group of automorphisms on M, and $\gamma_t(x) = q^{\mathrm{i}t}xq^{-\mathrm{i}t}$ defines a one-parameter group of automorphisms on $\widehat{M}'$.

LEMMA 4.9. (i) *For $x \in N$, we have $\alpha(\sigma^N_t(x)) = (\sigma^N_t \otimes \kappa_t)(\alpha(x))$.*
(ii) *For $m \in \widehat{M}'$, we have $\sigma^{N_2}_t(\widehat{\pi}_1(m)) = \widehat{\pi}_1(\gamma_t(m))$.*
(iii) *For $m \in \widehat{M}'$, we have $\widehat{\theta}_{\mathrm{r}}(m) = \widehat{\pi}_1(\widehat{R}'(m))$.*

Proof. The first two statements follow straightforwardly from Lemma 3.5 and Lemma 4.2. The final statement was noted at the beginning of this section. ∎

In particular, $\sigma^{N_2}_t(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{\mathrm{i}s}\widehat{J})) = \widehat{\pi}_1(\widehat{J}\widehat{\delta}^{\mathrm{i}s}\widehat{J})$ for each $s, t \in \mathbb{R}$, since an easy computation shows that each $\widehat{J}\widehat{\delta}^{\mathrm{i}s}\widehat{J}$ is invariant under γ_t.

COROLLARY 4.10. *The one-parameter groups $\nabla^{\mathrm{i}t}_N$ and $\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{\mathrm{i}t}\widehat{J})$ commute.*

We denote the resulting one-parameter group of unitaries by

$$P^{\mathrm{i}t}_N = \nabla^{\mathrm{i}t}_N \widehat{\pi}_1(\widehat{J}\widehat{\delta}^{-\mathrm{i}t}\widehat{J}).$$

PROPOSITION 4.11. *N is invariant under $\mathrm{Ad}(P^{\mathrm{i}t}_N)$.*

Proof. We only have to show that N is invariant under $\mathrm{Ad}(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{-\mathrm{i}t}\widehat{J}))$. But for any group-like element $u \in \widehat{M}'$, we have, denoting by $\widehat{\alpha}$ the dual coaction, that

$$((\rho \otimes \iota)\widehat{\alpha}\rho^{-1})(\widehat{\pi}_1(u)x\widehat{\pi}_1(u)^*) = (\widehat{\pi}_1(u) \otimes u)(x \otimes 1)(\widehat{\pi}_1(u)^* \otimes u^*) = \widehat{\pi}_1(u)x\widehat{\pi}_1(u)^* \otimes 1$$

for $x \in N$, and so, by the bi-duality theorem of [6], we get $\widehat{\pi}_1(u)x\widehat{\pi}_1(u)^* \in N$. ∎

DEFINITION 4.12. We call the resulting one-parameter group

$$\tau^N_t : N \to N : x \to P^{\mathrm{i}t}_N x P^{-\mathrm{i}t}_N$$

the *scaling group* of (N, α).

PROPOSITION 4.13. *The following identities hold for $x \in N$:*

$$\alpha(\tau_t^N(x)) = (\tau_t^N \otimes \tau_t)(\alpha(x)), \quad \alpha(\tau_t^N(x)) = (\sigma_t^N \otimes \sigma'_{-t})(\alpha(x)),$$
$$\alpha(\sigma_t^N(x)) = (\tau_t^N \otimes \sigma_t)(\alpha(x)).$$

Recall that τ_t denotes the scaling group of (M, Δ), while σ'_t denotes the modular one-parameter group of the right invariant weight ψ.

Proof. By Lemma 4.9, we have

$$\alpha \circ \sigma_t^N = (\sigma_t^N \otimes \mathrm{Ad}(\delta^{-\mathrm{i}t})\tau_{-t}) \circ \alpha.$$

Further, since G is a left $N \rtimes M$-module map, we have

$$\begin{aligned}\alpha(\mathrm{Ad}(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{-\mathrm{i}t}\widehat{J}))(x)) &= G((\mathrm{Ad}(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{-\mathrm{i}t}\widehat{J}))(x)) \otimes 1)G^* \\ &= (\iota \otimes \mathrm{Ad}(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{-\mathrm{i}t}\widehat{J})))(G(x \otimes 1)G^*) \\ &= (\iota \otimes \mathrm{Ad}(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{-\mathrm{i}t}\widehat{J})))(\alpha(x)).\end{aligned}$$

Now by the first formula of Theorem 4.17 in [27], we have $(\widehat{J}\widehat{\delta}^{-\mathrm{i}t}\widehat{J})P^{-\mathrm{i}t} = \nabla^{-\mathrm{i}t}$, where $P^{\mathrm{i}t}$ denotes the standard unitary implementation of the scaling group of (M, Δ), so $\mathrm{Ad}(\delta^{-\mathrm{i}t})\tau_{-t}\mathrm{Ad}(\widehat{J}\widehat{\delta}^{-\mathrm{i}t}\widehat{J})$ reduces to σ'_{-t} on M. This proves the second formula.

As for the first identity, we have, using the second identity, the coaction property of α and the identity $\Delta \circ \sigma'_{-t} = (\sigma'_{-t} \otimes \tau_t) \circ \Delta$, that

$$\begin{aligned}(\alpha \otimes \iota) \circ (\tau_t^N \otimes \tau_t) \circ \alpha &= (\sigma_t^N \otimes \sigma'_{-t} \otimes \tau_t) \circ (\iota \otimes \Delta) \circ \alpha = (\iota \otimes \Delta) \circ (\sigma_t^N \otimes \sigma'_{-t}) \circ \alpha \\ &= (\alpha \otimes \iota) \circ \alpha \circ \tau_t^N.\end{aligned}$$

Thus the first identity follows by the injectivity of α.

The third identity now easily follows from the first identity and, for $x \in N$,

$$\alpha(\mathrm{Ad}(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{\mathrm{i}t}\widehat{J}))(x)) = (\iota \otimes \sigma_t\tau_{-t})(\alpha(x)) \quad \blacksquare$$

For the next result, recall that $\nu > 0$ denotes the scaling constant of (M, Δ).

LEMMA 4.14. *The one-parameter group τ_t^N satisfies $\varphi_N \circ \tau_t^N = \nu^{-t}\varphi_N$, and if $x \in \mathcal{N}_{\varphi_N}$, then*

$$P_N^{\mathrm{i}t}\Lambda_N(x) = \nu^{t/2}\Lambda_N(\tau_t^N(x)).$$

Proof. The first statement easily follows since

$$\varphi_N \circ \tau_t^N = ((\iota \otimes \varphi) \circ \alpha) \circ \tau_t^N = \tau_t^N \circ ((\iota \otimes \varphi \circ \tau_t) \circ \alpha) = \nu^{-t}\varphi_N.$$

By the first statement, $\mathrm{Ad}(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{\mathrm{i}t}\widehat{J}))(x) \in \mathcal{N}_{\varphi_N}$ for $x \in \mathcal{N}_{\varphi_N}$, and the second statement is equivalent with

$$\nu^{t/2}\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{\mathrm{i}t}\widehat{J})\Lambda_N(x) = \Lambda_N(\mathrm{Ad}(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{\mathrm{i}t}\widehat{J}))(x)).$$

Taking an arbitrary $y \in \mathcal{N}_{\varphi_N}$, we have

$$G(\nu^{t/2}\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{\mathrm{i}t}\widehat{J})\Lambda_N(x) \otimes \Lambda_N(y)) = \nu^{t/2}(1 \otimes \widehat{J}\widehat{\delta}^{\mathrm{i}t}\widehat{J})(\Lambda_N \otimes \Lambda)(\alpha(x)(y \otimes 1)).$$

Since $\widehat{J}\widehat{\delta}^{\mathrm{it}}\widehat{J} = \nabla^{\mathrm{it}}P^{-\mathrm{it}}$ and $\alpha(\mathrm{Ad}(\widehat{\pi}_1(\widehat{J}\widehat{\delta}^{\mathrm{it}}\widehat{J}))(x)) = (\iota\otimes\mathrm{Ad}(\widehat{J}\widehat{\delta}^{\mathrm{it}}\widehat{J}))(\alpha(x))$, the result follows. ∎

COROLLARY 4.15. *We have the following commutation relations:*
(i) $\widetilde{G}(\nabla_N^{\mathrm{it}}\otimes\nabla_N^{\mathrm{it}}) = (\delta^{-\mathrm{it}}\widehat{\nabla}^{-\mathrm{it}}\otimes\nabla_N^{\mathrm{it}})\widetilde{G}$;
(ii) $\widetilde{G}(\nabla_N^{\mathrm{it}}\otimes P_N^{\mathrm{it}}) = (\nabla^{\mathrm{it}}\otimes P_N^{\mathrm{it}})\widetilde{G}$;
(iii) $\widetilde{G}(P_N^{\mathrm{it}}\otimes P_N^{\mathrm{it}}) = (P^{\mathrm{it}}\otimes P_N^{\mathrm{it}})\widetilde{G}$.

Proof. The first identity follows immediately from Lemma 3.5, while the other two follow by using the definition of G, the implementation of Lemma 4.14 and the identities in Lemma 4.13. ∎

Now consider $H^{\mathrm{it}} = \widetilde{G}^*(J\delta^{\mathrm{it}}J\otimes 1)\widetilde{G}$ in $N'\otimes N$.

PROPOSITION 4.16. *There exist non-singular $h,k\geqslant 0$ affiliated with respectively N' and N such that $H^{\mathrm{it}} = h^{\mathrm{it}}\otimes k^{\mathrm{it}}$ for all $t\in\mathbb{R}$. Moreover, $\alpha(k^{\mathrm{it}}) = k^{\mathrm{it}}\otimes\delta^{\mathrm{it}}$ for $t\in\mathbb{R}$.*

Proof. We show that

$$H^{\mathrm{it}}(B(\mathcal{L}^2(N))\otimes 1)H^{-\mathrm{it}} = B(\mathcal{L}^2(N))\otimes 1.$$

Since $B(\mathcal{L}^2(N)) = \rho(N\rtimes M)$, we only have to show that

$$H^{\mathrm{it}}(N\otimes 1)H^{-\mathrm{it}} = (N\otimes 1),\quad H^{\mathrm{it}}(\widehat{\pi}_1(\widehat{M}')\otimes 1)H^{-\mathrm{it}} = (\widehat{\pi}_1(\widehat{M}')\otimes 1).$$

Now the first equality is clear as the first leg of H^{it} lies in N'. As for the second equality, applying $\mathrm{Ad}(\widetilde{G})$, this is equivalent with $\mathrm{Ad}(J\delta^{\mathrm{it}}J)(\widehat{M}') = \widehat{M}'$, which is easily seen to be true.

Denote by h a positive (possibly unbounded) operator which implements the automorphism group $\mathrm{Ad}(H^{\mathrm{it}})$ on $B(\mathcal{L}^2(N))$, so

$$\mathrm{Ad}(H^{\mathrm{it}})(x\otimes 1) = (\mathrm{Ad}(h^{\mathrm{it}})(x))\otimes 1\quad\text{for all } x\in B(\mathcal{L}^2(N)).$$

Then h is non-singular, with h affiliated with N', and $H^{\mathrm{it}} = h^{\mathrm{it}}\otimes k^{\mathrm{it}}$ for a positive non-singular k affiliated with N.

Note now that $\widehat{W}^*(J\delta^{\mathrm{it}}J\otimes 1)\widehat{W} = J\delta^{\mathrm{it}}J\otimes\delta^{\mathrm{it}}$, which can be computed for example by Lemma 4.14 and the formulas in Proposition 4.17 of [27]. Then using the pentagon equation for $\widetilde{G}$, we have

$$\begin{aligned}(\iota\otimes\alpha^{\mathrm{op}})(H^{\mathrm{it}}) &= \widetilde{G}_{23}H_{12}^{\mathrm{it}}\widetilde{G}_{23}^* = \widetilde{G}_{23}\widetilde{G}_{12}^*(J\delta^{\mathrm{it}}J\otimes 1\otimes 1)\widetilde{G}_{12}\widetilde{G}_{23}^*\\ &= \widetilde{G}_{13}^*\widehat{W}_{12}^*\widetilde{G}_{23}(J\delta^{\mathrm{it}}J\otimes 1\otimes 1)\widetilde{G}_{23}^*\widehat{W}_{12}\widetilde{G}_{13}\\ &= \widetilde{G}_{13}^*(J\delta^{\mathrm{it}}J\otimes\delta^{\mathrm{it}}\otimes 1)\widetilde{G}_{13} = h^{\mathrm{it}}\otimes\delta^{\mathrm{it}}\otimes k^{\mathrm{it}},\end{aligned}$$

so that $\alpha(k^{\mathrm{it}}) = k^{\mathrm{it}}\otimes\delta^{\mathrm{it}}$. ∎

The operator k which appears in the proposition is determined up to a positive scalar. We will now fix some k, and call it δ_N.

