DERIVATIONS IN ALGEBRAS OF OPERATOR-VALUED FUNCTIONS

A.F. BER, B. DE PAGTER and F.A. SUKOCHEV

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ABSTRACT. In this paper we study derivations in subalgebras of $L_0^{wo}(\nu; \mathcal{L}(X))$, the algebra of all weak operator measurable functions $f : S \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the Banach algebra of all bounded linear operators on a Banach space X. It is shown, in particular, that all derivations on $L_0^{wo}(\nu; \mathcal{L}(X))$ are inner whenever X is separable and infinite dimensional. This contrasts strongly with the fact that $L_0^{wo}(\nu; \mathcal{L}(X))$ admits non-trivial non-inner derivations whenever X is finite dimensional and the measure ν is non-atomic. As an application of our approach, we study derivations in various algebras of measurable operators affiliated with von Neumann algebras.

KEYWORDS: *Derivations, operator valued functions, non-commutative integration, measurable operators.*

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INTRODUCTION

It follows from recent results ([4], [5]) that the algebra $L_0(\nu)$ of all (equivalence classes of) ν-measurable functions on a measure space (S, Σ, ν) does not admit non-trivial derivations if and only if the Boolean algebra of all ν-measurable subsets of S is atomic. Thus, if ν is atomless, then the algebra $L_0(\nu)$ admits non-trivial derivations δ which are obviously not inner and also not continuous with respect to the topology of convergence in measure. If X is a finite dimensional Banach space, then it is straightforward that any such derivation extends "coordinatewise" to a non-trivial non-inner derivation on the algebra $L_0(\nu; \mathcal{L}(X))$ of all (Bochner) ν-measurable functions $f : S \to \mathcal{L}(X)$. However, if X is an infinite dimensional Banach space, then the situation changes drastically.

Suppose now that X is an infinite dimensional separable Banach space and let $L_0^{wo}(\nu; \mathcal{L}(X))$ be the algebra of all weak operator ν-measurable functions $f : S \to \mathcal{L}(X)$ (see Section 2 for precise definitions). It follows from one of the main results of the present paper (see Theorem 4.17) that every derivation in

$L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ is necessarily inner and hence continuous with respect to the topology of convergence in measure. Actually, Theorem 4.17 shows that similar results hold for a large class of appropriate subalgebras (so-called admissible subalgebras) of $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$. It should be pointed out that, specializing to the case that the underlying set S consists of one point, we recover the classical result of P. Chernoff [7], who showed that every derivation on $\mathcal{L}(X)$ is inner (answering a question raised by S. Sakai in Section 4.1 of [18]).

Interesting applications of our approach and techniques lie in the realm of the theory of non-commutative integration on semi-finite von Neumann algebras, initiated by I.E. Segal [20]. The classical spaces of measurable operators associated with an arbitrary pair $(\mathcal{M}, \tau)$ of a semi-finite von Neumann algebra $\mathcal{M}$ and a faithful normal semi-finite trace τ are the following:

  (i) the space of all measurable operators $S(\mathcal{M})$, introduced by Segal [20];
  (ii) the space of all locally measurable operators $LS(\mathcal{M})$, introduced by S. Sankaran ([19]; see also F.J. Yeadon's paper [24]);
  (iii) the space $S(\tau) = S(\mathcal{M}, \tau)$ of all τ-measurable operators, first considered by E. Nelson [14].

It should be noted that always $S(\tau) \subseteq S(\mathcal{M}) \subseteq LS(\mathcal{M})$ (some of the relevant definitions may be found at the beginning of Section 5). For some of the simplest examples of von Neumann algebras of type I, we have

$$(0.1) \qquad\qquad S(\tau) = S(\mathcal{M}) = LS(\mathcal{M}).$$

For example, the equalities (0.1) hold in the special case $(\mathcal{M}, \tau) = (L_\infty(\nu), \nu)$, where the trace ν is given by integration with respect to a finite measure ν. Similarly, if H is a Hilbert space and $\mathcal{M} = B(H)$ is the von Neumann algebra of all bounded linear operators in H (so, $B(H) = \mathcal{L}(X)$ as Banach algebras, with $X = H$), equipped with the standard trace $\tau = \mathrm{tr}$, then the equalities (0.1) hold (and these algebras coincide with the von Neumann algebra $B(H)$ itself). However, these equalities do not hold any longer for the von Neumann algebra $\mathcal{M} = L_\infty(\nu) \overline{\otimes} B(H)$, equipped with the tensor product trace $\tau = \nu \otimes \mathrm{tr}$, whenever ν is non-atomic and H is infinite dimensional (see Remark 5.20). As will be seen in Section 5, in this latter case, our general results for derivations on subalgebras of $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ have implications for each of the algebras listed above.

Firstly, it will be shown that the algebra $L_0^{\mathrm{wo}}(\nu; B(H))$ may be identified with the algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with $\mathcal{M} = L_\infty(\nu) \overline{\otimes} B(H)$ (see Proposition 5.3) and therefore, every derivation on $LS(\mathcal{M})$ is inner. This result has already been obtained in [1], [2], through methods based on the theory of Kaplansky–Hilbert modules. Our approach is completely different and we believe that the methods based on the realization of the algebra $LS(\mathcal{M})$ as a space of operator-valued functions provide additional insight into the matter. For example, the following observation, which seems to have escaped the attention, exhibits itself: the topology of convergence locally in measure in $LS(\mathcal{M})$, as

introduced by Yeadon in [24], coincides with the natural topology of convergence in measure on sets of finite measure in $L_0^{wo}(\nu; B(H))$ (see Remark 5.4). In the setting when $\mathcal{M}$ is an arbitrary von Neumann algebra of type I, the description of all derivations in algebras $S(\mathcal{M})$, $S(\mathcal{M}, \tau)$, $LS(\mathcal{M})$ may be found in [1], [2], [3].

The effectiveness of our approach is illustrated further by showing that the results of Section 4 may be applied to show that derivations on the algebras $S(\mathcal{M})$ and $S(\tau)$ are also inner (see Corollary 5.18). These applications are not straightforward and require some additional techniques (see Section 5, in particular, Proposition 5.17 and its proof), which may be of independent potential interest. Furthermore, the notion of admissible subalgebra of $L_0^{wo}(\nu; \mathcal{L}(X))$ (see Section 4) is flexible enough to include also the ideal $S_0(\tau)$ in $S(\tau)$ of all τ-compact operators (see Proposition 5.7). Consequently, also derivations on $S_0(\tau)$ are discussed (see Corollary 5.8).

In Section 1 some basic notation and terminology is introduced and in Section 2 some necessary results concerning vector-valued and operator valued functions are established. Section 3 exhibits some elementary algebraic properties of derivations, which will be used in the subsequent sections. In Section 4, which contains one of the main results of the paper (Theorem 4.17), we study in detail derivations on admissible subalgebras of the algebra $L_0^{wo}(\nu; \mathcal{L}(X))$. Finally, in the last section, the results of Section 4 will be applied to derivations on algebras of measurable operators affiliated with a von Neumann algebra $\mathcal{M} = L_\infty(\nu) \overline{\otimes} B(H)$.

1. NOTATION AND TERMINOLOGY

If X is a (complex) Banach space, then X^* denotes its Banach dual space. The value of a functional $x^* \in X^*$ at $x \in X$ is denoted by $\langle x, x^* \rangle$. The Banach algebra of all bounded linear operators in X is denoted by $\mathcal{L}(X)$ and its multiplicative unit is the identity operator $\mathbf{1} = \mathbf{1}_X$. The subalgebra of $\mathcal{L}(X)$ consisting of all finite rank operators is denoted by $\mathcal{F}(X)$. Furthermore, $\mathcal{K}(X)$ is the Banach algebra of all compact operators. Evidently, $\mathcal{F}(X) \subseteq \mathcal{K}(X) \subseteq \mathcal{L}(X)$. If $p \in \mathcal{L}(X)$ is a projection (that is, $p^2 = p$), then its range $\mathrm{Ran}(p)$ will also be denoted by X_p. Hence, $X = X_p \oplus X_{p^\perp}$, where $p^\perp = \mathbf{1} - p$. The algebra $\mathcal{L}(X_p)$ may be identified with a subalgebra of $\mathcal{L}(X)$ as follows. Given $T \in \mathcal{L}(X_p)$, define $\widetilde{T} \in \mathcal{L}(X)$ by setting $\widetilde{T}x = Tx$, $x \in X_p$, and $\widetilde{T}x = 0$, $x \in X_{p^\perp}$. Note that

$$\|T\|_{\mathcal{L}(X_p)} \leqslant \|\widetilde{T}\|_{\mathcal{L}(X)} \leqslant \|p\|_{\mathcal{L}(X)}\|T\|_{\mathcal{L}(X_p)}.$$

The map $T \longmapsto \widetilde{T}$ is an algebraic isomorphism from $\mathcal{L}(X_p)$ onto the subalgebra

$$p\mathcal{L}(X)p = \{pSp : S \in \mathcal{L}(X)\} = \{S \in \mathcal{L}(X) : S = pSp\}$$

of $\mathcal{L}(X)$. It should be observed that the identification of the algebra $\mathcal{L}(X_p)$ with a subalgebra of $\mathcal{L}(X)$ depends on the choice of the specific projection p onto X_p.

2. SPACES OF VECTOR-VALUED FUNCTIONS

Next, we recall some facts concerning vector-valued and operator-valued functions, and introduce some relevant notation. For the general theory of vector-valued measurable functions, the reader is referred to [9]. Given a non-empty set S and a (complex) vector space V, if $f : S \to \mathbb{C}$ and $v \in V$, then the function $f \otimes v : S \to V$ is defined by setting $(f \otimes v)(s) = f(s)v$, $s \in S$. If, in addition, Σ is a σ-algebra of subsets of S, then $\mathrm{sim}(\Sigma; V)$ denotes the vector space of all Σ-simple V-valued functions, that is, $f \in \mathrm{sim}(\Sigma; V)$ if and only if f is of the form $f = \sum\limits_{j=1}^{n} \chi_{A_j} \otimes v_j$, with $A_j \in \Sigma$ and $v_j \in V$ $(j = 1, \ldots, n; n \in \mathbb{N})$. Note that, without loss of generality, it may be assumed that $A_i \cap A_j = \varnothing$ whenever $i \neq j$.

Let Y be a (complex) Banach space and (S, Σ, ν) be a complete σ-finite measure space (so, Σ equals the σ-algebra of all ν-measurable sets). Assume that Γ is a linear subspace of Y^* which contains a countable norming subset for Y (that is, there exists a sequence $\{y_n^*\}_{n=1}^{\infty}$ in Γ such that $\|y\|_Y = \sup\limits_{n} |\langle y, y_n^* \rangle|$ for all $y \in Y$; note that this implies that $\|y_n^*\|_{Y^*} \leqslant 1$ for all n). A function $f : S \to Y$ is called ν-measurable with respect to Γ if the scalar functions $s \longmapsto \langle f(s), y^* \rangle$, $s \in S$, are ν-measurable for all $y^* \in \Gamma$ (see e.g. Section II.1 of [9]). The linear space of all such Y-valued measurable functions is denoted by $\mathcal{L}_0^{\Gamma}(\nu; Y)$ and $\mathcal{L}_{\infty}^{\Gamma}(\nu; Y)$ is the subspace of $\mathcal{L}_0^{\Gamma}(\nu; Y)$ consisting of all ν-essentially bounded functions. The following simple observation will be used.

