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QUANTUM STOCHASTIC INTEGRALS AND DOOB–MEYER DECOMPOSITION

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ABSTRACT. We show that for a quantum L^p-martingale $(X(t))$, $p > 2$, there exists a Doob–Meyer decomposition of the submartingale $(|X(t)|^2)$. A non-commutative counterpart of a classical process continuous with probability one is introduced, and a quantum stochastic integral of such a process with respect to an L^p-martingale, $p > 2$, is constructed. Using this construction, the uniqueness of the Doob–Meyer decomposition for a quantum martingale "continuous with probability one" is proved, and explicit forms of this decomposition and the quadratic variation process for such a martingale are obtained.

KEYWORDS: *Quantum stochastic integrals, Doob–Meyer decomposition, quantum martingales, von Neumann algebras.*

MSC (2000): Primary 81S25; Secondary 46L53, 60G44, 60H05.

INTRODUCTION

In the existing theories of quantum stochastic integration we face a problem which is similar to that described on page 148 of [17] as follows: "We know all too well that it is one thing to develop a theory of integration in some reasonable generality and a completely different task to *compute* the integral in any specific case of interest". In quantum stochastic integration we are concerned not so much with computing the integral but rather with numerous important examples which do not fit into the nice theory that we have at our disposal. The origin of this problem lies in rather narrow classes of "theoretically admissible" integrands. Indeed, if for example $(X(t))$ is an L^2-martingale then the integral $\int f(t)\,\mathrm{d}X(t)$ is in general defined for adapted processes f satisfying pretty strong conditions such as e.g. being *norm* limits of simple processes. On the other hand it looks quite reasonable to define a stochastic integral in some natural way in many concrete situations making it possible to integrate a broader class of processes. This approach has already been taken in [10], [11] in several cases, where in particular it is shown how integration with respect to a quantum random time can be performed, or how one can integrate predictable processes. In the first part of the paper we follow the same idea and construct a stochastic integral of "continuous with probability one" noncommutative stochastic process with respect to an L^p-martingale for $p > 2$. The second part is devoted to the problem of a Doob–Meyer decomposition of the submartingale $(|X(t)|^2)$, where $(X(t))$ is an L^p-martingale for $p \geqslant 2$. This problem has a long history, see for example [2], [4], [5], [7], [8], [9], [10], [11], where this decomposition has been obtained in several concrete situations. However, the existence of such a decomposition in general remained an open question. We show that for $p > 2$ a Doob–Meyer decomposition always exists, and is unique for a martingale "continuous with probability one". In this case we give also explicit forms of the quadratic variation process of the martingale and the Doob–Meyer decomposition, using the construction of the integral given in the first part of the paper. As seen from the above, the notion of "continuity with probability one" for a noncommutative stochastic process plays an important role in our considerations, and we explain how it can be generalised from the classical context to the noncommutative one.

1. PRELIMINARIES AND NOTATION

A *noncommutative stochastic base* which is a basic object of our considerations consists of the following elements: a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$, a normal faithful unital trace τ on $\mathcal{A}$, a filtration $(\mathcal{A}_t\colon t \in [0, +\infty))$, which is an increasing ($s \leqslant t$ implies $\mathcal{A}_s \subset \mathcal{A}_t$) family of von Neumann subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_\infty = \Big(\bigcup\limits_{t \geqslant 0} \mathcal{A}_t \Big)''$ and $\mathcal{A}_s = \bigcap\limits_{t>s} \mathcal{A}_t$ (right-continuity). Moreover, for each t, there exists a normal conditional expectation $\mathbb{E}_t$ from $\mathcal{A}$ onto $\mathcal{A}_t$ such that $\tau \circ \mathbb{E}_t = \tau$.

For each $t \in [0, +\infty]$ we write $L^p(\mathcal{A}_t)$ for the non-commutative Lebesgue space associated with $\mathcal{A}_t$ and τ. The theory of such spaces is described e.g. in [22]; for our purposes we only recall that $L^p(\mathcal{A})$ (respectively $L^p(\mathcal{A}_t)$) consists of densely defined operators on $\mathcal{H}$, affiliated with $\mathcal{A}$, and that $L^p(\mathcal{A})$ is the completion of $\mathcal{A}$ with respect to the norm

$$\|X\|_p = [\tau(|X|^p)]^{1/p};$$

moreover, for $a \in \mathcal{A}$, $X \in L^p(\mathcal{A})$ the operators aX and Xa belong to $L^p(\mathcal{A})$. For each t the conditional expectation $\mathbb{E}_t$ extends to a projection of norm one from $L^p(\mathcal{A})$ onto $L^p(\mathcal{A}_t)$, for which we use the same notation. Notice that the conditional expectation, being a bounded operator on $L^p(\mathcal{A})$, is weakly continuous. Since the conditional expectation is completely positive we have

$$\mathbb{E}_t|x|^2 = \mathbb{E}_t(x^*x) \geqslant \mathbb{E}_t x^* \mathbb{E}_t x = |\mathbb{E}_t x|^2.$$

The following simple property is often useful. Let $x \in L^p(\mathcal{A})$, $y \in L^q(\mathcal{A})$, $p, q \in [0, +\infty]$, $1/p + 1/q = 1$. Then

$$\tau((\mathbb{E}_t x)y) = \tau(\mathbb{E}_t((\mathbb{E}_t x)y)) = \tau((\mathbb{E}_t x)(\mathbb{E}_t y)) = \tau(x\mathbb{E}_t y). \tag{1.1}$$

By an $\mathcal{A}$- (respectively L^p-) valued *process* we mean a map from $[0, +\infty)$ into $\mathcal{A}$ (respectively $L^p(\mathcal{A})$). $\mathcal{A}$-valued processes will be usually denoted by f, g or $(f(t))$, $(g(t))$, while for L^p-processes we shall use symbols $(X(t))$, $(Y(t))$. An $\mathcal{A}$- (respectively L^p-) valued process f (respectively X) is called *adapted* if $f(t) \in \mathcal{A}_t$ (respectively $X(t) \in L^p(\mathcal{A}_t)$).

An L^p-process $(X(t)\colon t \in [0, +\infty))$ is called a *martingale* if for each $s, t \in [0, \infty)$, $s \leqslant t$, we have $\mathbb{E}_s X(t) = X(s)$. It follows that a martingale is an adapted process. If the inequality $X(s) \leqslant \mathbb{E}_s X(t)$ (respectively $\mathbb{E}_s X(t) \leqslant X(s)$) holds for $s \leqslant t$, then the process is called a *submartingale* (respectively *supermartingale*). Let us notice that according to [1] the martingale $(X(t))$ is right-continuous in $\|\cdot\|_p$-norm. Moreover, for each $p \in [0, +\infty]$ and $s \leqslant t$ we have

$$\|X(s)\|_p = \|\mathbb{E}_s X(t)\|_p \leqslant \|X(t)\|_p.$$

If $(X(t)\colon t \in [0, +\infty))$ is an L^p-martingale, for $p \geqslant 2$, then $(|X(t)|^2\colon t \in [0, +\infty))$ is an $L^{p/2}$-submartingale since for any $s \leqslant t$ the above-mentioned property of conditional expectation yields

$$\mathbb{E}_s|X(t)|^2 \geqslant |\mathbb{E}_s X(t)|^2 = |X(s)|^2.$$

The submartingale $(|X(t)|^2\colon t \in [0, +\infty))$ is right-continuous in $\|\cdot\|_{p/2}$-norm. Indeed, we have

$$|X(t)|^2 - |X(s)|^2 = X(t)^*[X(t) - X(s)] + [X(t) - X(s)]^* X(s),$$

so using Hölder's inequality we get

$$\begin{aligned} \||X(t)|^2 - |X(s)|^2\|_{p/2} &\leqslant \|X(t)^*[X(t) - X(s)]\|_{p/2} + \|[X(t) - X(s)]^* X(s)\|_{p/2} \\ &\leqslant \|X(t)\|_p \|X(t) - X(s)\|_p + \|X(t) - X(s)\|_p \|X(s)\|_p, \end{aligned}$$

which on account of the right-continuity of $(X(t))$ in $\|\cdot\|_p$-norm shows that for $t \searrow s$, $|X(t)|^2 \to |X(s)|^2$ in $\|\cdot\|_{p/2}$-norm.

Let $(X(t)\colon t \in [0, +\infty))$ be a process, and let $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_m < +\infty$ be a sequence of points. To simplify the notation we put

$$\Delta X(t_k) = X(t_k) - X(t_{k-1}), \quad k = 1, \ldots, m.$$

Let $(X(t)\colon t \in [0, +\infty))$, $(Y(t)\colon t \in [0, +\infty))$ be arbitrary processes, and let $[a, b]$ be a subinterval of $[0, +\infty)$. For a partition $\theta = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a, b]$ we form left and right integral sums

$$S^{\mathrm{l}}_\theta = \sum_{k=1}^m \Delta X(t_k) Y(t_{k-1}), \quad S^{\mathrm{r}}_\theta = \sum_{k=1}^m Y(t_{k-1}) \Delta X(t_k).$$

If there exist limits (in any sense) of the above sums as θ refines, we call them respectively the *left* and *right stochastic integrals* of $(Y(t))$ with respect to $(X(t))$, and denote

$$\lim_{\theta} S_{\theta}^{\mathrm{l}} = \int_a^b \mathrm{d}X(t)\, Y(t), \quad \lim_{\theta} S_{\theta}^{\mathrm{r}} = \int_a^b Y(t)\, \mathrm{d}X(t).$$

This notion of integral is a weaker one. Indeed, we could define the integrals as the limits

$$\int_a^b \mathrm{d}X(t)\, Y(t) = \lim_{\|\theta\|\to 0} S_{\theta}^{\mathrm{l}}, \quad \int_a^b Y(t)\, \mathrm{d}X(t) = \lim_{\|\theta\|\to 0} S_{\theta}^{\mathrm{r}},$$

where $\|\theta\|$ stands for the mesh of the partition θ. A definition of this kind is standard in the classical theories of Riemann–Stieltjes as well as stochastic integral; it is worth noticing that in noncommutative integration theory, whenever this Riemann–Stieltjes type integral is considered, its definition refers to the weaker form of the limit with the refining net of partitions (cf. [3], [10], [11]). However, in our case we shall be able to obtain the integral in the stronger sense thus making it similar to the classical stochastic integral.

Let $(X(k)\colon k = 0, 1, \ldots, n)$ be a finite martingale. We have

$$\begin{aligned}
&\mathbb{E}_{k-1}|\Delta X(k)|^2 \\
&=\mathbb{E}_{k-1}(|X(k)|^2 - X(k)^*X(k-1) - X(k-1)^*X(k) + |X(k-1)|^2) \\
&=\mathbb{E}_{k-1}(|X(k)|^2) - (\mathbb{E}_{k-1}X(k)^*)X(k-1) \\
&\qquad\qquad - X(k-1)^*(\mathbb{E}_{k-1}X(k)) + |X(k-1)|^2 \\
&=\mathbb{E}_{k-1}|X(k)|^2 - |X(k-1)|^2 = \mathbb{E}_{k-1}(|X(k)|^2 - |X(k-1)|^2),
\end{aligned} \tag{1.2}$$

by martingale property. From the above we obtain on account of the $\mathbb{E}_k$-invariance of τ

$$\begin{aligned}
\sum_{k=1}^n \||\Delta X(k)|^2\|_1 &= \sum_{k=1}^n \tau(|\Delta X(k)|^2) \\
&= \sum_{k=1}^n \tau(\mathbb{E}_{k-1}(|X(k)|^2)) = \sum_{k=1}^n \tau(\mathbb{E}_{k-1}(|X(k)|^2 - |X(k-1)|)) \\
&= \sum_{k=1}^n \tau(|X(k)|^2 - |X(k-1)|^2) = \tau(|X(n)|^2) - \tau(|X(0)|^2).
\end{aligned} \tag{1.3}$$

The equality above gives the obvious estimation

$$\Big\|\Big(\sum_{k=1}^n |\Delta X(k)|^2\Big)^{1/2}\Big\|_2 = \Big(\sum_{k=1}^n \tau(|\Delta X(k)|)^2\Big)^{1/2} = \Big(\sum_{k=1}^n \||\Delta X(k)|^2\|_1\Big)^{1/2} \leqslant \|X(n)\|_2.$$

A fundamental result from [19], Theorem 2.1, says that the estimation of this type is valid for each $p > 1$. We shall use this for $p > 2$, in which case it has the form:

there exists a constant α_p depending only on p, such that for each L^p-martingale $(X(k)\colon k=0,1,\dots,n)$ we have

$$\Big\|\Big(\sum_{k=1}^{n}|\Delta X(k)|^2\Big)^{1/2}\Big\|_p \leqslant \alpha_p\|X(n)\|_p. \tag{1.4}$$

2. CONTINUITY OF A NONCOMMUTATIVE STOCHASTIC PROCESS

Let $(X(t,\cdot)\colon t\in[a,b])$ be a stochastic process over a probability space $(\Omega,\mathcal{F},P)$. Consider the following condition: for each $\varepsilon>0$ there is $\Omega_\varepsilon\in\mathcal{F}$ with $P(\Omega_\varepsilon)>1-\varepsilon$, such that the trajectories $\{X(\cdot,\omega)\colon\omega\in\Omega_\varepsilon\}$ are equally uniformly continuous. This can be rewritten as:

for each $\varepsilon>0$ there is $\Omega_\varepsilon\in\mathcal{F}$ with $P(\Omega_\varepsilon)>1-\varepsilon$, having the property:

$(*)$ for each $\eta>0$ there is $\delta>0$ such that for any $\omega\in\Omega_\varepsilon$ and any $s,t\in[a,b]$ with $|t-s|<\delta$, we have $|X(t,\omega)-X(s,\omega)|\leqslant\eta$.

