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AN SOT-DENSE PATH OF CHAOTIC OPERATORS WITH SAME HYPERCYCLIC VECTORS

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ABSTRACT. Recently many authors have obtained interesting results on the existence of a dense G_δ set of common hypercyclic vectors for a path of operators. We show that on a separable infinite dimensional Hilbert space, there is a path of chaotic operators that is dense in the operator algebra with the strong operator topology, and yet each operator along the path has the exact same dense G_δ set of hypercyclic vectors. As a corollary, the operators having that particular set of hypercyclic vectors form a connected subset of the operator algebra with the strong operator topology.

KEYWORDS: *Hypercyclic operator, hypercyclic vector, unilateral weighted backward shift.*

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1. INTRODUCTION

Let H be a separable, infinite dimensional Hilbert space over the complex field $\mathbb{C}$, and let $B(H)$ denote the algebra of all bounded linear operators $T : H \to H$. An operator T in $B(H)$ is *hypercyclic* if there is a vector x in H for which its orbit, $\mathrm{Orb}(T, x) = \{T^n x : n \geqslant 0\}$, is dense in H. Such a vector x is called a *hypercyclic vector* for T, and we use the notation $\mathcal{HC}(T)$ to denote the set of hypercyclic vectors for T.

The behavior of the orbit of a hypercyclic vector is wild. On the other hand, even when the operator is hypercyclic, the orbit of a certain vector may be finite. A vector x in H is called a *periodic point* for the operator T if $T^n x = x$ for some positive integer n. The operator T is *chaotic* if it is hypercyclic and has a dense set of periodic points. Godefroy and Shapiro [20] showed this definition of chaos is equivalent to the notion of chaos proposed by Devaney [18].

Whenever the operator T is hypercyclic, the set $\mathcal{HC}(T)$ of hypercyclic vectors is a dense G_δ set; see Kitai [23]. It follows from the Baire Category Theorem that if we have a countable collection $\{T_n \in B(H) : n \geqslant 1\}$ of hypercyclic operators, then the set $\bigcap_{n=1}^{\infty} \mathcal{HC}(T_n)$ of their common hypercyclic vectors is still a dense G_δ set. Since this argument fails when the collection of operators is uncountable, it becomes interesting to study their common hypercyclic vectors, especially when the operators in the uncountable collection are in fact related by continuity. This leads to the following definition. A collection of operators $\{F_t \in B(H) : t \in I\}$ is a *path of operators* if the map $F : I \to (B(H), \|\cdot\|)$, defined on an interval I by $F(t) = F_t$, is continuous with respect to the operator norm topology of $B(H)$ and the usual topology of the interval I. The set $\bigcap_{t\in I} \mathcal{HC}(F_t)$ is referred to as the *set of common hypercyclic vectors* for the whole path, and any vector in the set is called a *common hypercyclic vector*. If I is $[a,b]$, then the collection $\{F_t \in B(H) : t \in I\}$ is called a *path of operators between F_a and F_b*.

Many results on common hypercyclic vectors were obtained by different authors in the past few years. For instance, León-Saavedra and Müller [25] showed that every operator in the path of rotations $\{e^{i\theta}T : \theta \in [0, 2\pi]\}$ of a single hypercyclic operator T has the exact same hypercyclic vectors as the operator T itself. Later, Conejero, Müller, and Peris [14] studied common hypercyclic vectors for a semigroup of operators. The first example of a specific class of operators with common hypercyclic vectors is perhaps the unilateral weighted backward shifts. We provide a formal definition here.

DEFINITION 1.1. An operator B in $B(H)$ is called a *unilateral weighted backward shift* if there is an orthonormal basis $\{e_0, e_1, e_2, \dots\}$ and a sequence of nonzero scalars $\{w_j : j \geqslant 1\}$ in $\mathbb{C}$ such that $Be_0 = 0$ and $Be_j = w_j e_{j-1}$ for each integer $j \geqslant 1$.

It is easy to check that B in $B(H)$ implies that the weight sequence $\{w_j : j \geqslant 1\}$ is bounded. This class of operators was well studied in a survey article of Shields [31]. If all the weights satisfy $w_j = 1$, then B is simply called the *unilateral backward shift*. This particular shift B was used to provide the first examples of hypercyclic operators on a Hilbert space, as Rolewicz [28] showed that tB is hypercyclic whenever $t > 1$. Then Abakumov and Gordon [1] showed the path $\{tB : t \in (1, \infty)\}$ indeed has a dense G_δ set of common hypercyclic vectors. This result was re-obtained by Costakis and Sambarino [16] who introduced a sufficient condition for a path of general operators to have such a dense set. In fact, they also provided many natural examples of paths of operators with common hypercyclic vectors, including a path of unilateral weighted backward shifts. Another sufficient condition was provided by Bayart and Matheron [6] with applications for which Costakis and Sambarino's condition does not apply. A necessary and sufficient condition was provided by Chan and Sanders [13] for a path of operators to have a dense G_δ set of common hypercyclic vectors. They used this condition to prove another sufficient condition that reduces to the well known Hypercyclicity Criterion for the case when the whole path contains exactly one operator. Chan and Sanders used their conditions to reprove the result of Abakumov and Gordon, and further showed that between any two hypercyclic unilateral weighted backward shifts, there is a path of such shifts whose common hypercyclic vectors form a dense G_δ set. Though the results in the present paper call for techniques that are totally different from those of Chan and Sanders [13], their study on unilateral weighted backward shifts help motivate some of the ideas here. Other works on the existence of common hypercyclic vectors include that of Bayart [3], and Bayart and Grivaux [5] who studied composition operators on spaces of analytic functions, and Costakis [15] who studied Cesaro hypercyclic operators. For nonexistence results, Aron, Bès, León, and Peris [2] showed in their Example 2.2 that there does not exist a vector in H which is hypercyclic for every hypercyclic operator in $B(H)$. They proved this by showing for any nonzero vector x in H, there is a hypercyclic operator T in $B(H)$ for which $Tx = 0$. Along this line, Chan and Sanders [13] showed there is a path of hypercyclic unilateral weighted backward shifts which fails to have a common hypercyclic vector.