DEFINITION 4.17. We call δ_N the *modular element* of (N,α).

LEMMA 4.18. *With the notation of the previous proposition, we have:*
(i) $h = J_N \delta_N^{-1} J_N$,
(ii) $\sigma_t^N(\delta_N^{\mathrm{i}s}) = \nu^{\mathrm{i}st} \delta_N^{\mathrm{i}s}$,
(iii) $\tau_t^N(\delta_N^{\mathrm{i}s}) = \delta_N^{\mathrm{i}s}$.

Proof. Denoting again $H^{\mathrm{i}t} = \widetilde{G}^*(J\delta^{\mathrm{i}t}J \otimes 1)\widetilde{G}$, we first prove that

$$\Sigma(J_N \otimes J_N) H^{\mathrm{i}t} (J_N \otimes J_N)\Sigma = H^{\mathrm{i}t}.$$

Using Lemma 4.4, the left hand side equals

$$\widetilde{G}^*(\widehat{J} \otimes J_N)\Sigma U^* \Sigma (J\delta^{\mathrm{i}t} J \otimes 1)\Sigma U \Sigma (\widehat{J} \otimes J_N)\widetilde{G}.$$

As $U \in B(\mathcal{L}^2(N)) \otimes M$, this reduces to $\widetilde{G}^*(\widehat{J} J \delta^{\mathrm{i}t} J \widehat{J} \otimes 1)\widetilde{G}$. Since J commutes with $\widehat{J}$ up to a scalar of modulus 1, and since $\delta^{\mathrm{i}t}$ commutes with $\widehat{J}$, we find that this expression reduces to $\widetilde{G}^*(J\delta^{\mathrm{i}t}J \otimes 1)\widetilde{G} = H^{\mathrm{i}t}$. So

$$J_N \delta_N^{\mathrm{i}t} J_N \otimes J_N h^{\mathrm{i}t} J_N = h^{\mathrm{i}t} \otimes \delta_N^{\mathrm{i}t},$$

which implies that there exists a positive scalar r such that $h^{\mathrm{i}t} = r^{\mathrm{i}t} J_N \delta_N^{\mathrm{i}t} J_N$. But plugging this back into the above equality, we find that $r^{2\mathrm{i}t} = 1$ for all t, hence $r = 1$.

For the second statement, we easily get, using the first commutation relation of Corollary 4.15, that

$$(\nabla_N^{\mathrm{i}t} \otimes \nabla_N^{\mathrm{i}t})(J_N \delta_N^{\mathrm{i}s} J_N \otimes \delta_N^{\mathrm{i}s})(\nabla_N^{-\mathrm{i}t} \otimes \nabla_N^{-\mathrm{i}t}) = (J_N \delta_N^{\mathrm{i}s} J_N \otimes \delta_N^{\mathrm{i}s}).$$

This implies that there exists a positive number $\widetilde{\nu}$ such that $\sigma_t^N(\delta_N^{\mathrm{i}s}) = \widetilde{\nu}^{\mathrm{i}st} \delta_N^{\mathrm{i}s}$. We must show that $\widetilde{\nu} = \nu$.

But we know now that $\delta_N^{\mathrm{i}s}$ is analytic with respect to σ_t^N. So if $x \in \mathcal{M}_{\varphi_N}$, then also $x\delta_N^{\mathrm{i}s}$ and $\delta_N^{\mathrm{i}s} x$ are integrable. We have for such x that, choosing some state $\omega \in N_*$,

$$\begin{aligned}
\varphi_N(\delta_N^{\mathrm{i}s} x) &= \varphi((\omega \otimes \iota)(\alpha(\delta_N^{\mathrm{i}s} x))) = \varphi(\delta^{\mathrm{i}s}(\omega(\delta_N^{\mathrm{i}s} \cdot) \otimes \iota)(\alpha(x))) \\
&= \nu^s \varphi((\omega(\delta_N^{\mathrm{i}s} \cdot) \otimes \iota)(\alpha(x))\delta^{\mathrm{i}s}) \\
&= \nu^s \varphi((\omega(\delta_N^{\mathrm{i}s} \cdot \delta_N^{-\mathrm{i}s}) \otimes \iota)(\alpha(x\delta_N^{\mathrm{i}s}))) = \nu^s \varphi_N(x\delta_N^{\mathrm{i}s}).
\end{aligned}$$

This shows $\sigma_{-\mathrm{i}}^N(\delta_N^{\mathrm{i}s}) = \nu^s \delta_N^{\mathrm{i}s}$, which implies $\widetilde{\nu} = \nu$.

As for the last statement, this follows from

$$\alpha(\tau_t^N \sigma_{-t}^N(\delta_N^{\mathrm{i}s})) = (\iota \otimes \tau_t \sigma_{-t})\alpha(\delta_N^{\mathrm{i}s}) = \delta_N^{\mathrm{i}s} \otimes \tau_t \sigma_{-t}(\delta^{\mathrm{i}s}) = \nu^{-\mathrm{i}st} \alpha(\delta_N^{\mathrm{i}s}).$$

This ends the proof. ∎

By Connes' cocycle derivative theorem, we can now construct the nsf weight

$$\psi_N = \varphi_N(\delta_N^{1/2} \cdot \delta_N^{1/2}),$$

which is the deformation of φ_N by the cocycle $w_t = \nu^{\mathrm{i}t^2/2} \delta_N^{\mathrm{i}t}$.

THEOREM 4.19. *The weight ψ_N is invariant with respect to α: if $x \in \mathcal{M}_{\psi_N}^+$ and $\omega \in M_*^+$, then*

$$\psi_N((\iota \otimes \omega)\alpha(x)) = \psi_N(x)\omega(1).$$

Proof. Let $x \in N$ be a left multiplier of $\delta_N^{1/2}$ such that the closure of $x\delta_N^{1/2}$ is an element of $\mathcal{N}_{\varphi_N}$. Then $x \in \mathcal{N}_{\psi_N}$, and $\Lambda_{N,\delta_N}(x) := \Lambda_N(\overline{x\delta_N^{1/2}})$ determines a GNS-construction for ψ_N (see the remark before Proposition 1.15 in [13]). Choose $\xi \in \mathcal{D}(\delta^{-1/2})$. Then for any $\eta \in \mathcal{L}^2(M)$, we have $(\iota \otimes \omega_{\xi,\eta})\alpha(x)$ a left multiplier of $\delta_N^{1/2}$, and the closure of $((\iota \otimes \omega_{\xi,\eta})\alpha(x))\delta_N^{1/2}$ equals $(\iota \otimes \omega_{\delta^{-1/2}\xi,\eta})\alpha(\overline{x\delta_N^{1/2}})$. By the formula (3.1) for U (after the statement of Theorem 3.1), we conclude that this last operator is in $\mathcal{D}(\Lambda_N)$, and that

$$\Lambda_N((\iota \otimes \omega_{\delta^{-1/2}\xi,\eta})\alpha(\overline{x\delta_N^{1/2}})) = (\iota \otimes \omega_{\xi,\eta})(U)\Lambda_{\psi_N}(x).$$

Then by the closedness of Λ_{N,δ_N}, we can conclude that for x of the above form, $(\iota \otimes \omega)(\alpha(x)) \in \mathcal{D}(\Lambda_{N,\delta_N})$ for every $\omega \in M_*$, with

$$\Lambda_{N,\delta_N}((\iota \otimes \omega)(\alpha(x))) = (\iota \otimes \omega)(U)\Lambda_{N,\delta_N}(x).$$

Since such x form a σ-strong-norm core for Λ_{N,δ_N}, the same statement holds for a general $x \in \mathcal{N}_{\psi_N}$. >From this, it is standard to conclude the invariance: take $\omega = \omega_{\xi,\xi} \in M_*^+$ and $x = y^*y \in \mathcal{M}_{\psi_N}^+$. Let ξ_i denote an orthonormal basis for $\mathcal{L}^2(M)$. Then by the lower-semi-continuity of ψ_N, we find

$$\begin{aligned}
\psi_N((\iota\otimes\omega_{\xi,\xi})(\alpha(y^*y))) &= \psi_N\Big(\sum_n (\iota \otimes \omega_{\xi,\xi_n})(\alpha(y))^*((\iota \otimes \omega_{\xi,\xi_n})(\alpha(y)))\Big) \\
&= \sum_n \psi_N((\iota \otimes \omega_{\xi,\xi_n})(\alpha(y))^*((\iota \otimes \omega_{\xi,\xi_n})(\alpha(y)))) \\
&= \sum_n \|\Lambda_{N,\delta_N}((\iota \otimes \omega_{\xi,\xi_n})(\alpha(y)))\|^2 \\
&= \sum_n \|(\iota \otimes \omega_{\xi,\xi_n})(U)\Lambda_{N,\delta_N}(y)\|^2 \\
&= \Big\langle \Lambda_{N,\delta_N}(y), \Big(\sum_n (\iota\otimes\omega_{\xi_n,\xi})(U^*)(\iota\otimes\omega_{\xi,\xi_n})(U)\Big)\Lambda_{N,\delta_N}(y)\Big\rangle \\
&= \langle \Lambda_{N,\delta_N}(y), (\iota\otimes\omega_{\xi,\xi})(U^*U)\Lambda_{N,\delta_N}(y)\rangle = \psi_N(y^*y)\omega_{\xi,\xi}(1),
\end{aligned}$$

hence $\psi_N((\iota \otimes \omega)(\alpha(x))) = \psi_N(x)\omega(1)$. ∎

REMARK 4.20. It is natural to ask if there is a corresponding result for general Galois coactions. We briefly show that one can not expect *too* much: for general Galois coactions, there does not have to exist an invariant nsf operator valued weight T_{ψ_N}, i.e. an operator valued weight $N^+ \to (N^\alpha)^{+,\mathrm{ext}}$ such that

$$T_{\psi_N}((\iota \otimes \omega)\alpha(x)) = \omega(1)T_{\psi_N}(x) \quad \text{for all } \omega \in M_*^+, x \in \mathcal{M}_{T_{\psi_N}}^+.$$

To give an explicit example, suppose α is an integrable outer left coaction of a von Neumann algebraic quantum group (M, Δ) on a factor N. Then by outerness, there is a *unique* nsf operator valued weight $(N \rtimes M)^+ \to \alpha(N)^{+,\mathrm{ext}}$ (up to a scalar), namely $(\iota \otimes \widehat{\varphi})\widehat{\alpha}$, where $\widehat{\alpha}$ is the dual right coaction. But if $(\widehat{M}, \widehat{\Delta})$ is not unimodular, then this operator valued weight is not invariant. On the other hand, this does *not* rule out the possibility that there exists an invariant nsf *weight*: for if the original coaction has an invariant nsf weight ψ_N (for example, the coactions occurring in [23]), then one checks that $x \in (N \rtimes M)^+ \to \psi_{\widehat{M}'}((\psi_N \otimes \iota)(x)) \in [0, +\infty]$ is a well-defined $\widehat{\alpha}$-invariant nsf weight on $N \rtimes M$. We do not know of any example of a Galois coaction without invariant weights.