LEMMA 2.1. *If $f \in \mathcal{L}_0^{\Gamma}(\nu; Y)$, then the function $s \longmapsto \|f(s)\|_Y$, $s \in S$, is ν-measurable.*

Proof. If $\{y_n^*\}_{n=1}^{\infty} \subseteq \Gamma$ is a norming subset, then

$$\|f(s)\|_Y = \sup_{n} |\langle f(s), y_n^* \rangle|, \quad s \in S.$$

Since the functions $s \longmapsto |\langle f(s), y_n^* \rangle|$, $s \in S$, are ν-measurable for all n, the statement of the lemma is now clear. ∎

A function $f \in \mathcal{L}_0^{\Gamma}(\nu; Y)$ is called a *ν-null function* if $f(s) = 0$ ν-a.e. on S. Using the assumption on Γ, the following lemma is easily proved.

LEMMA 2.2. *If $f \in \mathcal{L}_0^{\Gamma}(\nu; Y)$, then the following two statements are equivalent:*
(i) *f is a ν-null function;*
(ii) *for every $y^* \in \Gamma$ the scalar function $s \longmapsto \langle f(s), y^* \rangle$, $s \in S$, is a ν-null function.*

The space of all ν-null functions is denoted by $\mathcal{N}(\nu; Y)$. Note that $\mathcal{N}(\nu; Y)$ is a linear subspace of both $\mathcal{L}_0^{\Gamma}(\nu; Y)$ and $\mathcal{L}_{\infty}^{\Gamma}(\nu; Y)$. The spaces $L_0^{\Gamma}(\nu; Y)$ and $L_{\infty}^{\Gamma}(\nu; Y)$ are defined by setting

$$L_0^{\Gamma}(\nu; Y) = \mathcal{L}_0^{\Gamma}(\nu; Y) \big/ \mathcal{N}(\nu; Y), \quad L_{\infty}^{\Gamma}(\nu; Y) = \mathcal{L}_{\infty}^{\Gamma}(\nu; Y) \big/ \mathcal{N}(\nu; Y),$$

respectively. Equipped with the norm $\| \cdot \|_\infty$, given by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{s \in S} \|f(s)\|_Y, \quad f \in L_\infty^\Gamma(\nu; Y),$$

it is easily verified that $L_\infty^\Gamma(\nu; Y)$ is a Banach space. Note that the algebraic tensor products $L_0(\nu) \otimes Y$ and $L_\infty(\nu) \otimes Y$ may be identified with linear subspaces of $L_0^\Gamma(\nu; Y)$ and $L_\infty^\Gamma(\nu; Y)$, respectively.

REMARK 2.3. In the present paper we shall be dealing with the following two special cases of this general setting. Suppose that X is a separable (complex) Banach space. Let $\{x_n\}_{n=1}^\infty$ be dense in the closed unit ball of X and suppose that $x_n^* \in X^*$ is such that $\|x_n^*\|_{X^*} = 1$ and $\langle x_n, x_n^* \rangle = \|x_n\|_X$, for each $n \in \mathbb{N}$.

(i) The sequence $\{x_n^*\}_{n=1}^\infty$ is norming for X. Hence, $\Gamma = X^*$ satisfies the requirements for $Y = X$. It follows from the Pettis measurability criterion (see e.g. Theorem II.1.2 of [9]) that the ν-measurable functions with respect to Γ are precisely the *Bochner ν-measurable functions*. In this case, the spaces $L_0^{X^*}(\nu; X)$ and $L_\infty^{X^*}(\nu; X)$ will be denoted simply by $L_0(\nu; X)$ and $L_\infty(\nu; X)$, respectively.

(ii) Let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators in X. For $x \in X$ and $x^* \in X^*$, define the linear functional $x \otimes x^* \in \mathcal{L}(X)^*$ by setting

$$\langle T, x \otimes x^* \rangle = \langle Tx, x^* \rangle, \quad T \in \mathcal{L}(X),$$

and let $\Gamma = X \otimes X^*$, the linear span of $\{x \otimes x^* : x \in X, x^* \in X^*\}$ in $\mathcal{L}(X)^*$. It is clear that $\{x_n \otimes x_m^* : n, m \in \mathbb{N}\}$ is a countable norming set for $Y = \mathcal{L}(X)$. The $\mathcal{L}(X)$-valued ν-measurable functions with respect to $X \otimes X^*$ will be called *weak operator ν-measurable functions* and the space $\mathcal{L}_0^{X \otimes X^*}(\nu; \mathcal{L}(X))$ is denoted by $\mathcal{L}_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$. The spaces $L_0^{X \otimes X^*}(\nu; \mathcal{L}(X))$ and $L_\infty^{X \otimes X^*}(\nu; \mathcal{L}(X))$ are denoted by $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ and $L_\infty^{\mathrm{wo}}(\nu; \mathcal{L}(X))$, respectively. Similar notation will be used if $\mathcal{L}(X)$ is replaced by any closed subspace $\mathfrak{B}$ of $\mathcal{L}(X)$. It should be observed that, by the Pettis measurability criterion, a function $f : S \to \mathcal{L}(X)$ is weak operator ν-measurable if and only if for each $x \in X$ the X-valued function $s \longmapsto f(s)x$, $s \in S$, is Bochner ν-measurable.

Returning to the general setting, the space $L_0^\Gamma(\nu; Y)$ is equipped with the vector space topology $\mathcal{T}_0$ of *convergence in measure on sets of finite measure*. A neighbourhood base at 0 for $\mathcal{T}_0$ is given by the sets

$$V(A; \varepsilon, \delta) = \{f \in L_0^\Gamma(\nu; Y) : \nu\{s \in A : \|f(s)\|_Y > \varepsilon\} < \delta\},$$

where $\varepsilon, \delta > 0$ and A ranges over all ν-measurable subsets of S satisfying $\nu(A) < \infty$. The topology $\mathcal{T}_0$ is metrizable, induced by the translation invariant metric d_0 given by

$$d_0(f, g) = \sum_{n=1}^\infty \frac{1}{2^n \mu(A_n)} \int_{A_n} \frac{\|f(s) - g(s)\|_Y}{1 + \|f(s) - g(s)\|_Y} d\nu(s), \quad f, g \in L_0^{\mathrm{wo}}(\nu; Y),$$

where $\{A_n\}_{n=1}^{\infty}$ is a partition of S into ν-measurable sets satisfying $0 < \nu(A_n) < \infty$ for all n (note: the metric d_0 depends on the choice of the partition $\{A_n\}_{n=1}^{\infty}$). A sequence $\{f_n\}_{n=1}^{\infty}$ in $L_0^{\Gamma}(\nu; Y)$ converges to $f \in L_0^{\Gamma}(\nu; Y)$ with respect to $\mathcal{T}_0$ if and only if every subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ has a subsequence $\{f_{n_{k_j}}\}_{j=1}^{\infty}$ such that $f_{n_{k_j}}(s) \to f(s)$ with respect to the norm in Y as $j \to \infty$, ν-a.e. on S. This is also equivalent to

$$\lim_{n\to\infty} \nu\{s \in A : \|f_n(s) - f(s)\|_Y \geqslant \varepsilon\} = 0$$

for every $\varepsilon > 0$ and every ν-measurable set $A \subseteq S$ with $\nu(A) < \infty$. The reader is advised to keep in mind that, in general, $L_0^{\Gamma}(\nu; Y)$ is not a Banach algebra. The following observation will be important.

LEMMA 2.4. *The space $L_0^{\Gamma}(\nu; Y)$ is complete with respect to the topology $\mathcal{T}_0$.*

Proof. For the sake of convenience, we present the proof for the case that $\nu(S) < \infty$. The extension to the general σ-finite case is easy. If $\nu(S) < \infty$, then it may be assumed that the metric d_0 is given by

$$d_0(f, g) = \int_S \frac{\|f(s) - g(s)\|_Y}{1 + \|f(s) - g(s)\|_Y} \, d\nu(s), \quad f, g \in L_0^{\mathrm{wo}}(\nu; Y).$$

Suppose that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L_0^{\Gamma}(\nu; Y)$. By passing to a subsequence, it may be assumed, without loss of generality, that $d_0(f_{n+1}, f_n) \leqslant 2^{-n}$ for all $n \in \mathbb{N}$. This implies that

$$\int_S \sum_{n=1}^{\infty} \frac{\|f_{n+1}(s) - f_n(s)\|_Y}{1 + \|f_{n+1}(s) - f_n(s)\|_Y} \, d\nu(s) < \infty$$

and so there exists a ν-measurable set $E \subseteq S$ such that $\nu(S \setminus E) = 0$ and

$$\sum_{n=1}^{\infty} \frac{\|f_{n+1}(s) - f_n(s)\|_Y}{1 + \|f_{n+1}(s) - f_n(s)\|_Y} < \infty, \quad s \in E.$$

It is easy to see that this implies that

$$\sum_{n=1}^{\infty} \|f_{n+1}(s) - f_n(s)\|_Y < \infty, \quad s \in E.$$

Since Y is a Banach space, the series $\sum_{n=1}^{\infty} (f_{n+1}(s) - f_n(s))$ is norm convergent in Y for all $s \in E$. Defining

$$f(s) = f_1(s) + \sum_{n=1}^{\infty} (f_{n+1}(s) - f_n(s)), \quad s \in E$$

and $f(s) = 0, s \in S \setminus E$, it is clear that $f_n(s) \to f(s)$ ν-a.e. on S and $f \in L_0^{\mathrm{wo}}(\nu; Y)$ (as f is the ν-a.e. pointwise limit of a sequence of functions in $L_0^{\mathrm{wo}}(\nu; Y)$). It will

be shown next that $d_0(f_n, f) \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\int_S \frac{\|f_n(s) - f_m(s)\|_Y}{1 + \|f_n(s) - f_m(s)\|_Y} d\nu(s) \leqslant \varepsilon$$

for all $n, m \geqslant N$. Letting $m \to \infty$, it follows from the dominated convergence theorem that

$$\int_S \frac{\|f_n(s) - f(s)\|_Y}{1 + \|f_n(s) - f(s)\|_Y} d\nu(s) \leqslant \varepsilon,$$

that is, $d_0(f_n, f) \leqslant \varepsilon$ for all $n \geqslant N$. This suffices to prove the lemma. ∎

If $a \in L_0(\nu)$ and $f \in L_0^\Gamma(\nu; Y)$, then the function $af \in L_0^\Gamma(\nu; Y)$ is defined by setting $(af)(s) = a(s)f(s)$, ν-a.e. on S. With respect to this action, $L_0^\Gamma(\nu; Y)$ is an $L_0(\nu)$-module. Similarly, $L_\infty^\Gamma(\nu; Y)$ has the structure of an $L_\infty(\nu)$-module.