Before we describe the results in our present paper, we need to turn our attention to the density of the hypercyclic operators in $B(H)$. Clearly the norm of any hypercyclic operator must be strictly greater than 1, nevertheless it is still easy to see that they collectively are not dense, with respect to the norm topology, in the complement of the unit ball of $B(H)$; see Chan [11]. As it turns out, they are dense in the whole operator algebra $B(H)$ with a weaker topology called the strong operator topology, abbreviated SOT in the rest of the paper. This result was first obtained by Chan [11], and was generalized to the Fréchet space case by Bès and Chan [8] using a fundamental property of the strong operator topology provided by Hadwin, Nordgren, Radjavi, and Rosenthal [22]. In fact, Bès and Chan showed that if T is a hypercyclic operator, then the set of conjugates $\{ATA^{-1} : A \text{ invertible in } B(H)\}$ is SOT-dense in $B(H)$. Even further, if the hypercyclic operator T is chaotic, this SOT-dense set of conjugates consists entirely of chaotic operators. On the other hand, that conjugate set is path connected because the collection of all invertible operators in $B(H)$ is indeed path connected; see Douglas ([19], Corollary 5.30). Hence, any single chaotic operator generates a path connected set of chaotic operators that is SOT-dense in $B(H)$. As a result, it is interesting to see whether we can improve the above results, by raising the following question: *Does there exist a path of chaotic operators which is* SOT*-dense in* $B(H)$*, and yet has a dense* G_δ *set of common hypercyclic vectors?*

The above question is answered in the positive with a constructive proof in Theorem 3.2 below. Even more interesting, we show each operator along this path has the exact same dense G_δ set of hypercyclic vectors. It should be noted that Bonet, Martínez, and Peris [10] have shown that there is a Banach space that fails to admit a chaotic operator, and so Theorem 3.2 is purely a Hilbert space result. Since the collection of all hypercyclic operators in $B(H)$ fails to have a common hypercyclic vector, the path that we construct is necessarily a proper subset of all hypercyclic operators. In fact, the path consists entirely of operators that satisfy the Hypercyclicity Criterion due to the result by Bès and Peris [9] that every chaotic operator satisfies the criterion. It should be pointed out here that de la Rosa and Read [17] showed that there is a Banach space which admits a hypercyclic operator that does not satisfy the Hypercyclicity Criterion. Inspired by de la Rosa and Read, Bayart and Matheron [7] were able to obtain an analogous result for a Hilbert space. We conclude in Section 3 with a discussion about the SOT-connectedness of the hypercyclic operators in $B(H)$.

Before we prove our main results in Section 3, we first examine unilateral weighted backward shifts in $B(H)$ in Section 2. In particular, we show that for any given orthonormal basis of H, there is a path of chaotic hypercyclic unilateral weighted backward shifts which is SOT-dense in the set of all unilateral weighted backward shifts on that particular basis and for which all operators along this path have the exact same set of hypercyclic vectors. As a consequence, the common hypercyclic vectors for the whole path is a dense G_δ set; see Theorem 2.2 below.

Since hypercyclic vectors may form some sort of linear structure, common hypercyclic vectors follow this natural pattern as well.

DEFINITION 1.2. By the term *hypercyclic subspace* for an operator T, we mean a closed, infinite dimensional subspace consisting entirely, except the zero vector, of hypercyclic vectors for T.

A sufficient condition for the existence of common hypercyclic subspaces was obtained by Bayart [4]. Different sufficient conditions were obtained by Aron, Bès, León, and Peris in [2], and by Sanders in [30]. We show the SOT-dense paths of chaotic operators given in Theorem 2.2 and Theorem 3.2 can be chosen to have common hypercyclic subspaces; see Corollary 2.3 and Corollary 3.3 below.

2. UNILATERAL WEIGHTED BACKWARD SHIFTS

Throughout this section, let $\mathcal{B}$ be the subset of $B(H)$ consisting of all unilateral weighted backward shifts of a fixed orthonormal basis $\{e_0, e_1, e_2, \dots\}$. The main goal of this section is to show there is a path of chaotic shifts in $\mathcal{B}$ which is SOT-dense in $\mathcal{B}$.

The first examples of hypercyclic operators on a Hilbert space, provided by Rolewicz [28], were unilateral weighted backward shifts. Salas [29] later completely characterized hypercyclic unilateral weighted backward shifts in terms of the weight sequences. His result was originally stated for positive weight sequences. Since a unilateral weighted backward shift with the complex weight sequence $\{w_j : j \geqslant 1\}$ is unitarily equivalent to one with the positive weight sequence $\{|w_j| : j \geqslant 1\}$; see Shields ([31], Corollary 1), Salas' characterization can be stated in term of complex weights: *A unilateral weighted backward shift is hypercyclic if and only if its weight sequence satisfies*

$$\sup\Big\{\prod_{j=1}^{n}|w_j| : n \geqslant 1\Big\} = \infty. \tag{2.1}$$

A more general version of Salas' characterization was established by Grosse-Erdmann [21]. Martínez and Peris ([26], Example 3.5) also characterized the hypercyclic unilateral weighted backward shifts which are chaotic in terms of weight sequences: *A unilateral weighted backward shift is chaotic if and only if its weight sequence satisfies*

$$\sum_{n=1}^{\infty}\frac{1}{\prod_{j=1}^{n}|w_j|^2} < \infty. \tag{2.2}$$

From Salas' characterization (2.1), it is easy to see that if one takes the weight sequence of a hypercyclic shift in $\mathcal{B}$, and multiply one of its weights by a nonzero complex scalar, the new resulting shift in $\mathcal{B}$ will also be hypercyclic. In fact, we now show they have the exact same set of hypercyclic vectors.

PROPOSITION 2.1. *Let $\lambda \in \mathbb{C}\setminus\{0\}$ and let m be a positive integer. If A is a unilateral weighted backward shift in $\mathcal{B}$ with weight sequence $\{w_j : j \geqslant 1\}$, and if B is another shift in $\mathcal{B}$ whose weight sequence $\{v_j : j \geqslant 1\}$ satisfies $v_j = w_j$ for any positive integer $j \neq m$ and $v_m = \lambda w_m$, then $\mathcal{HC}(A) = \mathcal{HC}(B)$.*

Proof. Observe that for any integer $n \geqslant m$, we have:

$$\begin{aligned}
A^n e_{j+n} &= \Big(\prod_{i=1}^{n} w_{j+i}\Big)e_j, \quad \text{for } j \geqslant 0,\\
B^n e_{j+n} &= \Big(\prod_{i=1}^{n} w_{j+i}\Big)e_j, \quad \text{for } j \geqslant m, \text{ and}\\
B^n e_{j+n} &= \lambda\Big(\prod_{i=1}^{n} w_{j+i}\Big)e_j, \quad \text{for } 0 \leqslant j \leqslant m-1.
\end{aligned}$$

Therefore, for any vector $g \in H$ and integer $n \geqslant m$, we have

$$\langle A^n g, e_j\rangle = \langle B^n g, e_j\rangle, \quad \text{for } j \geqslant m, \text{ and} \tag{2.3}$$

$$\langle A^n g, e_j\rangle = \lambda^{-1}\langle B^n g, e_j\rangle, \quad \text{for } 0 \leqslant j \leqslant m-1. \tag{2.4}$$