If the above condition is satisfied, then the trajectories of the process are uniformly continuous with probability one. Indeed, take $\varepsilon=1/n$, and let $\Omega_\varepsilon=\Omega_{1/n}$ be as above. Put

$$\Omega_0=\bigcup_{n=1}^{\infty}\Omega_{1/n}.$$

Then $P(\Omega_0)=1$, and for each $\omega\in\Omega_0$ we have $\omega\in\Omega_{1/n}$ for some n, which means that the trajectory $X(\cdot,\omega)$ is uniformly continuous.

Now let us assume that the trajectories are uniformly continuous with probability one, and let $\Omega_0=\{\omega\colon X(\cdot,\omega)$ is uniformly continuous$\}$. We have $P(\Omega_0)=1$, and

$$\begin{aligned}\Omega_0&=\bigcap_{r=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}}\Big\{\omega\colon|X(t,\omega)-X(s,\omega)|\leqslant\frac{1}{r}\Big\}\\ &=\bigcap_{r=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}}\Big\{\omega\in\Omega_0\colon|X(t,\omega)-X(s,\omega)|\leqslant\frac{1}{r}\Big\}.\end{aligned}$$

The continuity of the trajectories for $\omega\in\Omega_0$ implies that

$$\bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}}\Big\{\omega\in\Omega_0\colon|X(t,\omega)-X(s,\omega)|\leqslant\frac{1}{r}\Big\}=\bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]\cap\mathbb{Q}}}\Big\{\omega\in\Omega_0\colon|X(t,\omega)-X(s,\omega)|\leqslant\frac{1}{r}\Big\},$$

where $\mathbb{Q}$ stands for the rational numbers. It follows that the set

$$\bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}}\Big\{\omega\in\Omega_0\colon|X(t,\omega)-X(s,\omega)|\leqslant\frac{1}{r}\Big\}$$

is measurable, and for each positive integer r we have

$$
\begin{aligned}
1 &= P\Big(\bigcup_{m=1}^{\infty} \bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}} \Big\{\omega\colon |X(t,\omega)-X(s,\omega)| \leqslant \frac{1}{r}\Big\}\Big)\\
&= \lim_{m\to\infty} P\Big(\bigcap_{\substack{|t-s|<1/m\\ s,t\in[a,b]}} \Big\{\omega\in\Omega_0\colon |X(t,\omega)-X(s,\omega)| \leqslant \frac{1}{r}\Big\}\Big).
\end{aligned}
$$

For any $\varepsilon > 0$ and positive integer r choose m_r such that

$$
P\Big(\bigcap_{\substack{|t-s|<1/m_r\\ s,t\in[a,b]}} \Big\{\omega\in\Omega_0\colon |X(t,\omega)-X(s,\omega)| \leqslant \frac{1}{r}\Big\}\Big) > 1 - \frac{\varepsilon}{2^r},
$$

and put

$$
\Omega_\varepsilon = \bigcap_{r=1}^{\infty} \bigcap_{\substack{|t-s|<1/m_r\\ s,t\in[a,b]}} \Big\{\omega\in\Omega_0\colon |X(t,\omega)-X(s,\omega)| \leqslant \frac{1}{r}\Big\}.
$$

Then $P(\Omega_\varepsilon) > 1-\varepsilon$. For arbitrary fixed $\eta > 0$ let r_0 be such that $1/r_0 \leqslant \eta$. Put $\delta = 1/m_{r_0}$. For each $\omega_0 \in \Omega_\varepsilon$ we have, in particular, that

$$
\omega_0 \in \Big\{\omega\in\Omega_0\colon |X(t,\omega)-X(s,\omega)| \leqslant \frac{1}{r_0}\Big\}
$$

for any $s,t\in[a,b]$ with $|t-s| < 1/m_{r_0} = \delta$, which means that

$$
|X(t,\omega_0) - X(s,\omega_0)| \leqslant \frac{1}{r_0} \leqslant \eta,
$$

showing that condition $(*)$ holds.

We have thus shown the equivalence of uniform continuity of trajectories of the process with probability one and condition $(*)$. Since in our case the uniform continuity of trajectories is equivalent to ordinary continuity, condition $(*)$ can be treated simply as another definition of the classical notion of a continuous stochastic process.

Let us observe that condition $(*)$ can be given the following form. Denote by χ_E the indicator function of the set E. Then condition $(*)$ becomes:

For each $\varepsilon > 0$ there is $\Omega_\varepsilon \in \mathcal{F}$ with $P(\Omega_\varepsilon) > 1-\varepsilon$, having the property: for each $\eta > 0$ there is $\delta > 0$ such that for any $s,t\in[a,b]$ with $|t-s| < \delta$, we have

$$
\begin{aligned}
\sup_{\omega\in\Omega_\varepsilon} |X(t,\omega)-X(s,\omega)| &= \sup_{\omega\in\Omega}[|X(t,\omega)-X(s,\omega)|\chi_{\Omega_\varepsilon}(\omega)]\\
&= \|[X(t,\cdot)-X(s,\cdot)]\chi_{\Omega_\varepsilon}\|_\infty \leqslant \eta.
\end{aligned}
$$

The above form is essentially *algebraic*, referring only to the algebra $L^\infty(\Omega)$, which becomes clear if we replace the inequality $P(\Omega_\varepsilon) > 1-\varepsilon$ by the equivalent inequality

$$\int_{\Omega} \chi_{\Omega_\varepsilon}\, \mathrm{d}P > 1 - \varepsilon.$$

Thus for a noncommutative process $(X(t)\colon t \in [a,b])$ it can be given either of the following two forms: "right" and "left", denoted respectively by (R) and (L).

For each $\varepsilon > 0$ there is a projection e in $\mathcal{A}$ with $\tau(e) > 1 - \varepsilon$, having the property: for each $\eta > 0$ there is $\delta > 0$ such that for any $s, t \in [a,b]$ with $|t - s| < \delta$, we have

$$\text{(R)} \qquad [X(t) - X(s)]e \in \mathcal{A} \quad \text{and} \quad \|[X(t) - X(s)]e\|_\infty \leqslant \eta,$$

or

$$\text{(L)} \qquad e[X(t) - X(s)] \in \mathcal{A} \quad \text{and} \quad \|e[X(t) - X(s)]\|_\infty \leqslant \eta,$$

where $\|\cdot\|_\infty$ denotes the norm in the algebra $\mathcal{A}$. This form of "noncommutative continuity of trajectories with probability one", in its right version, has already been considered before in [14], [15], where it was given the name of "Segal's uniform continuity", and some theorems on this continuity were obtained. However, it is easily seen that the "right Segal's uniform continuity" which appears in the conclusions of those theorems can be changed to the "left Segal's uniform continuity", so the results in [14], [15] give in fact both forms of this continuity. We shall call a process *uniformly continuous in Segal's sense* if it satisfies both (R) and (L) conditions. It is obvious that for a selfadjoint process conditions (R) and (L) are equivalent. Let now ε, e and δ be as above. For arbitrary $s, t \in [a,b]$, $s < t$ choose points $s = t_0 < t_1 < \cdots < t_m = t$ such that $\max\limits_{1 \leqslant k \leqslant m} (t_k - t_{k-1}) < \delta$. Then

$$[X(t) - X(s)]e = [X(t) - X(t_{m-1})]e + \cdots + [X(t_1) - X(s)]e,$$

and since all the summands on the right hand side belong to $\mathcal{A}$ we get that $[X(t) - X(s)]e \in \mathcal{A}$. In particular, if $X(s_0) \in \mathcal{A}$ for some $s_0 \in [a,b]$ then right Segal's uniform continuity means that for each $\varepsilon > 0$ there is a projection $e \in \mathcal{A}$ with $\tau(e) > 1 - \varepsilon$ such that the process $(X(t)e\colon t \in [a,b]) \subset \mathcal{A}$ is uniformly continuous in $\|\cdot\|_\infty$-norm. The same holds of course for left Segal's uniform continuity.

It seems worthwhile to say a few words about the terminology. The term "Segal's convergence" was introduced by E.C. Lance in [18] in honour of I. Segal who first considered this mode of convergence in his celebrated paper [21]. This notion consists in the following: $x_n \to x$ *in Segal's sense* if for each $\varepsilon > 0$ there is a projection $e \in \mathcal{A}$ with $\tau(e^\perp) < \varepsilon$ such that $(x_n - x)e \in \mathcal{A}$ for sufficiently large n, and $\|(x_n - x)e\|_\infty \to 0$. In the definition above it is assumed that τ is a faithful normal *semifinite* trace on $\mathcal{A}$. If τ is finite (as in our case) then Segal's convergence becomes the so-called *almost uniform convergence* (which in the commutative case is *via* Egorov's theorem equivalent to convergence almost everywhere). Now the similarity between Segal's (or in other words: almost uniform) convergence and

Segal's continuity is obvious and goes (essentially) like this: in Segal's convergence we can find an "arbitrarily large" projection e such that $x_n e \to xe$ in $\|\cdot\|_\infty$-norm, while in Segal's continuity we can find an "arbitrarily large" projection e such that the process $(X(t)e\colon t \in [a,b])$ is uniformly continuous in $\|\cdot\|_\infty$-norm. Accordingly, Segal's continuity might also be called *almost uniform continuity*.

REMARK 2.1. It takes little effort to show that Segal's uniform continuity can be given the following, equivalent but technically simpler, form:

$$\text{(S-cont)}\qquad \begin{aligned} &\text{For each } \varepsilon > 0 \text{ there are a projection } e \in \mathcal{A} \text{ with } \tau(e) > 1-\varepsilon,\\ &\text{and } \delta > 0, \text{ such that for any } s,t \in [a,b] \text{ with } |t-s| < \delta, \text{ we have}\\ &[X(t)-X(s)]e \in \mathcal{A}, \qquad e[X(t)-X(s)] \in \mathcal{A},\\ &\text{and}\\ &\|[X(t)-X(s)]e\|_\infty \leqslant \varepsilon, \qquad \|e[X(t)-X(s)]\|_\infty \leqslant \varepsilon. \end{aligned}$$

Consider now a process $(X(t)\colon t \in [0,+\infty))$. It is easily seen that the trajectories of this process are continuous with probability one if and only if for each bounded interval $[a,b]$ contained in $[0,+\infty)$ the trajectories of the process $(X(t)\colon t \in [a,b])$ are uniformly continuous. In accordance with the above observation we adopt the following definition.

DEFINITION 2.2. Let $(X(t)\colon t \in [0,+\infty))$ be a noncommutative stochastic process. We say that it is *continuous in Segal's sense* if for any subinterval $[a,b]$ of the interval $[0,+\infty)$ the process $(X(t)\colon t \in [a,b])$ is uniformly continuous in Segal's sense, i.e., condition (S-cont) is satisfied.

The considerations above lead to one more notion of continuity. Namely, the projection e occurring in the definitions of left and right Segal's continuity can be put on both sides. Accordingly, we have

DEFINITION 2.3. Let $(X(t)\colon t \in [0,+\infty))$ be a noncommutative stochastic process. We say that it is *weakly continuous in Segal's sense* if for any subinteval $[a,b]$ of the interval $[0,+\infty)$ the process $(X(t)\colon t \in [a,b])$ is *weakly uniformly continuous in Segal's sense*, i.e., for each $\varepsilon > 0$ there is a projection e in $\mathcal{A}$ with $\tau(e) > 1-\varepsilon$, having the property: for each $\eta > 0$ there is $\delta > 0$ such that for any $s,t \in [a,b]$ with $|t-s| < \delta$, we have

$$e[X(t)-X(s)]e \in \mathcal{A} \quad \text{and} \quad \|e[X(t)-X(s)]e\|_\infty \leqslant \eta,$$

or equivalently, for each $\varepsilon > 0$ there are a projection $e \in \mathcal{A}$ with $\tau(e) > 1-\varepsilon$, and $\delta > 0$, such that for any $s,t \in [a,b]$ with $|t-s| < \delta$, we have

$$e[X(t)-X(s)]e \in \mathcal{A} \quad \text{and} \quad \|e[X(t)-X(s)]e\|_\infty \leqslant \varepsilon.$$

It is clear that both left and right Segal's uniform continuity imply weak Segal's uniform continuity; moreover, if $(X(t))$ is left and $(Y(t))$ is right uniformly continuous in Segal's sense then $(X(t)+Y(t))$ is weakly uniformly continuous in

Segal's sense. Obviously, in the commutative case all three modes of continuity are equivalent.