PROPOSITION 4.21. *Denote* $\widehat{\nabla}_N^{\mathrm{i}t} = P_N^{\mathrm{i}t} J_N \delta_N^{\mathrm{i}t} J_N$. *Then*

$$\widehat{\nabla}_N^{-\mathrm{i}t} \widehat{\pi}_1(m) \widehat{\nabla}_N^{\mathrm{i}t} = \widehat{\pi}_1(\sigma_t^{\widehat{\varphi}^{\mathrm{op}}}(m)) \quad \textit{for } m \in \widehat{M}'.$$

Proof. First note that $\widehat{\nabla}_N^{\mathrm{i}t}$ is well-defined, since $P_N^{\mathrm{i}t}$ is easily seen to commute with J_N and $\delta_N^{\mathrm{i}t}$. Then also

$$\widehat{\nabla}_N^{\mathrm{i}t} \Lambda_{\psi_N}(x) = \Lambda_{\psi_N}(\tau_t^N(x)\delta_N^{-\mathrm{i}t}) \quad \text{for } x \in \mathcal{N}_{\psi_N},$$

by an easy adjustment of Lemma 4.14 and using the relative invariance property of $\delta_N^{\mathrm{i}t}$. If we apply $(\iota \otimes \omega)(U)$ to this with $\omega \in M_*$, then, using the commutation rules between α, τ_t^N and $\delta_N^{\mathrm{i}t}$, we get

$$(\iota \otimes \omega)(U) \widehat{\nabla}_N^{\mathrm{i}t} \Lambda_{\psi_N}(x) = \Lambda_{\psi_N}(\tau_t^N((\iota \otimes \omega(\tau_t(\cdot)\delta^{-\mathrm{i}t}))\alpha(x))\delta_N^{-\mathrm{i}t}).$$

This shows

$$\widehat{\nabla}_N^{-\mathrm{i}t} \widehat{\pi}_1((\iota \otimes \omega)(V)) \widehat{\nabla}_N^{\mathrm{i}t} = \widehat{\pi}_1((\iota \otimes \omega(\tau_t(\cdot)\delta^{-\mathrm{i}t}))(V)).$$

But this is exactly $\widehat{\pi}_1(\sigma_t^{\widehat{\varphi}^{\mathrm{op}}}((\iota \otimes \omega)(V)))$. Then of course the same holds with $(\iota \otimes \omega)(V)$ replaced by a general element of $\widehat{M}'$, thus proving the proposition. ∎

PROPOSITION 4.22. *The following commutation relations hold:*

(i) $(\nabla^{\mathrm{i}t} \otimes \widehat{\nabla}_N^{\mathrm{i}t})\widetilde{G} = \widetilde{G}(\nabla_N^{\mathrm{i}t} \otimes \widehat{\nabla}_N^{\mathrm{i}t})$;

(ii) $(\widehat{\nabla}^{\mathrm{i}t} \otimes P_N^{\mathrm{i}t})\widetilde{G} = \widetilde{G}(\widehat{\nabla}_N^{\mathrm{i}t} \otimes P_N^{\mathrm{i}t}\delta_N^{\mathrm{i}t})$.

Proof. The first formula follows by the second formula in Corollary 4.15, and the fact that the second leg of $\widetilde{G}$ lies in N. The second formula follows from the fact that also $\widehat{\nabla}^{\mathrm{i}t} = J\delta^{\mathrm{i}t}JP^{\mathrm{i}t}$, then using the third formula of Corollary 4.15 and the first formula in Lemma 4.18 together with the definition of δ_N. ∎

THEOREM 4.23. *Up to a positive constant, ψ_N is the only invariant, and φ_N the only δ-invariant weight on N.*

Proof. The claim about φ_N follows immediately by Lemma 3.9 of [22] and the fact that α is ergodic. The second statement can be proven in the same fashion. ∎

Before going over to the next section, we remark that of course all results hold as well in the context of *left Galois coactions*: if (P, Δ_P) is a von Neumann algebraic quantum group, N a von Neumann algebra, and γ an integrable ergodic left coaction of (P,Δ_P) on N, we call (N,γ) a *left Galois object* if, with $\psi_N = (\psi\otimes\iota)\gamma$, the *Galois map*

$$\begin{aligned}&\widetilde{H}: \mathcal{L}^2(N)\otimes\mathcal{L}^2(N)\to\mathcal{L}^2(N)\otimes\mathcal{L}^2(P),\\ &\Lambda_{\psi_N}(x)\otimes\Lambda_{\psi_N}(y)\to(\Lambda_\psi\otimes\Lambda_{\psi_N})(\gamma(x)(1\otimes y)),\quad x,y\in\mathcal{N}_{\psi_N},\end{aligned}$$

is a unitary. We will therefore use the proper analogous statements of this section in the left context without further proof.

5. REFLECTING ACROSS A GALOIS OBJECT

In this section, we will construct another von Neumann algebraic quantum group given a Galois object (N,α) for a von Neumann algebraic quantum group (M,Δ). In fact, the new quantum group will be a corner of a special kind of quantum groupoid, with $(\widehat{M},\widehat{\Delta})$ in the other corner. This quantum groupoid picture turns out to be very useful, providing one with the right intuition on how to proceed. We use notation as in the previous section. For convenience, we will now treat also $\mathcal{L}^2(M)\otimes\mathcal{L}^2(N)$ as an $N\rtimes M$-bimodule (by applying $\mathrm{Ad}(\Sigma)$ to the previous representations), so that we can call $\widetilde{G}$ a bimodule map.

Denote as before by $\widehat{Q} = \begin{pmatrix}\widehat{Q}_{11} & \widehat{Q}_{12}\\ \widehat{Q}_{21} & \widehat{Q}_{22}\end{pmatrix} = \begin{pmatrix}\widehat{P} & \widehat{N}\\ \widehat{O} & \widehat{M}\end{pmatrix}$ the linking algebra between the right $\widehat{M}$-modules $\mathcal{L}^2(M)$ and $\mathcal{L}^2(N)$ (see the remark before Corollary 4.3). We will sometimes denote the natural inclusion $\widehat{Q}\subseteq B\begin{pmatrix}\mathcal{L}^2(N)\\ \mathcal{L}^2(M)\end{pmatrix}$ by $\pi^{\widehat{Q},2} = (\pi^2_{ij})_{i,j}$ for emphasis. We will identify the $\widehat{Q}_{ij}$ with their parts in $\widehat{Q}$ (so for example if $x\in\widehat{Q}_{12}$, we identify it with $\left(\begin{smallmatrix}0 & x\\ 0 & 0\end{smallmatrix}\right)$), *except* that we will write the unit 1 of $\widehat{Q}_{22} = \widehat{M}$ as $1_{\widehat{M}}$ when we see it as a projection in $\widehat{Q}$ (likewise for $\widehat{Q}_{11} = \widehat{P}$). As before, we denote the right $\widehat{M}$-module structure on $\mathcal{L}^2(N)$ by $\widehat{\pi}_{\mathrm{r}}$, i.e. $\widehat{\pi}_{\mathrm{r}}(m) = \widehat{\pi}_{\mathrm{l}}(\widehat{J}m^*\widehat{J}) = \rho((1\otimes\widehat{J}m^*\widehat{J}))$ for $m\in\widehat{M}$, where ρ is the Galois homomorphism for α. By $\widehat{\pi}_{\mathrm{r}}$, we also denote the map $\widehat{\pi}_{\mathrm{r}}$ with respect to the Galois object (M,Δ), i.e. the standard right representation $\widehat{\pi}_{\mathrm{r}}(m) = \widehat{J}m^*\widehat{J}$, and by $\widehat{\theta}_{\mathrm{r}}$ also the right representation $m\to\widehat{R}(m)$ of $\widehat{M}$ on $\mathcal{L}^2(M)$. This will not lead to any ambiguities, as we will in fact only use this double notation $\widehat{\pi}_{\mathrm{r}}$ in the proof of the following lemma.

LEMMA 5.1. *We have* $\widetilde{G}^*(1\otimes\widehat{N})\widehat{W}\subseteq\widehat{N}\otimes\widehat{N}$.

REMARK 5.2. By $\widehat{N}\otimes\widehat{N}$, we mean the σ-weak closure of the algebraic tensor product of $\widehat{N}$ with itself inside $\widehat{Q}\otimes\widehat{Q}$. By the commutation theorem for tensor products of von Neumann algebras, this coincides with the space of intertwiners for the right $\widehat{M}\otimes\widehat{M}$-modules $\mathcal{L}^2(M)\otimes\mathcal{L}^2(M)$ and $\mathcal{L}^2(N)\otimes\mathcal{L}^2(N)$.

Proof. Let x be an element of $\widehat{N}$. As the first leg of $\widehat{W}$ lies in $\widehat{M}$, and $\widetilde{G}$ is a left $\widehat{M}'$-module morphism, it is clear that for any $m \in \widehat{M}$, we have

$$\widetilde{G}^*(1 \otimes x)\widehat{W}(\widehat{\pi}_r(m) \otimes 1) = (\widehat{\pi}_r(m) \otimes 1)\widetilde{G}^*(1 \otimes x)\widehat{W}.$$

On the other hand, we have to prove that for all $m \in \widehat{M}$,

$$\widetilde{G}^*(1 \otimes x)\widehat{W}(1 \otimes \widehat{\pi}_r(m)) = (1 \otimes \widehat{\pi}_r(m))\widetilde{G}^*(1 \otimes x)\widehat{W}. \tag{5.1}$$

Now as $\widetilde{G}$ is a right $N \rtimes M$-map, we have

$$(1 \otimes \widehat{\pi}_r(m))\widetilde{G}^* = \widetilde{G}^*((\widehat{\theta}_r \otimes \widehat{\pi}_r)\widehat{\Delta}(m)),$$

using the fourth commutation relation of Lemma 4.2 in a slightly adapted form. Since also

$$\widehat{W}(1 \otimes \widehat{\pi}_r(m)) = ((\widehat{\theta}_r \otimes \widehat{\pi}_r)(\widehat{\Delta}(m))\widehat{W},$$

the stated commutation follows from the intertwining property of x, as $x\widehat{\pi}_r(m) = \widehat{\pi}_r(m)x$. ∎

Denote the corresponding map by

$$\Delta_{\widehat{N}} : \widehat{N} \to \widehat{N} \otimes \widehat{N} : x \to \widetilde{G}^*(1 \otimes x)\widehat{W}.$$

Then we can also define

$$\Delta_{\widehat{O}} : \widehat{O} \to \widehat{O} \otimes \widehat{O} : x \to \Delta_{\widehat{N}}(x^*)^*, \quad \text{and} \quad \Delta_{\widehat{P}} : \widehat{P} \to \widehat{P} \otimes \widehat{P} : x \to \widetilde{G}^*(1 \otimes x)\widetilde{G},$$

since $\widehat{Q}_{21} = (\widehat{Q}_{12})^*$ and the span of $\widehat{Q}_{12}\widehat{Q}_{21}$ is σ-weakly dense in $\widehat{P}$. Finally, we denote by $\Delta_{\widehat{Q}}$ the map

$$\widehat{Q} \to \widehat{Q} \otimes \widehat{Q} : x_{ij} \to \widehat{\Delta}_{ij}(x_{ij}), \quad x_{ij} \in \widehat{Q}_{ij},$$

where we denote $\widehat{\Delta}_{11} = \Delta_{\widehat{P}}, \ldots$ (in the following, we will use both notations without further comment). Then $\Delta_{\widehat{Q}}$ is easily seen to be a $*$-homomorphism. *However*, it is *not* unital: $\Delta_{\widehat{Q}}(1_{\widehat{Q}}) = 1_{\widehat{M}} \otimes 1_{\widehat{M}} + 1_{\widehat{P}} \otimes 1_{\widehat{P}}$ does not equal $(1_{\widehat{M}} + 1_{\widehat{P}}) \otimes (1_{\widehat{M}} + 1_{\widehat{P}}) = 1_{\widehat{Q} \otimes \widehat{Q}}$.

LEMMA 5.3. *The map $\Delta_{\widehat{Q}}$ is coassociative.*

The proof follows trivially by Proposition 4.5.

Since $J_N\widehat{\pi}_l(m)^*J_N = \widehat{\pi}_l(Jm^*J)$ for $m \in \widehat{M}'$, we can define an anti-$*$-isomorphism $R_{\widehat{Q}} : \widehat{Q} \to \widehat{Q}$ by sending $x \in \widehat{Q}_{12}$ to $(J_N xJ)^*$, and then extending it in the natural way.