DEFINITION 2.5. A linear operator $T : L_0^\Gamma(\nu; Y) \to L_0^\Gamma(\nu; Y)$ is called $L_0(\nu)$-linear if $T(af) = aT(f)$ for all $f \in L_0^\Gamma(\nu; Y)$ and all $a \in L_0(\nu)$. The unital algebra of all $L_0(\nu)$-linear operators in $L_0^\Gamma(\nu; Y)$ which are continuous with respect to $\mathcal{T}_0$ is denoted by $\mathcal{L}_{L_0}(L_0^\Gamma(\nu; Y))$. Similarly, the unital algebra of all $L_\infty(\nu)$-linear operators in $L_\infty^\Gamma(\nu; Y)$ which are continuous (with respect to the norm in $L_\infty^\Gamma(\nu; Y)$), is denoted by $\mathcal{L}_{L_\infty}(L_\infty^\Gamma(\nu; Y))$.

REMARK 2.6. It may be of some interest to observe that $\mathcal{L}_{L_\infty}(L_\infty^\Gamma(\nu; Y))$ coincides with the algebra of all $L_\infty(\nu)$-linear operators in $L_\infty^\Gamma(\nu; Y)$ which are continuous with respect to the topology $\mathcal{T}_0$. Indeed, if T is an $L_\infty(\nu)$-linear operator which is $\mathcal{T}_0$-continuous, then it is an immediate consequence of the closed graph theorem that T is continuous with respect to $\| \cdot \|_\infty$.

For the proof of the converse implication, suppose that T is an $L_\infty(\nu)$-linear operator in $L_\infty^\Gamma(\nu; Y)$ which is bounded with respect to $\| \cdot \|_\infty$. We claim that

(2.1) $\qquad \|(Tf)(s)\|_Y \leqslant \|T\|_{\mathcal{L}(L_\infty^\Gamma(\nu; Y))} \|f(s)\|_Y, \quad \nu$-a.e. on S,

for all $f \in L_\infty^\Gamma(\nu; Y)$. Indeed, suppose that $f \in L_\infty^\Gamma(\nu; Y)$ is such that (2.1) does not hold. Then there exists $\alpha > \|T\|_{\mathcal{L}(L_\infty^\Gamma(\nu; Y))}$ such that the set

$$A = \{s \in S : \|(Tf)(s)\|_Y > \alpha \|f(s)\|_Y\}$$

satisfies $\nu(A) > 0$. Since $T(\chi_A f) = \chi_A Tf$, it is clear that $\chi_A f \neq 0$ and

$$\|(T(\chi_A f))(s)\|_Y \geqslant \alpha \|(\chi_A f)(s)\|_Y, \quad s \in S.$$

Consequently,

$$\alpha \|(\chi_A f)(s)\|_Y \leqslant \|T(\chi_A f)\|_\infty \leqslant \|T\|_{\mathcal{L}(L_\infty^\Gamma(\nu; Y))} \|\chi_A f\|_\infty, \quad \nu\text{-a.e. on } S,$$

and so

$$\alpha \|\chi_A f\|_\infty \leqslant \|T\|_{\mathcal{L}(L_\infty^\Gamma(\nu; Y))} \|\chi_A f\|_\infty.$$

This implies that $\alpha \leqslant \|T\|_{\mathcal{L}(L_\infty^\Gamma(\nu; Y))}$, which is a contradiction. Therefore, (2.1) holds, from which it is clear that T is continuous with respect to $\mathcal{T}_0$.

It should be observed that it follows from (2.1) that any $T \in \mathcal{L}_{L_\infty}(L_\infty^\Gamma(\nu; Y))$ has the property that $Tf_n \to Tf$ ν-a.e. on S, whenever $f \in L_\infty^\Gamma(\nu; Y)$ and $\{f_n\}_{n=1}^\infty$ is a sequence in $L_\infty^\Gamma(\nu; Y)$ satisfying $f_n \to f$ ν-a.e. on S.

For the proof of the next proposition, the following simple and well known observation is needed (cf. Lemma 96.6 of [25]). Recall that a set $D \subseteq L_0(\nu)^+$ is called order bounded if there exists $a \in L_0(\nu)^+$ such that $f \leqslant a$ for all $f \in D$ (here $L_0(\nu)^+$ denotes the positive cone of $L_0(\nu)$).

LEMMA 2.7. *Suppose that $D \subseteq L_0(\nu)^+$ is upwards directed. If D is not order bounded in $L_0(\nu)^+$, then there exists a sequence $\{f_n\}_{n=1}^\infty$ in D and a ν-measurable set $E \subseteq S$ such that $\nu(E) > 0$ and $f_n \geqslant n\chi_E$ for all $n \in \mathbb{N}$.*

Proof. Since $L_0(\nu)$ is order separable (see e.g. Example 23.3(iv) of [13]), there exists an increasing sequence $\{g_n\}_{n=1}^\infty$ in D which has the same upper bounds as the set D. Hence, $\{g_n\}_{n=1}^\infty$ is not order bounded in $L_0(\nu)$. Define the function $h : S \to [0, \infty]$ by setting $h(s) = \sup_{n \in \mathbb{N}} g_n(s)$, $s \in S$. Since $\{g_n\}_{n=1}^\infty$ is not order bounded in $L_0(\nu)$, there exists a ν-measurable set H such that

$$H \subseteq \{s \in S : h(s) = \infty\}$$

and $0 < \nu(H) < \infty$. For $k, n \in \mathbb{N}$ define

$$G_{n,k} = \{s \in H : g_n(s) \leqslant k\}.$$

Since $G_{n,k} \downarrow_n \varnothing$ for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\nu(G_{n_k,k}) \leqslant 2^{-(k+1)}\nu(H)$. Defining

$$G = \bigcup_{k=1}^\infty G_{n_k,k},$$

it follows that $\nu(G) \leqslant (1/2)\nu(H)$ and so $\nu(H \setminus G) \geqslant (1/2)\nu(H) > 0$. Furthermore,

$$g_{n_k} \geqslant k\chi_{H \setminus G_{n_k,k}} \geqslant k\chi_{H \setminus G}, \quad k \in \mathbb{N}.$$

Hence, the set $E = H \setminus G$ and the functions $f_k = g_{n_k}$, $k = 1, 2, \ldots$, have the desired properties. ∎

PROPOSITION 2.8. *An $L_0(\nu)$-linear operator $T : L_0^\Gamma(\nu; Y) \to L_0^\Gamma(\nu; Y)$ is continuous with respect to $\mathcal{T}_0$ if and only if there exists a function $0 \leqslant a \in L_0(\nu)$ such that, for all $f \in L_0^\Gamma(\nu; Y)$,*

(2.2) $\|(Tf)(s)\|_Y \leqslant a(s)\|f(s)\|_Y, \quad \nu\text{-a.e. on } S,$

Proof. If there exists a function $0 \leqslant a \in L_0(\nu)$ satisfying (2.2), then it is clear that T is continuous with respect to the topology $\mathcal{T}_0$ (see the discussion preceding Lemma 2.4).

Assuming that T is $\mathcal{T}_0$-continuous, define the subset $D \subseteq L_0(\nu)^+$ by setting

$$D = \{\|(Tf)(\cdot)\|_Y : f \in L_\infty^\Gamma(\nu; Y), \|f\|_\infty \leqslant 1\}.$$

It should be observed that D is upwards directed. Indeed, suppose that $f, g \in L_\infty^\Gamma(\nu; Y)$ with $\|f\|_\infty, \|g\|_\infty \leqslant 1$ and let

$$A = \{s \in S : \|(Tf)(s)\|_Y \geqslant \|(Tg)(s)\|_Y\}.$$

Defining $h \in L_\infty^\Gamma(\nu; Y)$ by setting $h = \chi_A f + \chi_{S \setminus A} g$, it is clear that $\|h\|_\infty \leqslant 1$. Since T is L_0-linear, it follows that $Th = \chi_A Tf + \chi_{S \setminus A} Tg$ and so

$$\|(Th)(s)\|_Y = \chi_A(s)\|(Tf)(s)\|_Y + \chi_{S \setminus A}(s)\|(Tg)(s)\|_Y, \quad s \in S.$$

This implies that $\|Th\|_Y \geqslant \|Tf\|_Y \vee \|Tg\|_Y$ in $L_0(\nu)$. Hence, D is upwards directed.

Suppose that D is not order bounded in $L_0(\nu)^+$. It follows from Lemma 2.7 that there exists a sequence $\{f_n\}_{n=1}^\infty$ in $L_\infty^\Gamma(\nu; Y)$, with $\|f_n\|_\infty \leqslant 1$ for all n, and a ν-measurable set $E \subseteq S$, with $0 < \nu(E) < \infty$, such that

$$(2.3) \qquad \qquad \|Tf_n\|_Y \geqslant n\chi_E, \quad n \in \mathbb{N}.$$

Defining $g_n = (1/n)f_n$, $n \in \mathbb{N}$, it is clear that $\|g_n\|_\infty \to 0$ and hence, $g_n \to 0$ with respect to $\mathcal{T}_0$. Since T is assumed to be $\mathcal{T}_0$-continuous, this implies that $Tg_n \to 0$ with respect to $\mathcal{T}_0$. On the other hand, it follows from (2.3) that $\|Tg_n\|_Y \geqslant \chi_E$ for all $n \in \mathbb{N}$, which clearly is a contradiction.

Consequently, there exists a function $0 \leqslant a \in L_0(\nu)$ such that

$$\|(Tf)(s)\|_Y \leqslant a(s), \quad \nu\text{-a.e. on } S,$$

for all $f \in L_\infty^\Gamma(\nu; Y)$ satisfying $\|f\|_\infty \leqslant 1$. If $f \in L_0^\Gamma(\nu; Y)$ is arbitrary, then f may be written as $f = b\widetilde{f}$, where $\widetilde{f} \in L_\infty^\Gamma(\nu; Y)$ satisfies $\|\widetilde{f}(s)\|_Y \leqslant 1$, $s \in S$, and $b \in L_0(\nu)$ satisfies $0 \leqslant b(s) \leqslant \|f(s)\|_Y$, $s \in S$ (indeed, define $\widetilde{f}(s) = f(s)/\|f(s)\|_Y$ and $b(s) = \|f(s)\|_Y$ whenever $f(s) \neq 0$ and let $\widetilde{f}(s) = 0$ and $b(s) = 0$ whenever $f(s) = 0$). Since T is L_0-linear, it follows that

$$\|(Tf)(s)\|_Y = b(s)\|(T\widetilde{f})(s)\|_Y \leqslant a(s)\|f(s)\|_Y, \quad s \in S.$$

The proof is complete. ∎

In the sequel, we will also use the following simple observation.