3. STOCHASTIC INTEGRAL

In this section we shall prove the following

THEOREM 3.1. *Let $(X(t)\colon t \in [0, +\infty))$ be an L^p-valued martingale, $p > 2$, and let $(f(t)\colon t \in [0, +\infty))$ be an $\mathcal{A}$-valued adapted norm-bounded on each interval $[a, b]$ continuous in Segal's sense process. Then for each $t > 0$ there exist stochastic integrals*

$$Y(t) = \int_0^t \mathrm{d}X(u)\, f(u) \quad \text{and} \quad Z(t) = \int_0^t f(u)\, \mathrm{d}X(u)$$

as elements of $L^2(\mathcal{A})$. Moreover, $(Y(t))$ and $(Z(t))$ are martingales, and if $(X(t))$ is continuous either in Segal's sense or in $\|\cdot\|_2$-norm then these martingales are L^2-continuous.

Proof. Restrict attention to the left integral. Let $\theta(t) = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ be a partition of $[0, t]$, and let

$$S^1_{\theta(t)}(t) = \sum_{k=1}^{m} \Delta X(t_k) f(t_{k-1}).$$

We shall show that for each $a > 0$

$$\lim_{\|\theta(t)\| \to 0} S^1_{\theta(t)}(t) = Y(t) \quad \text{uniformly in } t \in [0, a],$$

that is, for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $t \in [0, a]$ and any $\theta(t)$ with $\|\theta(t)\| < \delta$ we have

$$\|S^1_{\theta(t)}(t) - Y(t)\|_2 \leqslant \varepsilon.$$

Put

$$M = \sup_{0 \leqslant u \leqslant a} \|f(u)\|_\infty, \tag{3.1}$$

$$K = \tau(|X(a)|^2) - \tau(|X(0)|^2), \tag{3.2}$$

$$q = \frac{p}{p-2} \quad \text{so that } \frac{1}{q} + \frac{2}{p} = 1, \tag{3.3}$$

and let α_p be as in (1.4). Take an arbitrary $t \in [0, a]$, and let $\varepsilon > 0$ be given. On account of Segal's continuity of f we can find a projection e in $\mathcal{A}$ with

$$\tau(e) > 1 - \Big[\frac{\varepsilon^2}{32M^2(\alpha_p \|X(a)\|_p)^2}\Big]^q \quad \text{i.e.,} \quad \tau(e^\perp) < \Big[\frac{\varepsilon^2}{32M^2(\alpha_p \|X(a)\|_p)^2}\Big]^q, \tag{3.4}$$

and $\delta > 0$ such that for any $t', t'' \in [0,t]$ with $|t'-t''| < \delta$, we have

$$\|e[f(t')-f(t'')]\|_\infty = \|[f(t')-f(t'')]^* e\|_\infty \leqslant \frac{\varepsilon^2}{32MK}, \tag{3.5}$$
$$\|[f(t')-f(t'')]e\|_\infty \leqslant \frac{\varepsilon^2}{32MK}.$$

Let $\theta'(t) = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ be an arbitrary partition of $[0,t]$ with $\|\theta'(t)\| < \delta$, and let $\theta''(t)$ be a partition of $[0,t]$ finer than $\theta'(t)$. Denote by $t_0^{(k)}, t_1^{(k)}, \ldots, t_{l_k}^{(k)}$ the points of $\theta''(t)$ lying between t_{k-1} and t_k, such that $t_{k-1} = t_0^{(k)} < t_1^{(k)} < \cdots < t_{l_k}^{(k)} = t_k$. We then have

$$S^1_{\theta''(t)} = \sum_{k=1}^m \sum_{i=1}^{l_k} \Delta X(t_i^{(k)}) f(t_{i-1}^{(k)}), \quad S^1_{\theta'(t)} = \sum_{k=1}^m \Delta X(t_k) f(t_{k-1}) = \sum_{k=1}^m \sum_{i=1}^{l_k} \Delta X(t_i^{(k)}) f(t_{k-1}),$$

so that

$$S^1_{\theta''(t)} - S^1_{\theta'(t)} = \sum_{k=1}^m \sum_{i=1}^{l_k} \Delta X(t_i^{(k)})[f(t_{i-1}^{(k)}) - f(t_{k-1})].$$

Consequently,

$$\begin{aligned}
&\|S^1_{\theta''(t)} - S^1_{\theta'(t)}\|_2^2 \\
&\quad = \tau\Big(\sum_{r=1}^m \sum_{j=1}^{l_r} [f(t_{j-1}^{(r)}) - f(t_{r-1})]^* \Delta X(t_j^{(r)})^* \sum_{k=1}^m \sum_{i=1}^{l_k} \Delta X(t_i^{(k)})[f(t_{i-1}^{(k)}) - f(t_{k-1})]\Big) \\
&\quad = \tau\Big(\sum_{r=1}^m \sum_{j=1}^{l_r} \sum_{k=1}^m \sum_{i=1}^{l_k} [f(t_{j-1}^{(r)}) - f(t_{r-1})]^* \Delta X(t_j^{(r)})^* \Delta X(t_i^{(k)})[f(t_{i-1}^{(k)}) - f(t_{k-1})]\Big) \\
&\quad = \sum_{r=1}^m \sum_{j=1}^{l_r} \sum_{k=1}^m \sum_{i=1}^{l_k} \tau([f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{j-1}^{(r)}) - f(t_{r-1})]^* \Delta X(t_j^{(r)})^* \Delta X(t_i^{(k)})).
\end{aligned}$$

For $k < r$ we have $t_{k-1} \leqslant t_{i-1}^{(k)} < t_i^{(k)} \leqslant t_{r-1} \leqslant t_{j-1}^{(r)} < t_j^{(r)}$, and the $\mathbb{E}_t$-invariance of τ together with the fact that $f(t_{i-1}^{(k)}) - f(t_{k-1})$, $[f(t_{j-1}^{(r)}) - f(t_{r-1})]^*$ and $\Delta X(t_i^{(k)})$ belong to $L^p(\mathcal{A}_{t_{j-1}^{(r)}})$ yield

$$\begin{aligned}
&\tau([f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{j-1}^{(r)}) - f(t_{r-1})]^* \Delta X(t_j^{(r)})^* \Delta X(t_i^{(k)})) \\
&\qquad = \tau(\mathbb{E}_{t_{j-1}^{(r)}} [f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{j-1}^{(r)}) - f(t_{r-1})]^* \Delta X(t_j^{(r)})^* \Delta X(t_i^{(k)})) \\
&\qquad = \tau([f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{j-1}^{(r)}) - f(t_{r-1})]^* [\mathbb{E}_{t_{j-1}^{(r)}} \Delta X(t_j^{(r)})]^* \Delta X(t_i^{(k)})) = 0,
\end{aligned}$$

since

$$\mathbb{E}_{t_{j-1}^{(r)}} \Delta X(t_j^{(r)}) = \mathbb{E}_{t_{j-1}^{(r)}} (X(t_j^{(r)}) - X(t_{j-1}^{(r)})) = \mathbb{E}_{t_{j-1}^{(r)}} X(t_j^{(r)}) - X(t_{j-1}^{(r)}) = 0$$

by martingale property. Analogously for $k > r$, thus we are left only with the case $k = r$. For $i < j$ we have $t_{k-1} \leqslant t_{i-1}^{(k)} < t_i^{(k)} \leqslant t_{j-1}^{(k)} < t_j^{(k)}$, and in a similar fashion as above we obtain

$$\begin{aligned}
&\tau([f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{j-1}^{(k)}) - f(t_{k-1})]^* \Delta X(t_j^{(k)})^* \Delta X(t_i^{(k)})) \\
&\qquad = \tau(\mathbb{E}_{t_{j-1}^{(k)}} [f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{j-1}^{(k)}) - f(t_{k-1})]^* \Delta X(t_j^{(k)})^* \Delta X(t_i^{(k)})) \\
&\qquad = \tau([f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{j-1}^{(k)}) - f(t_{k-1})]^* [\mathbb{E}_{t_{j-1}^{(k)}} \Delta X(t_j^{(k)})]^* \Delta X(t_i^{(k)})) = 0.
\end{aligned}$$

The same goes for $i > j$, so finally we get

$$\begin{aligned}
&\|S^1_{\theta''(t)} - S^1_{\theta'(t)}\|_2^2 \\
&= \tau\Big(\sum_{k=1}^{m} \sum_{i=1}^{l_k} [f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{i-1}^{(k)}) - f(t_{k-1})]^* \Delta X(t_i^{(k)})^* \Delta X(t_i^{(k)}) \Big) = I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \tau\Big(\sum_{k=1}^{m} \sum_{i=1}^{l_k} e[f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{i-1}^{(k)}) - f(t_{k-1})]^* |\Delta X(t_i^{(k)}|^2 \Big), \\
I_2 &= \tau\Big(\sum_{k=1}^{m} \sum_{i=1}^{l_k} e^{\perp}[f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{i-1}^{(k)}) - f(t_{k-1})]^* |\Delta X(t_i^{(k)}|^2 \Big).
\end{aligned}$$

For I_1 we have using (3.1), (3.2) and (3.5) together with (1.3)

$$\begin{aligned}
|I_1| &\leqslant \sum_{k=1}^{m} \sum_{i=1}^{l_k} |\tau(e[f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{i-1}^{(k)}) - f(t_{k-1})]^* |\Delta X(t_i^{(k)})|^2)| \\
&\leqslant \sum_{k=1}^{m} \sum_{i=1}^{l_k} \|e[f(t_{i-1}^{(k)}) - f(t_{k-1})]\|_\infty \|[f(t_{i-1}^{(k)}) - f(t_{k-1})]^*\|_\infty \\
&\qquad\qquad \cdot \||\Delta X(t_i^{(k)})|^2\|_1 \leqslant \frac{\varepsilon^2}{32MK} 2M \sum_{k=1}^{m} \sum_{i=1}^{l_k} \||\Delta X(t_i^{(k)})|^2\|_1 \\
&\leqslant \frac{\varepsilon^2}{16K} \tau(|X(t)|^2 - |X(0)|^2) \leqslant \frac{\varepsilon^2}{16}.
\end{aligned}$$

Divide I_2 into two parts: $I_2 = I_2' + I_2''$, where

$$\begin{aligned}
I_2' &= \tau\Big(\sum_{k=1}^{m} \sum_{i=1}^{l_k} e^{\perp}[f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{i-1}^{(k)}) - f(t_{k-1})]^* e^{\perp} |\Delta X(t_i^{(k)})|^2 \Big), \\
I_2'' &= \tau\Big(\sum_{k=1}^{m} \sum_{i=1}^{l_k} e^{\perp}[f(t_{i-1}^{(k)}) - f(t_{k-1})][f(t_{i-1}^{(k)}) - f(t_{k-1})]^* e |\Delta X(t_i^{(k)})|^2 \Big).
\end{aligned}$$

Then we have by (3.1)

$$
\begin{aligned}
|I_2'| &\leqslant \sum_{k=1}^{m}\sum_{i=1}^{l_k} |\tau(e^{\perp}[f(t_{i-1}^{(k)})-f(t_{k-1})][f(t_{i-1}^{(k)})-f(t_{k-1})]^* e^{\perp}|\Delta X(t_i^{(k)})|^2)| \\
&= \sum_{k=1}^{m}\sum_{i=1}^{l_k} |\tau([f(t_{i-1}^{(k)})-f(t_{k-1})][f(t_{i-1}^{(k)})-f(t_{k-1})]^* e^{\perp}|\Delta X(t_i^{(k)})|^2 e^{\perp})| \\
&\leqslant \sum_{k=1}^{m}\sum_{i=1}^{l_k} \|([f(t_{i-1}^{(k)})-f(t_{k-1})][f(t_{i-1}^{(k)})-f(t_{k-1})]^*\|_\infty \|e^{\perp}|\Delta X(t_i^{(k)})|^2 e^{\perp})\|_1 \\
&\leqslant 4M^2 \sum_{k=1}^{m}\sum_{i=1}^{l_k} \tau(e^{\perp}|\Delta X(t_i^{(k)})|^2 e^{\perp}) = 4M^2\tau\Big(e^{\perp}\sum_{k=1}^{m}\sum_{i=1}^{l_k}|\Delta X(t_i^{(k)})|^2\Big) \\
&\leqslant 4M^2\Big\|e^{\perp}\sum_{k=1}^{m}\sum_{i=1}^{l_k}|\Delta X(t_i^{(k)})|^2)\Big\|_1 .
\end{aligned}
$$

From Hölder's inequality applied to the last term above we get, with q given by (3.3),

$$
|I_2'| \leqslant 4M^2\|e^{\perp}\|_q\Big\|\sum_{k=1}^{m}\sum_{i=1}^{l_k}|\Delta X(t_i^{(k)})|^2\Big\|_{p/2} = 4M^2[\tau(e^{\perp})]^{1/q}\Big\|\Big(\sum_{k=1}^{m}\sum_{i=1}^{l_k}|\Delta X(t_i^{(k)})|^2\Big)^{1/2}\Big\|_p^2,
$$

and taking into account inequalities (1.4), (3.4) we finally obtain

$$
|I_2'| \leqslant 4M^2\frac{\varepsilon^2}{32M^2(\alpha_p\|X(a)\|_p)^2}(\alpha_p\|X(t)\|_p)^2 \leqslant \frac{\varepsilon^2}{8}.
$$