Let $P : H \to H$ be the orthogonal projection onto the subspace span$\{e_j : 0 \leqslant j \leqslant m-1\}$. That is, for any vector $g \in H$, we have $Pg = \sum_{j=0}^{m-1}\langle g, e_j\rangle e_j$. Hence, by equations (2.3) and (2.4), we have that for any integer $n \geqslant m$,

$$A^n g = \lambda^{-1}P(B^n g) + (I-P)B^n g, \quad \text{and} \quad B^n g = \lambda P(A^n g) + (I-P)A^n g.$$

It follows that the orbit $\operatorname{Orb}(A, g)$ is dense if and only if the orbit $\operatorname{Orb}(B, g)$ is dense. ∎

Using induction with Proposition 2.1, we get that if A and B are two hypercyclic unilateral weighted backward shifts in $\mathcal{B}$ whose weight sequences differ by only a finite number of members, then the two shifts satisfy $\mathcal{HC}(A) = \mathcal{HC}(B)$. This observation together with Martínez and Peris' necessary and sufficient condition (2.2) allows us to create a path of chaotic shifts in $\mathcal{B}$ which is SOT-dense in $\mathcal{B}$, and each operator along the path has the exact same set of hypercyclic vectors.

THEOREM 2.2. *There is a path $\{F_t \in B(H) : t \in [1, \infty)\}$ of chaotic unilateral weighted backward shifts in $\mathcal{B}$ which is* SOT*-dense in $\mathcal{B}$. Moreover, for each $t \in [1, \infty)$, we have $\mathcal{HC}(F_t) = \mathcal{HC}(F_1)$; that is, each operator along the path has the exact same dense G_δ set of hypercyclic vectors.*

Proof. Let B_0 be a chaotic unilateral weighted backward shift in $\mathcal{B}$ with weight sequence $\{\widetilde{w}_j : j \geqslant 1\}$ satisfying $\widetilde{w}_j \in (\mathbb{Q} + \mathrm{i}\mathbb{Q}) \setminus \{0\}$ for each integer $j \geqslant 1$. Consider the collection $\mathcal{A}$ of all weight sequences $w = \{w_j : j \geqslant 1\}$ satisfying that each weight $w_j \in (\mathbb{Q} + \mathrm{i}\mathbb{Q}) \setminus \{0\}$ and $w_j = \widetilde{w}_j$ for all but finitely many positive integers j. The collection $\mathcal{A}$ is countable, and so let $w^{(n)} = \{w_j^{(n)} : j \geqslant 1\}$, where $n = 1, 2, 3, \ldots$, be an enumeration of $\mathcal{A}$. From the definition of the weight sequences in $\mathcal{A}$, there is a sequence $(k_n)_{n=1}^{\infty}$ of positive integers such that $w_j^{(n)} = w_j^{(n+1)} = \widetilde{w}_j$ for any integer $j \geqslant k_n + 1$. For integers n, j with $n \geqslant 1$ and $1 \leqslant j \leqslant k_n$, write $w_j^{(n)} = r_j^{(n)} \exp(\mathrm{i}\theta_j^{(n)})$ where $r_j^{(n)} > 0$ and $0 \leqslant \theta_j^{(n)} < 2\pi$.

To define the path of operators, for each integer $n \geqslant 1$ and for each $t \in [0, 1]$, let $G_{t,n}$ be the unilateral weighted backward shift in $\mathcal{B}$ whose weight sequence $\{v_j^{(t)} : j \geqslant 1\}$ is given by

$$v_j^{(t)} = \widetilde{w}_j \quad \text{if } j \geqslant k_n + 1, \text{ and}$$

$$v_j^{(t)} = [(1-t) r_j^{(n)} + t r_j^{(n+1)}] \mathrm{e}^{\mathrm{i}[(1-t)\theta_j^{(n)} + t\theta_j^{(n+1)}]} \quad \text{if } 1 \leqslant j \leqslant k_n.$$

For each $t \in [n, n+1]$, define $F_t = G_{t-n,n}$. Since $G_{1,n} = G_{0,n+1}$ for each integer $n \geqslant 1$, the map $F : [1, \infty) \to (B(H), \|\cdot\|)$, given by $F(t) = F_t$, is well defined. Moreover, the map $F : [1, \infty) \to (B(H), \|\cdot\|)$ is continuous because the map $t \mapsto G_{t,n}$ is continuous on $[0, 1]$ for each integer $n \geqslant 1$. Lastly, note that the series $\sum\limits_{n=1}^{\infty} \prod\limits_{j=1}^{n} |v_j^{(t)}|^{-2}$ converges if and only if the series $\sum\limits_{n=1}^{\infty} \prod\limits_{j=1}^{n} |\widetilde{w}_j|^{-2}$ converges. Thus, from condition (2.2), we get each operator F_t is chaotic. Therefore, $\{F_t \in B(H) : t \in [1, \infty)\}$ is a path of chaotic unilateral weighted backward shifts in $\mathcal{B}$.

To show $\mathcal{HC}(F_t) = \mathcal{HC}(F_1)$ for each $t \in [1, \infty)$, observe that the weights of each F_t are the same as F_1 except at most a finite number of them. Thus, by Proposition 2.1, we get $\mathcal{HC}(F_t) = \mathcal{HC}(F_1)$.

To show the path $\{F_t \in B(H) : t \in [1, \infty)\}$ is SOT-dense in $\mathcal{B}$, let B be a shift in $\mathcal{B}$ with the weight sequence $\{v_j : j \geqslant 1\}$. Let $f_1, \dots, f_r$ be r nonzero vectors in H, and let $\varepsilon > 0$. Consider an SOT-basic open set U given by

$$U = \{A \in B(H) : \|(A - B)f_k\| < \varepsilon, \text{ whenever } 1 \leqslant k \leqslant r\}.$$

Let $\gamma_1 = \sup\{(|\widetilde{w}_j| + |v_j|)^2 : j \geqslant 1\}$, and choose an integer $N \geqslant 1$ such that for each vector f_k, we have

$$\sum_{j=N+1}^{\infty} |\langle f_k, e_j\rangle|^2 < \frac{\varepsilon^2}{2\gamma_1}.$$