PROPOSITION 2.9. *Suppose that $\mathfrak{A}$ is a linear subspace of $L_0^\Gamma(\nu; Y)$, which is also a module over $L_\infty(\nu)$ (that is, $af \in \mathfrak{A}$ for all $a \in L_\infty(\nu)$ and $f \in \mathfrak{A}$). Furthermore, assume that for every $f \in L_0^\Gamma(\nu; Y)$ there exists a ν-measurable partition $\{A_n\}_{n=1}^\infty$ of S such that $\chi_{A_n} f \in \mathfrak{A}$ for all $n \in \mathbb{N}$. If $T_0 : \mathfrak{A} \to L_0^\Gamma(\nu; Y)$ is an $L_\infty(\nu)$-linear operator, then T_0 has a unique extension to an L_0-linear operator $T : L_0^\Gamma(\nu; Y) \to L_0^\Gamma(\nu; Y)$. The operator T is uniquely determined by the property that $\chi_A Tf = T_0(\chi_A f)$ for all $f \in L_0^\Gamma(\nu; Y)$ and all ν-measurable sets $A \subseteq S$ satisfying $\chi_A f \in \mathfrak{A}$.*

Proof. It will be shown first that for every $f \in L_0^\Gamma(\nu; Y)$ there exists a unique $Tf \in L_0^\Gamma(\nu; Y)$ such that $\chi_A Tf = T_0(\chi_A f)$ for all ν-measurable sets $A \subseteq S$ satisfying $\chi_A f \in \mathfrak{A}$. To this end, let $f \in L_0^\Gamma(\nu; Y)$ be given. By the assumption on $\mathfrak{A}$,

there exists a ν-measurable partition $\{A_n\}_{n=1}^{\infty}$ of S such that $\chi_{A_n} f \in \mathfrak{A}$ for all n. Since T_0 is L_∞-linear, it follows that

$$T_0(\chi_{A_n} f) = \chi_{A_n} T_0(\chi_{A_n} f)$$

and so it is clear that $T_0(\chi_{A_n} f)$ is supported on A_n. Therefore,

$$Tf = \sum_{n=1}^{\infty} T_0(\chi_{A_n} f)$$

is well defined as a pointwise sum and $Tf \in L_0^\Gamma(\nu; Y)$. If $A \subseteq S$ is any ν-measurable set such that $\chi_A f \in \mathfrak{A}$, then

$$\chi_A Tf = \sum_{n=1}^{\infty} \chi_A T_0(\chi_{A_n} f) = \sum_{n=1}^{\infty} \chi_{A_n} T_0(\chi_A f) = T_0(\chi_A f).$$

To prove the uniqueness of Tf, suppose that $g \in L_0^\Gamma(\nu; Y)$ is such that $\chi_A g = T_0(\chi_A f)$ for all ν-measurable sets $A \subseteq S$ satisfying $\chi_A f \in \mathfrak{A}$. This implies, in particular, that $\chi_{A_n} g = T_0(\chi_{A_n} f)$ for all n and so

$$g = \sum_{n=1}^{\infty} \chi_{A_n} g = \sum_{n=1}^{\infty} T_0(\chi_{A_n} f) = Tf.$$

It follows, in particular, that the definition of Tf does not depend on the particular choice of the partition $\{A_n\}_{n=1}^{\infty}$ with the property that $\chi_{A_n} f \in \mathfrak{A}$ for all n.

To show that T is linear, let $f, g \in L_0^\Gamma(\nu; Y)$ be given. A moment's reflection shows that there exists a ν-measurable partition $\{A_n\}_{n=1}^{\infty}$ of S with the property that $\chi_{A_n} f, \chi_{A_n} g \in \mathfrak{A}$ for all n. If $\alpha, \beta \in \mathbb{C}$, then

$$\chi_{A_n} T(\alpha f + \beta g) = T_0(\alpha \chi_{A_n} f + \beta \chi_{A_n} g) = \alpha T_0(\chi_{A_n} f) + \beta T_0(\chi_{A_n} g)$$
$$= \alpha \chi_{A_n} Tf + \beta \chi_{A_n} Tg = \chi_{A_n}(\alpha Tf + \beta Tg)$$

for all n and hence, $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$.

Finally, it will be shown that T is L_0-linear. Suppose that $g \in L_0(\nu)$ and that $f \in L_0^\Gamma(\nu; Y)$ and let $\{A_n\}_{n=1}^{\infty}$ be a ν-measurable partition of S such that $\chi_{A_n} g \in L_\infty(\nu)$ and $\chi_{A_n} f \in \mathfrak{A}$ for all n. Since $\chi_{A_n} g f \in \mathfrak{A}$, it follows that

$$\chi_{A_n} T(gf) = T_0(\chi_{A_n} g f) = \chi_{A_n} g T_0(\chi_{A_n} f) = \chi_{A_n} g Tf$$

for all n. Hence, $T(gf) = gTf$, which completes the proof of the proposition. ∎

It should be observed that, in particular, the subspace $\mathfrak{A} = L_\infty^\Gamma(\nu; Y)$ satisfies the hypothesis of the above proposition. Indeed, if $f \in L_0^\Gamma(\nu; Y)$, then the sets

$$A_n = \{s \in S : n-1 \leqslant \|f(s)\|_Y < n\}, \quad n \in \mathbb{N}$$

are ν-measurable (see Lemma 2.1) and satisfy $f\chi_{A_n} \in L_\infty^\Gamma(\nu; Y)$ for all n.

It is assumed now that X is a *separable Banach* space and the notation and terminology introduced in Remark 2.3 will be employed.

LEMMA 2.10. *If $F : S \to X$ is Bochner ν-measurable and $f : S \to \mathcal{L}(X)$ is weak operator ν-measurable, then the function $s \longmapsto f(s)F(s)$, $s \in S$, is Bochner ν-measurable.*

Proof. First suppose that $F \in \text{sim}(\Sigma; X)$, that is, there exist $A_1, \ldots, A_n \in \Sigma$ and $x_1, \ldots, x_n \in X$ such that $F = \sum_{j=1}^{n} \chi_{A_j} \otimes x_j$. This implies that

$$f(s)F(s) = \sum_{j=1}^{n} \chi_{A_j}(s) f(s) x_j, \quad s \in S.$$

Since the functions $s \longmapsto f(s)x_j$, $s \in S$, are Bochner ν-measurable (see Remark 2.3(ii)), it is clear that the function $s \longmapsto f(s)F(s)$, $s \in S$, is Bochner ν-measurable. The general result now follows from the fact that the pointwise limit of a sequence of Bochner ν-measurable functions is again Bochner ν-measurable (see Section II.1 of [9]). ∎

COROLLARY 2.11. *With respect to pointwise multiplication, the space $\mathcal{L}_0^{\text{wo}}(\nu; \mathcal{L}(X))$ is a unital algebra. Consequently, $L_0^{\text{wo}}(\nu; \mathcal{L}(X))$ is also a unital algebra.*

Proof. Let $f, g \in \mathcal{L}_0^{\text{wo}}(\nu; \mathcal{L}(X))$ be given. If $x \in X$, then the function $s \longmapsto g(s)x, s \in S$, is Bochner ν-measurable (see Remark 2.3(ii)). Hence, by Lemma 2.10, the function $s \longmapsto f(s)g(s)x$, $s \in S$, is Bochner ν-measurable. Consequently (see Remark 2.3(ii)), the function $s \longmapsto f(s)g(s)$ is weak operator ν-measurable, that is, $fg \in \mathcal{L}_0^{\text{wo}}(\nu; \mathcal{L}(X))$. Since the ν-null functions form a two-sided ideal in $\mathcal{L}_0^{\text{wo}}(\nu; \mathcal{L}(X))$, it is now evident that $L_0^{\text{wo}}(\nu; \mathcal{L}(X))$ is an algebra. ∎

Next, the algebras $L_\infty^{\text{wo}}(\nu; \mathcal{L}(X))$ and $L_0^{\text{wo}}(\nu; \mathcal{L}(X))$ will be identified with algebras of linear operators on the spaces $L_\infty(\nu; X)$ and $L_0(\nu; X)$, respectively. Suppose that $f \in L_\infty^{\text{wo}}(\nu; \mathcal{L}(X))$. For $F \in L_\infty(\nu; X)$, define $m_f F : S \to X$ by setting

$$(m_f F)(s) = f(s)F(s), \quad s \in S.$$

It follows from Lemma 2.10 that $m_f F$ is Bochner measurable. Moreover,

$$(2.4) \qquad \|(m_f F)(s)\|_X \leqslant \|f(s)\|_{\mathcal{L}(X)} \|F(s)\|_X \leqslant \|f\|_\infty \|F\|_\infty, \quad \nu\text{-a.e. on } S.$$

Consequently, $m_f : L_\infty(\nu; X) \to L_\infty(\nu; X)$ is a bounded linear map satisfying $\|m_f\|_{\mathcal{L}(L_\infty(\nu, X))} \leqslant \|f\|_\infty$. The next lemma shows that the latter inequality is actually an equality.

LEMMA 2.12. *If $f \in L_\infty^{\text{wo}}(\nu; \mathcal{L}(X))$, then $\|m_f\|_{\mathcal{L}(L_\infty(\nu, X))} = \|f\|_\infty$.*

Proof. If $x \in X$, then $m_f(\mathbf{1} \otimes x)(s) = f(s)x$ ν-a.e. on S. Hence, if $\|x\|_X \leqslant 1$, then

$$\|f(s)x\|_X \leqslant \|m_f(\mathbf{1} \otimes x)\|_\infty \leqslant \|m_f\|_{\mathcal{L}(L_\infty(\nu, X))} \|(\mathbf{1} \otimes x)\|_\infty$$
$$= \|m_f\|_{\mathcal{L}(L_\infty(\nu, X))} \|x\|_X \leqslant \|m_f\|_{\mathcal{L}(L_\infty(\nu, X))}, \quad \nu\text{-a.e. on } S,$$

where the exceptional set depends on x. Since X is separable, there exists a countable set $\{x_n\}_{n=1}^\infty$ which is dense in the unit ball of X. It follows from the above that $\|f(s)x_n\|_X \leqslant \|m_f\|_{\mathcal{L}(L_\infty(\nu,X))}$ for all n, ν-a.e. on S. Consequently,

$$\|f(s)\|_{\mathcal{L}(X)} = \sup_n \|f(s)x_n\|_X \leqslant \|m_f\|_{\mathcal{L}(L_\infty(\nu,X))}, \quad \nu\text{-a.e. on } S,$$

and so $\|f\|_\infty \leqslant \|m_f\|_{\mathcal{L}(L_\infty(\nu,X))}$. The proof is complete. ∎

If $f \in L_\infty^{\mathrm{wo}}(\nu; \mathcal{L}(X))$, then it is clear that $m_f \in \mathcal{L}_{L_\infty}(L_\infty(\nu; X))$ (see Definition 2.5) and, by Lemma 2.12, it is clear that the map $f \longmapsto m_f$ is a linear isometry from $L_\infty^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ into $\mathcal{L}_{L_\infty}(L_\infty(\nu; X))$. The next objective is to show that this isometry is surjective. The proof of this proposition is modeled after the proof of Lemma 6 in [15], where $L_\infty^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ is considered as an algebra of operators on the space $L_p(\nu; X)$, $1 \leqslant p < \infty$.

