J. OPERATOR THEORY
66:1(2011), 193–207

ALGEBRAIC PROPERTIES OF TOEPLITZ OPERATORS WITH SEPARATELY QUASIHOMOGENEOUS SYMBOLS ON THE BERGMAN SPACE OF THE UNIT BALL

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Communicated by William Arveson

ABSTRACT. In this paper we discuss some algebraic properties of Toeplitz operators with separately quasihomogeneous symbols (i.e., symbols being of the form $\xi^k\varphi(|z_1|,\dots,|z_n|)$) on the Bergman space of the unit ball in $\mathbb{C}^n$. We provide a decomposition of $L^2(B_n, \mathrm{d}v)$, then we use it to show that the zero product of two Toeplitz operators has only a trivial solution if one of the symbols is separately quasihomogeneous and the other is arbitrary. Also, we describe the commutant of a Toeplitz operator whose symbol is radial.

KEYWORDS: *Toeplitz operators, Bergman space, separately quasihomogeneous symbols.*

MSC (2000): Primary 47B35; Secondary 32A36.

1. INTRODUCTION

Toeplitz operators, or localization operators, appear in the literature in many different situations. They were introduced in time-frequency analysis by Daubechies in [8] as certain filters for signals. In [10], Lieb and Solovej reformulated problems in quantum mechanics in terms of coherent state transforms, where Toeplitz operators appeared in a natural way. Since then, boundedness and compactness properties of such operators have been investigated in many works (see e.g. [6], [14], [15], [20], [21], [22]). The aim of this paper is to study algebraic properties of Toeplitz operators with separately quasihomogeneous symbols on the Bergman space of the unit ball.

Let $\mathrm{d}v$ denote the Lebesgue volume measure on the unit ball B_n of $\mathbb{C}^n$, normalized so that the measure of B_n equals 1. The Bergman space $L_a^2(B_n)$ is the Hilbert space consisting of holomorphic functions on B_n which are also in $L^2(B_n, \mathrm{d}v)$. Let P be the orthogonal projection from $L^2(B_n, \mathrm{d}v)$ onto $L_a^2(B_n)$. For a function $\varphi \in L^\infty(B_n, \mathrm{d}v)$, the Toeplitz operator $T_\varphi : L_a^2(B_n) \mapsto L_a^2(B_n)$ with symbol φ is defined by

$$T_\varphi(f) = P(\varphi f).$$

Since for any $w \in B_n$, the pointwise evaluation of functions in $L_a^2(B_n)$ at w is a bounded functional, there is a unique function $K_w \in L_a^2(B_n)$ which has the following reproducing property:

$$f(w) = \langle f, K_w \rangle, \quad f \in L_a^2(B_n),$$

where the notation $\langle\ ,\ \rangle$ denotes the inner product in $L^2(B_n, \mathrm{d}v)$. The function K_w is the well-known Bergman kernel and its explicit formula is given by

$$K_w(z) = \frac{1}{(1 - \langle z, w \rangle)^{n+1}}, \quad z \in B_n.$$

Then

$$T_\varphi(f)(z) = \int_{B_n} f(w)\varphi(w)\overline{K_z(w)}\mathrm{d}v(w), \quad z \in B_n.$$

Quiroga-Barranco and Vasilevski [15] gave some basic results concerning Toeplitz operators on $L_a^2(B_n)$ with separately radial symbols (i.e., symbols depending only on $|z_1|, \ldots, |z_n|$), then they proved that the C^*-algebra generated by Toeplitz operators with bounded separately radial symbols is commutative. In [24], we studied algebraic properties of Toeplitz operators with radial symbols on $L_a^2(B_n)$. In Section 2 of this paper, we will introduce some properties of separately radial functions and characterize when a bounded holomorphic function on $\mathbb{C}^n$ is equal to zero.

In 1964, Brown and Halmos [4] proved that $T_f T_g = T_h$ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ of $\mathbb{C}$ if and only if either $\overline{f}$ or g is holomorphic and $h = fg$. On the Bergman space $L_a^2(D)$ of the unit disk D, Ahern and Čučković [1] showed that a similar result holds for Toeplitz operators with harmonic symbols. Louhichi, Strouse and Zakariasy [12] determined when the product of two Toeplitz operators with quasihomogeneous symbols (i.e., symbols being of the form $\mathrm{e}^{ik\theta}\phi(|z|)$) is a Toeplitz operator. Inspired by [12], we discuss the same question for more general symbols on B_n in Section 3. We will give a necessary condition for the product of two Toeplitz operators whose symbols are separately quasihomogeneous to be equal to a Toeplitz operator.

In Section 4, we will consider the so-called "zero-product" problem. Brown and Halmos [4] easily deduced that if $T_f T_g = 0$ on $H^2(\mathbb{T})$ then either f or g must be identically zero. Ahern and Čučković [1] showed that the same result holds for $L_a^2(D)$ if the symbols f and g are harmonic. Moreover in [2], they proved that if $T_f T_g = 0$, where f is arbitrary bounded and g is radial, then either $f \equiv 0$ or $g \equiv 0$. Choe and Koo [5] got a result on B_n which is analogous to [1] with an assumption about the continuity of the symbols on an open subset of the boundary, and solved the zero-product problem for several Toeplitz operators with harmonic symbols that have Lipschitz continuous extensions to the whole boundary. In spite of all these mentioned former works, the zero-product problem for Toeplitz operators remains open and is far from resolved. In Section 4, we first provide a decomposition of $L^2(B_n, \mathrm{d}v)$, then we investigate the zero-product problem for two Toeplitz operators, when one of the two symbols is separately quasihomogeneous and the other is arbitrary bounded function.