Let $\gamma_2 = \max\{\|f_k\|^2 : 1 \leqslant k \leqslant r\}$, and choose $w_1, \dots, w_N \in (\mathbb{Q} + \mathrm{i}\mathbb{Q}) \setminus \{0\}$ to satisfy

$$|w_j - v_j|^2 < \frac{\varepsilon^2}{2\gamma_2}.$$

Consider the weight sequence $\{w_1, w_2, \dots, w_N, \widetilde{w}_{N+1}, \widetilde{w}_{N+2}, \widetilde{w}_{N+3}, \dots\}$. From the definition of the collection $\mathcal{A}$, there exists an integer $n_0 \geqslant 1$ such that

$$\{w_j^{(n_0)} : j \geqslant 1\} = \{w_1, w_2, \dots, w_N, \widetilde{w}_{N+1}, \widetilde{w}_{N+2}, \dots\}.$$

Thus, for each vector f_k, we have

$$\begin{aligned}
\|(F_{n_0} - B)f_k\|^2 &= \sum_{j=1}^{\infty} |w_j^{(n_0)} - v_j|^2 |\langle f_k, e_j\rangle|^2 \\
&= \sum_{j=1}^{N} |w_j - v_j|^2 |\langle f_k, e_j\rangle|^2 + \sum_{j=N+1}^{\infty} |\widetilde{w}_j - v_j|^2 |\langle f_k, e_j\rangle|^2 \\
&< \frac{\varepsilon^2}{2\gamma_2} \sum_{j=1}^{N} |\langle f_k, e_j\rangle|^2 + \sum_{j=N+1}^{\infty} (|\widetilde{w}_j| + |v_j|)^2 |\langle f_k, e_j\rangle|^2 \\
&< \frac{\varepsilon^2}{2\gamma_2} \|f_k\|^2 + \gamma_1 \sum_{j=N+1}^{\infty} |\langle f_k, e_j\rangle|^2 < \frac{\varepsilon^2}{2} + \gamma_1 \frac{\varepsilon^2}{2\gamma_1} = \varepsilon^2,
\end{aligned}$$

and so, $F_{n_0} \in U$. ∎

In fact, the above proof can be modified to show the following interesting connection with some linear structure of the hypercyclic vectors.

COROLLARY 2.3. *There is a path of chaotic shifts in $\mathcal{B}$ that is* SOT-*dense in $\mathcal{B}$, and the shifts along the whole path have a common hypercyclic subspace.*

To see that, we first note that León and Montes [24] completely characterized, in terms of weight sequences, the unilateral weighted backward shifts which possess a hypercyclic subspace. Their characterization, expressed in terms of complex weights, states that a unilateral weighted backward shift has a hypercyclic subspace if and only if its weight sequence satisfies

$$\sup\Big\{\prod_{j=1}^{n}|w_j| : n \geqslant 1\Big\} = \infty, \quad \text{and} \quad \lim_{n\to\infty}\Big(\inf_{k\geqslant 1}\prod_{j=0}^{n-1}|w_{k+j}|\Big)^{1/n} \leqslant 1.$$

Using this condition, we see that each operator along the path given in the proof of Theorem 2.2 may fail to have a hypercyclic subspace. However, if we select the chaotic shift B_0 in the proof of Theorem 2.2 to also have a hypercyclic subspace, then each operator F_t along the corresponding path of shifts in $\mathcal{B}$ is chaotic and has a hypercyclic subspace. Furthermore, since $\mathcal{HC}(F_t) = \mathcal{HC}(F_1)$ for each t, the whole path has a common hypercyclic subspace within the dense G_δ set of common hypercyclic vectors. The shift in $\mathcal{B}$ with the weight sequence $\{\frac{j+1}{j} : j \geqslant 1\}$ is an example of such a chaotic shift with a hypercyclic subspace.

3. AN SOT-DENSE PATH

In Section 2, we focused on the collection of all unilateral weighted backward shifts of a fixed orthonormal basis in the Hilbert space H. However, this collection fails to be SOT-dense in $B(H)$. To construct the path of chaotic operators desired in Theorem 3.2, we turn our attention to generalized backward shifts. An operator T in $B(H)$ is a *generalized backward shift* if the kernel, $\mathrm{Ker}(T)$, of T is one dimensional, and the set $\bigcup\{\mathrm{Ker}(T^n) : n \geqslant 1\}$ is dense in H. Godefroy and Shapiro ([20], Proposition 3.3) showed that if the operator T is a generalized backward shift, then there is a sequence $\{x_j : j \geqslant 0\}$ of vectors in H for which $Tx_j = x_{j-1}$ for each integer $j \geqslant 1$, and $\mathrm{Ker}(T) = \mathrm{span}\{x_0\}$. For more on generalized backward shifts, see Godefroy and Shapiro [20].

Proposition 3.1 below is the major building block of Theorem 3.2. By using Proposition 2.1 and the fact that the invertible operators on the Hilbert space H are path connected, Proposition 3.1 creates a path between a hypercyclic unilateral weighted backward shift and a specific generalized backward shift. To be more precise, let $\{e_j : j \geqslant 0\}$ be any orthonormal basis of the Hilbert space H, and let $B_0 : H \to H$ be a hypercyclic unilateral weighted backward shift of the basis $\{e_j : j \geqslant 0\}$ with weight sequence $\{w_j : j \geqslant 1\}$. Let $\{g_j : j \geqslant 0\}$ be a sequence of vectors in H and $\{v_j : j \geqslant 1\}$ a weight sequence for which there is an integer $N \geqslant 0$ such that

$$\mathrm{span}\{g_j : 0 \leqslant j \leqslant N\} = \mathrm{span}\{e_j : 0 \leqslant j \leqslant N\}, \tag{3.1}$$

and for any integer $n \geqslant N+1$, we have

$$g_j = e_j \quad \text{and} \quad v_j = w_j. \tag{3.2}$$

Define an operator $B_1 : H \to H$ by $B_1 g_j = v_j g_{j-1}$ for each integer $j \geqslant 1$, and $B_1 g_0 = 0$. The operator B_1 is a generalized backward shift with $x_0 = g_0$, and $x_j = \Big[\prod_{i=1}^{j} v_i \Big]^{-1} g_j$ for integers $j \geqslant 1$.

PROPOSITION 3.1. *There is a path $\{G_t \in B(H) : t \in [0,1]\}$ of hypercyclic operators between B_0 and B_1 such that for each t in $[0,1]$, we have $\mathcal{HC}(G_t) = \mathcal{HC}(B_0)$. Furthermore, if the operator B_0 is chaotic, then this path may be chosen to consist entirely of chaotic operators.*

Proof. To begin our proof, first observe we can assume $v_j = w_j$ for each integer $j \geqslant 1$. To see this, note that in the general case where $v_j = w_j$ for each integer $j \geqslant N+1$, we can use the argument in the first half of the proof of Theorem 2.2 to create a path of hypercyclic operators between B_0 and the unilateral weighted backward shift of the basis $\{e_j : j \geqslant 0\}$ with weight sequence $\{v_j : j \geqslant 1\}$ where each operator along the path has the same set of hypercyclic vectors as the operator B_0. Moreover, if the operator B_0 is chaotic, then this path can be chosen so each operator along the path is chaotic.