LEMMA 5.4. *We have $\Delta_{\widehat{Q}}(R_{\widehat{Q}}(x)) = (R_{\widehat{Q}} \otimes R_{\widehat{Q}})\Delta^{\mathrm{op}}_{\widehat{Q}}(x)$ for $x \in \widehat{Q}$.*

Proof. We only have to check whether

$$\widetilde{G}^*(1 \otimes J_N xJ)\widehat{W} = (J_N \otimes J_N)\Sigma\widetilde{G}^*(1 \otimes x)\widehat{W}\Sigma(J \otimes J)$$

for $x \in \widehat{Q}_{12}$. But using Lemma 4.4 twice, once for N and once for M itself, the right hand side simplifies as follows:

$$\begin{aligned}(J_N \otimes J_N)\Sigma\widetilde{G}^*(1\otimes x)\widehat{W}\Sigma(J\otimes J) &= \widetilde{G}^*(\widehat{J}\otimes J_N)\Sigma U^*\Sigma(1\otimes x)\widehat{W}\Sigma(J\otimes J)\\ &= \widetilde{G}^*(\widehat{J}\otimes J_N)(1\otimes x)\Sigma V^*\Sigma\widehat{W}\Sigma(J\otimes J)\\ &= \widetilde{G}^*(1\otimes J_N x J)\widehat{W}. \quad \blacksquare\end{aligned}$$

This $(\widehat{Q},\Delta_{\widehat{Q}})$ could be called a coinvolutive Hopf–von Neumann algebraic quantum groupoid, and in particular $(\widehat{P},\Delta_{\widehat{P}})$ is a coinvolutive Hopf–von Neumann algebra. We proceed to show that $(\widehat{Q},\Delta_{\widehat{Q}})$ is a measured quantum groupoid ([15]), and in particular that $(\widehat{P},\Delta_{\widehat{P}})$ is a von Neumann algebraic quantum group. However, we first briefly return to the situation of a general Galois coaction: it is not difficult to see that up to this point, everything in this section could be done without assuming α ergodic. Of course, $\widehat{P}$ will then not be a quantum group, but a quantum groupoid. More precisely: we will have that $(\widehat{P},N^\alpha,\pi_{\mathrm{l}},\pi_{\mathrm{r}},\Delta_{\widehat{P}})$ is a Hopf bimodule (in the sense of Definition 3.1 of [4]), with $\Delta_{\widehat{P}}(x)=\widetilde{G}^*(1\otimes x)\widetilde{G}\in P \underset{N^\alpha}{{}_{\pi_{\mathrm{r}}}*_{\pi_{\mathrm{l}}}} P$ for $x\in\widehat{P}$. We can even equip it with a "scaling group" and a unitary antipode. However, we do not know if $\widehat{P}$ can actually be made into a measured quantum groupoid in general.

We have shown in Proposition 4.21 that the modular automorphism group of $\widehat{\varphi}^{\mathrm{op}}$ on $\widehat{M}'$ can be implemented on $\mathcal{L}^2(N)$ by the one-parameter group $\widehat{\nabla}_N^{it}$. Then by Theorem IX.3.11 in [21], we can construct an nsf weight $\varphi_{\widehat{P}}$ on $\widehat{P}$ which has $\widehat{\nabla}_N$ as spatial derivative with respect to $\widehat{\varphi}^{\mathrm{op}}$. Then we can also consider the balanced weight $\varphi_{\widehat{Q}}=\varphi_{\widehat{P}}\oplus\widehat{\varphi}$ on $\widehat{Q}$. Its modular automorphism group $\sigma_t^{\widehat{Q}}$ is then implemented by $\widehat{\nabla}_N^{it}\oplus\widehat{\nabla}^{it}$ if we use the faithful representation $\pi^{\widehat{Q},2}$ of $\widehat{Q}$ on $\mathcal{L}^2(N)\oplus\mathcal{L}^2(M)$.

We make the identification

$$(\mathcal{L}^2(\widehat{Q}),\pi^{\widehat{Q}},\Lambda_{\widehat{Q}})\cong\left(\begin{pmatrix}\mathcal{L}^2(\widehat{P}) & \mathcal{L}^2(N)\\ \mathcal{L}^2(N) & \mathcal{L}^2(M)\end{pmatrix},\pi^{\widehat{Q}},(\widehat{\Lambda}_{ij})\right)$$

of the natural semi-cyclic representations of $\widehat{Q}$ with respect to $\varphi_{\widehat{Q}}$, as in Lemma IX.3.5 of [21] and the remark above it. Here $\begin{pmatrix}\mathcal{L}^2(\widehat{P}) & \mathcal{L}^2(N)\\ \mathcal{L}^2(N) & \mathcal{L}^2(M)\end{pmatrix}$ is just the direct sum Hilbert space of its entries, written as a matrix to emphasize its left $\widehat{Q}$-module structure. Further, $\widehat{\Lambda}_{11}$ and $\widehat{\Lambda}_{22}$ are the GNS-constructions for the weights $\varphi_{\widehat{P}}$ and $\widehat{\varphi}$. The map $\widehat{\Lambda}_{12}:\widehat{Q}_{12}\cap\mathcal{N}_{\varphi_{\widehat{Q}}}\to\mathcal{L}^2(N)$ is determined by $\widehat{\Lambda}_{12}(L_\xi)=\xi$ for $\xi\in\mathcal{L}^2(N)$ left-bounded, i.e. those ξ for which the closure L_ξ of the map

$$\widehat{\Lambda}^{\mathrm{op}}(m)=\widehat{\Lambda}^{\mathrm{op}}(\widehat{J}m^*\widehat{J})\to\widehat{\pi}_{\mathrm{r}}(m)\xi=\widehat{\pi}_{\mathrm{l}}(\widehat{J}m^*\widehat{J})\xi \quad \text{for } m\in\mathcal{N}_{\widehat{\varphi}}^*$$

is bounded. The map $\widehat{\Lambda}_{21}$ is determined by $\widehat{\Lambda}_{21}(L_\xi^*) = \overline{\widehat{\nabla}_N^{1/2}\xi}$ for $\xi \in \mathcal{L}^2(N)$ left-bounded *and* in the domain of $\widehat{\nabla}_N^{1/2}$. Then the restriction $\widehat{J}_{21}$ of the modular conjugation $J_{\widehat{Q}}$ of $\varphi_{\widehat{Q}}$ to a map

$$\Lambda_{\widehat{Q}}(\mathcal{N}_{\varphi_{\widehat{Q}}} \cap \widehat{Q}_{21}) \to \Lambda_{\widehat{Q}}(\mathcal{N}_{\varphi_{\widehat{Q}}} \cap \widehat{Q}_{12})$$

is simply the natural anti-unitary map $\overline{\mathcal{L}^2(N)} \to \mathcal{L}^2(N) : \overline{\xi} \to \xi$. We will denote the inverse of this map by $\widehat{J}_{12}$. Finally, we note that $\pi^{\widehat{Q}}$ decomposes as $\pi^{\widehat{Q},1} \oplus \pi^{\widehat{Q},2}$, with $\pi^{\widehat{Q},i}$ acting on the i-th column, and we will then also write $\pi^{\widehat{Q},1} = (\pi^1_{ij})_{i,j}$.

We will now provide another formula for $\widetilde{G}^*$.

LEMMA 5.5. *If* $m \in \mathcal{N}_{\widehat{\varphi}}$ *and* $x \in \widehat{N} \cap \mathcal{N}_{\varphi_{\widehat{Q}}}$, *then*

$$\Delta_{\widehat{N}}(x)(m\otimes 1) \in \mathcal{D}(\Lambda_{\widehat{N}} \otimes \Lambda_{\widehat{N}}) \quad \textit{and} \quad (\Lambda_{\widehat{N}} \otimes \Lambda_{\widehat{N}})(\Delta_{\widehat{N}}(x)(m\otimes 1)) = \widetilde{G}^*(\widehat{\Lambda}(m) \otimes \Lambda_{\widehat{N}}(x)).$$

Proof. Since

$$\begin{aligned}(\iota \otimes \varphi_{\widehat{Q}})((m^* \otimes 1)\widehat{\Delta}_{12}(x)^*\widehat{\Delta}_{12}(x)(m\otimes 1)) &= (\iota \otimes \widehat{\varphi})((m^* \otimes 1)\widehat{\Delta}(x^*x)(m\otimes 1)) \\ &= \varphi_{\widehat{Q}}(x^*x)m^*m\end{aligned}$$

for $x \in \widehat{Q}_{12}$ and $m \in \widehat{M}$, it is clear that $\widehat{\Delta}_{12}(x)(m\otimes 1) \in \mathcal{D}(\widehat{\Lambda}_{12} \otimes \widehat{\Lambda}_{12})$ for $m \in \mathcal{N}_{\widehat{\varphi}}$ and $x \in \widehat{Q}_{12} \cap \mathcal{N}_{\varphi_{\widehat{Q}}}$, and that the map

$$\widehat{\Lambda}(m) \otimes \widehat{\Lambda}_{12}(x) \to (\widehat{\Lambda}_{12} \otimes \widehat{\Lambda}_{12})(\widehat{\Delta}_{12}(x)(m\otimes 1))$$

extends to a well-defined isometry. We now show that it coincides with $\widetilde{G}^*$.

Let z be an element of $\mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}}$. Then it is sufficient to prove that

$$\widehat{\Delta}_{12}(x)(\widehat{\Lambda}(m) \otimes \widehat{\Lambda}^{\mathrm{op}}(z)) = (1 \otimes \widehat{\pi}_1(z))\widetilde{G}^*(\widehat{\Lambda}(m) \otimes \widehat{\Lambda}_{12}(x)).$$

But $\widehat{\Delta}_{12}(x) = \widetilde{G}^*(1\otimes x)\widehat{W}$, and bringing $\widetilde{G}$ to the other side, $\widetilde{G}(1 \otimes \widehat{\pi}_1(z))\widetilde{G}^*$ can be written as $\Sigma U(1 \otimes \widehat{J}\widehat{R}'(z)^*\widehat{J})U^*\Sigma$. Taking a scalar product in the first factor, it is then sufficient to prove that for $\omega \in \widehat{M}'_*$, we have

$$x(\omega\otimes\iota)(\widehat{W})\widehat{\Lambda}^{\mathrm{op}}(z) = (\iota\otimes\omega)(U(1 \otimes \widehat{J}\widehat{R}'(z)\widehat{J})U^*)\widehat{\Lambda}_{12}(x).$$

But now using again that $(\widehat{\pi}_1 \otimes \iota)(V) = U$, it is sufficient to show that

$$(\iota\otimes\omega)(V(1\otimes \widehat{J}\widehat{R}'(z)\widehat{J})V^*) \in \mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}}$$

and that applying $\widehat{\Lambda}^{\mathrm{op}}$ to it gives $(\omega\otimes\iota)(\widehat{W})\widehat{\Lambda}^{\mathrm{op}}(z)$. We could check this directly, but we can just as easily backtrack our arguments: we only have to see if for $y \in \mathcal{N}_{\widehat{\varphi}}$, we have

$$y(\omega\otimes\iota)(\widehat{W})\widehat{\Lambda}^{\mathrm{op}}(z) = (\iota\otimes\omega)(V(1\otimes \widehat{J}\widehat{R}'(z)\widehat{J})V^*)\widehat{\Lambda}(y)$$

for any $z \in \mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}}$. This is then seen to be the same as saying that

$$(\widehat{\Lambda}\otimes\widehat{\Lambda})(\widehat{\Delta}(y)(m\otimes 1)) = \widehat{W}^*(\widehat{\Lambda}(m)\otimes\widehat{\Lambda}(y)),$$

which is of course true by definition. ∎

LEMMA 5.6. *Let x be in $\mathcal{N}_{\varphi_N}\cap\mathcal{N}_{\varphi_N}^*$, and $a\in\mathcal{T}_\varphi$, the Tomita algebra for φ. Then*

$$(\omega_{\Lambda_N(x^*),\Lambda(\sigma_{\mathrm{i}}(a)^*)}\otimes\iota)(\widetilde{G})=(\omega_{\Lambda(a),\Lambda_N(x)}\otimes\iota)(\widetilde{G}^*).$$

Proof. Choose $\omega\in N_*$. Then we have the following which ends the proof:

$$\begin{aligned}\omega((\omega_{\Lambda_N(x^*),\Lambda(\sigma_{\mathrm{i}}(a)^*)}\otimes\iota)(\widetilde{G}))&=\varphi(\sigma_i(a)((\omega\otimes\iota)(\alpha(x)^*)))=\varphi(((\omega\otimes\iota)(\alpha(x)^*))a)\\&=\langle\Lambda(a),\Lambda((\overline{\omega}\otimes\iota)\alpha(x))\rangle=\langle\Lambda(a),(\iota\otimes\overline{\omega})(\widetilde{G})\Lambda_N(x)\rangle\\&=\omega((\omega_{\Lambda(a),\Lambda_N(x)}\otimes\iota)(\widetilde{G}^*)).\end{aligned}$$ ∎

LEMMA 5.7. *Let $x\in\mathcal{N}_{\varphi_N}$ and $y\in\mathcal{T}_{\varphi_N}$. Then writing $w=x\sigma_{-\mathrm{i}}^N(y^*)$, we have that $\Lambda_N(w)$ is left-bounded, and*

$$L_{\Lambda_N(w)}=(\iota\otimes\omega_{\Lambda_N(x),\Lambda_N(y)})(\widetilde{G}^*).$$