Brown and Halmos [4] showed that two bounded Toeplitz operators T_f and T_g commute on $H^2(\mathbb{T})$ if and only if:

(I) both f and g are analytic, or

(II) both $\overline{f}$ and $\overline{g}$ are analytic, or

(III) one is a linear function of the other.

Axler and Čučković [3] characterized commutativity of Toeplitz operators with harmonic symbols on $L_a^2(D)$. In [7] Čučković and Rao used the Mellin transform to study the commutativity of two Toeplitz operators on $L_a^2(D)$ and describe those operators which commute with $T_{\mathrm{e}^{\mathrm{i}p\theta}r^m}$. Later in [13], Louhichi and Zakariasy gave a partial characterization of commuting Toeplitz operators on $L_a^2(D)$ with quasihomogeneous symbols. The main reason for them to study such family of symbols is that any function f in $L^2(D, \mathrm{d}A)$, where $\mathrm{d}A$ is the Lebesgue area measure, has the following polar decomposition

$$f(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}k\theta} f_k(r),$$

where f_k are radial functions in $L^2([0,1], r\mathrm{d}r)$. Zheng [23] studied commuting Toeplitz operators with pluriharmonic symbols on $L_a^2(B_n)$. Recently, Quiroga-Barranco and Vasilevski gave the description of many (geometrically defined) classes of commuting Toeplitz operators on the unit ball (see [16], [17]). In the last section, we will give necessary and sufficient conditions for a symbol that produces a Toeplitz operator commuting with another such operator whose symbol is a radial function.

2. PRELIMINARIES

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, where $\mathbb{N}$ denotes the set of all nonnegative integers, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \alpha! = \alpha_1! \cdots \alpha_n!.$$

We will also write, for $z = (z_1, \ldots, z_n) \in B_n$:

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

For two multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, the notation $\alpha \succeq \beta$ means that

$$\alpha_i \geqslant \beta_i, \quad i = 1, \ldots, n$$

and $\alpha \perp \beta$ means that

$$\alpha_1\beta_1 + \cdots + \alpha_n\beta_n = 0.$$

We also define

$$\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n).$$

Moreover, if $\alpha \succeq \beta$ then we obtain

$$|\alpha - \beta| = |\alpha| - |\beta|.$$

Let $\mathrm{d}\sigma$ be the surface area measure on the unit sphere S_n, then it is known that the holomorphic monomials z^α ($\alpha \in \mathbb{N}^n$) are orthogonal to each other in $L^2(S_n, \mathrm{d}\sigma)$. We show a more general result in the following lemma.

LEMMA 2.1. *Let α, $\beta \in \mathbb{N}^n$ with $\alpha \neq \beta$ and let $t \in [0,1)$. If φ is a bounded separately radial function on B_n, then*

$$\int_{S_n} \varphi(t\xi)\xi^\alpha\overline{\xi}^\beta \mathrm{d}\sigma(\xi) = 0.$$

Proof. The result is obvious when $n = 1$. Now suppose $n > 1$. Without loss of generality, we may assume $\alpha_n \neq \beta_n$. Since φ is a bounded separately radial function, it follows from Proposition 1.4.7 of [19] that

$$\begin{aligned}\int_{S_n} \varphi(t\xi)\xi^\alpha\overline{\xi}^\beta \mathrm{d}\sigma(\xi) &= \int_{B_{n-1}} \mathrm{d}v_{n-1}(\xi')\frac{1}{2\pi}\int_0^{2\pi} \varphi(t\xi', t\mathrm{e}^{\mathrm{i}\theta}\xi_n)(\xi', \mathrm{e}^{\mathrm{i}\theta}\xi_n)^\alpha\overline{(\xi', \mathrm{e}^{\mathrm{i}\theta}\xi_n)}^\beta \mathrm{d}\theta \\ &= \int_{B_{n-1}} \varphi(t\xi', t\xi_n)(\xi', \xi_n)^\alpha\overline{(\xi', \xi_n)}^\beta \mathrm{d}v_{n-1}(\xi')\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(\alpha_n - \beta_n)\theta}\mathrm{d}\theta = 0.\end{aligned}$$

This completes the proof. ∎

We recall some useful results from [15]. Denote by $\tau(B_n)$ the base of B_n, i.e.,

$$\tau(B_n) = \{r = (r_1, \ldots, r_n) = (|z_1|, \ldots, |z_n|) : z = (z_1, \ldots, z_n) \in B_n\}.$$

Then for any $\alpha \in \mathbb{N}^n$ and $r \in \tau(B_n)$, we will write

$$r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n} \quad \text{and} \quad |r| = \sqrt{r_1^2 + \cdots + r_n^2}.$$

Note that $\mathrm{d}v(z) = \frac{n!}{\pi^n}\mathrm{d}m_{2n}$, where $\mathrm{d}m_{2n} = \mathrm{d}x_1\mathrm{d}y_1 \cdots \mathrm{d}x_n\mathrm{d}y_n$ is the usual Lebesgue measure in $\mathbb{C}^n$, and for any bounded separately radial function φ, we get

$$\int_{B_n} \varphi(z)\mathrm{d}v(z) = 2^n n! \int_{\tau(B_n)} \varphi(r)r\mathrm{d}r, \tag{2.1}$$

where $r\mathrm{d}r = \prod_{i=1}^n r_i\mathrm{d}r_i$. Corollary 3.2 of [15] tells that the Toeplitz operator T_φ with bounded separately radial symbol φ is diagonal with respect to the orthogonal basis $\{z^\alpha : \alpha \in \mathbb{N}^n\}$. Conversely, the following lemma will show that the symbol on any diagonal Toeplitz operator must be separately radial.