To create the path of operators described in our proposition, let $H_N = \operatorname{span}\{g_j : 0 \leqslant j \leqslant N\} = \operatorname{span}\{e_j : 0 \leqslant j \leqslant N\}$, and let $A : H_N \to H_N$ be the invertible operator satisfying

$$Ae_j = g_j \quad \text{for } 0 \leqslant j \leqslant N. \tag{3.3}$$

Since the invertible operators on H_N are path connected, see Douglas ([19], Corollary 5.30), there exists a path $\{A_t \in B(H_N) : t \in [0,1]\}$ of invertible operators such that $A_0 = I$ and $A_1 = A$. For each $t \in [0,1]$ and each integer $j \geqslant 0$, let

$$g_{t,j} = \begin{cases} A_t e_j & \text{if } 0 \leqslant j \leqslant N, \\ e_j & \text{if } j \geqslant N+1, \end{cases} \tag{3.4}$$

and define the operator $G_t : H \to H$ by

$$G_t g_j = \begin{cases} v_j g_{t,j-1} & \text{if } j \geqslant 1, \\ 0 & \text{if } j = 0. \end{cases}$$

Then $\{G_t \in B(H) : t \in [0,1]\}$ is a path of operators between $G_0 = B_0$ and $G_1 = B_1$.

To show $\mathcal{HC}(G_t) = \mathcal{HC}(B_0)$ for each $t \in [0,1]$, let $P : H \to H$ be the orthogonal projection onto the closed subspace H_N. For each $t \in [0,1]$, define the operator $S_t : H \to H$ by $S_t = A_t P + (I - P)$. Since the operator S_t is invertible, it suffices to show $G_t^n = S_t B_0^n$ for each integer $n \geqslant N+1$. For such an integer n, we observe that by (3.1), (3.2), (3.4), and the definition of B_0, we get $\operatorname{Ker}(B_0^n) = \operatorname{span}\{g_{t,j} : 0 \leqslant j \leqslant n-1\}$ which gives us

$$S_t B_0^n g_{t,j} = 0 = G_t^n g_{t,j} \quad \text{for } 0 \leqslant j \leqslant n-1. \tag{3.5}$$

From (3.4) and the definition of S_t, we get $S_t e_j = g_{t,j}$ for each integer $j \geqslant 0$. Thus, for any integer $j \geqslant n$, we have

$$S_t B_0^n g_{t,j} = S_t B_0^n e_j \quad \text{(by (3.4))}$$
$$= \prod_{i=0}^{n-1} w_{j-i} S_t e_{j-n} = \prod_{i=0}^{n-1} v_{j-i} g_{t,j-n} = G_t^n g_{t,j}.$$

Therefore, $G_t^n = S_t B_0^n$ whenever $n \geqslant N+1$.

To complete the proof of our proposition, it remains to show that if the operator B_0 has a dense set of periodic points, then so has each operator G_t. For that we observe that if f is a periodic point of B_0, then we choose an integer $n \geqslant N+1$ such that $B_0^n f = f$. Now, the vectors $A_t P f$ and $Pf \in H_N \subseteq \mathrm{Ker}(B_0^n)$, and so

$$B_0^n S_t f = B_0^n (A_t P f + (I-P)f) = B_0^n (Pf + (I-P)f) = B_0^n f = f.$$

It follows that $G_t^n S_t f = S_t B_0^n S_t f = S_t f$. Hence $S_t f$ is a periodic point of G_t if the f is a periodic point of B_0. Since the operator S_t is invertible, it takes the dense set of periodic points of B_0 to a dense set of periodic points of G_t. ∎

We should note here that each operator along the path $\{G_t : t \in [0,1]\}$ given in the proof of Proposition 3.1 is, in fact, a generalized backward shift. We now use Proposition 3.1 to establish the main result of the paper.

THEOREM 3.2. *Let H be a separable, infinite dimensional Hilbert space over $\mathbb{C}$. Then there is a path $\{F_t \in B(H) : t \in [1,\infty)\}$ of chaotic operators which is* SOT*-dense in $B(H)$. Furthermore, for each t in $[0,\infty]$, we have $\mathcal{HC}(F_t) = \mathcal{HC}(F_1)$; that is, each operator along the path has the same dense G_δ set of hypercyclic vectors.*

Proof. To start, fix an orthonormal basis $\{e_j : j \geqslant 0\}$ of the Hilbert space H. Let $\mathcal{D}$ be the collection of all nonzero finite rank operators $D \in B(H)$ each of which has an integer $n \geqslant 1$ such that $De_j \in \left\{ \sum_{k=0}^{n} a_k e_k : a_k \in \mathbb{Q} + \mathrm{i}\mathbb{Q} \right\}$ whenever $0 \leqslant j \leqslant n$, and $De_j = 0$ whenever $j \geqslant n+1$. Clearly $\mathcal{D}$ is a countable collection. It is also easy to see that $\mathcal{D}$ is SOT-dense in $B(H)$ because if $T \in B(H)$ and $P_n : H \to H$ is the orthogonal projection onto $\mathrm{span}\{e_j : 0 \leqslant j \leqslant n\}$, then $P_n T P_n \to T$ in the strong operator topology. Let $\{D_\alpha : \alpha \geqslant 1\}$ be an enumeration of the collection $\mathcal{D}$ such that

$$D_\alpha e_j \in \left\{ \sum_{k=0}^{\alpha} a_k e_k : a_k \in \mathbb{Q} + \mathrm{i}\mathbb{Q} \right\}, \quad \text{whenever } 0 \leqslant j \leqslant \alpha,$$

and

$$D_\alpha e_j = 0, \quad \text{whenever } j \geqslant \alpha + 1. \tag{3.6}$$

Let $B_0 : H \to H$ be a chaotic unilateral weighted backward shift of the basis $\{e_j : j \geqslant 0\}$ with weight sequence $\{w_j : j \geqslant 1\}$. For each D_α and each pair of integers $\beta, \gamma \geqslant 1$, we define the linear operator $T_{\alpha,\beta,\gamma} : H \to H$ in the following manner:

$$T_{\alpha,\beta,\gamma} e_j = D_\alpha e_j + \frac{1}{\gamma} e_{\alpha+\beta+1+j}, \quad \text{for } 0 \leqslant j \leqslant \alpha; \tag{3.7}$$