For I_2'' we have using again (3.1), (3.5) and (3.2) together with (1.3)

$$
\begin{aligned}
|I_2''| &\leqslant \sum_{k=1}^{m}\sum_{i=1}^{l_k} |\tau(e^{\perp}[f(t_{i-1}^{(k)})-f(t_{k-1})][f(t_{i-1}^{(k)})-f(t_{k-1})]^* e|\Delta X(t_i^{(k)})|^2)| \\
&\leqslant \sum_{k=1}^{m}\sum_{i=1}^{l_k} \|e^{\perp}[f(t_{i-1}^{(k)})-f(t_{k-1})][f(t_{i-1}^{(k)})-f(t_{k-1})]^* e\|_\infty \||\Delta X(t_i^{(k)})|^2\|_1 \\
&\leqslant \sum_{k=1}^{m}\sum_{i=1}^{l_k} 2M\frac{\varepsilon^2}{32MK}\||\Delta X(t_i^{(k)})|^2\|_1 = \frac{\varepsilon^2}{16K}\sum_{k=1}^{m}\sum_{i=1}^{l_k}\||\Delta X(t_i^{(k)})|^2\|_1 \\
&= \frac{\varepsilon^2}{16K}\tau(|X(t)|^2-|X(0)|^2) \leqslant \frac{\varepsilon^2}{16}.
\end{aligned}
$$

Thus

$$
|I_2| \leqslant |I_2'| + |I_2''| \leqslant \frac{\varepsilon^2}{8}+\frac{\varepsilon^2}{16},
$$

and consequently,

$$
\|S^1_{\theta''(t)} - S^1_{\theta'(t)}\|_2^2 = I_1 + I_2 \leqslant \frac{\varepsilon^2}{16}+\frac{\varepsilon^2}{8}+\frac{\varepsilon^2}{16} = \frac{\varepsilon^2}{4}. \tag{3.6}
$$

Let now $\theta_1(t)$ and $\theta_2(t)$ be arbitrary partitions of $[0,t]$ such that $\|\theta_1(t)\|<\delta$, $\|\theta_2(t)\| < \delta$, and let $\theta''(t) = \theta_1(t) \cup \theta_2(t)$. Then we have by (3.6)

$$\|S^{\mathrm{l}}_{\theta''(t)}(t) - S^{\mathrm{l}}_{\theta_1(t)}(t)\|_2 \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad \|S^{\mathrm{l}}_{\theta''(t)}(t) - S^{\mathrm{l}}_{\theta_2(t)}(t)\|_2 \leqslant \frac{\varepsilon}{2},$$

so

$$\|S^{\mathrm{l}}_{\theta_1(t)}(t) - S^{\mathrm{l}}_{\theta_2(t)}(t)\|_2 \leqslant \varepsilon,$$

which means that the net $\{S^{\mathrm{l}}_{\theta(t)}(t)\}$ satisfies the Cauchy condition as $\|\theta(t)\| \to 0$ uniformly in $t \in [0,a]$, proving the existence of the left integral $Y(t) = \int_0^t \mathrm{d}X(t)\, f(t)$, together with the uniform convergence of $S_{\theta(t)}(t)$ to $Y(t)$ in $\|\cdot\|_2$-norm. The existence of the right integral $Z(t) = \int_0^t f(t)\,\mathrm{d}X(t)$ is proved in virtually the same way.

Now we shall show that $(Y(t)\colon t \in [0,+\infty))$ is a martingale. Fix $t > 0$ and take an arbitrary $s < t$ (t being fixed, so we suppress in our notation the dependence of θ and S^{l}_θ on t). We have

$$\int_0^t \mathrm{d}X(u)\, f(u) = \lim_{\|\theta\|\to 0} S^{\mathrm{l}}_\theta.$$

We may assume that s is one of the points of each partition $\theta = \{0 = t_0 < t_1 < \cdots < t_m = t\}$, say $s = t_k$. Then we have

$$\begin{aligned}\mathbb{E}_s S^{\mathrm{l}}_\theta &= \mathbb{E}_s\Big(\sum_{i=1}^{k}[X(t_i) - X(t_{i-1})]f(t_{i-1}) + \sum_{i=k+1}^{m}[X(t_i) - X(t_{i-1})]f(t_{i-1})\Big)\\ &= \sum_{i=1}^{k}\mathbb{E}_s[X(t_i) - X(t_{i-1})]f(t_{i-1}) + \sum_{i=k+1}^{m}\mathbb{E}_s[X(t_i) - X(t_{i-1})]f(t_{i-1}).\end{aligned}$$

For $i \leqslant k$ we have $t_i \leqslant s$, and thus

$$\mathbb{E}_s[X(t_i) - X(t_{i-1})]f(t_{i-1}) = [X(t_i) - X(t_{i-1})]f(t_{i-1}),$$

while for $i > k$ we have $t_{i-1} \geqslant s$, and thus

$$\begin{aligned}\mathbb{E}_s[X(t_i) - X(t_{i-1})]f(t_{i-1}) &= \mathbb{E}_s\mathbb{E}_{t_{i-1}}[X(t_i) - X(t_{i-1})]f(t_{i-1})\\ &= \mathbb{E}_s(\mathbb{E}_{t_{i-1}}[X(t_i) - X(t_{i-1})])f(t_{i-1}) = 0\end{aligned}$$

by martingale property. Consequently,

$$\mathbb{E}_s S^{\mathrm{l}}_\theta = \sum_{i=1}^{k}[X(t_i) - X(t_{i-1})]f(t_{i-1}). \tag{3.7}$$

But the sum on the right hand side of (3.7) is an integral sum for the integral $\int_0^s \mathrm{d}X(u)\, f(u)$, and passing to the limit in (3.7) yields

$$\mathbb{E}_s \int_0^t \mathrm{d}X(u)\, f(u) = \int_0^s \mathrm{d}X(u)\, f(u),$$

which shows that $(Y(t))$ is a martingale. Analogously for $(Z(t))$.

Now we shall prove the $\|\cdot\|_2$-continuity of $(Y(t))$ in an arbitrary interval $[0,a]$. Let $\theta_n = \{0 = t_1^{(n)} < t_2^{(n)} < \cdots < t_{m_n}^{(n)} = a\}$ be a sequence of partitions of $[0,a]$ such that $\theta_n \subset \theta_{n+1}$ and $\|\theta_n\| \to 0$. For an arbitrary $t \in [0,a]$ put $\theta_n(t) = (\theta_n \cap [0,t]) \cup \{t\}$, and

$$S_n(t) = \sum_{0 \leqslant t_{k-1}^{(n)} < t_k^{(n)} \leqslant t} \Delta X(t_k^{(n)}) f(t_{k-1}^{(n)}) + [X(t) - X(t_n')] f(t_n'),$$

where $t_n' = \max\{t_k^{(n)} : t_k^{(n)} \leqslant t\}$. Then

$$S_n(t) = S^{\mathrm{l}}_{\theta_n(t)}(t)$$

in the notation from the first part of the proof, and

$$\lim_{n\to\infty} S_n(t) = Y(t) \qquad \text{uniformly in } t \in [0,a].$$

First we show the $\|\cdot\|_2$-continuity of S_n. Take arbitrary $s, t \in [0,a]$, $s < t$, such that $|t-s| < \|\theta_n\|$. We have three possibilities:

(i) $s \in \theta_n,\ t \notin \theta_n$.
Then $t_n' = s$, and

$$S_n(t) - S_n(s) = [X(t) - X(s)] f(s).$$

(ii) $s \notin \theta_n,\ t \in \theta_n$.
Then $t_n' = t$, and

$$S_n(t) - S_n(s) = [X(t) - X(s_n')] f(s_n') - [X(s) - X(s_n')] f(s_n') = [X(t) - X(s)] f(s_n'),$$

where $s_n' = \max\{t_k^{(n)} : t_k^{(n)} \leqslant s\}$.

(iii) $s \notin \theta_n,\ t \notin \theta_n$.
Then $s_n' < s < t_n' < t$, and

$$\begin{aligned} S_n(t) - S_n(s) &= [X(t) - X(t_n')] f(t_n') + [X(t_n') - X(s_n')] f(s_n') - [X(s) - X(s_n')] f(s_n') \\ &= [X(t) - X(t_n')] f(t_n') + [X(t_n') - X(s)] f(s_n'). \end{aligned} \tag{3.8}$$

Note that case (iii) contains the other two, so we have in general

$$\|S_n(t) - S_n(s)\|_2 \leqslant M \|X(t) - X(t_n')\|_2 + M \|X(t_n') - X(s)\|_2. \tag{3.9}$$

Consequently, we only need to estimate the expression $\|X(u) - X(v)\|_2$ for $s \leqslant v < u \leqslant t$.

If $(X(t))$ is $\|\cdot\|_2$-continuous then obviously this can be made arbitrarily small for s, t sufficiently close to each other.

If $(X(t))$ is continuous in Segal's sense then for each $\varepsilon > 0$ we can find a projection e in $\mathcal{A}$ with

$$\tau(e^{\perp}) < \Big[\frac{\varepsilon}{8M\|X(a)\|_p}\Big]^{q'}, \quad \text{where } \frac{1}{q'} + \frac{1}{p} = \frac{1}{2},$$

and $\delta > 0$ such that for $|t' - t''| < \delta$ we have

$$\|e[X(t') - X(t'')]\|_\infty < \frac{\varepsilon}{4M}.$$

If $|t - s| < \delta$ then also $|u - v| < \delta$, and

$$\begin{aligned}\|X(u) - X(v)\|_2 &\leqslant \|e[X(u) - X(v)]\|_2 + \|e^{\perp}[X(u) - X(v)]\|_2 \\ &\leqslant \|e[X(u) - X(v)]\|_\infty + \|e^{\perp}[X(u) - X(v)]\|_2 \\ &\leqslant \frac{\varepsilon}{4M} + \|e^{\perp}[X(u) - X(v)]\|_2.\end{aligned}$$

From Hölder's inequality we obtain

$$\|e^{\perp}[X(u) - X(v)]\|_2 \leqslant \|e^{\perp}\|_{q'}\|X(u) - X(v)\|_p \leqslant 2\|X(a)\|_p[\tau(e^{\perp})]^{1/q'} < \frac{\varepsilon}{4M},$$

thus finally

$$\|X(u) - X(v)\|_2 < \frac{\varepsilon}{2M}.$$

This estimation shows that for $|t - s| < \delta \wedge \|\theta_n\|$ the inequality in (3.9) takes the form

$$\|S_n(t) - S_n(s)\|_2 < M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon,$$

showing the uniform $\|\cdot\|_2$-continuity of S_n. Since $S_n \rightrightarrows Y$ in $[0, a]$ we obtain the $\|\cdot\|_2$-continuity of Y. ∎

REMARK 3.2. It is seen from the above proof that for the existence of the left integral it suffices that f be "left Segal's continuous" while for the existence of the right integral it suffices that f be "right Segal's continuous".

Our next aim is to show a noncommutative counterpart of the known classical result saying that a stochastic integral with the integrator being a continuous martingale is continuous. To this end, we begin with a result which may be looked upon as a noncommutative generalisation of one of the classical martingale inequalities. Our attention will be restricted to its simplest version for a finite martingale, which suffices for the purposes of this paper; however, it is worth mentioning that a result of this type can be obtained also in a more general setting. The idea of the proof is an adaptation of the classical method to the noncommutative setup, and has already been used by C.J.K. Batty in a slightly different context for proving "noncommutative Kolmogorov's inequality" for sums of "independent noncommutative random variables" (cf. Proposition 5.1 in [12]).

PROPOSITION 3.3. *Let $(X_1, \dots, X_m)$ be an L^2-martingale. Then for each $\varepsilon > 0$ there is a projection $e \in \mathcal{A}$ such that*

$$\tau(e^{\perp}) < \frac{\|X_m\|_2^2}{\varepsilon^2}, \quad \textit{and} \quad \|eX_n\|_\infty \leqslant \varepsilon \quad \textit{for each} \;\; n = 1, \dots, m.$$

The same conclusion holds also in the "right version" with the projection e put to the right of X_n.