LEMMA 2.2. *Let φ be a bounded function on B_n. If for each $\alpha \in \mathbb{N}^n$, there exists $\lambda_\alpha \in \mathbb{C}$ such that $T_\varphi(z^\alpha) = \lambda_\alpha z^\alpha$, then φ is separately radial.*

Proof. Assume $T_\varphi(z^\alpha) = \lambda_\alpha z^\alpha$, then for any unitary transformation U of $\mathbb{C}^n$ with a diagonal matrix, i.e., $U = \operatorname{diag}\{\mathrm{e}^{\mathrm{i}\theta_1}, \dots, \mathrm{e}^{\mathrm{i}\theta_n}\}$, we have

$$\begin{aligned} T_{\varphi\circ U}(z^\alpha)(w) &= T_\varphi(U^{-1}z)^\alpha(Uw) = \mathrm{e}^{-\mathrm{i}\alpha_1\theta_1}\cdots\mathrm{e}^{-\mathrm{i}\alpha_n\theta_n}T_\varphi(z^\alpha)(Uw) \\ &= \mathrm{e}^{-\mathrm{i}\alpha_1\theta_1}\cdots\mathrm{e}^{-\mathrm{i}\alpha_n\theta_n}\lambda_\alpha(z^\alpha)(Uw) = \lambda_\alpha w^\alpha = T_\varphi(z^\alpha)(w), \end{aligned}$$

which implies $T_{\varphi\circ U} = T_\varphi$. Thus $\varphi\circ U = \varphi$ and therefore φ is a separately radial function. This completes the proof. ∎

It is well known that a bounded analytic function on $\Pi_+ = \{z : \operatorname{Re} z > 0\}$ is uniquely determined by its value on an arithmetic sequence of integers. In fact, we have the following classical theorem (see p. 102 of [18]).

THEOREM 2.3. *Suppose that f is a bounded analytic function on Π_+ which vanishes at the pairwise distinct points $z_1, z_2, \dots,$ where*
(i) $\inf\{|z_n|\} > 0$ *and*
(ii) $\sum\limits_{n\geqslant 1} \operatorname{Re}(\frac{1}{z_n}) = \infty.$
Then f vanishes identically on Π_+.

In this paper, we will need a similar result in higher dimensions. First, we give the following definition.

DEFINITION 2.4. Let E be a subset of $\mathbb{Z}_+^2$, we say that E satisfies condition (I) if the following statement holds:

(I) there exists a sequence $\{\alpha_i^{(1)}\}_{i=1}^\infty$ such that $\sum\limits_{i=1}^{\infty} \frac{1}{\alpha_i^{(1)}} = \infty$, and for every fixed $\alpha_i^{(1)}$, there also exists a sequence $\{\alpha_{j(i)}^{(2)}\}_{j(i)=1}^\infty$ such that $\sum\limits_{j(i)=1}^{\infty} \frac{1}{\alpha_{j(i)}^{(2)}} = \infty$ and $\{(\alpha_i^{(1)}, \alpha_{j(i)}^{(2)}) : j(i) = 1, 2, \dots\} \subset E$.

REMARK 2.5. According to Definition 2.4, one can immediately get that, for a multi-index $\gamma \in \mathbb{N}^2$, if E is a subset of $\{\alpha \in \mathbb{Z}_+^2 : \alpha \succeq \gamma\}$ and if E' is the complement of E in $\{\alpha \in \mathbb{Z}_+^2 : \alpha \succeq \gamma\}$, then either E or E' satisfies condition (I).

Now we are ready to show when a bounded holomorphic function on Π_+^2 is identically zero.

PROPOSITION 2.6. *Let f be a bounded holomorphic function on Π_+^2. If the set $E = \{\alpha \in \mathbb{Z}_+^2 : f(\alpha) = 0\}$ satisfies condition* (I)*, then f vanishes identically on Π_+^2.*

Proof. If E satisfies condition (I), then for any arbitrary but fixed $\alpha_i^{(1)} \in \{\alpha_i^{(1)}\}_{i=1}^{\infty}$, there exists a sequence $\{\alpha_{j(i)}^{(2)}\}_{j(i)=1}^{\infty} \subset \mathbb{Z}_+$ such that $\sum_{j(i)=1}^{\infty} \frac{1}{\alpha_{j(i)}^{(2)}} = \infty$ and $f(\alpha_i^{(1)}, \alpha_{j(i)}^{(2)}) = 0$. By Theorem 2.3, we can get

$$f(\alpha_i^{(1)}, z_2) = 0, \quad \forall z_2 \in \Pi_+.$$

Condition (I) also gives $\sum_{i=1}^{\infty} \frac{1}{\alpha_i^{(1)}} = \infty$. Applying Theorem 2.3 to the first variable, then we have $f(z_1, z_2) = 0$ on Π_+^2. This completes the proof. ∎

COROLLARY 2.7. *Let $p,\ s \in \mathbb{N}^2$ and let $g(r)$ be a bounded function on $\tau(B_2)$. If the set*

$$E = \Big\{\alpha \in \mathbb{Z}_+^2 : \alpha \succeq s \text{ and } \int\limits_{\tau(B_2)} g(r) r^{2\alpha+p-s} r \mathrm{d}r = 0\Big\}$$

satisfies condition (I), *then $g = 0$.*

Proof. Let $f(z) = \int\limits_{\tau(B_2)} g(r) r^z r \mathrm{d}r$. Obviously, f is a bounded holomorphic function on Π_+^2 since g is bounded. By Proposition 2.6, it is easy to see that $f(z) = 0$ on Π_+^2. Therefore $g = 0$ on $\tau(B_2)$. This completes the proof. ∎

3. PRODUCTS OF SEPARATELY QUASIHOMOGENEOUS TOEPLITZ OPERATORS

In this section, we will show some basic results concerning Toeplitz operators with separately quasihomogeneous symbols on $L_a^2(B_n)$. The definition of quasihomogeneous functions on D can be found in [12]. Below we give an analogous definition on B_n.