$$T_{\alpha,\beta,\gamma} e_{\alpha+1} = \frac{1}{\gamma} e_0; \tag{3.8}$$

$$T_{\alpha,\beta,\gamma} e_{\alpha+1+j} = \frac{1}{\gamma^{j+1}} e_{\alpha+j}, \quad \text{for } 1 \leqslant j \leqslant \beta - 1; \tag{3.9}$$

$$T_{\alpha,\beta,\gamma} e_{\alpha+\beta+1+j} = -\gamma T_{\alpha,\beta,\gamma} D_\alpha e_j + \gamma e_{j+1}, \quad \text{for } 0 \leqslant j \leqslant \alpha - 1; \tag{3.10}$$

$$T_{\alpha,\beta,\gamma} e_{2\alpha+\beta+1} = -\gamma T_{\alpha,\beta,\gamma} D_\alpha e_\alpha; \tag{3.11}$$

$$T_{\alpha,\beta,\gamma} e_{2\alpha+\beta+2} = w_{2\alpha+\beta+2} e_{\alpha+\beta}; \tag{3.12}$$

$$T_{\alpha,\beta,\gamma} e_j = w_j e_{j-1}, \quad \text{for } j \geqslant 2\alpha + \beta + 3. \tag{3.13}$$

Equations (3.10) and (3.11) define $T_{\alpha,\beta,\gamma}$ because $\operatorname{Ran}(D_\alpha) \subseteq \operatorname{span}\{e_k : 0 \leqslant k \leqslant \alpha\}$. In fact, the operator $T_{\alpha,\beta,\gamma}$ is a compact perturbation of a chaotic unilateral weighted backward shift, and hence a bounded linear operator on H.

Claim 1. *The set* $\{T_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma \geqslant 1\}$ *is* SOT-*dense in* $B(H)$.

Proof of Claim 1. Let U be a nonempty SOT-open set in $B(H)$. Since $\mathcal{D}$ is SOT-dense in $B(H)$, there is an $\alpha \geqslant 1$, nonzero vectors $f_1, \ldots, f_r \in H$, and $\varepsilon > 0$ for which the basic SOT-open set

$$\{A \in B(H) : \|(A - D_\alpha) f_k\| < \varepsilon, \text{ whenever } 1 \leqslant k \leqslant r\} \subseteq U.$$

Let $M_1 = \max\{\|f_k\| : 1 \leqslant k \leqslant r\}$, and let $M_2 = (\alpha+1)^2(\|D_\alpha\| + 2)^2$. Choose an integer $\gamma \geqslant 2$ such that

$$\frac{(\alpha+1)M_1}{\gamma} < \frac{\varepsilon}{4}, \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{1}{\gamma^{j+1}} < \frac{\varepsilon}{4M_1}. \tag{3.14}$$

Since $\sup\{|w_j| : j \geqslant 1\} = \|B_0\| < \infty$, we can then choose an integer $\beta \geqslant 1$ such that for each vector f_k,

$$|\langle f_k, e_j \rangle| < \frac{\varepsilon}{4\gamma M_2}, \quad \text{if } j \geqslant \alpha + \beta + 1; \tag{3.15}$$

$$\Big(\sum_{j=\alpha+\beta+1}^{\infty} |w_j|^2 |\langle f_k, e_j \rangle|^2 \Big)^{1/2} < \frac{\varepsilon}{4}. \tag{3.16}$$

To show $T_{\alpha,\beta,\gamma} \in U$, let $T = T_{\alpha,\beta,\gamma}$, and let k be an integer with $1 \leqslant k \leqslant r$. Observe that

$$\|(T-D_\alpha)f_k\| = \Big\| \sum_{j=0}^{\infty} \langle f_k, e_j\rangle (T-D_\alpha)e_j \Big\|$$

$$\leqslant \sum_{j=0}^{\alpha} |\langle f_k, e_j\rangle| \|(T-D_\alpha)e_j\| + \sum_{j=0}^{\beta-1} |\langle f_k, e_{\alpha+1+j}\rangle| \|(T-D_\alpha)e_{\alpha+1+j}\| \tag{3.17}$$

$$+ \sum_{j=0}^{\alpha} |\langle f_k, e_{\alpha+\beta+1+j}\rangle| \|(T-D_\alpha)e_{\alpha+\beta+1+j}\| + \Big\| \sum_{j=2\alpha+\beta+2}^{\infty} \langle f_k, e_j\rangle (T-D_\alpha)e_j \Big\|.$$

To estimate each of the above four summations, we note that for the first term

$$\begin{aligned} \sum_{j=0}^{\alpha} |\langle f_k, e_j\rangle| \|(T-D_\alpha)e_j\| &= \sum_{j=0}^{\alpha} |\langle f_k, e_j\rangle| \frac{1}{\gamma} \quad \text{(by (3.7))} \\ &\leqslant \frac{1}{\gamma} \sum_{j=0}^{\alpha} \|f_k\| \\ &\leqslant \frac{M_1(\alpha+1)}{\gamma} \quad \text{(by definition of } M_1) \\ &< \frac{\varepsilon}{4} \quad \text{(by (3.14)).} \end{aligned} \tag{3.18}$$

To estimate the second summation, observe that

$$\begin{aligned} \sum_{j=0}^{\beta-1} |\langle f_k, e_{\alpha+1+j}\rangle| \|(T-D_\alpha)e_{\alpha+1+j}\| &= \sum_{j=0}^{\beta-1} |\langle f_k, e_{\alpha+1+j}\rangle| \frac{1}{\gamma^{j+1}} \quad \text{(by (3.6), (3.8), (3.9))} \\ &\leqslant \|f_k\| \sum_{j=0}^{\infty} \frac{1}{\gamma^{j+1}} \\ &< M_1 \frac{\varepsilon}{4M_1} \quad \text{(by definition of } M_1 \text{ and (3.14))} \\ &= \frac{\varepsilon}{4}. \end{aligned} \tag{3.19}$$

To estimate the fourth summation, observe that by equalities (3.6), (3.12), and (3.13), we have

$$\begin{aligned} \Big\| \sum_{j=2\alpha+\beta+2}^{\infty} \langle f_k, e_j\rangle (T-D_\alpha)e_j \Big\|^2 &= \Big\| w_{2\alpha+\beta+2} \langle f_k, e_{2\alpha+\beta+2}\rangle e_{\alpha+\beta} + \sum_{j=2\alpha+\beta+3}^{\infty} w_j \langle f_k, e_j\rangle e_{j-1} \Big\|^2 \\ &= \sum_{j=2\alpha+\beta+2}^{\infty} |w_j|^2 |\langle f_k, e_j\rangle|^2 \\ &< \frac{\varepsilon^2}{16} \quad \text{(by (3.16)).} \end{aligned} \tag{3.20}$$