Proof. We shall prove the "left version", the proof of the other one being *mutatis mutandis* the same. Let us start with some simple remarks. Let $x \in L^1(\mathcal{A})$ be a positive operator with its spectral decomposition

$$x = \int_0^\infty \lambda \, e(\mathrm{d}\lambda).$$

For each $\eta > 0$ we have

$$x = \int_{[0,\eta)} \lambda \, e(\mathrm{d}\lambda) + \int_{[\eta,+\infty)} \lambda \, e(\mathrm{d}\lambda) \geqslant \int_{[\eta,+\infty)} \lambda \, e(\mathrm{d}\lambda) \geqslant \eta e([\eta, +\infty)), \tag{3.10}$$

and

$$e([0,\eta))xe([0,\eta)) = \int_{[0,\eta)} \lambda \, e(\mathrm{d}\lambda).$$

In particular, we obtain "noncommutative Chebyshev's inequality"

$$\tau(e([\eta, +\infty))) \leqslant \frac{\tau(x)}{\eta},$$

and the estimation

$$\|e([0,\eta))xe([0,\eta))\|_\infty \leqslant \eta.$$

Now let f be an arbitrary projection in $\mathcal{A}$. Write the spectral decomposition of fxf

$$fxf = \int_0^\infty \lambda \, h(\mathrm{d}\lambda).$$

Since f commutes with fxf it follows that f commutes with the spectral measure h of fxf, consequently

$$\|fh([0,\eta))xh([0,\eta))f\|_\infty = \|h([0,\eta))fxfh([0,\eta))\|_\infty \leqslant \eta. \tag{3.11}$$

To facilitate notation, we agree to denote the value of the spectral measure of the operator $x \geqslant 0$ on a Borel set $Z \subset \mathbb{R}$ by $e_Z(x)$. Let an arbitrary $\varepsilon > 0$ be given.

Define inductively projections e_n, f_n for $n = 1, \dots, m$, by

$$\begin{aligned} &e_1 = e_{[0,\varepsilon^2)}(|X_1^*|^2), \quad f_1 = e_1; \\ &e_2 = e_{[0,\varepsilon^2)}(f_1|X_2^*|^2 f_1), \quad f_2 = e_1 \wedge e_2; \\ &\vdots \\ &e_n = e_{[0,\varepsilon^2)}(f_{n-1}|X_n^*|^2 f_{n-1}), \quad f_n = e_1 \wedge \cdots \wedge e_n. \end{aligned}$$

We have $e_n \in \mathcal{A}_n$, and for $i < j$

$$e_j^\perp \leqslant f_{j-1} \leqslant f_i \leqslant e_i, \tag{3.12}$$

thus, in particular, $e_i^\perp e_j^\perp = 0$ for $i \neq j$. Put $e = f_m = e_1 \wedge \cdots \wedge e_m$. Then

$$e^\perp = \bigvee_{n=1}^m e_n^\perp = \sum_{n=1}^m e_n^\perp.$$

Inequality (3.10) yields $\varepsilon^2 e_n^\perp = \varepsilon^2 e_{[\varepsilon^2,+\infty)}(f_{n-1}|X_n^*|^2 f_{n-1}) \leqslant f_{n-1}|X_n^*|^2 f_{n-1}$, and thus multiplying both sides of the above inequality by $e_n^\perp$ we obtain from (3.12) and the fact that $(|X_n^*|^2 \colon n = 1, \dots, m)$ is a submartingale

$$\varepsilon^2 e_n^\perp \leqslant e_n^\perp |X_n^*|^2 e_n^\perp \leqslant e_n^\perp \mathbb{E}_n |X_m^*|^2 e_n^\perp = \mathbb{E}_n(e_n^\perp |X_m^*|^2 e_n^\perp).$$

Consequently, $\varepsilon^2 \tau(e_n^\perp) \leqslant \tau(\mathbb{E}_n(e_n^\perp |X_m^*|^2 e_n^\perp)) = \tau(e_n^\perp |X_m^*|^2 e_n^\perp) = \tau(e_n^\perp |X_m^*|^2)$. Thus

$$\begin{aligned} \varepsilon^2 \tau(e^\perp) &= \varepsilon^2 \tau\Big(\bigvee_{n=1}^m e_n^\perp\Big) = \varepsilon^2 \sum_{n=1}^m \tau(e_n^\perp) \leqslant \tau\Big(\Big(\sum_{n=1}^m e_n^\perp\Big)|X_m^*|^2\Big) \\ &= \tau(e^\perp |X_m^*|^2) \leqslant \tau(|X_m^*|^2) = \|X_m^*\|_2^2 = \|X_m\|_2^2, \end{aligned}$$

which gives the desired estimation of $\tau(e^\perp)$. For the norm we have on account of (3.11)

$$\begin{aligned} \|eX_n\|_\infty^2 &= \|X_n^* e\|_\infty^2 = \|e|X_n^*|^2 e\|_\infty^2 = \|f_{n-1} e|X_n^*|^2 e f_{n-1}\|_\infty^2 \leqslant \|f_{n-1} e_n |X_n^*|^2 e_n f_{n-1}\|_\infty^2 \\ &= \|f_{n-1} e_{[0,\varepsilon^2)}(f_{n-1}|X_n^*|^2 f_{n-1})|X_n^*|^2 e_{[0,\varepsilon^2)}(f_{n-1}|X_n^*|^2 f_{n-1}) f_{n-1}\|_\infty^2 \leqslant \varepsilon^2, \end{aligned}$$

which proves the claim. ∎

THEOREM 3.4. *Let $(X(t))$ and $(f(t))$ be as in Theorem 3.1. If $(X(t))$ is left continuous in Segal's sense then $(Y(t))$ is left continuous in Segal's sense. If $(X(t))$ is right continuous in Segal's sense then $(Z(t))$ is right continuous in Segal's sense.*

Proof. Again we restrict attention to the "left" case. We shall prove the left uniform Segal's continuity of $(Y(t))$ in an arbitrary interval $[0, a]$.

Fix a positive integer n. It is easily seen that $(S_n(t) \colon t \in [0, a])$ is a martingale; moreover, equality (3.8) shows that if $(X(t) \colon t \in [0, a])$ is uniformly left continuous in Segal's sense then $(S_n(t) \colon t \in [0, a])$ is also uniformly left continuous in Segal's sense.

Let now m, n be arbitrary fixed positive integers. The process $(S_n(t) - S_m(t)\colon t \in [0,a])$ is a uniformly left continuous in Segal's sense martingale. For any given $\varepsilon_{n,m} > 0$ let $f_{n,m}$ be a projection in $\mathcal{A}$ such that $\tau(f_{n,m}^{\perp}) < \varepsilon_{n,m}$, and the processes $(f_{n,m}[S_n(t) - S_m(t)]\colon t \in [0,a])$, $(f_{n,m}S_n(t)\colon t \in [0,a])$, $(f_{n,m}S_m(t)\colon t \in [0,a]) \subset \mathcal{A}$ are uniformly continuous in $\|\cdot\|_\infty$-norm. In particular, there is $\delta_{n,m} > 0$ such that for all $t', t'' \in [0,a]$ with $|t' - t''| < \delta_{n,m}$ we have

$$\|f_{n,m}([S_n(t') - S_m(t')] - [S_n(t'') - S_m(t'')])\|_\infty \leqslant \frac{\varepsilon_{n,m}}{2}. \tag{3.13}$$

Choose points $0 = t_0 < t_1 < \cdots < t_r = a$ such that $\max\limits_{1\leqslant i\leqslant r}(t_i - t_{i-1}) < \delta_{n,m}$. For the martingale $(S_n(t_i) - S_m(t_i)\colon i = 0,1,\ldots,r)$ we infer on account of Proposition 3.3 that there exists a projection $q_{n,m} \in \mathcal{A}$ with

$$\tau(q_{n,m}^{\perp}) < \frac{4\|S_n(a) - S_m(a)\|_2^2}{\varepsilon_{n,m}^2}$$

such that

$$\|q_{n,m}[S_n(t_i) - S_m(t_i)]\|_\infty \leqslant \frac{\varepsilon_{n,m}}{2}, \quad \text{for } i = 0,1,\ldots,r. \tag{3.14}$$

Put $e_{n,m} = f_{n,m} \wedge q_{n,m}$. Then

$$\tau(e_{n,m}) < \varepsilon_{n,m} + \frac{4\|S_n(a) - S_m(a)\|_2^2}{\varepsilon_{n,m}^2},$$

and for each $t \in [0,a]$ there is t_i such that $|t - t_i| < \delta_{n,m}$, so from (3.13) and (3.14) we get

$$\begin{aligned}
&\|e_{n,m}[S_n(t) - S_m(t)]\|_\infty \\
&\quad\leqslant \|e_{n,m}([S_n(t) - S_m(t)] - [S_n(t_i) - S_m(t_i)])\|_\infty + \|e_{n,m}[S_n(t_i) - S_m(t_i)]\|_\infty \\
&\quad\leqslant \|f_{n,m}([S_n(t) - S_m(t)] \\
&\qquad - [S_n(t_i) - S_m(t_i)])\|_\infty + \|q_{n,m}[S_n(t_i) - S_m(t_i)]\|_\infty \leqslant \frac{\varepsilon_{n,m}}{2} + \frac{\varepsilon_{n,m}}{2} = \varepsilon_{n,m}.
\end{aligned}$$

Moreover, the processes $(e_{n,m}S_n(t)\colon t \in [0,a])$ and $(e_{n,m}S_m(t)\colon t \in [0,a])$ are uniformly continuous in $\|\cdot\|_\infty$-norm.

Let an arbitrary $\varepsilon > 0$ be given. We have $S_n(a) \to Y(a)$ in $\|\cdot\|_2$-norm, so we can find a subsequence $\{n_k\}$ such that

$$\|S_{n_{k+1}}(a) - S_{n_k}(a)\|_2 < \frac{\varepsilon^3}{2^{3k+5}}, \quad k = 1,2,\ldots.$$

Apply our previous considerations to the martingale $(S_{n_{k+1}}(t) - S_{n_k}(t)\colon t \in [0,a])$ (i.e., we take $n = n_{k+1}$, $m = n_k$). For $(\varepsilon_{n,m} =)\ \varepsilon_k = \varepsilon/2^{k+1}$ there is a projection $e_k \in \mathcal{A}$ with

$$\tau(e_k^{\perp}) < \frac{\varepsilon}{2^{k+1}} + \frac{4(2^{k+1})^2\|S_{n_{k+1}}(a) - S_{n_k}(a)\|_2^2}{\varepsilon^2} < \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2^k},$$

such that for each $t \in [0, a]$

$$\|e_k[S_{n_{k+1}}(t) - S_{n_k}(t)]\|_\infty \leqslant \frac{\varepsilon}{2^{k+1}}.$$

Put

$$e = \bigwedge_{k=1}^{\infty} e_k.$$

Then $\tau(e^\perp) < \varepsilon$, and for each $t \in [0, a]$

$$\|e[S_{n_{k+1}}(t) - S_{n_k}(t)]\|_\infty \leqslant \frac{\varepsilon}{2^{k+1}}. \tag{3.15}$$

Since the processes $(e_k S_{n_k}(t) \colon t \in [0, a])$, $k = 1, 2, \ldots$, are uniformly continuous in $\|\cdot\|_\infty$-norm it follows that the processes $(eS_{n_k}(t) \colon t \in [0, a])$, $k = 1, 2, \ldots$, are also uniformly continuous in $\|\cdot\|_\infty$-norm. Condition (3.15) says that the sequence of processes $(eS_{n_k}(t) \colon t \in [0, a])$, $k = 1, 2, \ldots$, is Cauchy in $\|\cdot\|_\infty$-norm uniformly for $t \in [0, a]$. Since

$$eS_{n_k}(t) \to eY(t) \quad \text{in } \|\cdot\|_2 \text{ -norm}$$

it follows that

$$eS_{n_k}(t) \to eY(t) \quad \text{in } \|\cdot\|_\infty \text{ -norm uniformly for } t \in [0, a],$$

and the $\|\cdot\|_\infty$-norm continuity of $(eS_{n_k}(t) \colon t \in [0, a])$ yields the norm continuity of $(eY(t) \colon t \in [0, a])$ which proves the claim. ∎

4. QUANTUM DOOB–MEYER DECOMPOSITION

Let $(X(t) \colon t \in [0, +\infty))$ be an L^2-martingale. Then the process $(|X(t)|^2 \colon t \in [0, +\infty))$ is a submartingale, and a Doob–Meyer decomposition is given by the representation

$$|X(t)|^2 = M(t) + A(t), \quad t \in [0, +\infty), \tag{4.1}$$

where $(M(t))$ is a martingale, and $(A(t))$ is an increasing positive process, i.e., $0 \leqslant A(s) \leqslant A(t)$ for $0 \leqslant s \leqslant t$. This decomposition has been obtained in many concrete situations (cf. [2], [4], [5], [7], [8], [9], [10], [11]); in general, a sufficient condition for the existence of a Doob–Meyer decomposition was given in [4] in the following form.

An L^1-process $(Y(t) \colon t \in [0, +\infty))$ is said to be of *class D* if the set

$$\Big\{ \sum_{k=1}^{m} \mathbb{E}_{t_{k-1}}(Y(t_k) - Y(t_{k-1})) \colon 0 \leqslant t_0 < t_1 < \cdots < t_m,\ m = 1, 2, \ldots \Big\}$$

is weakly relatively compact. If $(|X(t)|^2)$ is of class D then it has a Doob–Meyer decomposition.

We shall refer to this condition in a slightly modified form; however, since in the proof some specific features of the construction of the decomposition will be exploited, it seems preferable to present it in full detail. First we shall show the crucial property of the submartingale $(|X(t)|^2)$, namely that it is of class D (in our case even with a stronger compactness requirement) on each finite interval.