DEFINITION 3.1. Let $k \in \mathbb{Z}^n$ and let $f \in L^\infty(B_n, \mathrm{d}v)$. Then we say that f is a *separately quasihomogeneous function of degree* k if it is of the form $\xi^k \varphi$ where φ is a separately radial function, i.e.,

$$f(z) = f(|r|\xi) = \xi^k \varphi(r)$$

for any $\xi \in S_n$ and $r \in \tau(B_n)$. In this case, the associated Toeplitz operator T_f is called a *separately quasihomogeneous Toeplitz operator of degree* k.

REMARK 3.2. Let $k \in \mathbb{Z}^n$ and let $\xi \in S_n$. It is obvious that k can be uniquely written as $p - s$, where $p,\ s \in \mathbb{N}^n$ such that $p \perp s$. Thus in this paper, the function ξ^k is always understood as

$$\xi^k = \xi^p \overline{\xi}^s.$$

A direct calculation gives the following lemma which we shall use often.

LEMMA 3.3. *Let $p,\ s \in \mathbb{N}^n$ and let $r \in \tau(B_n)$. If $\xi^p\overline{\xi}^s\varphi(r)$ is a bounded function on B_n, then for each $\alpha \in \mathbb{N}^n$*

$$T_{\xi^p\overline{\xi}^s\varphi(r)}(z^\alpha) = \begin{cases} 0 & \text{if } \alpha+p \not\succeq s; \\ \frac{2^n(n+|\alpha|+|p|-|s|)!}{(\alpha+p-s)!} \int\limits_{\tau(B_n)} \varphi(r) r^{2\alpha+2p}|r|^{-(|p|+|s|)} r\mathrm{d}r z^{\alpha+p-s} & \text{if } \alpha+p \succeq s. \end{cases}$$

Proof. For each $\alpha \in \mathbb{N}^n$, if $\alpha + p \not\succeq s$, i.e., there is an integer $1 \leqslant j \leqslant n$ so that $\alpha_j + p_j < s_j$, then $\alpha + p \neq \beta + s$ for any $\beta \in \mathbb{N}^n$. Thus by Lemma 2.1 we can get

$$\langle T_{\xi^p\overline{\xi}^s\varphi(r)}(z^\alpha), z^\beta\rangle = \int\limits_{B_n} \varphi(r)\xi^p\overline{\xi}^s z^\alpha \overline{z}^\beta \mathrm{d}v(z) = 0,$$

which implies that $T_{\xi^p\overline{\xi}^s\varphi(r)}(z^\alpha) = 0$.

On the other hand, if $\alpha + p \succeq s$, then using Lemma 2.1 and (2.1) again, we can get

$$\begin{aligned}\langle T_{\xi^p\overline{\xi}^s\varphi(r)}(z^\alpha), z^\beta\rangle &= \int\limits_{B_n} \varphi(r)\xi^p\overline{\xi}^s z^\alpha \overline{z}^\beta \mathrm{d}v(z) \\ &= \begin{cases} 0 & \text{if } \beta \neq \alpha+p-s; \\ 2^n n! \int\limits_{\tau(B_n)} \varphi(r) r^{2\alpha+2p}|r|^{-(|p|+|s|)} r\mathrm{d}r & \text{if } \beta = \alpha+p-s. \end{cases}\end{aligned}$$

Then by the well-known identity (see Proposition 1.4.9 of [19])

$$\langle z^{\alpha+p-s}, z^\beta\rangle = \begin{cases} 0 & \text{if } \beta \neq \alpha+p-s, \\ \frac{n!(\alpha+p-s)!}{(n+|\alpha|+|p|-|s|)!} & \text{if } \beta = \alpha+p-s, \end{cases}$$

we obtain

$$\langle T_{\xi^p\overline{\xi}^s\varphi(r)}(z^\alpha), z^\beta\rangle = \frac{2^n(n+|\alpha|+|p|-|s|)!}{(\alpha+p-s)!} \int\limits_{\tau(B_n)} \varphi(r) r^{2\alpha+2p}|r|^{-(|p|+|s|)} r\mathrm{d}r \langle z^{\alpha+p-s}, z^\beta\rangle,$$

which implies that

$$T_{\xi^p\overline{\xi}^s\varphi(r)}(z^\alpha) = \frac{2^n(n+|\alpha|+|p|-|s|)!}{(\alpha+p-s)!} \int\limits_{\tau(B_n)} \varphi(r) r^{2\alpha+2p}|r|^{-(|p|+|s|)} r\mathrm{d}r z^{\alpha+p-s}.$$

So the desired result is obvious. ∎

PROPOSITION 3.4. *Let $p,\ s \in \mathbb{N}^n$ and let f be a bounded function on B_n. Then the following assertions are equivalent:*

(i) *For each $\alpha \in \mathbb{N}^n$, there exists $\lambda_\alpha \in \mathbb{C}$ such that*

$$T_f(z^\alpha) = \begin{cases} 0 & \text{if } \alpha+p \not\succeq s; \\ \lambda_\alpha z^{\alpha+p-s} & \text{if } \alpha+p \succeq s. \end{cases}$$

(ii) *f is a separately quasihomogeneous function of degree $p - s$.*

Proof. If f is a separately quasihomogeneous function of degree $p-s$, then there exists a separately radial function φ such that $f=\xi^p\overline{\xi}^s\varphi$. Then (i) is a direct consequence of Lemma 3.3.