Proof. We have to prove that for $m\in\mathcal{N}_{\widehat{\varphi}^{\mathrm{op}}}$, we have

$$(\iota\otimes\omega_{\Lambda_N(x),\Lambda_N(y)})(\widetilde{G}^*)\widehat{\Lambda}^{\mathrm{op}}(m)=\widehat{\pi}_1(m)\Lambda_N(x\sigma_{-\mathrm{i}}^N(y^*)).$$

But using the square (3.2) at the end of Section 1 and Lemma 2.6, we get for any $z\in\mathcal{N}_{\varphi_N}$ the following which ends the proof:

$$\begin{aligned}\langle(\iota\otimes\omega_{\Lambda_N(x),\Lambda_N(y)})(\widetilde{G}^*)\widehat{\Lambda}^{\mathrm{op}}(m),\Lambda_N(z)\rangle&=\langle\widetilde{G}^*(\widehat{\Lambda}^{\mathrm{op}}(m)\otimes\Lambda_N(x)),\Lambda_N(z)\otimes\Lambda_N(y)\rangle\\&=\langle\Lambda_{N_2}(\widehat{\pi}_1(m)x),\Lambda_N(z)\otimes\Lambda_N(y)\rangle\\&=\langle\widehat{\pi}_1(m)x\Lambda_N(\sigma_{-\mathrm{i}}^N(y^*)),\Lambda_N(z)\rangle.\end{aligned}$$ ∎

PROPOSITION 5.8. *If $x\in\widehat{N}\cap\mathcal{N}_{\varphi_{\widehat{Q}}}$ and $y\in\widehat{O}\cap\mathcal{N}_{\varphi_{\widehat{Q}}}$, then $\Delta_{\widehat{O}}(y)(x\otimes1)$ in $\mathcal{D}(\widehat{\Lambda}\otimes\Lambda_{\widehat{O}})$, and*

$$(\widehat{\Lambda}\otimes\Lambda_{\widehat{O}})(\Delta_{\widehat{O}}(y)(x\otimes1))=(J\otimes\widehat{J}_{12})\widetilde{G}(J_N\otimes\widehat{J}_{21})(\Lambda_{\widehat{N}}(x)\otimes\Lambda_{\widehat{O}}(y)).$$

REMARK 5.9. Compare this formula with the identity $(\widehat{J}\otimes J)W(\widehat{J}\otimes J)=W^*$.

Proof. This statement is equivalent with proving for sufficiently many y in $\widehat{Q}_{21}\cap\mathcal{N}_{\varphi_{\widehat{Q}}}$ and $\omega\in\widehat{Q}_{12,*}$ that $(\omega\otimes\iota)(\widehat{\Delta}_{21}(y))\in\mathcal{N}_{\varphi_{\widehat{Q}}}$, and

$$\widehat{\Lambda}_{21}((\omega\otimes\iota)(\widehat{\Delta}_{21}(y)))=(\omega\otimes\iota)((J\otimes\widehat{J}_{12})\widetilde{G}(J_N\otimes\widehat{J}_{21}))\widehat{\Lambda}_{21}(y),$$

which can be written as

$$\widehat{J}_{21}\widehat{\Lambda}_{21}((\omega\otimes\iota)(\widehat{\Delta}_{21}(y)))=(\overline{\omega}(J(\cdot)^*J_N)\otimes\iota)(\widetilde{G})\widehat{J}_{21}\widehat{\Lambda}_{21}(y).\tag{5.2}$$

Let $y\in\widehat{Q}_{21}\cap\mathcal{N}_{\varphi_{\widehat{Q}}}$ be in the Tomita algebra of $\varphi_{\widehat{Q}}$. Let ω be of the form $\omega_{\Lambda_N(x),\Lambda(a)}$ with x,a in the Tomita algebra of respectively φ_N and φ. Then by the first formula of Lemma 4.22 (used both in the general case and the case where $N=M$), we have that $(\omega\otimes\iota)(\widehat{\Delta}_{21}(y))$ will also be analytic for $\sigma_t^{\widehat{Q}}$, with

$$\sigma_{-\mathrm{i}/2}^{\widehat{Q}}((\omega_{\Lambda_N(x),\Lambda(a)}\otimes\iota)(\widehat{\Delta}_{21}(y)))=(\omega_{\nabla_N^{1/2}\Lambda_N(x),\nabla^{-1/2}\Lambda(a)}\otimes\iota)(\widehat{\Delta}_{12}(\sigma_{-\mathrm{i}/2}^{\widehat{Q}}(y))).$$

Further, $(\omega \otimes \iota)(\widehat{\Delta}_{21}(y))^* = (\overline{\omega} \otimes \iota)(\widehat{\Delta}_{12}(y^*))$, which will be in $\mathcal{D}(\widehat{\Lambda}_{12})$ by Lemma 5.5, with

$$\widehat{\Lambda}_{12}((\overline{\omega} \otimes \iota)(\widehat{\Delta}_{12}(y^*))) = (\overline{\omega} \otimes \iota)(\widetilde{G}^*)\widehat{\Lambda}_{12}(y^*).$$

This shows that $(\omega \otimes \iota)(\widehat{\Delta}_{21}(y)) \in \mathcal{D}(\widehat{\Lambda}_{21})$.

Now by Lemma 5.6, we have then also

$$\widehat{\Lambda}_{12}((\overline{\omega} \otimes \iota)(\widehat{\Delta}_{12}(y^*))) = (\omega_{\Lambda_N(x^*),\Lambda(\sigma_{-\mathrm{i}}(a^*))} \otimes \iota)(\widetilde{G})\widehat{\Lambda}_{12}(y^*),$$

and by Lemma 4.22, we have that $(\omega_{\Lambda_N(x^*),\Lambda(\sigma_{-\mathrm{i}}(a^*))} \otimes \iota)(\widetilde{G})$ is analytic for $\chi_t = \mathrm{Ad}(\widehat{\nabla}_N^{\mathrm{i}t})$, with

$$\chi_{-\mathrm{i}/2}((\omega_{\Lambda_N(x^*),\Lambda(\sigma_{-\mathrm{i}}(a^*))} \otimes \iota)(\widetilde{G})) = (\omega_{J_N\Lambda_N(x),J\Lambda(a)} \otimes \iota)(\widetilde{G}).$$

So combining all this, we get

$$\begin{aligned}
\widehat{J}_{21}\widehat{\Lambda}_{21}((\omega \otimes \iota)(\widehat{\Delta}_{21}(y))) &= \nabla_{\widehat{Q}}^{1/2}\widehat{\Lambda}_{12}((\omega \otimes \iota)(\widehat{\Delta}_{21}(y))^*) \\
&= (\nabla_{\widehat{Q}}^{1/2}(\omega_{\Lambda_N(x^*),\Lambda(\sigma_{-\mathrm{i}}(a^*))} \otimes \iota)(\widetilde{G})\nabla_{\widehat{Q}}^{-1/2})\nabla_{\widehat{Q}}^{1/2}\widehat{\Lambda}_{12}(y^*) \\
&= (\omega_{J_N\Lambda_N(x),J\Lambda(a)} \otimes \iota)(\widetilde{G})\widehat{J}_{21}\widehat{\Lambda}_{21}(y) \\
&= (\overline{\omega}(J(\cdot)^*J_N) \otimes \iota)(\widetilde{G})\widehat{J}_{21}\widehat{\Lambda}_{21}(y).
\end{aligned}$$

Now by closedness of $\Lambda_{\widehat{Q}}$, this equality remains true for ω arbitrary. Since such y's form a σ-strong$*$-norm core for $\widehat{\Lambda}_{12}$, the equality is true for any $y \in \widehat{Q}_{21} \cap \mathcal{N}_{\varphi_{\widehat{Q}}}$. ∎

THEOREM 5.10. *The weight $\varphi_{\widehat{P}}$ is left invariant.*

Proof. It follows from the last proposition that

$$(\iota \otimes \varphi_{\widehat{P}})(\Delta_{\widehat{P}}(L_\xi L_\xi^*)) = \varphi_{\widehat{P}}(L_\xi L_\xi^*)$$

for ξ right-bounded and in the domain of $\widehat{\nabla}_N^{1/2}$. From Lemma IX.3.9 of [21], it follows that also $(\iota \otimes \varphi_{\widehat{P}})(\Delta_{\widehat{P}}(b)) = \varphi_{\widehat{P}}(b)$ for $b \in \mathcal{M}_{\varphi_{\widehat{P}}}^+$. Indeed: that lemma implies that b can be approximated from below by elements of the form $\sum_{i=1}^{n} L_{\xi_i} L_{\xi_i}^*$ with ξ_i right-bounded, and since b is integrable, every ξ_i must be in $\mathcal{D}(\widehat{\nabla}_N^{1/2})$. So we can conclude by lower-semi-continuity. ∎

This proves that $(\widehat{P}, \Delta_{\widehat{P}})$ is a von Neumann algebraic quantum group, since $\varphi_{\widehat{P}}$ is a left invariant weight, and by Lemma 5.4, $\psi_{\widehat{P}} := \varphi_{\widehat{P}} \circ R_{\widehat{Q}}$ will be a right invariant weight.

DEFINITION 5.11. If (N, α) is a Galois object for a von Neumann algebraic quantum group (M, Δ), and $(\widehat{P}, \Delta_{\widehat{P}})$ the von Neumann algebraic quantum group constructed from it in the foregoing manner, then we call $(\widehat{P}, \Delta_{\widehat{P}})$ the *reflected von Neumann algebraic quantum group* (or just the reflection) of $(\widehat{M}, \Delta_{\widehat{M}})$ across (N, α).

To end this section, we show that $(\widehat{Q},\Delta_{\widehat{Q}})$ is a measured quantum groupoid. In fact, our set-up is closer in spirit to the formulation of the generalized Kac algebras of [32], but this theory has no full generalization to the "locally compact" world. It is however well known that these approaches are equivalent in the finite-dimensional Kac case (cf. [17]).

Let d be the natural imbedding of $\mathbb{C}^2$ in $\widehat{Q}$:

$$d:\mathbb{C}^2\rightarrow\widehat{Q}:(w,z)\rightarrow\begin{pmatrix} w & 0\\ 0 & z\end{pmatrix}.$$

Let ε denote the map

$$\varepsilon:\mathbb{C}^2\rightarrow\mathbb{C}:(w,z)\rightarrow w+z.$$

Then we have natural identifications

$$\begin{aligned}\mathcal{L}^2(\widehat{Q})\;{}_d\underset{\varepsilon}{\otimes}{}_d\;\mathcal{L}^2(\widehat{Q}) &= \Big(\bigoplus_{i,j}\mathcal{L}^2(\widehat{Q}_{ij})\Big)\;{}_d\underset{\varepsilon}{\otimes}{}_d\;\Big(\bigoplus_{l,k}\mathcal{L}^2(\widehat{Q}_{lk})\Big)\\ &\cong\bigoplus_{i,j,k}^{2}(\mathcal{L}^2(\widehat{Q}_{ij})\otimes\mathcal{L}^2(\widehat{Q}_{ik}))=\Delta_{\widehat{Q}}(1)(\mathcal{L}^2(\widehat{Q})\otimes\mathcal{L}^2(\widehat{Q})),\end{aligned}$$

since ${}_d\underset{\varepsilon}{\otimes}{}_d$ is just the ordinary balanced tensor product of two $\mathbb{C}^2$-modules. (Note that $\mathbb{C}^2$ acts on the left on both the $\mathcal{L}^2(\widehat{Q})$ spaces, so we don't get ordinary "matrix multiplication compatibility" on the summands!) Under this identification we have

$$\widehat{Q}\;{}_d\underset{\mathbb{C}^2}{*}{}_d\;\widehat{Q}\;\cong\;\Delta_{\widehat{Q}}(1)(\widehat{Q}\otimes\widehat{Q})\Delta_{\widehat{Q}}(1),$$

where the expression left is the fibred product. Thus $\Delta_{\widehat{Q}}$ can be seen as a map

$$\Delta_{\widehat{Q}}:\widehat{Q}\rightarrow\widehat{Q}\;{}_d\underset{\mathbb{C}^2}{*}{}_d\;\widehat{Q}.$$