PROPOSITION 2.13. *If $T \in \mathcal{L}_{L_\infty}(L_\infty(\nu; X))$, then there exists a unique $f \in L_\infty^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ such that $T = m_f$.*

Proof. To avoid any possible confusion, in the present proof we use the pedantic notation $\dot{F}$ for the equivalence class (with respect to the measure ν) of a ν-measurable function F on S.

Let $T \in \mathcal{L}_{L_\infty}(L_\infty(\nu; X))$ be given. Since

$$\|T(\mathbf{1} \otimes x)\|_\infty \leqslant \|T\|_{\mathcal{L}(L_\infty(\nu;X))}\|x\|_X, \quad x \in X,$$

we may choose $F_x \in \mathcal{L}_\infty(\nu; X)$ for each $x \in X$, such that

$$\dot{F}_x = T(\mathbf{1} \otimes x) \quad \text{and} \quad \|F_x(s)\|_X \leqslant \|T\|_{\mathcal{L}(L_\infty(\nu;X))}\|x\|_X, \quad s \in S.$$

Since the Banach space X is assumed to be separable, there exists a countable linearly independent subset $\{x_n\}_{n=1}^\infty$ such that $\overline{\mathrm{span}}\{x_n\}_{n=1}^\infty = X$. Let W be the $\mathbb{Q} + i\mathbb{Q}$-linear span of the set $\{x_n\}_{n=1}^\infty$ and note that W is countable and dense in X. Since the vectors $\{x_n\}_{n=1}^\infty$ are linearly independent, for each $s \in S$ there exists a unique $\mathbb{Q} + i\mathbb{Q}$-linear map $f_0(s) : W \to X$ such that $f_0(s)x_n = F_{x_n}(s)$ for all $n \in \mathbb{N}$. If $w \in W$, then $w = \sum_{j=1}^n \alpha_j x_j$ with $\alpha_j \in \mathbb{Q} + i\mathbb{Q}$, $1 \leqslant j \leqslant n$. Hence,

$$(f_0(\cdot)w)^{\cdot} = \sum_{j=1}^n \alpha_j \dot{F}_{x_n} = \sum_{j=1}^n \alpha_j T(\mathbf{1} \otimes x_n) = T(\mathbf{1} \otimes w) = \dot{F}_w.$$

Since W is countable, this implies that there exists a ν-measurable set $A \subseteq S$ such that $\nu(S \setminus A) = 0$ and

(2.5) $$f_0(s)w = F_w(s), \quad s \in A, w \in W.$$

In particular,

$$\|f_0(s)w\|_X \leqslant \|T\|_{\mathcal{L}(L_\infty(\nu;X))}\|w\|_X, \quad s \in A, w \in W,$$

which implies that for each $s \in A$, the $\mathbb{Q} + i\mathbb{Q}$-linear map $f_0(s) : W \to X$ extends uniquely to a bounded ($\mathbb{C}$-) linear map $f(s) : X \to X$ satisfying $\|f(s)\|_{\mathcal{L}(X)} \leqslant$

$\|T\|_{\mathcal{L}(L_\infty(\nu;X))}$. Setting $f(s) = 0$ whenever $s \in S \setminus A$, this defines a function $f : S \to \mathcal{L}(X)$. A moment's reflection shows that the function f is weak operator ν-measurable and so $f \in \mathcal{L}^{\mathrm{wo}}_\infty(\nu; \mathcal{L}(X))$.

We claim that

$$(2.6) \qquad (f(\cdot)x)^{\cdot} = T(\mathbf{1} \otimes x), \quad x \in X.$$

Indeed, given $x \in X$, there exists a sequence $\{w_k\}_{k=1}^{\infty}$ in W such that $\|x - w_k\|_X \to 0$ as $k \to \infty$. Since $f(s) \in \mathcal{L}(X)$, this implies that $f(s)w_k \to f(s)x$ for all $s \in S$. On the other hand, $w_k \to x$ in X implies that $\mathbf{1} \otimes w_k \to \mathbf{1} \otimes x$ in $L_\infty(\nu; X)$ and so $T(\mathbf{1} \otimes w_k) \to T(\mathbf{1} \otimes x)$, that is, $\dot{F}_{w_k} \to \dot{F}_x$ in $L_\infty(\nu; X)$. Consequently, $F_{w_k}(s) \to F_x(s)$ ν-a.e. on S. It follows from (2.5) that $f(s)w_k = f_0(s)w_k = F_{w_k}(s)$ for all $s \in A$ and all k. Therefore, $F_x(s) = f(s)x$ ν-a.e. on S, that is, $(f(\cdot)x)^{\cdot} = \dot{F}_x = T(\mathbf{1} \otimes x)$, by which the claim (2.6) is proved.

Let $m_f \in \mathcal{L}_{L_\infty}(L_\infty(\nu; X))$ be the operator corresponding to $\dot{f} \in L^{\mathrm{wo}}_\infty(\nu; \mathcal{L}(X))$. It will be shown next that $T = m_f$. It follows from (2.6) that $m_f(\mathbf{1} \otimes x) = T(\mathbf{1} \otimes x)$ for all $x \in X$. By the $L_\infty(\nu)$-linearity of both operators, this implies that

$$(2.7) \qquad m_f(g \otimes x) = T(g \otimes x), \quad g \in L_\infty(\nu), \ x \in X.$$

Consequently, $m_f G = TG$ for all $G \in L_\infty(\nu) \otimes X$. Given $F \in L_\infty(\nu; X)$, it follows from the definition of Bochner measurability that there exists a sequence $\{F_n\}_{n=1}^{\infty}$ in $L_\infty(\nu) \otimes X$ such that $F_n(s) \to F(s)$ ν-a.e. on S. as $n \to \infty$. This clearly implies that $m_f F_n \to m_f F$ ν-a.e. Furthermore, as observed at the end of Remark 2.6, also $TF_n \to TF$ ν-a.e. It follows from (2.7) that, $m_f F_n = TF_n$ for all n and hence, $m_f F = TF$. This shows that $T = m_f$ and the proof is complete. ∎

In combination with Lemma 2.12, the above proposition yields immediately the following result.

COROLLARY 2.14. *The map* $f \longmapsto m_f$, $f \in L^{\mathrm{wo}}_\infty(\nu; \mathcal{L}(X))$, *is an isometric unital algebra isomorphism from* $L^{\mathrm{wo}}_\infty(\nu; \mathcal{L}(X))$ *onto* $\mathcal{L}_{L_\infty}(L_\infty(\nu; X))$.

Next, operators on the space $L_0(\nu; X)$ will be considered. If $f \in L^{\mathrm{wo}}_0(\nu; \mathcal{L}(X))$, then it follows from Lemma 2.10 that, for every $F \in L_0(\nu; X)$, the function $s \longmapsto f(s)F(s), s \in S$, defines an element of $L_0(\nu; X)$. Defining the map $m_f : L_0(\nu; X) \to L_0(\nu; X)$ by setting

$$(m_f F)(s) = f(s)F(s), \quad s \in S,$$

it is clear that m_f is an L_0-linear operator. Moreover,

$$\|(m_f F)(s)\|_X \leqslant \|f(s)\|_{\mathcal{L}(X)} \|F(s)\|_X, \quad s \in S,$$

and so it follows from Proposition 2.8 that $m_f \in \mathcal{L}_{L_0}(L_0(\nu; X))$. If $f \in L^{\mathrm{wo}}_0(\nu; \mathcal{L}(X))$ and $m_f = 0$, then $f = 0$. Indeed, $m_f = 0$ implies that $m_f(\mathbf{1} \otimes x) = 0$ for all $x \in X$, that is, $f(s)x = 0$ ν-a.e. on S for all $x \in X$. Since X is separable, it follows that f is a ν-null function (cf. Lemma 2.2). Consequently, the map $f \longmapsto m_f$ is

a unital algebra isomorphism from $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ into $\mathcal{L}_{L_0}(L_0(\nu; X))$. It will be shown next that this map is actually surjective.

PROPOSITION 2.15. *If* $T \in \mathcal{L}_{L_0}(L_0(\nu; X))$, *then there exists a unique* $f \in L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ *such that* $T = m_f$.

Proof. Let $T \in \mathcal{L}_{L_0}(L_0(\nu; X))$ be given. It follows from Proposition 2.8 that there exists $0 \leqslant a \in L_0(\nu)$ such that $\|(TF)(s)\|_X \leqslant a(s)\|F(s)\|_X$ ν-a.e. on S for all $F \in L_0(\nu; X)$. Defining

$$A_n = \{s \in S : n - 1 \leqslant a(s) < n\}, \quad n \in \mathbb{N},$$

it is clear that $\{A_n\}_{n=1}^\infty$ is a partition of S and $a(s) \leqslant n$ ν-a.e. on S. Since $T(\chi_{A_n} F) = \chi_{A_n} T(F)$ for all $F \in L_0(\nu; X)$, it follows that T leaves the subspace $L_0(A_n, \nu; X)$ invariant. Moreover,

$$\|(TF)(s)\|_X \leqslant n\|F(s)\|_X, \quad s \in A_n, F \in L_0(A_n, \nu; X).$$

This implies that T leaves $L_\infty(A_n, \nu; X)$ invariant and its restriction to $L_\infty(A_n, \nu; X)$ is bounded (and evidently, $L_\infty(A_n, \nu)$-linear). Hence, it follows from Proposition 2.13 that there exists $f_n \in L_\infty^{\mathrm{wo}}(A_n, \nu; \mathcal{L}(X))$ such that $\|f_n\|_\infty \leqslant n$ and

(2.8) $(TF)(s) = f_n(s)F(s), \quad s \in A_n, F \in L_\infty(A_n, \nu; X).$

It should be observed that this implies that

(2.9) $(TF)(s) = f_n(s)F(s), \quad s \in A_n, F \in L_0(A_n, \nu; X).$

Indeed, let $F \in L_0(A_n, \nu; X)$ be given and define

$$B_k = \{s \in A_n : \|F(s)\|_X \leqslant k\}, \quad k \in \mathbb{N}.$$

Since $\chi_{B_k} F \in L_\infty(A_n, \nu; X)$, it follows from (2.8) that

$$\chi_{B_k}(s)(TF)(s) = T(\chi_{B_k} F)(s) = \chi_{B_k}(s)f_n(s)F(s), \quad s \in A_n.$$

Since $B_k \uparrow A_n$ as $k \to \infty$, this implies that (2.9) holds.