Lastly, to estimate the third summation, we observe that $\mathrm{Ran}(D_\alpha) \subseteq \mathrm{span}\{e_k : 0 \leqslant k \leqslant \alpha\}$, and so for any vector $g \in H$, we have

$$\begin{aligned}\|TD_\alpha g\| &= \Big\| \sum_{j=0}^{\alpha} \langle D_\alpha g, e_j\rangle T e_j \Big\| \\ &\leqslant \sum_{j=0}^{\alpha} |\langle D_\alpha g, e_j\rangle| \Big\| D_\alpha e_j + \frac{1}{\gamma} e_{\alpha+\beta+1+j} \Big\| \quad \text{(by (3.7))} \\ &\leqslant \sum_{j=0}^{\alpha} \|D_\alpha g\| \Big(\|D_\alpha e_j\| + \frac{1}{\gamma}\Big) \leqslant \sum_{j=0}^{\alpha} (\|D_\alpha\|+1)^2 \|g\| = (\alpha+1)(\|D_\alpha\|+1)^2\|g\|.\end{aligned}$$

Thus, for any integer j with $0 \leqslant j \leqslant \alpha - 1$,

$$\begin{aligned}\|(T-D_\alpha)e_{\alpha+\beta+1+j}\| &= \| -\gamma T D_\alpha e_j + \gamma e_{\alpha+1}\| \quad \text{(by (3.6), (3.10))} \\ &\leqslant \gamma(\alpha+1)(\|D_\alpha\|+1)^2 + \gamma \leqslant \gamma(\alpha+1)(\|D_\alpha\|+2)^2.\end{aligned}$$

For similar reasons,

$$\|(T-D_\alpha)e_{2\alpha+\beta+1}\| \leqslant \gamma(\alpha+1)(\|D_\alpha\|+2)^2.$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{\alpha} |\langle f_k, e_{\alpha+\beta+1+j}\rangle| \|(T-D_\alpha)e_{\alpha+\beta+1+j}\| &< \sum_{j=0}^{\alpha} \frac{\varepsilon}{4\gamma M_2} \|(T-D_\alpha)e_{\alpha+\beta+1+j}\| \quad \text{(by (3.15))} \\ &< \frac{\varepsilon}{4M_2\gamma} \sum_{j=0}^{\alpha} \gamma(\alpha+1)(\|D_\alpha\|+2)^2 = \frac{\varepsilon}{4M_2} M_2 = \frac{\varepsilon}{4}. \quad (3.21)\end{aligned}$$

Combining inequalities (3.18), (3.19), (3.20), and (3.21) with (3.17) yields

$$\|(T-D_\alpha)f_k\| < \varepsilon,$$

which completes the proof of Claim 1. ∎

We now use Proposition 3.1 to connect the chaotic shift B_0 with the operator $T_{\alpha,\beta,\gamma}$.

Claim 2. *For each triple of integers $\alpha, \beta, \gamma \geqslant 1$, there is a path $\{G_t \in B(H) : t \in [0,1]\}$ of chaotic operators between B_0 and $T_{\alpha,\beta,\gamma}$ for which $\mathcal{HC}(G_t) = \mathcal{HC}(B_0)$ for each $t \in [0,1]$.*

Proof of Claim 2. To prove Claim 2 using Proposition 3.1, we define a sequence $\{g_j : j \geqslant 0\}$ of vectors in H and a weight sequence $\{v_j : j \geqslant 1\}$ in the following manner. For each integer j with $0 \leqslant j \leqslant \alpha$, let

$$g_{2j} = D_\alpha e_{\alpha-j} + \frac{1}{\gamma} e_{\alpha+\beta+1+(\alpha-j)}, \tag{3.22}$$

$$g_{2j+1} = e_{\alpha-j}, \text{ and} \tag{3.23}$$

$$v_{2j} = v_{2j+1} = 1. \tag{3.24}$$

For each integer j with $0 \leqslant j \leqslant \beta - 1$, let

$$g_{2\alpha+2+j} = e_{\alpha+1+j} \quad \text{and} \quad v_{2\alpha+2+j} = \frac{1}{\gamma^{j+1}} \tag{3.25}$$

and for integers $j \geqslant 2\alpha + \beta + 2$, let

$$g_j = e_j \quad \text{and} \quad v_j = w_j. \tag{3.26}$$

Since $\text{Ran}(D_\alpha) \subseteq \text{span}\{e_j : 0 \leqslant j \leqslant \alpha\}$, we get $\text{span}\{g_j : 0 \leqslant j \leqslant 2\alpha + \beta + 1\} = \text{span}\{e_j : 0 \leqslant j \leqslant 2\alpha + \beta + 1\}$. To show $T_{\alpha,\beta,\gamma} g_j = v_j g_{j-1}$ for each integer $j \geqslant 1$ and $T_{\alpha,\beta,\gamma} g_0 = 0$, observe that

$$\begin{aligned} T_{\alpha,\beta,\gamma} g_0 &= T_{\alpha,\beta,\gamma} D_\alpha e_\alpha + \frac{1}{\gamma} T_{\alpha,\beta,\gamma} e_{2\alpha+\beta+1} \quad \text{(by (3.22))} \\ &= T_{\alpha,\beta,\gamma} D_\alpha e_\alpha + \frac{1}{\gamma}(-\gamma T_{\alpha,\beta,\gamma} D_\alpha e_\alpha) \quad \text{(by (3.11))} \\ &= 0, \text{ and} \\ T_{\alpha,\beta,\gamma} g_1 &= T_{\alpha,\beta,\gamma} e_\alpha \quad \text{(by (3.23))} \\ &= D_\alpha e_\alpha + \frac{1}{\gamma} e_{2\alpha+\beta+1} \quad \text{(by (3.7))} \\ &= v_1 g_0 \quad \text{(by (3.22) and (3.24))}. \end{aligned}$$

Using a similar argument, for integers j with $1 \leqslant j \leqslant \alpha$,

$$\begin{aligned} T_{\alpha,\beta,\gamma} g_{2j} &= T_{\alpha,\beta,\gamma} D_\alpha e_{\alpha-j} + \frac{1}{\gamma} T_{\alpha,\beta,\gamma} e_{\alpha+\beta+1+(\alpha-j)} \quad \text{(by (3.22))} \\ &= e_{\alpha-j+1} \quad \text{(by (3.10))} \\ &= v_{2j} g_{2j-1} \quad \text{(by (3.23) and (3.24)), and} \\ T_{\alpha,\beta,\gamma} g_{2j+1} &= T_{\alpha,\beta,\gamma} e_{\alpha-j} \quad \text{(by (3.23))} \\ &= D_\alpha e_{\alpha-j} + \frac{1}{\gamma} e_{\alpha+\beta+1+(\alpha-j)} \quad \text{(by (3.7))} \\ &= v_{2j+1} g_{2j} \quad \text{(by (3.22) and (3.24))}. \end{aligned}$$