Note now that the expressions $1\;{}_d\underset{\mathbb{C}^2}{\otimes}{}_d\;d(x)$ and $d(x)\;{}_d\underset{\mathbb{C}^2}{\otimes}{}_d\;1$ coincide with respectively $(1\otimes d(x))\Delta_{\widehat{Q}}(1)$ and $(d(x)\otimes 1)\Delta_{\widehat{Q}}(1)$ for $x\in\mathbb{C}^2$. Using also that $\iota\;{}_d\underset{\mathbb{C}^2}{*}{}_d\;\Delta_{\widehat{Q}}$ is just the restriction of $(\iota\otimes\Delta_{\widehat{Q}})$ to $\Delta_{\widehat{Q}}(1)(\widehat{Q}\otimes\widehat{Q})\Delta_{\widehat{Q}}(1)$, it is easy to see that $\Delta_{\widehat{Q}}$ satisfies the coassociativity conditions for a Hopf bimodule as in Definition 3.1 of [4]. Now the octuple

$$\left(\mathbb{C}^2,\widehat{Q},d,d,\Delta_{\widehat{Q}},\begin{pmatrix}\varphi_{\widehat{P}} & 0\\ 0 & \widehat{\varphi}\end{pmatrix},\begin{pmatrix}\psi_{\widehat{P}} & 0\\ 0 & \widehat{\psi}\end{pmatrix},\varepsilon\right)$$

will form a measured quantum groupoid as in Definition 3.7 of [5]. First of all, after the proper identifications, it is easy to see that $T_{\widehat{Q}}=\left(\begin{smallmatrix}\varphi_{\widehat{P}} & 0\\ 0 & \widehat{\varphi}\end{smallmatrix}\right)$ is a left invariant nsf operator valued weight onto $d(\mathbb{C}^2)$, and that $T'_{\widehat{Q}}=\left(\begin{smallmatrix}\psi_{\widehat{P}} & 0\\ 0 & \widehat{\psi}\end{smallmatrix}\right)$ is a right invariant nsf operator valued weight onto $d(\mathbb{C}^2)$. So we only have to check whether ε is relatively invariant with respect to $T_{\widehat{Q}}$ and $T'_{\widehat{Q}}$. Now since $\psi_{\widehat{P}} \oplus \widehat{\psi} = (\varphi_{\widehat{P}} \oplus \widehat{\varphi}) \circ R_{\widehat{Q}}$, we have

$$\sigma_t^{\psi_{\widehat{P}} \oplus \widehat{\psi}} = R_{\widehat{Q}} \circ \sigma_t^{\varphi_{\widehat{P}} \oplus \widehat{\varphi}} \circ R_{\widehat{Q}}.$$

If we look at the faithful representation $\pi^{\widehat{Q},2}$ of $\widehat{Q}$ on $\mathcal{L}^2(N) \oplus \mathcal{L}^2(M)$, then:

$$(\widehat{\nabla}_N^{\mathrm{i}t} \oplus \widehat{\nabla}^{\mathrm{i}t})\pi^{\widehat{Q},2}(x)(\widehat{\nabla}_N^{-\mathrm{i}t} \oplus \widehat{\nabla}^{-\mathrm{i}t}) = \pi^{\widehat{Q},2}(\sigma_t^{\varphi_{\widehat{P}} \oplus \widehat{\varphi}}(x)),$$
$$(J_N \oplus J)\pi^{\widehat{Q},2}(x)^*(J_N \oplus J) = \pi^{\widehat{Q},2}(R_{\widehat{Q}}(x)),$$

for $x \in \widehat{Q}$, so that $\sigma_t^{\psi_{\widehat{P}} \oplus \widehat{\psi}}$ is implemented on $\mathcal{L}^2(N) \oplus \mathcal{L}^2(M)$ by $\widehat{\mathbb{V}}_N^{\,it} \oplus \widehat{\nabla}'^{\,it}$, where $\widehat{\mathbb{V}}_N^{\,it} = J_N \widehat{\nabla}_N^{-it} J_N$. Using the definition of $\widehat{\nabla}_N$ and the commutation rules between δ_N, $J_N \delta_N J_N$ and P_N, it is then easy to see that indeed $\sigma_t^{\psi_{\widehat{P}} \oplus \widehat{\psi}}$ commutes with $\sigma_s^{\varphi_{\widehat{P}} \oplus \widehat{\varphi}}$.

6. TWISTING BY 2-COCYCLES

We now treat a specific method to create non-trivial Galois objects, namely the twisting by cocycles. Let (M, Δ) be a von Neumann algebraic quantum group, and let $\Omega \in \widehat{M} \otimes \widehat{M}$ be a unitary 2-cocycle, i.e. a unitary element satisfying

$$(1 \otimes \Omega)(\iota \otimes \widehat{\Delta})(\Omega) = (\Omega \otimes 1)(\widehat{\Delta} \otimes \iota)(\Omega).$$

Denote by $\check{\alpha}$ the trivial coaction $\mathbb{C} \to \widehat{M} \otimes \mathbb{C}$ of $\widehat{M}$. The following definitions and propositions will refer to [24]. So $(\check{\alpha}, \Omega)$ is a cocycle action in the terminology of Definition 1.1 in that paper. Let

$$N = \widehat{M} \underset{\Omega}{\ltimes} \mathbb{C} := [(\omega \otimes \iota)(\widehat{W}\Omega^*) : \omega \in \widehat{M}_*]^{\sigma-\text{weak}}$$

be the cocycle crossed product as in Definition 1.3 (actually, one should take the von Neumann algebra generated by elements of this last set, in stead of just the σ-weak closure, but it will follow from our Proposition 4.7 and the following proposition that this is the same). Then, by Proposition 1.4 of [24], there is a canonical *right* coaction α of M on N, determined by

$$\alpha((\omega \otimes \iota)(\widehat{W}\Omega^*)) = (\omega \otimes \iota \otimes \iota)(\widehat{W}_{13}\widehat{W}_{12}\Omega_{12}^*) \quad \text{for } \omega \in \widehat{M}_*.$$

By Theorem 1.11.1 of [24] it is ergodic. By the remark after Lemma 1.12 of [24] it is integrable, and by Proposition 1.15 of [24] we can take the GNS-construction for φ_N in $\mathcal{L}^2(M)$, by defining $\Lambda_N((\omega \otimes \iota)(\widehat{W}\Omega^*)) = \Lambda((\omega \otimes \iota)(\widehat{W}))$ for $\omega \in \widehat{M}_*$ well-behaved. Finally, (N, α) is a Galois object, since the unitary

$$\widehat{W}\Omega^* \in B(\mathcal{L}^2(M)) \otimes N$$

satisfies

$$(\iota \otimes \alpha)(\widehat{W}\Omega^*) = \widehat{W}_{13}\widehat{W}_{12}\Omega_{12}^*,$$

so that α is semi-dual (see Proposition 5.12 of [22] in the setting of left coactions). In fact, not surprisingly, we have the following proposition.

PROPOSITION 6.1. *The Galois map* $\widetilde{G}$ *of* (N, α) *equals* $\widehat{W}\Omega^*$.

Proof. Choose $\xi, \eta, \zeta \in \mathcal{L}^2(M)$, and an orthonormal basis ξ_i of $\mathcal{L}^2(M)$. Further, let $m \in \widehat{M}$ be a Tomita element for $\widehat{\varphi}$, and denote $\omega' = \omega_{\zeta, \widehat{\Lambda}(m)}$. Then by Proposition 1.15 of [24], $(\omega' \otimes \iota)(\widehat{W}\Omega^*) \in \mathcal{N}_{\varphi_N}$, $(\omega' \otimes \iota)(\widehat{W}) \in \mathcal{N}_{\varphi}$ and

$$\Lambda_N((\omega' \otimes \iota)(\widehat{W}\Omega^*)) = \Lambda((\omega' \otimes \iota)(\widehat{W})).$$

So

$$\begin{aligned}(\iota \otimes \omega_{\xi,\eta})(\widetilde{G})\Lambda((\omega' \otimes \iota)(\widehat{W})) &= (\iota \otimes \omega_{\xi,\eta})(\widetilde{G})\Lambda_N((\omega' \otimes \iota)(\widehat{W}\Omega^*)) \\ &= \Lambda((\omega_{\xi,\eta} \otimes \iota)(\alpha((\omega' \otimes \iota)(\widehat{W}\Omega^*)))) \\ &= \Lambda((\omega' \otimes \omega_{\xi,\eta} \otimes \iota)(\widehat{W}_{13}\widehat{W}_{12}\Omega_{12}^*)) \\ &= \Lambda\Big(\sum_i(\omega' \otimes \omega_{\xi,\xi_i} \otimes \omega_{\xi_i,\eta} \otimes \iota)(\widehat{W}_{14}\widehat{W}_{13}\Omega_{12}^*)\Big),\end{aligned}$$

where the sum is taken in the σ-strong-topology.

On the other hand, using Result 8.6 of [13], adapted to the von Neumann algebra setting, we get

$$\begin{aligned}(\iota \otimes \omega_{\xi,\eta})(\widehat{W}\Omega^*)\Lambda((\omega'\otimes\iota)(\widehat{W})) &= \sum_i(\iota\otimes\omega_{\xi_i,\eta})(\widehat{W})(\iota\otimes\omega_{\xi,\xi_i})(\Omega^*)\Lambda((\omega'\otimes\iota)(\widehat{W})) \\ &= \sum_i \Lambda((\omega_{\xi_i,\eta}\otimes\iota)\Delta((\omega'(\cdot(\iota\otimes\omega_{\xi,\xi_i})(\Omega^*))\otimes\iota)(\widehat{W}))) \\ &= \sum_i \Lambda((\omega' \otimes \omega_{\xi,\xi_i} \otimes \omega_{\xi_i,\eta} \otimes \iota)(\widehat{W}_{14}\widehat{W}_{13}\Omega_{12}^*)),\end{aligned}$$

so that the result follows by the closedness of Λ and the density of elements of the form $\Lambda((\omega' \otimes \iota)(\widehat{W}))$ in $\mathcal{L}^2(M)$. ∎

THEOREM 6.2. *The* Ω*-twisted Hopf–von Neumann algebra* $(\widehat{M}, \widehat{\Delta}_\Omega)$ *is a von Neumann algebraic quantum group.*

Proof. Recall that the Ω-twisted Hopf–von Neumann algebra is the algebra $\widehat{M}$ with the comultiplication $\widehat{\Delta}_\Omega(m)=\Omega\widehat{\Delta}(m)\Omega^*$. But the representation of $\widehat{M}'$ on $\mathcal{L}^2(N)$ equals the ordinary representation on $\mathcal{L}^2(M)$ (since it's easy to see that the unitary implementation of the coaction *equals* the right regular representation V), so we can identify the underlying algebra of the reflected quantum group $(\widehat{P}, \Delta_{\widehat{P}})$ with $\widehat{M}$, and then we have the following which proves the theorem:

$$\Delta_{\widehat{P}}(m) = \widetilde{G}^*(1 \otimes m)\widetilde{G} = \Omega\widehat{W}^*(1 \otimes m)\widehat{W}\Omega^* = \widehat{\Delta}_\Omega(m). \quad ∎$$

We will keep notation as in the previous sections, so we keep writing $(\widehat{P}, \Delta_{\widehat{P}})$ for $(\widehat{M}, \widehat{\Delta}_\Omega)$.

Denote by $u_t = \widehat{\nabla}_N^{it}\widehat{\nabla}^{-it} \in \widehat{M}$ the cocycle derivative of $\varphi_{\widehat{P}}$ with respect to $\widehat{\varphi}$, so that $u_{s+t} = u_s\widehat{\sigma}_s(u_t)$. Denote $v_t = \nabla_N^{it}\nabla^{-it}$. Then also $v_t \in \widehat{M}$, since ∇_N^{it} and ∇^{it} implement the same automorphism on $\widehat{M}'$. Finally, denote $X = J_N J$, then $X \in \widehat{M}$ for the same reason.

PROPOSITION 6.3. (i) *The one-parameter group v_t is a cocycle with respect to $\widehat{\tau}_t$.*
(ii) *The 2-cocycles Ω and $(\widehat{\tau}_t \otimes \widehat{\tau}_t)(\Omega)$ are cohomologous by the coboundary v_t.*
(iii) *The 2-cocycles Ω and $\widetilde{\Omega}=(\widehat{R}\otimes\widehat{R})(\Sigma\Omega^*\Sigma)$ are cohomologous by the coboundary X.*

REMARK 6.4. The third statement of this proposition was noted for 2-cocycles in the group von Neumann algebra of a compact group in [30].