Conversely, suppose (i) is true. Then for any multi-indexes $\alpha,\ \beta\in\mathbb{N}^n$,

$$\begin{aligned}\langle T_{\overline{z}^p z^s f}(z^\alpha), z^\beta\rangle &= \langle T_f(z^{\alpha+s}), z^{\beta+p}\rangle = \langle \lambda_{\alpha+s} z^{\alpha+p}, z^{\beta+p}\rangle\\ &= \begin{cases} 0 & \text{if } \beta\neq\alpha,\\ \frac{\lambda_{\alpha+s} n!(\alpha+p)!}{(n+|\alpha|+|p|)!} & \text{if } \beta=\alpha;\end{cases}\end{aligned}$$

and consequently

$$T_{\overline{z}^p z^s f}(z^\alpha) = \frac{\lambda_{\alpha+s}(\alpha+p)!(n+|\alpha|)!}{\alpha!(n+|\alpha|+|p|)!} z^\alpha.$$

Thus, it follows from Lemma 2.2 that $\overline{z}^p z^s f$ is a separately radial function, which implies that f is a separately quasihomogeneous function of degree $p-s$. This completes the proof. ∎

THEOREM 3.5. *Let f_1 and f_2 be two bounded separately quasihomogeneous functions on B_n of degrees k and k' respectively. If there exists a bounded function h such that $T_{f_1}T_{f_2}=T_h$, then h is a separately quasihomogeneous function of degree $k+k'$.*

Proof. Suppose $k=p-s$ and $k'=p'-s'$, where $p,\ p',\ s,\ s'\in\mathbb{N}^n$ such that $p\perp s$ and $p'\perp s'$, then by Definition 3.1 and Remark 3.2, we have

$$f_1=\xi^k\varphi_1=\xi^p\overline{\xi}^s\varphi_1 \quad\text{and}\quad f_2=\xi^{k'}\varphi_2=\xi^{p'}\overline{\xi}^{s'}\varphi_2,$$

for two separately radial functions φ_1 and φ_2. By Lemma 3.3, for each $\alpha\in\mathbb{N}^n$, there exists $\lambda'_\alpha\in\mathbb{C}$ such that

$$\begin{aligned}T_{f_1}T_{f_2}(z^\alpha) &= T_{\xi^p\overline{\xi}^s\varphi_1} T_{\xi^{p'}\overline{\xi}^{s'}\varphi_2}(z^\alpha)\\ &= \begin{cases} 0 & \text{if } \alpha+p'\not\succeq s';\\ 0 & \text{if } \alpha+p'\succeq s' \text{ and } \alpha+p'+p\not\succeq s'+s;\\ \lambda'_\alpha z^{\alpha+p+p'-s-s'} & \text{if } \alpha+p'\succeq s' \text{ and } \alpha+p'+p\succeq s'+s.\end{cases}\end{aligned}$$

Let

$$\lambda_\alpha = \begin{cases} 0 & \text{if } \alpha+p'\not\succeq s' \text{ and } \alpha+p'+p\succeq s'+s,\\ \lambda'_\alpha & \text{if } \alpha+p'\succeq s' \text{ and } \alpha+p'+p\succeq s'+s;\end{cases}$$

then it follows that

$$T_{f_1}T_{f_2}(z^\alpha) = \begin{cases} 0 & \text{if } \alpha+p'+p\not\succeq s'+s;\\ \lambda_\alpha z^{\alpha+p+p'-s-s'} & \text{if } \alpha+p'+p\succeq s'+s.\end{cases}$$

Hence, if $T_{f_1}T_{f_2}=T_h$ then by Proposition 3.4, h is a separately quasihomogeneous function of degree $p'+p-s'-s=k+k'$. This completes the proof. ∎

REMARK 3.6. Theorem 3.5 only gives a necessary condition for $T_{f_1}T_{f_2}$ to be equal to a Toeplitz operator T_h. However, it is not known for which bounded separately quasihomogenous f_1 and f_2 there exists a bounded h with $T_{f_1}T_{f_2} = T_h$. The problem of determining when the product of two separately quasihomogenous Toeplitz operators is a Toeplitz operator still remains open, even in the one variable case. Louhichi, Strouse and Zakariasy [12] gave lots of examples of functions ψ_1 and ψ_2 such that $T_{\mathrm{e}^{\mathrm{i}p\theta}\psi_1}T_{\mathrm{e}^{-\mathrm{i}s\theta}\psi_2}$ on $L^2_a(D)$ is not a Toeplitz operator. Moreover, Theorem 6.7 of [12] illustrated the difficulty in characterizing more precisely those pairs of Toeplitz operators whose product is a Toeplitz operator.

4. ON THE ZERO-PRODUCT PROBLEM FOR TOEPLITZ OPERATORS

We start this section with a decomposition of the space $L^2(B_n, \mathrm{d}v)$. Let

$$\Re = \Big\{\varphi : B_n \to \mathbb{C} \text{ separately radial} \ \Big| \int\limits_{\tau(B_n)} |\varphi(r)|^2 r\mathrm{d}r < \infty\Big\},$$

and let $\Re_k = \xi^k\Re$, $k \in \mathbb{Z}^n$. Each $\Re_k$ is a subspace of $L^2(B_n, \mathrm{d}v)$ since for $f \in \Re_k$, $f(z) = f(|r|\xi) = \xi^k\varphi(r)$ and

$$\int\limits_{B_n} |f(z)|^2\mathrm{d}v(z) = 2^n n! \int\limits_{\tau(B_n)} |\xi^k|^2|\varphi(r)|^2 r\mathrm{d}r < \infty.$$