Define $f \in L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ by setting $f(s) = f_n(s)$ whenever $s \in A_n$. If $F \in L_0(\nu; X)$, then $\chi_{A_n} F \in L_0(A_n, \nu; X)$ for all n and so it follows from (2.9) that

$$\chi_{A_n}(s)(TF)(s) = T(\chi_{A_n} F)(s) = \chi_{A_n}(s)f_n(s)F(s) = \chi_{A_n}(s)f(s)F(s), \quad s \in S,$$

for all $n \in \mathbb{N}$. Since $\bigcup_{n=1}^\infty A_n = S$, this implies that $(TF)(s) = f(s)F(s)$, ν-a.e. on S. The proof is complete. ∎

COROLLARY 2.16. *The map* $f \longmapsto m_f$, $f \in L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ *is an algebra isomorphism from* $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ *onto* $\mathcal{L}_{L_0}(L_0(\nu; X))$.

3. DERIVATIONS

In this section, some simple and well known facts concerning derivations are recalled. For convenience of the reader, we include their short proofs. For an account of derivations in Banach algebras, the reader is referred to [8]. Additional details concerning derivations in C^* and W^*-algebras may be found in [17], [18] and [6].

Let A be a complex algebra. For $x, y \in A$, the *commutator* $[x, y]$ is defined by setting

$$[x, y] = xy - yx.$$

The *center* of A is denoted by $\mathcal{Z}(A)$, that is,

$$\mathcal{Z}(A) = \{x \in A : [x, y] = 0 \; \forall y \in A\}.$$

Evidently, $\mathcal{Z}(A)$ is an abelian subalgebra of A.

A linear map $\delta : A \to A$ is called a *derivation* if

$$\delta(xy) = \delta(x)y + x\delta(y), \quad x, y \in A.$$

If $w \in A$, then the map $\delta_w : A \to A$, given by $\delta_w(x) = [w, x]$, $x \in A$, is a derivation. A derivation of this form is called *inner*. It is clear that any inner derivation on A vanishes on the center. Furthermore, if $w_1, w_2 \in A$, then $\delta_{w_1} = \delta_{w_2}$ if and only if $w_1 - w_2 \in \mathcal{Z}(A)$.

LEMMA 3.1. *Suppose that A is unital (with unit element **1**). If $\delta : A \to A$ is a derivation, then the following two statements are equivalent:*
  (i) $\delta(z) = 0$ *for all $z \in \mathcal{Z}(A)$;*
  (ii) δ *is $\mathcal{Z}$-linear, that is, $\delta(zx) = z\delta(x)$ for all $x \in A$ and $z \in \mathcal{Z}(A)$.*

Proof. If $\delta(z) = 0$ for all $z \in \mathcal{Z}(A)$, then $\delta(zx) = z\delta(x)$ for all $x \in A$ and $z \in \mathcal{Z}(A)$. Hence, (i) implies (ii). If δ is $\mathcal{Z}$-linear and $z \in \mathcal{Z}(A)$, then $\delta(z)x = \delta(zx) - z\delta(x) = 0$ for all $x \in A$. Taking $x = \mathbf{1}$, this implies that $\delta(z) = 0$. Hence, (ii) implies (i). ∎

LEMMA 3.2. *If $\delta : A \to A$ is a derivation, then $\delta(z) \in \mathcal{Z}(A)$ for all $z \in \mathcal{Z}(A)$.*

Proof. If $z \in \mathcal{Z}(A)$, then

$$x\delta(z) = \delta(xz) - \delta(x)z = \delta(zx) - z\delta(x) = \delta(z)x$$

for all $x \in A$. Hence, $\delta(z) \in \mathcal{Z}(A)$. ∎

The set of all *idempotents* in A is denoted by $\mathcal{I}(A)$, that is,

$$\mathcal{I}(A) = \{p \in A : p^2 = p\}.$$

In particular, the set of all *central idempotents* in A is denoted by $\mathcal{I}(\mathcal{Z}(A))$.

LEMMA 3.3. *Let $\delta : A \to A$ be a derivation.*
  (i) *If $p \in \mathcal{I}(A)$, then $p\delta(p)p = 0$.*
  (ii) *If $p \in \mathcal{I}(\mathcal{Z}(A))$, then $\delta(p) = 0$.*

(iii) *If $p \in \mathcal{I}(\mathcal{Z}(A))$, then $\delta(px) = p\delta(x)$ for all $x \in A$.*

(iv) *If $p \in \mathcal{I}(A)$ and $x \in A$ are such that $px = xp$, then $p\delta(x)p = p\delta(xp)p$.*

Proof. (i) If $p \in \mathcal{I}(A)$, then

$$(3.1) \qquad\qquad \delta(p) = \delta(p^2) = \delta(p)p + p\delta(p).$$

Multiplying both sides of this equation by p yields that $p\delta(p)p = 2p\delta(p)p$ and so $p\delta(p)p = 0$.

(ii) If $p \in \mathcal{I}(\mathcal{Z}(A))$, then it follows from (i) that $p\delta(p) = \delta(p)p = 0$ and so (3.1) implies that $\delta(p) = 0$.

(iii) Since $\delta(px) = \delta(p)x + p\delta(x)$, statement (iii) follows immediately from (ii).

(iv) It follows from $\delta(xp) = \delta(x)p + x\delta(p)$ that

$$p\delta(xp)p = p\delta(x)p + px\delta(p)p.$$

Since $px\delta(p)p = xp\delta(p)p$, it follows from (i) that $px\delta(p)p = 0$ and so $p\delta(xp)p = p\delta(x)p$. ∎

The following lemma may be considered as an extension of Lemma 3.3(iii).

LEMMA 3.4. *Suppose that A is an algebra which is also a module over a commutative unital algebra B, such that $\mathbf{1}_B x = x$, $x \in A$, and $b(xy) = (bx)y = x(by)$ for all $b \in B$ and $x, y \in A$. Assume furthermore that $x \in A$ and $xy = 0$ for all $y \in A$ imply that $x = 0$. If $\delta : A \to A$ is a derivation, then $\delta(px) = p\delta(x)$ for all $x \in A$ and $p \in \mathcal{I}(B)$.*

Proof. Let $p \in \mathcal{I}(B)$ and $x \in A$ be given. If $y \in A$, then $px(\mathbf{1}_B - p)y = 0$ and so,

$$(\mathbf{1}_B - p)\delta(px)y + px\delta((\mathbf{1}_B - p)y) = 0.$$

Multiplying this identity on the left by $\mathbf{1}_B - p$ shows that $(\mathbf{1}_B - p)\delta(px)y = 0$. Since this holds for all $y \in A$, it follows that $(\mathbf{1}_B - p)\delta(px) = 0$, that is $\delta(px) = p\delta(px)$. Interchanging p and $\mathbf{1}_B - p$, this also implies that $p\delta((\mathbf{1}_B - p)x) = 0$. Furthermore,

$$\delta(x) = \delta(\mathbf{1}_B x) = \delta(px) + \delta((\mathbf{1}_B - p)x).$$

Multiplying this identity by p, we have the following and the proof of the lemma is complete:

$$p\delta(x) = p\delta(px) + p\delta((\mathbf{1}_B - p)x) = \delta(px). \quad ∎$$

Also, the following general observation will be used.

LEMMA 3.5. *Let A be an algebra and $\delta : A \to A$ be a derivation. For $p \in \mathcal{I}(A)$ define the subalgebra A_p of A by setting*

$$A_p = pAp = \{pxp : x \in A\}.$$

Defining $\delta_p : A_p \to A_p$ by setting $\delta_p(x) = p\delta(x)p$, $x \in A_p$, the map δ_p is a derivation in A_p.

Proof. It should be noted that an element $x \in A$ belongs to A_p if and only if $x = px = xp$. Therefore, if $x, y \in A_p$, then we have the following and this proof is complete:

$$\delta_p(xy) = p\delta(xy)p = p\delta(x)yp + px\delta(y)p = p\delta(x)py + xp\delta(y)p = \delta_p(x)y + x\delta_p(y). \quad \blacksquare$$

The following observation will be used in the next section.

LEMMA 3.6. *Let (S, Σ, ν) be a finite measure space. If $0 \neq \delta : L_\infty(\nu) \to L_0(\nu)$ is a linear map satisfying $\delta(ab) = \delta(a)b + a\delta(b)$, $a, b \in L_\infty(\nu)$, then there exists a ν-measurable set $B \subseteq S$ with $\nu(B) > 0$ such that for every $\varepsilon > 0$ there exists $a_\varepsilon \in L_\infty(\nu)$ satisfying $|a_\varepsilon| \leqslant \varepsilon\mathbf{1}$ and $|\delta(a_\varepsilon)| \geqslant \chi_B$.*

Proof. If $A \subseteq S$ is ν-measurable, then $\delta(\chi_A) = \delta(\chi_A^2) = 2\chi_A\delta(\chi_A)$ and so $\delta(\chi_A) = 0$. Since $\delta \neq 0$, there exists $a \in L_\infty(\nu)$ such that $\delta(a) \neq 0$. Multiplying a by an appropriate scalar, it may be assumed that $|\delta(a)| \geqslant \chi_B$ for some ν-measurable set $B \subseteq S$ with $\nu(B) > 0$. Given $\varepsilon > 0$, there exist disjoint ν-measurable sets $\{A_n\}_{n=1}^N$ and scalars $\{\alpha_n\}_{n=1}^N$ such that the function $b = \sum_{n=1}^N \alpha_n \chi_{A_n}$ satisfies $|a - b| \leqslant \varepsilon\mathbf{1}$. From the observations above, it follows that $\delta(b) = 0$. Consequently, the function $a_\varepsilon = a - b$ satisfies the desired conditions. $\quad \blacksquare$

4. ADMISSIBLE SUBALGEBRAS OF $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$

Following P.R. Chernoff [7], a subalgebra $\mathfrak{U}$ of $\mathcal{L}(X)$ is termed *standard* whenever $\mathcal{F}(X) \subseteq \mathfrak{U}$. If $\mathfrak{U} \subseteq \mathcal{L}(X)$ is a standard subalgebra, then $\mathcal{I}(\mathfrak{U})$ denotes the set of all projections in $\mathfrak{U}$, that is,

$$\mathcal{I}(\mathfrak{U}) = \{p \in \mathfrak{U} : p^2 = p\}.$$

Throughout the present section, it is assumed that X is a *separable Banach space* and that (S, Σ, ν) is a σ-finite measure space.

DEFINITION 4.1. *Let $\mathfrak{U}$ be a closed standard algebra of $\mathcal{L}(X)$. A subalgebra $\mathfrak{A}$ of $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ is called $\mathfrak{U}$-admissible if:*

(i) $\mathfrak{A}$ is an $L_\infty(\nu)$-submodule of $L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$, that is, $\mathfrak{A} \subseteq L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$ and $gf \in \mathfrak{A}$ whenever $g \in L_\infty(\nu)$ and $f \in \mathfrak{A}$;

(ii) if $f \in L_\infty^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ and if for every $\varepsilon > 0$ there exists a projection $p \in \mathcal{I}(\mathcal{F}(X))$ such that $\|f(\mathbf{1} \otimes p^\perp)\|_\infty \leqslant \varepsilon$, then $\chi_A f \in \mathfrak{A}$ for all ν-measurable sets $A \subseteq S$ with $\nu(A) < \infty$;

(iii) for every $f \in L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$ there exists a ν-measurable partition $\{A_n\}_{n=1}^\infty$ of S such that $\chi_{A_n} f \in \mathfrak{A}$ for all $n \in \mathbb{N}$.