Proof. By Lemma 4.22, we have

$$(\nabla^{it} \otimes u_t\widehat{\nabla}^{it})(\widehat{W}\Omega^*) = (\widehat{W}\Omega^*)(\nabla_N^{it} \otimes u_t\widehat{\nabla}^{it}).$$

Since $\nabla^{it} \otimes \widehat{\nabla}^{it}$ commutes with $\widehat{W}$ and ∇^{it} implements $\widehat{\tau}_t$ on $\widehat{M}$, the left hand side can be rewritten as $(1 \otimes u_t)\widehat{W}(\widehat{\tau}_t \otimes \widehat{\sigma}_t)(\Omega^*)(\nabla^{it} \otimes \widehat{\nabla}^{it})$, and so, bringing $\widehat{W}$ and $(\nabla^{it} \otimes \widehat{\nabla}^{it})$ to the other side, we obtain

$$\widehat{\Delta}(u_t)(\widehat{\tau}_t \otimes \widehat{\sigma}_t)(\Omega^*) = \Omega^*(v_t \otimes u_t).$$

Hence

$$\begin{aligned} v_{s+t} \otimes u_{s+t} &= \Omega\widehat{\Delta}(u_{s+t})(\widehat{\tau}_{s+t} \otimes \widehat{\sigma}_{s+t})(\Omega^*) = \Omega\widehat{\Delta}(u_s\widehat{\sigma}_s(u_t))(\widehat{\tau}_{s+t} \otimes \widehat{\sigma}_{s+t})(\Omega^*) \\ &= \Omega\widehat{\Delta}(u_s)(\widehat{\tau}_s \otimes \widehat{\sigma}_s)(\Omega^*) \cdot (\widehat{\tau}_s \otimes \widehat{\sigma}_s)(\Omega\widehat{\Delta}(u_t)(\widehat{\tau}_t \otimes \widehat{\sigma}_t)(\Omega^*)) \\ &= v_s\widehat{\tau}_s(v_t) \otimes u_s\widehat{\sigma}_s(u_t), \end{aligned}$$

from which the cocycle property of v_t follows.

Now note that v_t also equals $P_N^{it}P^{-it}$ (by definition of P_N). So using the third equality of Corollary 4.15,

$$\widehat{W}\Omega^*(v_t \otimes v_t)(P^{it} \otimes P^{it}) = (P^{it} \otimes v_tP^{it})\widehat{W}\Omega^*.$$

Using that $P^{it} = \widehat{P}^{it}$, taking $\widehat{W}$ and $P^{it} \otimes P^{it}$ to the other side, we arrive at

$$\Omega^*(v_t \otimes v_t) = \widehat{\Delta}(v_t)(\widehat{\tau}_t \otimes \widehat{\tau}_t)(\Omega^*),$$

which proves the second statement.

Finally, as mentioned already, the unitary implementation of α is just V itself. So by Lemma 4.4, we have $\widehat{W}\Omega^*(J_N \otimes J_N)\Sigma = \Sigma V\Sigma(\widehat{J} \otimes J_N)\widehat{W}\Omega^*$. Multiplying to the right with $(J \otimes J)\Sigma$, we get

$$\begin{aligned} \widehat{W}\Omega^*(X\otimes X) &= \Sigma V\Sigma(1\otimes X)(\widehat{J} \otimes J)\widehat{W}\Omega^*(J\otimes J)\Sigma = \Sigma V\Sigma(1\otimes X)(\widehat{J}\otimes J)\widehat{W}(J\otimes J)\Sigma\widetilde{\Omega}^* \\ &= \Sigma V\Sigma(1 \otimes X)(\widehat{J} \otimes J)\Sigma V\Sigma(\widehat{J} \otimes J)\widehat{W}\widetilde{\Omega}^* \\ &= \Sigma V\Sigma(1 \otimes X)\Sigma V^*\Sigma\widehat{W}\widetilde{\Omega}^* = (1 \otimes X)\widehat{W}\widetilde{\Omega}^*, \end{aligned}$$

from which $\Omega^*(X \otimes X) = \widehat{\Delta}(X)\widetilde{\Omega}^*$ immediately follows. ∎

We have the next formula for the multiplicative unitary $\widehat{W}_\Omega$ for $(\widehat{M}, \widehat{\Delta}_\Omega)$:

PROPOSITION 6.5. $\widehat{W}_{\Omega} = (J_N \otimes \widehat{J})\Omega\widehat{W}^*(J \otimes \widehat{J})\Omega^*$.

REMARK 6.6. This is to be compared with the formula for the multiplicative unitary in [8].

Proof. We will use notation as in the previous section. As already noted, the unitary implementation of α is the multiplicative unitary V, and the identification of $\mathcal{L}^2(N)$ with $\mathcal{L}^2(M)$ is an identification of left $\widehat{M}'$-modules. Hence the left module structure of $\begin{pmatrix} \widehat{M} & \widehat{M} \\ \widehat{M} & \widehat{M} \end{pmatrix}$ on $\begin{pmatrix} \mathcal{L}^2(M) & \mathcal{L}^2(M) \\ \overline{\mathcal{L}^2(M)} & \mathcal{L}^2(M) \end{pmatrix}$ is explicitly known: if we identify $\overline{\mathcal{L}^2(M)}$ with $\mathcal{L}^2(M)$ by the map $\widehat{J}\widehat{J}_{21}$, then the module structure is just ordinary matrix multiplication, the module structure on all summands being the standard one. Also, $\widehat{\Lambda}_{12}$ becomes $\widehat{\Lambda}$, and $\widehat{\Lambda}_{21}$ becomes $\Lambda_{\varphi_{\widehat{P}}}$. Then from Lemma 5.8, and the fact that $\widehat{\Delta}_{21}(x) = \widehat{\Delta}(x)\Omega^*$, it is easy to conclude that $\widehat{W}_{\Omega}\Omega = \widetilde{G}_J^* = (J_N \otimes \widehat{J})(\Omega\widehat{W}^*)(J \otimes \widehat{J})$. The proposition follows. ∎

7. GALOIS OBJECTS AND COACTIONS ON TYPE I FACTORS

We now look at another possible way to create examples. One of the major motivations for the present paper was the article [2]. There the authors consider examples of Galois objects (which they call "ergodic coactions of full quantum multiplicity") which were not induced by a 2-cocycle. This was surprising, as Wassermann had shown in [30] that for compact groups, any Galois object for the function algebra must come from a 2-cocycle of the dual (a result which was in turn based on the work in [29], and ultimately on the fundamental results of [10]). In fact, in [2] all Galois objects for the compact quantum groups $SU_q(2)$ are classified. There is a whole family of them, parametrized by orthogonal matrices which satisfy some relation with respect to q, even though there are no non-trivial cocycles for the dual of $SU_q(2)$.

To obtain examples in our wider setting, the following construction would seem to be very helpful. It is a generalization of the fact that any action (and by the work of A. Wassermann, any coaction ([30], Theorem 3)) of a compact group on a type I-factor comes from a cocycle representation. We need some terminology.

DEFINITION 7.1. Let (N, α) be a (right) Galois object for a von Neumann algebraic quantum group (M, Δ). Denote again by $\widehat{N}$ the space of intertwiners between $\mathcal{L}^2(M)_{\widehat{M}}$ and $\mathcal{L}^2(N)_{\widehat{M}}$. Let $\mathcal{H}$ be a Hilbert space. A (unitary) *left* (N, α)-*corepresentation* for $(\widehat{M}, \widehat{\Delta})$ is a unitary $\mathcal{G} \in \widehat{N} \otimes B(\mathcal{H})$ such that

$$(\Delta_{\widehat{N}} \otimes \iota)(\mathcal{G}) = \mathcal{G}_{13}\mathcal{G}_{23}.$$

By a *projective corepresentation* for $(\widehat{M}, \widehat{\Delta})$, we mean a left (N, α)-corepresentation for $(\widehat{M}, \widehat{\Delta})$ and some Galois object (N, α).

For any Galois object (N, α), there is a regular left (N, α)-corepresentation, namely the unitary $(J_N \otimes \widehat{J}_{12})\widetilde{G}^*(J \otimes \widehat{J}_{21})$. In case $(M, \Delta) = (\mathcal{L}(\mathfrak{G}), \Delta)$ is the group von Neumann algebra of an ordinary locally compact group $\mathfrak{G}$, and (N, α) is the twisted convolution algebra by a cocycle $\Omega \in \mathcal{L}^\infty(\mathfrak{G}) \otimes \mathcal{L}^\infty(\mathfrak{G})$, we just get back the ordinary notion of a cocycle representation. Of course, one can also easily adapt the definition to find the notion of a right (N, α)-corepresentation.

THEOREM 7.2. *Let (M, Δ) be a von Neumann algebraic quantum group. If (N, α) is a Galois object for (M, Δ), then any left (N, α)-corepresentation of $(\widehat{M}, \widehat{\Delta})$ gives rise to a left coaction of $(\widehat{M}, \widehat{\Delta})$ on a type-I-factor. Conversely, any left coaction on a type-I-factor is induced by a projective corepresentation.*

Proof. The first statement is easy: if $\mathcal{G}$ is such a corepresentation, define

$$\Upsilon : B(\mathcal{H}) \to \widehat{M} \otimes B(\mathcal{H}) : x \to \mathcal{G}^*(1 \otimes x)\mathcal{G},$$

then this is a coaction by the defining property of $\mathcal{G}$.

Now let $\mathcal{H}$ be a Hilbert space, and $\Upsilon : B(\mathcal{H}) \to \widehat{M} \otimes B(\mathcal{H})$ be a coaction of $(\widehat{M}, \widehat{\Delta})$. Denote by N the relative commutant of $\Upsilon(B(\mathcal{H}))$ inside $\widehat{M} \ltimes B(\mathcal{H})$. Then we have a canonical isomorphism $\Phi : \widehat{M} \ltimes B(\mathcal{H}) \to N \otimes B(\mathcal{H})$. We claim that the dual (right) coaction $\widehat{\Upsilon} : \widehat{M} \ltimes B(\mathcal{H}) \to (\widehat{M} \ltimes B(\mathcal{H})) \otimes M$ restricts to a coaction of M on N. Indeed: choose an orthonormal basis ξ_i of $\mathcal{H}$, with respective matrix unit system $\{e_{ij}\}$. Then for $x \in N$, we have $x = \sum_k \Upsilon(e_{k1})x\Upsilon(e_{1k})$ σ-strongly. Applying $\widehat{\Upsilon}$, we get $\widehat{\Upsilon}(x) = \sum_k (\Upsilon(e_{k1}) \otimes 1)\widehat{\Upsilon}(x)(\Upsilon(e_{1k}) \otimes 1)$, whose first leg clearly commutes with $\Upsilon(B(\mathcal{H}))$.

We now show that (N, α) is a Galois object. Ergodicity is clear, since $1 \otimes B(\mathcal{H})$ is the fixed point algebra of $\mathrm{Ad}(\Sigma)_{23}(\alpha \otimes \iota) = (\Phi \otimes \iota)\widehat{\Upsilon} \circ \Phi^{-1}$. Also integrability follows easily by this, $\widehat{\Upsilon}$ being integrable. Since we have a canonical isomorphism $(\widehat{M} \ltimes B(\mathcal{H})) \rtimes M \cong (N \rtimes M) \otimes B(\mathcal{H})$, and the first space is $\cong B(\mathcal{H}) \otimes B(\mathcal{L}^2(M))$, also $N \rtimes M$ must be a type I factor, from which it follows that the Galois homomorphism for α is necessarily an isomorphism.