It is also clear that $\Re_k \perp \Re_l$ if $k \neq l$. Every polynomial p in z and $\overline{z}$ can be written as

$$p(z, \overline{z}) = \sum_{k=-\gamma_0}^{\gamma_0} \Big(\sum_{\alpha-\beta=k} a_{\alpha,\beta} z^\alpha \overline{z}^\beta \Big)$$

for some $\gamma_0 \in \mathbb{N}^n$. In polar coordinates, if $z = |r|(\xi_1, \dots, \xi_n)$, then $p(z, \overline{z})$ has the following form

$$\sum_{k=-\gamma_0}^{\gamma_0} \xi^k \Big\{ \sum_{\alpha-\beta=k} a_{\alpha,\beta} |r|^{|\alpha|+|\beta|} |\xi_1|^{2\min\{\alpha_1,\beta_1\}} \cdots |\xi_n|^{2\min\{\alpha_n,\beta_n\}} \Big\} \in \sum_{k\in\mathbb{Z}^n} \xi^k \Re.$$

Since the polynomials $p(z, \overline{z})$ are dense in $\mathcal{C}(\overline{B_n})$ and $\mathcal{C}(\overline{B_n})$ is dense in $L^2(B_n, \mathrm{d}v)$, we can conclude that

$$L^2(B_n, \mathrm{d}v) = \bigoplus_{k\in\mathbb{Z}^n} \xi^k \Re,$$

i.e., every function $f \in L^2(B_n, \mathrm{d}v)$ has the decomposition

$$f(|r|\xi) = \sum_{k\in\mathbb{Z}^n} \xi^k f_k(r), \quad f_k \in \Re.$$

Moreover, if $f \in L^\infty(B_n, \mathrm{d}v) \subset L^2(B_n, \mathrm{d}v)$, the following lemma will show that the functions $\xi^k f_k(r)$ are also bounded.

LEMMA 4.1. *Let* $f(z) = \sum\limits_{k\in\mathbb{Z}^n} \xi^k f_k(r) \in L^\infty(B_n, \mathrm{d}v)$*, then the functions* $\xi^k f_k(r)$ *are bounded on* B_n *for all* $k \in \mathbb{Z}^n$*.*

Proof. If U is a unitary transformation of $\mathbb{C}^n$ with a diagonal matrix, i.e.,

$$U = \mathrm{diag}\{\mathrm{e}^{\mathrm{i}\theta_1}, \dots, \mathrm{e}^{\mathrm{i}\theta_n}\},$$

then we obtain

$$f(Uz) = f(|r|\mathrm{e}^{\mathrm{i}\theta_1}\xi_1, \dots, |r|\mathrm{e}^{\mathrm{i}\theta_n}\xi_n) = \sum_{k\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}k_1\theta_1} \cdots \mathrm{e}^{\mathrm{i}k_n\theta_n} \xi^k f_k(r).$$

So for any $k \in \mathbb{Z}^n$,

$$\begin{aligned}\xi^k f_k(r) &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{l\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}l_1\theta_1} \cdots \mathrm{e}^{\mathrm{i}l_n\theta_n} \xi^l f_l(r) \mathrm{e}^{-\mathrm{i}k_1\theta_1} \cdots \mathrm{e}^{-\mathrm{i}k_n\theta_n} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(Uz) \mathrm{e}^{-\mathrm{i}k_1\theta_1} \cdots \mathrm{e}^{-\mathrm{i}k_n\theta_n} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n,\end{aligned}$$

which implies that

$$|\xi^k f_k(r)| \leqslant \sup_{z\in B_n} |f(z)|, \quad \forall k \in \mathbb{Z}^n.$$

Hence, the functions $\xi^k f_k(r)$ are bounded on B_n. ∎

Our next result is a partial answer to the zero-product problem for two Toeplitz operators on $L_a^2(B_n)$.

THEOREM 4.2. *Let* $f \in L^\infty(B_n, \mathrm{d}v)$ *and let* g *be a bounded separately quasihomogeneous function on* B_n*. If* $T_f T_g = 0$*, then either* $f = 0$ *or* $g = 0$*.*

Proof. Suppose $f(z) = \sum\limits_{k\in\mathbb{Z}^n} \xi^k f_k(r)$ and $g = \xi^{k'}\varphi(r)$. Then for any $k \in \mathbb{Z}^n$, Lemma 4.1 implies that $\xi^k f_k$ is bounded. Let $k = p - s$ where $p,\ s \in \mathbb{N}^n$ such that $p \perp s$. Here p and s depend only on k. Thus for $\alpha \in \mathbb{N}^n$, it follows from Lemma 3.3 that

$$T_{\xi^k f_k}(z^\alpha) = c(k) \int_{\tau(B_n)} f_k(r) r^{2\alpha+2p} |r|^{-(|p|+|s|)} r \mathrm{d}r z^{\alpha+k},$$

where

$$c(k) = \begin{cases} 0 & \text{if } \alpha \not\succeq s, \\ \frac{2^n(n+|\alpha|+|p|-|s|)!}{(\alpha+k)!} & \text{if } \alpha \succeq s; \end{cases}$$

and consequently

$$T_f(z^\alpha) = \sum_{k\in\mathbb{Z}^n} c(k) \int_{\tau(B_n)} f_k(r) r^{2\alpha+2p} |r|^{-(|p|+|s|)} r \mathrm{d}r z^{\alpha+k}. \tag{4.1}$$

Assume $k' = p' - s'$ for two multi-indexes $p', s' \in \mathbb{N}^n$ such that $p' \perp s'$, then the equality $T_f T_g(z^\alpha) = 0$ for each $\alpha \in \mathbb{N}^n$ together with Lemma 3.3 and (4.1) give

$$\int\limits_{\tau(B_n)} f_k(r) r^{2\alpha+2k'+2p} |r|^{-(|p|+|s|)} r\mathrm{d}r \int\limits_{\tau(B_n)} \varphi(r) r^{2\alpha+2p'} |r|^{-(|p'|+|s'|)} r\mathrm{d}r = 0 \tag{4.2}$$

for each $\alpha \in \mathbb{N}^n$ such that $\alpha \succeq s + s'$.