It should be noted that condition (ii) implies that $L_\infty^{\mathrm{wo}}(A, \nu; \mathcal{L}(X_p)) \subseteq \mathfrak{A}$ for all ν-measurable sets $A \subseteq S$ with $\nu(A) < \infty$ and all $p \in \mathcal{I}(\mathcal{F}(X))$ (where $\mathcal{L}(X_p)$ is identified with a closed subalgebra of $\mathcal{L}(X)$, as described in Section 1).

Furthermore, if $\mathfrak{U} \subseteq \mathcal{L}(X)$ is a closed standard algebra, then it is evident that $L_0^{\text{wo}}(\nu; \mathfrak{U})$ and $L_\infty^{\text{wo}}(\nu; \mathfrak{U})$ are $\mathfrak{U}$-admissible subalgebras of $L_0^{\text{wo}}(\nu; \mathcal{L}(X))$. The main objective in the present section is to show that, if X is *infinite dimensional*, then any derivation $\delta : \mathfrak{A} \to \mathfrak{A}$ is given by $\delta(f) = [w, f]$, $f \in \mathfrak{A}$, for some $w \in L_0^{\text{wo}}(\nu; \mathcal{L}(X))$.

REMARK 4.2. Let (S, Σ, ν) be a σ-finite measure space and H be a separable Hilbert space. The von Neumann algebra $\mathcal{M} = L_\infty(\nu) \overline{\otimes} B(H)$ is equipped with the tensor product trace τ (with respect to the integral in $L_\infty(\nu)$ and the standard trace in $B(H)$). As will be shown in Section 5, the algebra $LS(\mathcal{M})$ of all locally measurable operators may be identified with the algebra $L_0^{\text{wo}}(\nu; B(H))$. As we shall see in Proposition 5.7, the above definition of admissible algebras has been chosen such that:

(i) The algebra $S(\mathcal{M})$ of all measurable operators affiliated with $\mathcal{M}$ and the algebra $S(\tau)$ of all τ-measurable operators are both $\mathfrak{U}$-admissible with respect to $\mathfrak{U} = B(H)$.

(ii) The algebra $S_0(\tau)$ of all τ-compact operators is $\mathfrak{U}$-admissible with respect to $\mathfrak{U} = K(H)$, the algebra of all compact operators in H.

Some simple observations first.

LEMMA 4.3. *Suppose that $\mathfrak{A} \subseteq L_0^{\text{wo}}(\nu; \mathcal{L}(X))$ satisfies* (i) *and* (ii) *of Definition 4.1. If $f \in \mathfrak{A}$ is such that $fg = 0$ for all $g \in \mathfrak{A}$, then $f = 0$.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a dense subset of X. For each $n \in \mathbb{N}$, let $x_n^* \in X^*$ satisfy $\langle x_n, x_n^* \rangle = 1$. Define the rank one projection p_n in X by setting $p_n(x) = \langle x, x_n^* \rangle x_n$, $x \in X$. Let the ν-measurable set $A \subseteq S$ with $\nu(A) < \infty$ be fixed. It follows from assumption (ii) that $\chi_A \otimes p_n \in \mathfrak{A}$ and so $f(\chi_A \otimes p_n) = 0$ for all n. This implies, in particular, that

$$f(s)x_n = f(s)\chi_A(s)p_n x_n = 0 \quad \nu\text{-a.e. on } A$$

for all $n \in \mathbb{N}$. Since $\{x_n\}_{n=1}^\infty$ dense in X, this implies that $f(s) = 0$ ν-a.e. on A. Using that ν is a σ-finite measure, it follows that $f = 0$. ∎

A combination of Lemmas 3.4 and 4.3 yields the following result.

COROLLARY 4.4. *If $\mathfrak{A} \subseteq L_0^{\text{wo}}(\nu; \mathcal{L}(X))$ satisfies* (i) *and* (ii) *and if $\delta : \mathfrak{A} \to \mathfrak{A}$ is a derivation, then $\delta(\chi_A f) = \chi_A \delta(f)$ for all $f \in \mathfrak{A}$ and all ν-measurable subsets $A \subseteq S$.*

Suppose now that $\mathfrak{A} \subseteq L_0^{\text{wo}}(\nu; \mathcal{L}(X))$ satisfies (i) and (ii) of Definition 4.1 and let $\delta : \mathfrak{A} \to \mathfrak{A}$ be a derivation. It will be shown that δ is L_∞-linear, that is, $\delta(af) = a\delta(f)$ for all $a \in L_\infty(\nu)$ and $f \in \mathfrak{A}$. *It is assumed, up to Lemma 4.8, in addition, that $\nu(S) < \infty$.*

If $p \in \mathcal{I}(\mathcal{F}(X))$, then it follows from (ii) and the discussion in Section 1 that

$$L_\infty^{\text{wo}}(\nu; \mathcal{L}(X_p)) \subseteq \mathfrak{A}_p = (\mathbf{1} \otimes p)\mathfrak{A}(\mathbf{1} \otimes p) \subseteq L_0^{\text{wo}}(\nu; \mathcal{L}(X_p)).$$

It is not difficult to show that the commutant of $L_\infty^{\text{wo}}(\nu; \mathcal{L}(X_p))$ in $L_0^{\text{wo}}(\nu; \mathcal{L}(X_p))$ is equal to $L_0(\nu) \otimes \mathbb{C}p$ and so the centre of $\mathfrak{A}_p$ is given by $Z(\mathfrak{A}_p) = Z_p \otimes \mathbb{C}p$,

where $Z_p \subseteq L_0(\nu)$ is given by

$$Z_p = \{a \in L_0(\nu) : a \otimes p \in \mathfrak{A}\}.$$

Note that Z_p is a subalgebra of $L_0(\nu)$ satisfying $L_\infty(\nu) \subseteq Z_p$. It follows from Lemma 3.5 that the map $\delta_p : f \longmapsto (\mathbf{1} \otimes p)\delta(f)(\mathbf{1} \otimes p)$, $f \in \mathfrak{A}_p$, defines a derivation on $\mathfrak{A}_p$. The derivation δ_p leaves the centre $Z_p \otimes \mathbb{C}p$ invariant (see Lemma 3.2). Consequently, there exists a unique linear map $\delta_p^z : L_\infty(\nu) \to L_0(\nu)$ such that

(4.1) $$\delta_p(a \otimes p) = \delta_p^z(a) \otimes p, \quad a \in L_\infty(\nu),$$

and satisfying

$$\delta_p^z(ab) = \delta_p^z(a)b + a\delta_p^z(a), \quad a, b \in L_\infty(\nu).$$

LEMMA 4.5. *If $0 \neq p, q \in \mathcal{I}(\mathcal{F}(X))$ are such that $q = pq = qp$, then $\delta_p^z(a) = \delta_q^z(a)$ for all $a \in L_\infty(\nu)$.*

Proof. If $a \in L_\infty(\nu)$, then

$$\delta_q^z(a) \otimes q = \delta_q(a \otimes q) = (\mathbf{1} \otimes q)\delta(a \otimes q)(\mathbf{1} \otimes q) = (\mathbf{1} \otimes q)\delta((\mathbf{1} \otimes q)(a \otimes p))(\mathbf{1} \otimes q)$$
$$= (\mathbf{1} \otimes q)[\delta(\mathbf{1} \otimes q)(a \otimes p) + (\mathbf{1} \otimes q)\delta(a \otimes p)](\mathbf{1} \otimes q).$$

Using Lemma 3.3 (i), it follows that

$$(\mathbf{1} \otimes q)\delta(\mathbf{1} \otimes q)(a \otimes p)(\mathbf{1} \otimes q) = (\mathbf{1} \otimes q)\delta(\mathbf{1} \otimes q)(\mathbf{1} \otimes q)(a \otimes p) = 0$$

and so

$$\delta_q^z(a) \otimes q = (\mathbf{1} \otimes q)\delta(a \otimes p)(\mathbf{1} \otimes q) = (\mathbf{1} \otimes q)(\mathbf{1} \otimes p)\delta(a \otimes p)(\mathbf{1} \otimes p)(\mathbf{1} \otimes q)$$
$$= (\mathbf{1} \otimes q)\delta_p(a \otimes p)(\mathbf{1} \otimes q) = (\mathbf{1} \otimes q)(\delta_p^z(a) \otimes p)(\mathbf{1} \otimes q) = \delta_p^z(a) \otimes q.$$

Since $q \neq 0$, this implies that $\delta_q^z(a) = \delta_p^z(a)$. The proof of the lemma is complete. ∎

Assume now that X is infinite dimensional. Since the Banach space X is separable, there exist sequences $\{x_n\}_{n=1}^\infty$ in X and $\{x_n^*\}_{n=1}^\infty$ in X^* such that (see e.g. Proposition 1.f.3 of [12]):

    (i) $\langle x_n, x_m^* \rangle = \delta_{nm}$ for all $n, m \in \mathbb{N}$;
    (ii) $\mathrm{span}\{x_n\}_{n=1}^\infty$ is dense in X;
    (iii) $\{x_n^*\}_{n=1}^\infty$ separates the points of X.

For $n \in \mathbb{N}$, define the projection $p_n \in \mathcal{I}(\mathcal{F}(X))$ by setting

$$p_n x = \sum_{k=1}^n \langle x, x_k^* \rangle x_k, \quad x \in X.$$

It should be observed that $p_m p_n = p_n p_m = p_m$ whenever $m \leqslant n$. Indeed, if $x \in X$, then

$$p_m p_n x = \sum_{k=1}^{m} \langle p_n x, x_k^* \rangle x_k = \sum_{k=1}^{m} \left\langle \sum_{j=1}^{n} \langle x, x_j^* \rangle x_j, x_k^* \right\rangle x_k = \sum_{k=1}^{m} \sum_{j=1}^{n} \langle x, x_j^* \rangle \langle x_j, x_k^* \rangle x_k$$

$$= \sum_{k=1}^{m} \sum_{j=1}^{n} \delta_{j,k} \langle x, x_j^* \rangle x_k = \sum_{k=1}^{m} \langle x, x_k^* \rangle x_k = p_m x,$$

and similarly, $p_n p_m x = p_m x$. In particular, that $\{p_n\}_{n=1}^{\infty}$ is a commuting family of projections. Also the following observation will be used.