We now show that the original coaction is implemented by an (N, α)-corepresentation. Denote by Tr the ordinary trace on $B(\mathcal{H})$, by $\widehat{\mathrm{Tr}}$ the dual weight on $\widehat{M} \ltimes B(\mathcal{H})$ with respect to Tr, and by φ_N the weight $(\iota \otimes \varphi)\alpha$ on N. Then we have

$$\widehat{\mathrm{Tr}} = (\varphi_N \otimes \mathrm{Tr}) \circ \Phi.$$

Hence we obtain a unitary

$$u : \mathcal{L}^2(M) \otimes \mathcal{L}^2(B(\mathcal{H})) \to \mathcal{L}^2(N) \otimes \mathcal{L}^2(B(\mathcal{H}))$$

which sends $\Lambda(m) \otimes \Lambda_{\mathrm{Tr}}(x)$ to $(\Lambda_N \otimes \Lambda_{\mathrm{Tr}})(\Phi(m \otimes 1)(1 \otimes x))$ for $m \in \mathcal{N}_\varphi$ and x Hilbert–Schmidt. But identifying $\mathcal{L}^2(B(\mathcal{H}), \mathrm{Tr})$ with $\mathcal{H} \otimes \overline{\mathcal{H}}$, and observing that u is right $B(\mathcal{H})$-linear, we must have that $u = \mathcal{G} \otimes 1$ for some unitary

$$\mathcal{G} : \mathcal{L}^2(M) \otimes \mathcal{H} \to \mathcal{L}^2(N) \otimes \mathcal{H}.$$

We proceed to show that $\mathcal{G}$ is indeed an (N,α)-corepresentation implementing Υ. First of all, it is not difficult to see that $\mathcal{G} \in \widehat{Q}_{12} \otimes B(\mathcal{H})$, since for $m \in \mathcal{N}_\varphi$ and x Hilbert–Schmidt, and $\xi, \eta \in \mathcal{L}^2(M)$ with $\xi \in \mathcal{D}(\delta^{-1/2})$, we have, putting $\omega = \omega_{\xi,\eta}$ and $\omega_\delta = \omega_{\delta^{-1/2}\xi,\eta}$, denoting by U the unitary corepresentation belonging to α and by V the right regular representation for (M,Δ),

$$\begin{aligned}
u((\iota\otimes\omega)(V)\otimes 1)(\Lambda(m)\otimes\Lambda_{\mathrm{Tr}}(x)) &= u(\Lambda((\iota\otimes\omega_\delta)(\Delta(m)))\otimes\Lambda_{\mathrm{Tr}}(x)) \\
&= (\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\Phi(((\iota\otimes\omega_\delta)(\Delta(m)))\otimes 1)(1\otimes x)) \\
&= (\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\Phi((((\iota\otimes\omega_\delta)(\Delta(m)))\otimes 1)\Upsilon(x))) \\
&= (\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\Phi((\iota\otimes\omega_\delta)(\widehat{\Upsilon}((m\otimes 1)\Upsilon(x))))) \\
&= (\Lambda_N\otimes\Lambda_{\mathrm{Tr}})((\iota\otimes\omega_\delta\otimes\iota)(\alpha\otimes\iota)\Phi((m\otimes 1)\Upsilon(x))) \\
&= ((\iota\otimes\omega)(U)\otimes 1)(\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\Phi((m\otimes 1)\Upsilon(x))) \\
&= ((\iota\otimes\omega)(U)\otimes 1)u(\Lambda(m)\otimes\Lambda_{\mathrm{Tr}}(x)),
\end{aligned}$$

so that $\mathcal{G}((\iota\otimes\omega)(V)\otimes 1) = ((\iota\otimes\omega)(U)\otimes 1)\mathcal{G}$, which is sufficient to conclude that the first leg of $\mathcal{G}$ is in $\widehat{Q}_{12}$.

Since $u\Upsilon(x) = (1\otimes x)u$ on $\mathcal{L}^2(M)\otimes\mathcal{L}^2(B(\mathcal{H}))$, we have $\mathcal{G}\Upsilon(x) = (1\otimes x)\mathcal{G}$ on $\mathcal{L}^2(M)\otimes\mathcal{H}$, so that $\mathcal{G}$ implements Υ.

The only thing left to show is that $\mathcal{G}$ satisfies

$$(\widehat{\Delta}_{12}\otimes\iota)(\mathcal{G}) = \mathcal{G}_{13}\mathcal{G}_{23}.$$

Writing out $\widehat{\Delta}_{12}$ and tensoring by $1_{\overline{\mathcal{H}}}$ to the right, this translates into proving that

$$\widetilde{G}_{12}^* u_{23}\widehat{W}_{12} = u_{13}u_{23},$$

with $\widetilde{G}$ the Galois unitary for (N,α). Moving $\widetilde{G}$ to the other side, and multiplying to the left with Σ_{12}, this becomes

$$u_{13}W_{12}^*\Sigma_{12} = \Sigma_{12}\widetilde{G}_{12}u_{13}u_{23}.$$

This can again be proven using a simple matrix algebra argument: we can write $\Phi(m\otimes 1) = \sum\limits_{i,j}\Phi_{ij}(m)\otimes e_{ij}$ with $\Phi_{ij}(m) = \sum\limits_k \Upsilon(e_{ki})(m\otimes 1)\Upsilon(e_{jk}) \in N$, where the sums are in the σ-strong topology. Then for $m,n\in\mathcal{N}_\varphi$ and x Hilbert–Schmidt, we have

$$\begin{aligned}
u_{13}W_{12}^*\Sigma_{12}(\Lambda(m)\otimes\Lambda(n)\otimes\Lambda_{\mathrm{Tr}}(x)) &= u_{13}(\Lambda\otimes\Lambda\otimes\Lambda_{\mathrm{Tr}})(\Delta(m)(n\otimes 1)\otimes x) \\
&= (\Lambda_N\otimes\Lambda_M\otimes\Lambda_{\mathrm{Tr}})\Big(\sum_{i,j}((\Phi_{ij}\otimes\iota)(\Delta(m)(n\otimes 1))\otimes e_{ij}x\Big),
\end{aligned}$$

while

$$\begin{aligned}
&\Sigma_{12}\widetilde{G}_{12}u_{13}u_{23}(\Lambda(m)\otimes\Lambda(n)\otimes\Lambda_{\mathrm{Tr}}(x)) \\
&\quad = \Sigma_{12}\widetilde{G}_{12}u_{13}(\Lambda\otimes\Lambda_N\otimes\Lambda_{\mathrm{Tr}})\Big(\sum_{i,j} m\otimes\Phi_{ij}(n)\otimes e_{ij}x\Big)
\end{aligned}$$

$$= \Sigma_{12}\widetilde{G}_{12}(\Lambda_N \otimes \Lambda_N \otimes \Lambda_{\mathrm{Tr}})\Big(\sum_{i,j,r}(\Phi_{ri}(m) \otimes \Phi_{ij}(n) \otimes e_{rj}x)\Big)$$
$$= (\Lambda_N \otimes \Lambda_M \otimes \Lambda_{\mathrm{Tr}})\Big(\sum_{i,j,r}((\alpha(\Phi_{ri}(m)) \otimes 1)(\Phi_{ij}(n) \otimes 1 \otimes e_{rj}x))\Big)$$
$$= (\Lambda_N \otimes \Lambda_M \otimes \Lambda_{\mathrm{Tr}})\Big(\sum_{i,j,r}((\Phi_{ri} \otimes \iota)(\Delta(m)) \otimes 1)(\Phi_{ij}(n) \otimes 1 \otimes e_{rj}x)\Big)$$
$$= (\Lambda_N \otimes \Lambda_M \otimes \Lambda_{\mathrm{Tr}})\Big(\sum_{j,r}((\Phi_{rj} \otimes \iota)(\Delta(m)(n \otimes 1)) \otimes e_{rj}x)\Big),$$

where we have used $\sum_{\mathrm{i}} \Phi_{ri}(m)\Phi_{ij}(n) = \Phi_{rj}(mn)$ for $m, n \in M$ in the last step. So we are done. ∎

REMARK 7.3. Starting from an (N_1, α_1)-corepresentation $\mathcal{G}_1$ and considering its associated coaction on $B(\mathcal{L}^2(N_1))$, one thus obtains a Galois object (N_2, α_2) and an (N_2, α_2)-corepresentation $\mathcal{G}_2$. As is to be expected, these Galois objects are isomorphic, in such a way that the corepresentations correspond to each other. Indeed: it's easy to see that $\mathcal{G}_1\mathcal{G}_2^* = v \otimes 1$ for some unitary $v : \mathcal{L}^2(N_2) \to \mathcal{L}^2(N_1)$. Since v is a right $\widehat{M}$-module map, we can extend the (well-defined) map $\widehat{Q}_{2,12} \to \widehat{Q}_{1,12} : z \to vz$ to an isomorphism Ψ of the linking algebras $\widehat{Q}_2$ and $\widehat{Q}_1$. From the fact that $\mathcal{G}_1$ and $\mathcal{G}_2$ are corepresentations, it is easy to deduce that $\widehat{\Delta}_{1,12}(vz) = (v \otimes v)\widehat{\Delta}_{2,12}(z)$ for $z \in \widehat{Q}_{2,12}$. Hence Ψ preserves the comultiplication structure, and thus (N_1, α_1) and (N_2, α_2) are isomorphic by a map $\widehat{\Psi}$, and moreover $(\Psi \otimes \iota)(\mathcal{G}_2) = \mathcal{G}_1$.

Recall that two coactions Υ_1 and Υ_2 of $(\widehat{M}, \widehat{\Delta})$ on a von Neumann algebra Y are called outer equivalent if there exists a unitary element $v \in \widehat{M} \otimes Y$ which satisfies
$$(\widehat{\Delta} \otimes \iota)(v) = v_{23}(\iota \otimes \Upsilon_1)(v),$$
(i.e., v is an Υ_1-cocycle) and such that $\Upsilon_2(x) = v\Upsilon_1(x)v^*$ for $x \in Y$. Then it is easy to see that also the following classical result still holds true.

THEOREM 7.4. *Suppose (M, Δ) is a von Neumann algebraic quantum group for which M has a separable predual. Then there is a natural one-to-one correspondence between outer equivalence classes of coactions of $(\widehat{M}, \widehat{\Delta})$ on $B(\mathcal{H})$, with $\mathcal{H}$ a separable infinite-dimensional Hilbert space, and isomorphism classes of right Galois objects (with separable predual) for (M, Δ).*

Proof. First suppose that Υ_1 and Υ_2 are two coactions on $B(\mathcal{H})$ which are outer equivalent by a unitary v. Then we get an isomorphism
$$\Phi : \widehat{M} \underset{\Upsilon_1}{\ltimes} B(\mathcal{H}) \to \widehat{M} \underset{\Upsilon_2}{\ltimes} B(\mathcal{H}) : z \to vzv^*,$$
which obviously sends $\Upsilon_1(B(\mathcal{H}))$ to $\Upsilon_2(B(\mathcal{H}))$. Hence if (N_i, α_i) denotes the Galois object constructed from Υ_i as in the previous theorem, N_1 gets sent to N_2 by

Φ. But Φ also preserves the dual right coaction, since $V_{13}v_{12} = v_{12}V_{13}$. So $\Phi_{|N_1}$ gives an (M,Δ)-equivariant isomorphism from (N_1,α_1) to (N_2,α_2).

Conversely, suppose that Υ_1 and Υ_2 are two coactions on $B(\mathcal{H})$, which are induced by respective (N,α)-corepresentations $\mathcal{G}_1$ and $\mathcal{G}_2$ for some Galois object (N,α) for (M,Δ). Put $v = \mathcal{G}_2^*\mathcal{G}_1 \in \widehat{M}\otimes B(\mathcal{H})$. Then v is an Υ_1-cocycle:

$$(\widehat{\Delta}\otimes\iota)(v) = \mathcal{G}_{2,23}^*\mathcal{G}_{2,13}^*\mathcal{G}_{1,13}\mathcal{G}_{1,23} = v_{23}\mathcal{G}_{1,23}^*v_{13}\mathcal{G}_{1,23} = v_{23}(\iota\otimes\Upsilon_1)(v),$$

and obviously $\Upsilon_2(x) = v\Upsilon_1(x)v^*$ for $x\in B(\mathcal{H})$. Hence Υ_1 and Υ_2 are outer equivalent.

Now for any right Galois object (N,α) with separable predual, there exists a coaction on $B(\mathcal{H})$ which has (N,α) as its associated Galois object: for example, one can take $\mathcal{H}\cong\overline{\mathcal{L}^2(N)}\otimes\mathcal{H}$ and equip it with the coaction

$$\Upsilon : B(\overline{\mathcal{L}^2(N)}\otimes\mathcal{H}) \to \widehat{M}\otimes\overline{\mathcal{L}^2(N)}\otimes\mathcal{H} :$$

$$\Upsilon(x) = (((J_N\otimes\widehat{J}_{21})\widetilde{G}(J\otimes\widehat{J}_{12}))\otimes 1)(1\otimes x)(((J_N\otimes\widehat{J}_{12})\widetilde{G}^*(J\otimes\widehat{J}_{21}))\otimes 1),$$

i.e., take an amplification of the coaction coming from the regular left projective corepresentation of a Galois object. This observation then ends the proof of the proposition. ∎

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