Now we are ready to show either $\varphi = 0$ or $f_k = 0$ for each $k \in \mathbb{Z}^n$. For the sake of simplicity, we will prove it in the case of $n = 2$. Let

$$E_k = \Big\{ \alpha \in \mathbb{Z}_+^2 : \alpha \succeq s + s' \text{ and } \int\limits_{\tau(B_2)} f_k(r) r^{2\alpha+2k'+2p} |r|^{-(|p|+|s|)} r\mathrm{d}r = 0 \Big\}.$$

If E_k satisfies condition (I), noting that

$$|f_k(r) r^{p+s} |r|^{-(|p|+|s|)}| = |\xi^k f_k(r)| < \infty,$$

then $f_k = 0$ by Corollary 2.7. Otherwise, if we denote by E_k' the complement of E_k in $\{\alpha \in \mathbb{Z}_+^2 : \alpha \succeq s + s'\}$, then E_k' must satisfy condition (I) by Remark 2.5. It follows from (4.2) that

$$\int\limits_{\tau(B_2)} \varphi(r) r^{2\alpha+2p'} |r|^{-(|p'|+|s'|)} r\mathrm{d}r = 0, \quad \forall \alpha \in E_k'.$$

Using Corollary 2.7 again, we get $\varphi = 0$. This completes the proof. ∎

REMARK 4.3. In the one-dimensional case, a radial function is a separately quasihomogeneous function of degree 0, therefore Theorem 4.2 implies the result of Ahern and Čučković in [2].

Using the same reasoning as in [4], we will show that there are no idempotent Toeplitz operators with separately quasihomogeneous symbols other than the identity and the zero operators.

COROLLARY 4.4. *Let f be a bounded separately quasihomogeneous function on B_n. Then $T_f^2 = T_f$ if and only if either $f = 0$ or $f = 1$.*

Proof. If $T_f^2 = T_f$, then $T_f T_{f-1} = 0$, and by Theorem 4.2, $f = 0$, or $f = 1$. The converse implication is clear. This completes the proof. ∎

5. COMMUTING TOEPLITZ OPERATORS

Recall that a function ϕ on B_n is radial if $\phi(z)$ depends only on $|z|$. Then for each radial function ϕ, we define the function $\widetilde{\phi}$ on $[0, 1)$ by $\widetilde{\phi}(s) = \phi(se)$ where e is a unit vector in $\mathbb{C}^n$. It is obvious that $\widetilde{\phi}$ is well-defined. In the following, we shall often identify an integrable radial function ϕ on B_n with the corresponding function $\widetilde{\phi}$ defined on the interval $[0, 1)$.

The Mellin transform $\widehat{\phi}$ of a function $\phi \in L^1([0,1], r\mathrm{d}r)$ is defined by the equation:

$$\widehat{\phi}(z) = \int_0^1 \phi(s) s^{z-1} \mathrm{d}s.$$

It is known that $\widehat{\phi}$ is a bounded analytic function in the half plane $\{z : \operatorname{Re} z > 2\}$. In addition, if ϕ is a radial function, then for each $\alpha \in \mathbb{N}^n$, one can easily get

$$T_\phi(z^\alpha) = (2n + 2|\alpha|)\widehat{\phi}(2n + 2|\alpha|) z^\alpha. \tag{5.1}$$

It was shown in [7], that a Toeplitz operator with a radial symbol on $L^2_a(D)$ commutes with another Toeplitz operator only if that operator also has a radial symbol. However, we will see that it is not true on B_n with $n > 1$, i.e., we can get a separately quasihomogenous Toeplitz operator commuting with another such operator whose symbol is a radial function.

THEOREM 5.1. *Let $p, s \in \mathbb{N}^n$ and let $\xi^p \overline{\xi}^s \psi(r)$ and $\phi(z) = \phi(|z|)$ be two bounded functions on B_n. If ϕ is nonconstant, then*

$$T_\phi T_{\xi^p \overline{\xi}^s \psi} = T_{\xi^p \overline{\xi}^s \psi} T_\phi$$

if and only if either $|p| = |s|$ or $\psi = 0$.

Proof. If $T_{\xi^p \overline{\xi}^s \psi}$ commutes with T_ϕ, then for each $\alpha \in \mathbb{N}^n$ such that $\alpha + p \succeq s$, from Lemma 3.3 and (5.1) we can deduce

$$\begin{aligned}(2n + 2|\alpha|)\widehat{\phi}(2n + 2|\alpha|) &\int_{\tau(B_n)} \psi(r) r^{2\alpha+2p} |r|^{-(|p|+|s|)} r\mathrm{d}r \\ &= (2n + 2|\alpha| + 2|p| - 2|s|)\widehat{\phi}(2n + 2|\alpha| + 2|p| - 2|s|) \\ &\quad \times \int_{\tau(B_n)} \psi(r) r^{2\alpha+2p} |r|^{-(|p|+|s|)} r\mathrm{d}r.\end{aligned} \tag{5.2}$$

Assume $|p| \neq |s|$, without loss of generality, we can assume that $|p| > |s|$, for otherwise we could take adjoints. Similarly as the proof of Theorem 4.2, let $n = 2$ and denote

$$E = \Big\{ \alpha \in \mathbb{Z}^2_+ : \alpha \succeq s \text{ and } \int_{\tau(B_2)} \psi(r) r^{2\alpha+2p} |r|^{-(|p|+|s|)} r\mathrm{d}r = 0 \Big\}.$$

Then we will show that E satisfies condition (I), which implies that $\psi = 0$ by Corollary 2.7.