Next, note that

$$\begin{aligned} T_{\alpha,\beta,\gamma} g_{2\alpha+2} &= T_{\alpha,\beta,\gamma} e_{\alpha+1} \quad \text{(by (3.25))} \\ &= \frac{1}{\gamma} e_0 \quad \text{(by (3.8))} \\ &= v_{2\alpha+2} g_{2\alpha+1} \quad \text{(by (3.23) and (3.25))}, \end{aligned}$$

and for integers j with $1 \leqslant j \leqslant \beta - 1$,

$$\begin{aligned} T_{\alpha,\beta,\gamma} g_{2\alpha+2+j} &= T_{\alpha,\beta,\gamma} e_{\alpha+1+j} \quad \text{(by (3.25))} \\ &= \frac{1}{\gamma^{j+1}} e_{\alpha+j} \quad \text{(by (3.9))} \\ &= v_{2\alpha+2+j} g_{2\alpha+1+j} \quad \text{(by (3.25))}. \end{aligned}$$

Lastly, observe that

$$\begin{aligned} T_{\alpha,\beta,\gamma} g_{2\alpha+\beta+2} &= T_{\alpha,\beta,\gamma} e_{2\alpha+\beta+2} \quad \text{(by (3.26))} \\ &= w_{2\alpha+\beta+2} e_{\alpha+\beta} \quad \text{(by (3.12))} \\ &= v_{2\alpha+\beta+2} g_{2\alpha+\beta+1} \quad \text{(by (3.25) and (3.26))}, \end{aligned}$$

and for integers $j \geqslant 2\alpha + \beta + 3$,

$$\begin{aligned} T_{\alpha,\beta,\gamma} g_j &= T_{\alpha,\beta,\gamma} e_j \quad \text{(by (3.26))} \\ &= w_j e_{j-1} \quad \text{(by (3.13))} \\ &= v_j g_{j-1} \quad \text{(by (3.26))}, \end{aligned}$$

which completes the proof of Claim 2. ∎

We are now ready to construct the desired SOT-dense path of chaotic operators. Let $\{T_{\alpha_k,\beta_k,\gamma_k} : k \geqslant 1\}$ be an enumeration of the countable set $\{T_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma \geqslant 1\}$. By Claim 2, for each integer $k \geqslant 1$, there is a path $\{G_{t,k} \in B(H) : t \in [0,1]\}$ of chaotic operators such that $G_{0,k} = G_{1,k} = B_0$ and $T_{\alpha_k,\beta_k,\gamma_k} \in \{G_{t,k} \in B(H) : t \in [0,1]\}$, and in addition $\mathcal{HC}(G_{t,k}) = \mathcal{HC}(B_0)$ for each $t \in [0,1]$. For each $t \in [k, k+1]$, let $F_t = G_{t-k,k}$. Then $\{F_t \in B(H) : t \in [1,\infty)\}$ is a path of chaotic operators which is SOT-dense in $B(H)$ by Claim 1, and for which $\mathcal{HC}(F_t) = \mathcal{HC}(B_0) = \mathcal{HC}(F_1)$ for each $t \in [1,\infty)$. ∎

If we choose the chaotic shift B_0 given within the proof of Theorem 3.2 to have a hypercyclic subspace, then the corresponding path of operators in the theorem maintains the linear structure.

COROLLARY 3.3. *There is a path of chaotic shifts that is* SOT-*dense in* $B(H)$, *and the shifts along the whole path have a common hypercyclic subspace.*

Not only does the strong operator topology play an important role in the density of the hypercyclic operators, it also plays a role in the connectedness of those operators. To explain, recall Bès and Chan [8] showed that if an operator T in $B(H)$ is hypercyclic, then its conjugate class $\{ATA^{-1} : A \text{ invertible in } B(H)\}$ is an SOT-dense collection of hypercyclic operators in $B(H)$. That conjugate class is also path connected because the invertible operators in $B(H)$ are path connected; see Douglas ([19], Corollary 5.30). Hence, the conjugate class consisting entirely of hypercyclic operators is SOT-dense and SOT-connected in $B(H)$. On the other hand, we observe that if Y and Z are two subsets of a topological space X satisfying $Y \subseteq Z \subseteq \overline{Y}$ and if Y is connected, then Z is connected; see Munkres ([27], Theorem 1.4, page 149). This observation and our discussion above lead to the following fact

PROPOSITION 3.4. *The set of all hypercyclic operators is* SOT-*connected in* $B(H)$.

Now, if the hypercyclic operator that generates the SOT-dense conjugate class is chaotic, then the conjugate class consists entirely of chaotic operator, which by the same discussion as above, implies that the set of all chaotic operators is also SOT-connected. Furthermore, one can easily verify that an operator satisfies the Hypercyclicity Criterion if and only if every operator in the conjugate class does. Hence, the same argument shows the set of operators satisfying the Hypercyclicity Criterion is SOT-connected. Similarly, the set of hypercyclic operators not satisfying the criterion is SOT-connected as well.

With the same topological argument, we see that if we let $\mathcal{G}$ be the dense G_δ set of common hypercyclic vectors in Theorem 3.2, then we have the following conclusion.

COROLLARY 3.5. *The set of operators T in $B(H)$ with $\mathcal{G} \subseteq \mathcal{HC}(T)$ is* SOT-*connected.*

Likewise, if we let $\mathcal{G}$ be the common hypercyclic subspace in Corollary 3.3, then the set of all operators T for which $\mathcal{G} \subseteq \mathcal{HC}(T)$ is also SOT-connected in $B(H)$.

Related to hypercyclicity are the concepts of supercyclicity and cyclicity. An operator T in $B(H)$ is *supercyclic* if there is a vector x in H for which the set $\{\lambda T^n x : n \geqslant 0, \lambda \in \mathbb{C}\}$, consisting of all scalar multiples of vectors from the orbit $\mathrm{Orb}(T, x)$, is dense in H. An operator T is *cyclic* if there is a vector x in H for which the linear span of the orbit, span $\mathrm{Orb}(T, x)$, is dense in H. Clearly, hypercyclicity implies supercyclicity, and supercyclicity implies cyclicity. From the above topological argument, the supercyclic operators in $B(H)$ are SOT-connected and SOT-dense. Furthermore, as in Corollary 3.5, the set of supercyclic operators T in $B(H)$ having the prescribed dense G_δ set of supercyclic vectors forms an SOT-connected subset of $B(H)$. Likewise, the same holds true for the cyclic operators in $B(H)$.

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