PROPOSITION 4.1. *Let $(X(t)\colon t \in [0,+\infty))$ be an L^p-martingale, $p > 2$. Then for each $a > 0$ the following set is weakly relatively compact:*

$$\mathcal{S} = \Big\{ \sum_{k=1}^{m} \mathbb{E}_{t_{k-1}}(|X(t_k)|^2 - |X(t_{k-1})|^2)\colon 0 \leqslant t_0 < t_1 < \cdots < t_m = a, \\ m = 1,2,\dots \Big\} \subset L^{p/2}(\mathcal{A}).$$

Proof. Take arbitrary $0 \leqslant t_0 < t_1 < \cdots < t_m = a$. On account of (1.2) we have

$$\sum_{k=1}^{m} \mathbb{E}_{t_{k-1}}(|X(t_k)|^2 - |X(t_{k-1})|^2) = \sum_{k=1}^{m} \mathbb{E}_{t_{k-1}}|\Delta X(t_k)|^2.$$

The main result of [16], Theorem 0.1, says that there exists a constant $c_{p/2}$ depending only on p such that

$$\Big\| \sum_{k=1}^{m} \mathbb{E}_{t_{k-1}}\Delta|X(t_k)|^2 \Big\|_{p/2} \leqslant c_{p/2} \Big\| \sum_{k=1}^{m} |\Delta X(t_k)|^2 \Big\|_{p/2},$$

consequently we obtain by (1.4)

$$\begin{aligned} &\Big\| \sum_{k=1}^{m} \mathbb{E}_{t_{k-1}}(|X(t_k)|^2 - |X(t_{k-1})|^2) \Big\|_{p/2} \\ &= \Big\| \sum_{k=1}^{m} \mathbb{E}_{t_{k-1}}|\Delta X(t_k)|^2 \Big\|_{p/2} \leqslant c_{p/2} \Big\| \sum_{k=1}^{m} |\Delta X(t_k)|^2 \Big\|_{p/2} \\ &= c_{p/2}\Big[\tau\Big(\Big(\sum_{k=1}^{m} |\Delta X(t_k)|^2\Big)^{p/2}\Big)\Big]^{2/p} = c_{p/2}\Big\|\Big(\sum_{k=1}^{m} |\Delta X(t_k)|^2\Big)^{1/2}\Big\|_p^2 \leqslant c_{p/2}\,\alpha_p^2 \|X(a)\|_p^2, \end{aligned}$$

which means that $\mathcal{S}$ is norm-bounded. Since $L^{p/2}(\mathcal{A})$ is the dual space to $L^q(\mathcal{A})$ with q given by (3.3), and vice versa, the conclusion follows. ∎

Now we are ready to prove the existence of a Doob–Meyer decomposition. The main idea of the proof is the same as in the classical Rao's proof (cf. Theorem 4.10 in [17] or [20]), however some additional refinements will be needed.

THEOREM 4.2 (Doob–Meyer decomposition). *Let $(X(t)\colon t \in [0,+\infty))$ be an L^p-martingale, $p > 2$. Then there exists a Doob–Meyer decomposition for $(|X(t)|^2\colon t \in [0,+\infty))$.*

Proof. Let $\mathbb{D} = \{t_k^{(n)} = k/2^n\colon k,n = 1,2,\dots\}$ be the set of dyadic numbers. In the remaining part of the proof, whenever the symbol $t_k^{(n)}$ is used, it will always be assumed that $t_k^{(n)} \in \mathbb{D}$. Fix an arbitrary positive integer m. For each $n =$

$1, 2, \ldots$, and each dyadic number u in $[m-1, m]$ put

$$S_n^{(m)}(u) = \sum_{m-1 \leqslant t_{k-1}^{(n)} < u} \mathbb{E}_{t_{k-1}^{(n)}}(|X(t_k^{(n)})|^2 - |X(t_{k-1}^{(n)})|^2),$$

in particular

$$S_n^{(m)}(m) = \sum_{k=(m-1)2^n+1}^{m2^n} \mathbb{E}_{t_{k-1}^{(n)}}(|X(t_k^{(n)})|^2 - |X(t_{k-1}^{(n)})|^2).$$

By Proposition 4.1 the sequence $\{S_n^{(m)}(m) \colon n = 1, 2, \ldots\}$ is weakly relatively compact. Let $A_m \in L^{p/2}(\mathcal{A}_m)$ be a limit point of this sequence. By the Eberlein-Šmulian theorem there is a subsequence $\{n_r\}$ (depending on m) such that

$$\lim_{r\to\infty} S_{n_r}^{(m)}(m) = A_m \quad \text{weakly.}$$

From the weak continuity of conditional expectation we obtain

$$\mathbb{E}_{m-1}A_m = \lim_{r\to\infty} \mathbb{E}_{m-1} \sum_{k=(m-1)2^{n_r}+1}^{m2^{n_r}} \mathbb{E}_{t_{k-1}^{(n_r)}}(|X(t_k^{(n_r)})|^2 - |X(t_{k-1}^{(n_r)})|^2)$$

$$= \lim_{r\to\infty} \mathbb{E}_{m-1}(|X(m)|^2 - |X(m-1)|^2) = \mathbb{E}_{m-1}|X(m)|^2 - |X(m-1)|^2. \tag{4.2}$$

Define a process $(A(t) \colon t \in [0, +\infty))$ as follows. For $m = 1, 2, \ldots$, and $t \in [0, +\infty)$ put

$$A(t) = |X(t)|^2 - \mathbb{E}_t|X(m)|^2 + \mathbb{E}_t(A_1 + \cdots + A_m), \quad t \in [m-1, m]. \tag{4.3}$$

Verify first the correctness of this definition. Computing $A(m)$ in the interval $[m-1, m]$, we obtain

$$A(m) = |X(m)|^2 - \mathbb{E}_m|X(m)|^2 + \mathbb{E}_m(A_1 + \cdots + A_m) = A_1 + \cdots + A_m.$$

For the interval $[m, m+1]$ we have

$$\begin{aligned} A(m) &= |X(m)|^2 - \mathbb{E}_m|X(m+1)|^2 + \mathbb{E}_m(A_1 + \cdots + A_{m+1}) \\ &= |X(m)|^2 - \mathbb{E}_m|X(m+1)|^2 + \mathbb{E}_m A_{m+1} + (A_1 + \cdots + A_m) = A_1 + \cdots + A_m, \end{aligned}$$

because

$$|X(m)|^2 - \mathbb{E}_m|X(m+1)|^2 + \mathbb{E}_m A_{m+1} = 0$$

by (4.2). Thus $(A(t))$ is well defined. Moreover, $(A(t))$ is right-continuous in $\|\cdot\|_{p/2}$-norm as a sum of the submartingale $(|X(t)|^2)$ and two martingales, all being $\|\cdot\|_{p/2}$-right-continuous as noted in Section 1. Putting $m = 1$, $t = 0$ in (4.3) gives

$$A(0) = |X(0)|^2 - \mathbb{E}_0|X(1)|^2 + \mathbb{E}_0 A_1,$$

and putting $m = 1$ in (4.2) gives

$$\mathbb{E}_0 A_1 = \mathbb{E}_0|X(1)|^2 - |X(0)|^2,$$

which shows that $A(0) = 0$.

Let u be an arbitrary dyadic number in $[m-1,m]$. For sufficiently large n (so large that $u \in \{t_k^{(n)} : k=(m-1)2^n+1,\dots,m2^n\}$) the following equality holds

$$\begin{aligned}\mathbb{E}_u S_n^{(m)}(m) &= \sum_{m-1\leqslant t_{k-1}^{(n)}<u} \mathbb{E}_{t_{k-1}^{(n)}}(|X(t_k^{(n)})|^2-|X(t_{k-1}^{(n)})|^2)+\sum_{u\leqslant t_{k-1}^{(n)}<m}\mathbb{E}_u(|X(t_k^{(n)})|^2-|X(t_{k-1}^{(n)})|^2)\\ &= \sum_{m-1\leqslant t_{k-1}^{(n)}<u} \mathbb{E}_{t_{k-1}^{(n)}}(|X(t_k^{(n)})|^2-|X(t_{k-1}^{(n)})|^2)+\mathbb{E}_u(|X(m)|^2-|X(u)|^2)\\ &= S_n^{(m)}(u)+\mathbb{E}_u|X(m)|^2-|X(u)|^2.\end{aligned}$$

Passing to the limit along the sequence $\{n_r\}$ in the above equality, we get

$$\mathbb{E}_u A_m = \lim_{r\to\infty} S_{n_r}^{(m)}(u)+\mathbb{E}_u|X(m)|^2-|X(u)|^2,$$

and thus

$$\begin{aligned}\lim_{r\to\infty} S_{n_r}^{(m)}(u) &= |X(u)|^2-\mathbb{E}_u|X(m)|^2+\mathbb{E}_u A_m\\ &= A(u)-\mathbb{E}_u(A_1+\dots+A_{m-1}) = A(u)-(A_1+\dots+A_{m-1}).\end{aligned} \tag{4.4}$$

Let now u,v be arbitrary dyadic numbers such that $m-1\leqslant u\leqslant v\leqslant m$. Since

$$\mathbb{E}_{t_{k-1}^{(n)}}(|X(t_k^{(n)})|^2-|X(t_{k-1}^{(n)})|^2) = \mathbb{E}_{t_{k-1}^{(n)}}(|X(t_k^{(n)})-X(t_{k-1}^{(n)})|^2)\geqslant 0,$$

we obtain that for sufficiently large n (again so large that $u,v\in\{t_k^{(n)} : k=(m-1)2^n+1,\dots,m2^n\}$)

$$S_n^{(m)}(u)\leqslant S_n^{(m)}(v),$$

and passing to the limit in the above inequality along the sequence $\{n_r\}$ yields on account of (4.4)

$$A(u)-(A_1+\dots+A_{m-1})\leqslant A(v)-(A_1+\dots+A_{m-1}),$$

so

$$A(u)\leqslant A(v).$$

The right continuity of the process $(A(t))$ implies that it is increasing in $[m-1,m]$, thus from the arbitrariness of m, $(A(t))$ is increasing on the whole of $[0,+\infty)$. Moreover, $(A(t))$ is positive since $A(0)=0$. The equality (4.3) gives

$$|X(t)|^2=\mathbb{E}_t(|X(m)|^2-(A_1+\dots+A_m))+A(t)=M(t)+A(t),$$

for $t\in[m-1,m]$, where

$$M(t)=\mathbb{E}_t(|X(m)|^2-(A_1+\dots+A_m)).$$

Take an arbitrary $t\in[0,+\infty)$, and let m be a positive integer such that $t\in[m-1,m]$. For $m-1\leqslant s\leqslant t$ we obviously have $\mathbb{E}_s M(t)=M(s)$. Assume that

$s \in [m-2, m-1]$. Then

$$\mathbb{E}_s M(t) = \mathbb{E}_s(|X(m)|^2 - (A_1 + \cdots + A_m)) = \mathbb{E}_s|X(m)|^2 - \mathbb{E}_s A_m - \mathbb{E}_s(A_1 + \cdots + A_{m-1}),$$

$$\begin{aligned} M(s) &= \mathbb{E}_s(|X(m-1)|^2 - (A_1 + \cdots + A_{m-1})) \\ &= \mathbb{E}_s|X(m-1)|^2 - \mathbb{E}_s(A_1 + \cdots + A_{m-1}), \end{aligned}$$

so we obtain

$$\mathbb{E}_s M(t) - M(s) = \mathbb{E}_s|X(m)|^2 - \mathbb{E}_s A_m - \mathbb{E}_s|X(m-1)|^2.$$

Applying $\mathbb{E}_s$ to both sides of (4.2) yields

$$\mathbb{E}_s A_m = \mathbb{E}_s|X(m)|^2 - \mathbb{E}_s|X(m-1)|^2,$$

which shows that

$$\mathbb{E}_s M(t) - M(s) = 0, \quad \text{i.e., } \mathbb{E}_s M(t) = M(s).$$

Now for an arbitrary $s < t$ choose $s_1 < \cdots < s_l$ between s and t lying in neighbouring intervals with the ends being positive integers. Then

$$\mathbb{E}_s M(t) = \mathbb{E}_s \mathbb{E}_{s_1} \cdots \mathbb{E}_{s_l} M(t) = \mathbb{E}_s \mathbb{E}_{s_1} \cdots \mathbb{E}_{s_{l-1}} M(s_l) = \cdots = \mathbb{E}_s M(s_1) = M(s),$$

proving that $(M(t) \colon t \in [0, +\infty))$ is a martingale. ∎

Let us now consider an important question concerning the uniqueness of a Doob–Meyer decomposition. Recall the following definition from Definition 2.2 of [7].