Suppose E does not satisfy condition (I). Denote by E' the complement of E in $\{\alpha \in \mathbb{Z}^2_+ : \alpha \succeq s\}$, then E' must satisfy condition (I) by Remark 2.5. Furthermore, we can get

$$\sum_{\alpha \in E'} \frac{1}{|\alpha|} = \infty.$$

On the other hand, for any $\alpha \in E'$, (5.2) gives

$$(2n+2|\alpha|)\widehat{\phi}(2n+2|\alpha|) = (2n+2|\alpha|+2|p|-2|s|)\widehat{\phi}(2n+2|\alpha|+2|p|-2|s|). \tag{5.3}$$

Denote

$$F(z) = \widehat{\phi}(z+2n+2|p|-2|s|) - \frac{z+2n}{z+2n+2|p|-2|s|}\widehat{\phi}(z+2n),$$

then F is analytic and bounded on Π^+. Moreover, it follows from (5.3) that

$$F(2|\alpha|) = 0, \quad \forall \alpha \in E'.$$

According to Theorem 2.3, F must be zero, which implies that

$$(z+2n+2|p|-2|s|)\widehat{\phi}(z+2n+2|p|-2|s|) = (z+2n)\widehat{\phi}(z+2n), \quad z \in \Pi^+.$$

As in the proof of Theorem 6 in [7], the above equation gives that ϕ is constant, which is a contradiction. Thus we conclude that either $|p| = |s|$ or $\psi = 0$.

Conversely, if $|p| = |s|$ or $\psi = 0$, then we can easily show that (5.2) holds, and consequently

$$T_{\xi^p \overline{\xi}^s \psi} T_\phi(z^\alpha) = T_\phi T_{\xi^p \overline{\xi}^s \psi}(z^\alpha)$$

for each $\alpha \in \mathbb{N}^n$, which implies that $T_{\xi^p \overline{\xi}^s \psi}$ and T_ϕ commute. This completes the proof. ∎

COROLLARY 5.2. *Let $f(z)$ and $\phi(z) = \phi(|z|)$ be two bounded functions on B_n. If ϕ is nonconstant, then $T_f T_\phi = T_\phi T_f$ if and only if $f(\mathrm{e}^{\mathrm{i}\theta} z) = f(z)$ for almost all $\theta \in \mathbb{R}$.*

Proof. Suppose $f(z) = \sum_{k \in \mathbb{Z}^n} \xi^k f_k(r) \in L^\infty(B_n, \mathrm{d}v)$. By (4.1) and (5.1), it is obvious that T_f commutes with T_ϕ if and only if

$$T_{\xi^k f_k} T_\phi = T_\phi T_{\xi^k f_k}$$

for all $k \in \mathbb{Z}^n$. Let $k = p - s$, where $p,\ s \in \mathbb{N}^n$ such that $p \perp s$, then by Theorem 5.1, the above equation is equivalent to

$$|p| = |s| \quad \text{or} \quad f_k = 0. \tag{5.4}$$

Obviously, (5.4) is equivalent to

$$(\xi^k f_k)(\mathrm{e}^{\mathrm{i}\theta} z) = (\xi^k f_k)(z), \quad \forall \theta \in \mathbb{R}.$$

Therefore T_f commutes with T_ϕ if and only if $f(\mathrm{e}^{\mathrm{i}\theta} z) = f(z)$ for almost all $\theta \in \mathbb{R}$. This completes the proof. ∎

REMARK 5.3. We noticed that Le [9] studied the commutants of certain diagonal Toeplitz operators on weighted Bergman spaces of the unit ball when we were revising the manuscript. Obviously, Corollary 5.2 coincides with Theorem 1.2 of [9], though they are proved by different methods.

I. Louhichi and N.V. Rao [11] showed that the commutant of a quasihomogenous Toeplitz operator on $L_a^2(D)$ is equal to its bicommutant. Moreover, they conjectured that if two Toeplitz operators commute with a third one, none of them being the identity, then they commute with each other. The next is a counterexample to their conjecture when formulated for Toeplitz operators on $L_a^2(B_n)$, with $n > 1$.

EXAMPLE 5.4. Let $n > 1$ and let $p = (1, 0, \dots, 0)$ and $s = (0, \dots, 0, 1)$. Then for each bounded nonconstant radial function ϕ, by Theorem 5.1, T_ϕ commutes with both $T_{\zeta^p\overline{\zeta}^s}$ and $T_{\zeta^s\overline{\zeta}^p}$. However, it follows from Lemma 3.3 that

$$T_{\zeta^p\overline{\zeta}^s} T_{\zeta^s\overline{\zeta}^p}(z^p) = \Big[2^n(n+1) \int\limits_{\tau(B_n)} r^{2p+2s} r\mathrm{d}r\Big]^2 z^p \quad \text{and} \quad T_{\zeta^s\overline{\zeta}^p} T_{\zeta^p\overline{\zeta}^s}(z^p) = 0.$$

Therefore $T_{\zeta^p\overline{\zeta}^s}$ and $T_{\zeta^s\overline{\zeta}^p}$ cannot commute.

Acknowledgements. We would like to thank the referee for his excellent suggestions. The second author is the corresponding author, and was partially supported by the National Natural Science Foundation of China (Grant Nos. 10971153, 10671141).

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Received November 10, 2008; revised June 30, 2009.