DEFINITION 4.3. An $L^1(\mathcal{A})$ process $(A(t) \colon t \in [0, +\infty))$ is natural if for each $t > 0$ and any sequence $\{\theta_n\}$ of partitions of $[0, t]$, $\theta_n = \{0 = t_0^{(n)} < t_1^{(n)} \cdots < t_{m_n}^{(n)} = t\}$ with $\|\theta_n\| \to 0$ we have, for all $y \in \mathcal{A}$,

$$\lim_{n\to\infty} \tau\Big(\sum_{k=1}^{m_n} \mathbb{E}_{t_{k-1}^{(n)}}(y)(A(t_k^{(n)}) - A(t_{k-1}^{(n)}))\Big) = \tau(yA(t)). \tag{4.5}$$

It was pointed out in [7] that, in full analogy with the classical case, if the process $(A(t))$ in decomposition (4.1) is natural then this decomposition is unique. Note that equality (4.5) may be, on account of (1.1), rewritten in the following form (this was also observed in Definition 5.3 in [9])

$$\begin{aligned} &\lim_{n\to\infty} \tau\Big(\sum_{k=1}^{m_n} \mathbb{E}_{t_{k-1}^{(n)}}(y)(A(t_k^{(n)}) - A(t_{k-1}^{(n)}))\Big) \\ &= \lim_{n\to\infty} \sum_{k=1}^{m_n} \tau(\mathbb{E}_{t_{k-1}^{(n)}}(y)(A(t_k^{(n)}) - A(t_{k-1}^{(n)}))) = \lim_{n\to\infty} \sum_{k=1}^{m_n} \tau(y\mathbb{E}_{t_{k-1}^{(n)}}((A(t_k^{(n)}) - A(t_{k-1}^{(n)})))) \\ &= \lim_{n\to\infty} \tau\Big(y \sum_{k=1}^{m_n} \mathbb{E}_{t_{k-1}^{(n)}}(A(t_k^{(n)}) - A(t_{k-1}^{(n)}))\Big) = \tau(yA(t)), \end{aligned}$$

which simply means that

$$\sum_{k=1}^{m_n} \mathbb{E}_{t_{k-1}^{(n)}} (A(t_k^{(n)}) - A(t_{k-1}^{(n)})) \to A(t) \quad \text{weakly.} \tag{4.6}$$

From decomposition (4.1) it follows that for any $0 \leqslant s \leqslant t$ we have

$$\mathbb{E}_s(A(t)-A(s))=\mathbb{E}_s(|X(t)|^2-|X(s)|^2)-\mathbb{E}_s(M(t)-M(s))=\mathbb{E}_s(|X(t)|^2-|X(s)|^2),$$

by martingale property, so condition (4.6) becomes

$$\sum_{k=1}^{m_n} \mathbb{E}_{t_{k-1}^{(n)}} (|X(t_k^{(n)})|^2 - |X(t_{k-1}^{(n)})|^2) \to A(t) \quad \text{weakly.} \tag{4.7}$$

In the next section we shall show that this condition is satisfied for a certain class of martingales, thus obtaining the uniqueness of a Doob–Meyer decomposition. Moreover, an explicit form of this decomposition will be given.

5. QUADRATIC VARIATION PROCESS

In this section we assume that $(X(t)\colon t \in [0,+\infty))$ is an $\mathcal{A}$-valued martingale continuous in Segal's sense. The following theorem is a noncommutative counterpart of Theorem 4.1 from [13].

THEOREM 5.1. *Let $t \geqslant 0$, and denote by $\theta = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ a partition of $[0,t]$. Put*

$$S_\theta(t) = \sum_{k=1}^{m} |\Delta X(t_k)|^2.$$

Then $\{S_\theta(t)\}$ converges in $\|\cdot\|_2$-norm as $\|\theta\| \to 0$ to $\langle X\rangle(t)$ defined as

$$\langle X\rangle(t) = |X(t)|^2 - |X(0)|^2 - \Big(\int_0^t \mathrm{d}X^*(u)\, X(u) + \int_0^t X^*(u)\, \mathrm{d}X(u)\Big). \tag{5.1}$$

Proof. We have, analogously as in the classical case,

$$\begin{aligned} S_\theta(t) &= \sum_{k=1}^{m} [X(t_k) - X(t_{k-1})]^*[X(t_k) - X(t_{k-1})] \\ &= \sum_{k=1}^{m} (|X(t_k)|^2 - |X(t_{k-1})|^2) - \sum_{k=1}^{m} [X^*(t_k) - X^*(t_{k-1})]X(t_{k-1}) \\ &\qquad - \sum_{k=1}^{m} X^*(t_{k-1})[X(t_k) - X(t_{k-1})] \end{aligned}$$

$$= |X(t)|^2 - |X(0)|^2 - \sum_{k=1}^{m} [X^*(t_k) - X^*(t_{k-1})]X(t_{k-1}) - \sum_{k=1}^{m} X^*(t_{k-1})[X(t_k) - X(t_{k-1})].$$

Now observe that the martingales $(X(t))$ and $(X^*(t))$ satisfy the assumption of Theorem 3.1 both as the integrators and the integrands, thus the two sums in the equation above, being respectively the left and right integral sum, tend to the integrals $\int_0^t \mathrm{d}X^*(u)\, X(u)$ and $\int_0^t X^*(u)\, \mathrm{d}X(u)$, which proves the theorem. ∎

For any $0 \leqslant s \leqslant t$ we clearly have $0 \leqslant S_\theta(s) \leqslant S_\theta(t)$, so the process $(\langle X\rangle(t) \colon t \in [0, +\infty))$ defined by equation (5.1), being the limit of $S_\theta(t)$, is positive and increasing; obviously $\langle X\rangle(0) = 0$. This process is called the *quadratic variation process* for the martingale $(X(t) \colon t \in [0, +\infty))$. Denote

$$M(t) = |X(0)|^2 + \int_0^t \mathrm{d}X^*(u)\, X(u) + \int_0^t X^*(u)\, \mathrm{d}X(u).$$

By virtue of Theorem 3.1 we obtain that $(M(t))$ is a martingale, thus (5.1) gives a Doob–Meyer decomposition

$$|X(t)|^2 = M(t) + \langle X\rangle(t). \tag{5.2}$$

From Theorem 3.4 it follows that $(M(t))$ is weakly continuous in Segal's sense. We shall show that the quadratic variation process $(\langle X\rangle(t))$ is natural. The following lemma is a noncommutative version of Lemma 5.10 from [17].

LEMMA 5.2. *Let $t > 0$, and denote by $\theta = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ a partition of $[0, t]$. Then*

$$\lim_{\|\theta\|\to 0} \tau\Big(\sum_{k=1}^{m} |\Delta X(t_k)|^4\Big) = 0.$$

Proof. The estimations in the proof are similar to those in Theorem 3.1. Put

$$M = \|X(t)\|_\infty = \sup_{0\leqslant s\leqslant t} \|X(s)\|_\infty, \tag{5.3}$$

$$K = \tau(|X(t)|^2 - |X(0)|^2), \tag{5.4}$$

and let α_4 be as in (1.4) with $p = 4$. Let $\varepsilon > 0$ be given. From Segal's continuity of the martingale $(X(u))$ there exist a projection e in $\mathcal{A}$ with

$$\tau(e^\perp) < \frac{\varepsilon^2}{64M^4\alpha_4^4\|X(t)\|_4^4}, \tag{5.5}$$

and $\delta > 0$ such that for any $t', t'' \in [0, t]$ with $|t' - t''| < \delta$, we have

$$\|[X(t') - X(t'')]e\|_\infty < \frac{\varepsilon}{4MK}. \tag{5.6}$$

Let $\theta = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ be a partition of $[0, t]$ such that $\|\theta\| < \delta$. We have

$$\tau\Big(\sum_{k=1}^{m} |\Delta X(t_k)|^4\Big) = \tau\Big(\sum_{k=1}^{m} e|\Delta X(t_k)|^4\Big) + \tau\Big(\sum_{k=1}^{m} e^{\perp}|\Delta X(t_k)|^4\Big) = I_1 + I_2,$$

where

$$I_1 = \tau\Big(\sum_{k=1}^{m} e|\Delta X(t_k)|^4\Big), \quad I_2 = \tau\Big(\sum_{k=1}^{m} e^{\perp}|\Delta X(t_k)|^4\Big).$$

For I_1 we have using (5.3), (5.4) and (5.6) together with (1.3)

$$\begin{aligned}
|I_1| &= \sum_{k=1}^{m} |\tau(e|\Delta X(t_k)|^4)| \leqslant \sum_{k=1}^{m} \|e|\Delta X(t_k)|^2\|_\infty \||\Delta X(t_k)|^2\|_1 \\
&\leqslant \sum_{k=1}^{m} \|e\Delta X(t_k)\|_\infty \||\Delta X(t_k)|\|_\infty \||\Delta X(t_k)|^2\|_1 \\
&< \sum_{k=1}^{m} 2M \frac{\varepsilon}{4MK} \||\Delta X(t_k)|^2\|_1 = \frac{\varepsilon}{2K} \sum_{k=1}^{m} \||\Delta X(t_k)|^2\|_1 = \frac{\varepsilon}{2}.
\end{aligned}$$

For I_2 we have

$$\begin{aligned}
|I_2| &= \sum_{k=1}^{m} |\tau(e^{\perp}|\Delta X(t_k)|^4)| = \sum_{k=1}^{m} |\tau(|\Delta X(t_k)|e^{\perp}|\Delta X(t_k)||\Delta X(t_k)|^2)| \\
&\leqslant \sum_{k=1}^{m} \||\Delta X(t_k)|e^{\perp}|\Delta X(t_k)|\|_1 \||\Delta X(t_k)|^2\|_\infty = \sum_{k=1}^{m} \tau(e^{\perp}|\Delta X(t_k)|^2) \||\Delta X(t_k)|^2\|_\infty \\
&\leqslant 4M^2 \tau\Big(\sum_{k=1}^{m} e^{\perp}|\Delta X(t_k)|^2\Big) \leqslant 4M^2 \Big\|e^{\perp} \sum_{k=1}^{m} |\Delta X(t_k)|^2\Big\|_1.
\end{aligned}$$

From Hölder's inequality we get

$$\Big\|e^{\perp} \sum_{k=1}^{m} |\Delta X(t_k)|^2\Big\|_1 \leqslant \|e^{\perp}\|_2 \Big\|\sum_{k=1}^{m} |\Delta X(t_k)|^2\Big\|_2 = [\tau(e^{\perp})]^{1/2} \Big\|\Big(\sum_{k=1}^{m} |\Delta X(t_k)|^2\Big)^{1/2}\Big\|_4^2,$$

and taking into account inequalities (1.4), (5.5) we finally obtain

$$\begin{aligned}
|I_2| &\leqslant 4M^2 \Big\|e^{\perp} \sum_{k=1}^{m} |\Delta X(t_k)|^2\Big\|_1 \leqslant 4M^2 [\tau(e^{\perp})]^{1/2} \Big\|\Big(\sum_{k=1}^{m} |\Delta X(t_k)|^2\Big)^{1/2}\Big\|_4^2 \\
&< 4M^2 \frac{\varepsilon}{8M^2 \alpha_4^2 \|X(t)\|_4^2} \alpha_4^2 \|X(t)\|_4^2 = \frac{\varepsilon}{2},
\end{aligned}$$

and thus we have the following which finishes the proof:

$$\tau\Big(\sum_{k=1}^{m} |\Delta X(t_k)|^4\Big) = I_1 + I_2 < \varepsilon. \quad ∎$$

THEOREM 5.3. *The process* $(\langle X\rangle(t) \colon t \in [0, +\infty))$ *is natural.*

Proof. Let $t > 0$, and denote by $\theta = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ a partition of $[0, t]$. We shall first show that

$$\lim_{\|\theta\| \to 0} \Big\| \sum_{k=1}^{m} (|\Delta X(t_k)|^2 - \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2) \Big\|_2 = 0.$$

We have

$$\begin{aligned}
&\Big\| \sum_{k=1}^{m} (|\Delta X(t_k)|^2 - \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2) \Big\|_2^2 \\
&\qquad = \tau\Big(\sum_{i,k=1}^{m} (|\Delta X(t_i)|^2 - \mathbb{E}_{t_{i-1}} |\Delta X(t_i)|^2)(|\Delta X(t_k)|^2 - \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2) \Big) \\
&\qquad = \sum_{i,k=1}^{m} \tau((|\Delta X(t_i)|^2 - \mathbb{E}_{t_{i-1}} |\Delta X(t_i)|^2)(|\Delta X(t_k)|^2 - \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2)).
\end{aligned}$$

For $i < k$ we obtain

$$\begin{aligned}
&\tau((|\Delta X(t_i)|^2 - \mathbb{E}_{t_{i-1}} |\Delta X(t_i)|^2)(|\Delta X(t_k)|^2 - \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2)) \\
&\qquad = \tau(\mathbb{E}_{t_{k-1}}(|\Delta X(t_i)|^2 - \mathbb{E}_{t_{i-1}} |\Delta X(t_i)|^2)(|\Delta X(t_k)|^2 - \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2)) \\
&\qquad = \tau((|\Delta X(t_i)|^2 - \mathbb{E}_{t_{i-1}} |\Delta X(t_i)|^2)\mathbb{E}_{t_{k-1}}(|\Delta X(t_k)|^2 - \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2)) = 0,
\end{aligned}$$

and analogously for $i > k$. Consequently, by Lemma 5.2

$$\begin{aligned}
&\Big\| \sum_{k=1}^{m} (|\Delta X(t_k)|^2 - \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2) \Big\|_2^2 \\
&= \tau\Big(\sum_{k=1}^{m} [|\Delta X(t_k)|^2 - \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2]^2 \Big) \leqslant \tau\Big(\sum_{k=1}^{m} 2[|\Delta X(t_k)|^4 + (\mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2)^2] \Big) \\
&\leqslant 2\tau\Big(\sum_{k=1}^{m} [|\Delta X(t_k)|^4 + \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^4] \Big) = 4\tau\Big(\sum_{k=1}^{m} |\Delta X(t_k)|^4 \Big) \to 0,
\end{aligned}$$

where in the last estimation we have used the inequality $(\mathbb{E}_t x)^2 \leqslant \mathbb{E}_t x^2$ valid for $x = x^*$ because $\mathbb{E}_t$ is positive.

From Theorem 5.1 we have $\lim_{\|\theta\| \to 0} \sum_{k=1}^{m} |\Delta X(t_k)|^2 = \langle X \rangle (t)$, which yields

$$\lim_{\|\theta\| \to 0} \sum_{k=1}^{m} \mathbb{E}_{t_{k-1}} |\Delta X(t_k)|^2 = \langle X \rangle (t) \quad \text{in } \| \cdot \|_2\text{-norm},$$

and thus condition (4.7) is satisfied. ∎

Our next aim is to show a noncommutative counterpart of the classical result saying that if the martingale in the Doob–Meyer decomposition is continuous with probability one then this decomposition is unique.

THEOREM 5.4. *Let*

$$|X(t)|^2 = M(t) + A(t)$$

be a Doob–Meyer decomposition of the submartingale $(|X(t)|^2)$ *such that the martingale* $(M(t))$ *is weakly continuous in Segal's sense and* $M(0) = |X(0)|^2$. *Then this decomposition is unique.*

Proof. Assume that there are decompositions

$$|X(t)|^2 = M_1(t) + A_1(t) = M_2(t) + A_2(t),$$

where $(M_1(t))$, $(M_2(t))$ are weakly continuous in Segal's sense martingales such that $M_1(0) = M_2(0) = |X(0)|^2$, and $(A_1(t))$, $(A_2(t))$ are increasing positive processes. Put

$$M(t) = M_1(t) - M_2(t).$$

Then $(M(t))$ is a weakly continuous in Segal's sense martingale such that

$$M(t) = A_2(t) - A_1(t),$$

in particular, $(M(t))$ is selfadjoint and $M(0) = 0$. Fix $t > 0$. Take an arbitrary $\varepsilon > 0$. From weak uniform continuity in Segal's sense of the martingale $(M(s) \colon s \in [0,t])$ it follows that there exist a projection $e \in \mathcal{A}$ with

$$\tau(e^\perp) < \frac{\varepsilon^2}{16\alpha_8^4 \|M(t)\|_8^4}, \tag{5.7}$$

where α_8 is as in (1.4) for $p = 8$, and $\delta > 0$ such that for any $t', t'' \in [0,t]$ with $|t' - t''| < \delta$ we have

$$\|e[M(t') - M(t'')]e\|_\infty \leqslant \frac{\varepsilon}{4(\|A_1(t)\|_1 + \|A_2(t)\|_1)}. \tag{5.8}$$

Let $0 = t_0 < t_1 < \cdots < t_m = t$ be a partition of the interval $[0,t]$ such that $\max\limits_{1\leqslant k\leqslant m}(t_k - t_{k-1}) < \delta$. We have

$$\begin{aligned}
\tau\Big(\sum_{k=1}^m [\Delta M(t_k)]^2\Big) &= \tau\Big(\sum_{k=1}^m e\Delta M(t_k)e\Delta M(t_k)\Big) + \tau\Big(\sum_{k=1}^m e^\perp \Delta M(t_k)e\Delta M(t_k)\Big) \\
&\quad + \tau\Big(\sum_{k=1}^m e\Delta M(t_k)e^\perp \Delta M(t_k)\Big) + \tau\Big(\sum_{k=1}^m e^\perp \Delta M(t_k)e^\perp \Delta M(t_k)\Big) \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$I_1 = \tau\Big(\sum_{k=1}^m e\Delta M(t_k)e\Delta M(t_k)\Big), \qquad I_2 = \tau\Big(\sum_{k=1}^m e^\perp \Delta M(t_k)e\Delta M(t_k)\Big),$$

$$I_3 = \tau\Big(\sum_{k=1}^m e\Delta M(t_k)e^\perp \Delta M(t_k)\Big), \qquad I_4 = \tau\Big(\sum_{k=1}^m e^\perp \Delta M(t_k)e^\perp \Delta M(t_k)\Big).$$

On account of (5.8) we get

$$\begin{aligned}|I_1| = I_1 &= \sum_{k=1}^{m} |\tau(e\Delta M(t_k)e\Delta M(t_k))| \leqslant \sum_{k=1}^{m} \|e\Delta M(t_k)e\|_\infty \|\Delta M(t_k)\|_1 \\ &\leqslant \frac{\varepsilon}{4(\|A_1(t)\|_1 + \|A_2(t)\|_1)} \sum_{k=1}^{m} \|\Delta M(t_k)\|_1,\end{aligned}$$

and since $(A_1(t))$ and $(A_2(t))$ are increasing and $A_1(0) = A_2(0) = 0$,

$$\begin{aligned}\sum_{k=1}^{m} \|\Delta M(t_k)\|_1 &= \sum_{k=1}^{m} \|\Delta A_2(t_k) - \Delta A_1(t_k)\|_1 \leqslant \sum_{k=1}^{m} \|\Delta A_2(t_k)\|_1 + \sum_{k=1}^{m} \|\Delta A_1(t_k)\|_1 \\ &= \sum_{k=1}^{m} \tau(A_2(t_k) - A_2(t_{k-1})) + \sum_{k=1}^{m} \tau(A_1(t_k) - A_1(t_{k-1})) \\ &= \tau(A_2(t)) + \tau(A_1(t)) = \|A_2(t)\|_1 + \|A_1(t)\|_1,\end{aligned}$$

which gives the estimation

$$I_1 \leqslant \frac{\varepsilon}{4}.$$

Furthermore on account of (5.7) we obtain

$$\begin{aligned}|I_2| = |I_3| = I_2 = I_3 &= \Big|\tau\Big(\sum_{k=1}^{m} e^\perp \Delta M(t_k)e\Delta M(t_k)\Big)\Big| \\ &= \Big|\tau\Big(e^\perp \sum_{k=1}^{m} \Delta M(t_k)e\Delta M(t_k)\Big)\Big| \leqslant \|e^\perp\|_2 \Big\|\sum_{k=1}^{m} \Delta M(t_k)e\Delta M(t_k)\Big\|_2 \\ &= \|e^\perp\|_2 \Big\|\sum_{k=1}^{m} (\Delta M(t_k)e)(\Delta M(t_k)e)^*\Big\|_2 \\ &= [\tau(e^\perp)]^{1/2} \Big\|\Big(\sum_{k=1}^{m} (\Delta M(t_k)e)(\Delta M(t_k)e)^*\Big)^{1/2}\Big\|_4^2 \\ &< \frac{\varepsilon}{4\alpha_8^2 \|M(t)\|_8^2} \Big\|\Big(\sum_{k=1}^{m} (\Delta M(t_k)e)(\Delta M(t_k)e)^*\Big)^{1/2}\Big\|_4^2.\end{aligned}$$

From Lemma 1.1 in [19] it follows that

$$\Big\|\Big(\sum_{k=1}^{m} (\Delta M(t_k)e)(\Delta M(t_k)e)^*\Big)^{1/2}\Big\|_4 \leqslant \Big\|\Big(\sum_{k=1}^{m} [\Delta M(t_k)]^2\Big)^{1/2}\Big\|_8 \|e\|_8,$$

and from inequality (1.4) we obtain

$$\Big\|\Big(\sum_{k=1}^{m} [\Delta M(t_k)]^2\Big)^{1/2}\Big\|_8 \leqslant \alpha_8 \|M(t)\|_8.$$

Consequently,

$$I_2 = I_3 < \frac{\varepsilon}{4\alpha_8^2\|M(t)\|_8^2}\Big\|\Big(\sum_{k=1}^{m}(\Delta M(t_k)e)(\Delta M(t_k)e)^*\Big)^{1/2}\Big\|_4^2$$
$$\leqslant \frac{\varepsilon}{4\alpha_8^2\|M(t)\|_8^2}\Big\|\Big(\sum_{k=1}^{m}[\Delta M(t_k)]^2\Big)^{1/2}\Big\|_8^2\|e\|_8^2 \leqslant \frac{\varepsilon}{4\alpha_8^2\|M(t)\|_8^2}\alpha_8^2\|M(t)\|_8^2 = \frac{\varepsilon}{4}.$$

The same estimation is obtained for $|I_4| = I_4$, so finally we have

$$\tau\Big(\sum_{k=1}^{m}[\Delta M(t_k)]^2\Big) \leqslant I_1 + I_2 + I_3 + I_4 < \varepsilon.$$

On the other side, from equality (1.3) and martingale property we get

$$\tau(|M(t)-M(0)|^2) = \tau(|M(t)|^2 - |M(0)|^2) = \sum_{k=1}^{m}\||\Delta M(t_k)|^2\|_1 = \tau\Big(\sum_{k=1}^{m}[\Delta M(t_k)]^2\Big) < \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary this yields

$$\tau(|M(t) - M(0)|^2) = 0.$$

The faithfulness of τ implies the following which gives the uniqueness of the decomposition:

$$M(t) = M(0) = 0. \quad ∎$$

REMARK 5.5. From the decomposition (5.2) and the uniqueness it follows that the process $(A(t))$ in the above theorem is actually equal to $(\langle X\rangle(t))$.

We finish with a brief discussion of the classical notion of the cross-variation process.

DEFINITION 5.6. Let $(X(t)\colon t \in [0,+\infty))$ and $(Y(t)\colon t \in [0,+\infty))$ be $\mathcal{A}$-valued continuous in Segal's sense martingales. The *cross-variation process* (called also the *bracket* or *quadratic covariation process*) is defined by

$$\langle X, Y\rangle(t) = \frac{1}{4}\{\langle X+Y\rangle(t) - \langle X-Y\rangle(t) + \mathrm{i}[\langle \mathrm{i}X+Y\rangle(t) - \langle \mathrm{i}X - Y\rangle(t)]\}.$$

This is of course the polarization formula for the "sesquilinear form" (linear in the second position) $\langle X, Y\rangle$ obtained from the "quadratic form" $\langle X\rangle$, so that

$$\langle X, X\rangle(t) = \langle X\rangle(t).$$

After straightforward calculations we obtain

$$\langle X, Y\rangle(t) = X(t)^*Y(t) - X(0)^*Y(0) - \Big(\int_0^t \mathrm{d}X^*(u)\,Y(u) + \int_0^t X^*(u)\,\mathrm{d}Y(u)\Big).$$

Our last theorem shows that the cross-variation process can be obtained in the way entirely analogous to the classical case.

THEOREM 5.7. *Let* $(X(t))$ *and* $(Y(t))$ *be as above, and let* $\theta = \{0 = t_0 < t_1 < \cdots < t_m = t\}$ *denote a partition of* $[0,t]$. *Then*

$$\langle X, Y\rangle(t) = \lim_{\|\theta\|\to 0} \sum_{k=1}^{m} [X(t_k) - X(t_{k-1})]^* [Y(t_k) - Y(t_{k-1})] \quad \textit{in } \|\cdot\|_2\textit{-norm}.$$

Proof. We have

$$\begin{aligned}
&\sum_{k=1}^{m} [X(t_k) - X(t_{k-1})]^* [Y(t_k) - Y(t_{k-1})] \\
&\quad = \sum_{k=1}^{m} X^*(t_k)Y(t_k) - \sum_{k=1}^{m} X^*(t_k)Y(t_{k-1}) - \sum_{k=1}^{m} X^*(t_{k-1})[Y(t_k) - Y(t_{k-1})] \\
&\quad = X^*(t)Y(t) - X^*(0)Y(0) + \sum_{k=1}^{m} X^*(t_{k-1})Y(t_{k-1}) - \sum_{k=1}^{m} X^*(t_k)Y(t_{k-1}) \\
&\qquad\qquad - \sum_{k=1}^{m} X(t_{k-1})[Y(t_k) - Y(t_{k-1})] \\
&\quad = X^*(t)Y(t) - X^*(0)Y(0) - \sum_{k=1}^{m} [X^*(t_k) - X^*(t_{k-1})]Y(t_{k-1}) \\
&\qquad\qquad - \sum_{k=1}^{m} X^*(t_{k-1})[Y(t_k) - Y(t_{k-1})].
\end{aligned}$$

The two sums on the right hand side of the above equation tend to $\int_0^t \mathrm{d}X^*(u)\, Y(u)$ and $\int_0^t X^*(u)\, \mathrm{d}Y(u)$, respectively, which shows the claim. ∎

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