LEMMA 4.6. *If $f \in L_0(\nu; \mathcal{L}(X))$ is such that $(\mathbf{1} \otimes p_n) f (\mathbf{1} \otimes p_n) = 0$ for all $n \in \mathbb{N}$, then $f = 0$.*

Proof. First assume that $f \in L_0(\nu; \mathcal{L}(X))$ is such that $f(\mathbf{1} \otimes p_n) = 0$ for all $n \in \mathbb{N}$, that is,

$$\sum_{k=1}^{n} \langle x, x_k^* \rangle f(s) x_k = 0, \quad \nu\text{-a.e. on } S, \ x \in X, n \in \mathbb{N}.$$

Taking $x = x_n$, this implies that $f(s) x_n = 0$ for all $n \in \mathbb{N}$ and ν-a.e. on S. Since $\mathrm{span}\{x_n\}_{n=1}^{\infty}$ is dense in X, it follows that $f(s) = 0$ ν-a.e. on S.

Suppose next that $f \in L_0(\nu; \mathcal{L}(X))$ is such that $(\mathbf{1} \otimes p_n) f = 0$ for all $n \in \mathbb{N}$, that is,

$$\sum_{k=1}^{n} \langle f(s) x, x_k^* \rangle x_k = 0, \quad \nu\text{-a.e. on } S, \ x \in X, n \in \mathbb{N}.$$

Since $\{x_k\}_{k=1}^{\infty}$ are linearly independent, this implies that $\langle f(s) x, x_k^* \rangle = 0$ ν-a.e. on S for all $k \in \mathbb{N}$. Using that $\{x_k^*\}_{k=1}^{\infty}$ separates the points of X, it follows that $f(s) x = 0$ ν-a.e. on S for all $x \in X$ and so $f = 0$ (since X is separable).

Assume now that $(\mathbf{1} \otimes p_n) f (\mathbf{1} \otimes p_n) = 0$ for all $n \in \mathbb{N}$. Since

$$(\mathbf{1} \otimes p_m)(\mathbf{1} \otimes p_n) = \mathbf{1} \otimes p_m$$

whenever $m \leqslant n$, it follows that $(\mathbf{1} \otimes p_m) f (\mathbf{1} \otimes p_n) = 0$ for all $m, n \in \mathbb{N}$. Fixing $m \in \mathbb{N}$, it follows from the first part of the proof that $(\mathbf{1} \otimes p_m) f = 0$ for all m. Applying the second part of the present proof, this implies that $f = 0$.   ∎

Since $p_m p_n = p_m$ whenever $m \leqslant n$, it follows from Lemma 4.5 that $\delta_{p_n}^z = \delta_{p_m}^z$ for all $m, n \in \mathbb{N}$. Consequently, there exists a linear map $\delta^z : L_\infty(\nu) \to L_0(\nu)$ satisfying

$$\delta^z(ab) = \delta^z(a)b + a\delta^z(b), \quad a, b \in L_\infty(\nu), \text{ and}$$

$$(\mathbf{1} \otimes p_n)\delta(a \otimes p_n)(\mathbf{1} \otimes p_n) = \delta^z(a) \otimes p_n, \quad a \in L_\infty(\nu), n \in \mathbb{N}.$$

LEMMA 4.7. *In the situation as above, it follows that $\delta^z = 0$.*

Proof. Suppose that $\delta^z \neq 0$. Let $q_n \in \mathcal{I}(\mathcal{F}(X))$ be defined by $q_n = p_{n+1} - p_n$ for all $n \in \mathbb{N}$. Note that $q_n \neq 0$ for all n. It follows from Lemma 3.6 that there

exists a ν-measurable subset $B \subseteq S$ with $\nu(B) > 0$ such that for every $n \in \mathbb{N}$ there exists $a_n \in L_\infty(\nu)$ satisfying $|a_n| \leqslant n^{-2}\|q_n\|_{\mathcal{L}(X)}^{-1}$ and

$$(4.2) \qquad\qquad |\delta^z(a_n)| \geqslant n\|q_n\|_{\mathcal{L}(X)}\chi_B.$$

Since $|a_n(s)| \leqslant n^{-2}\|q_n\|_{\mathcal{L}(X)}^{-1}, s \in S$, it follows that the series

$$f(s) = \sum_{n=1}^{\infty} a_n(s)q_n$$

defines an element $f(s) \in \mathcal{K}(X)$. It is also clear that $f \in L_0(\nu; \mathcal{K}(X)) \subseteq L_0(\nu; \mathcal{L}(X))$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} |a_n(s)|\|q_n\|_{\mathcal{L}(X)} \leqslant \varepsilon, \quad s \in S,$$

and hence, the projection $p_N \in \mathcal{I}(\mathcal{F}(X))$ satisfies $\|f(\mathbf{1} \otimes p_N^\perp)\|_\infty \leqslant \varepsilon$. Therefore, it follows from (ii) that $f \in \mathfrak{A}$ (recall that it is assumed at present that $\nu(S) < \infty$).

Since $(\mathbf{1} \otimes q_n)f = (a_n \otimes q_n) = f(\mathbf{1} \otimes q_n)$, it follows that

$$(\mathbf{1} \otimes q_n)\delta(f)(\mathbf{1} \otimes q_n) = (\mathbf{1} \otimes q_n)\delta(f(\mathbf{1} \otimes q_n))(\mathbf{1} \otimes q_n) = (\mathbf{1} \otimes q_n)\delta(a_n \otimes q_n)(\mathbf{1} \otimes q_n)$$
$$= \delta_{q_n}(a_n \otimes q_n) = \delta_{q_n}^z(a_n) \otimes q_n$$

for all $n \in \mathbb{N}$. Observing that $q_n p_{n+1} = p_{n+1} q_n = q_n$, it follows from Lemma 4.5 that $\delta_{q_n}^z = \delta_{p_{n+1}}^z$, that is, $\delta_{q_n}^z = \delta^z$ on $L_\infty(\nu)$. Consequently,

$$(\mathbf{1} \otimes q_n)\delta(f)(\mathbf{1} \otimes q_n) = \delta^z(a_n) \otimes q_n, \quad n \in \mathbb{N}.$$

If $s \in B$, then

$$|\delta^z(a_n)(s)|\|q_n\|_{\mathcal{L}(X)} = \|q_n\delta(f)(s)q_n\|_{\mathcal{L}(X)} \leqslant \|q_n\|_{\mathcal{L}(X)}^2\|\delta(f)(s)\|_{\mathcal{L}(X)}$$

and so (4.2) implies that

$$n\|q_n\|_{\mathcal{L}(X)}^2 \leqslant \|q_n\|_{\mathcal{L}(X)}^2\|\delta(f)(s)\|_{\mathcal{L}(X)}, \quad n \in \mathbb{N}.$$

Since $q_n \neq 0$, this contradicts the fact that $\|\delta(f)(s)\|_{\mathcal{L}(X)} < \infty$ ν-a.e. on S. Therefore, it may be concluded that $\delta^z = 0$. ∎

Note that the result of Lemma 4.7 may be formulated as

$$(4.3) \qquad (\mathbf{1} \otimes p_n)\delta(a \otimes p_n)(\mathbf{1} \otimes p_n) = 0, \quad a \in L_\infty(\nu), n \in \mathbb{N}.$$

LEMMA 4.8. *Same situation as above. The derivation δ is $L_\infty(\nu)$-linear on $\mathfrak{A}$, that is, $\delta(af) = a\delta(f)$ for all $f \in \mathfrak{A}$ and $a \in L_\infty(\nu)$.*

Proof. It should be observed first that $\delta(a \otimes p_n) = a\delta(\mathbf{1} \otimes p_n)$ for all $a \in L_\infty(\nu)$ and $n \in \mathbb{N}$. Indeed, if $m \geqslant n$, then

$$\delta(a \otimes p_n) = \delta((\mathbf{1} \otimes p_n)(a \otimes p_m)) = \delta(\mathbf{1} \otimes p_n)(a \otimes p_m) + (\mathbf{1} \otimes p_n)\delta(a \otimes p_m)$$

and so

$$\delta(a \otimes p_n)(\mathbf{1} \otimes p_m) = a\delta(\mathbf{1} \otimes p_n)(\mathbf{1} \otimes p_m) + (\mathbf{1} \otimes p_n)\delta(a \otimes p_m)(\mathbf{1} \otimes p_m).$$

It follows from (4.3) that

$$(\mathbf{1} \otimes p_n)\delta(a \otimes p_m)(\mathbf{1} \otimes p_m) = (\mathbf{1} \otimes p_n)(\mathbf{1} \otimes p_m)\delta(a \otimes p_m)(\mathbf{1} \otimes p_m) = 0$$

and hence,

$$\delta(a \otimes p_n)(\mathbf{1} \otimes p_m) = a\delta(\mathbf{1} \otimes p_n)(\mathbf{1} \otimes p_m)$$

for all $m \geqslant n$. Now, Lemma 4.6 implies that $\delta(a \otimes p_n) = a\delta(\mathbf{1} \otimes p_n)$.

Suppose next that $f \in \mathfrak{A}$ and $a \in L_\infty(\nu)$. If $n \in \mathbb{N}$, then

$$(4.4) \quad \delta(a(\mathbf{1} \otimes p_n)f) = \delta(a(\mathbf{1} \otimes p_n))f + a(\mathbf{1} \otimes p_n)\delta(f) = a\delta(\mathbf{1} \otimes p_n)f + a(\mathbf{1} \otimes p_n)\delta(f).$$

On the other hand,

$$(4.5) \qquad \delta(a(\mathbf{1} \otimes p_n)f) = \delta((\mathbf{1} \otimes p_n)af) = \delta(\mathbf{1} \otimes p_n)af + (\mathbf{1} \otimes p_n)\delta(af)$$
$$= a\delta(\mathbf{1} \otimes p_n)f + (\mathbf{1} \otimes p_n)\delta(af).$$

Comparing (4.4) and (4.5), it follows that

$$(\mathbf{1} \otimes p_n)(a\delta(f)) = (\mathbf{1} \otimes p_n)\delta(af), \quad n \in \mathbb{N}.$$

Consequently, Lemma 4.6 implies that $a\delta(f) = \delta(af)$. The proof is complete. ∎

PROPOSITION 4.9. *Suppose that X is an infinite dimensional separable Banach space and that $\mathfrak{A}$ is a subalgebra of $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ satisfying conditions* (i) *and* (ii) *of Definition 4.1. If $\delta : \mathfrak{A} \to \mathfrak{A}$ is a derivation, then δ is $L_\infty(\nu)$-linear, that is, $\delta(af) = a\delta(f)$ for all $f \in \mathfrak{A}$ and $a \in L_\infty(\nu)$.*

Proof. The case that $\nu(S) < \infty$ has been obtained in Lemma 4.8. Using Corollary 4.4, the extension to the case that ν is σ-finite follows via a standard argument. ∎

REMARK 4.10. In the situation of Proposition 4.9, define the subalgebra $M_{\mathfrak{A}}$ of $L_0(\nu)$ by

$$M_{\mathfrak{A}} = \{a \in L_0(\nu) : af \in \mathfrak{A} \text{ for all } f \in \mathfrak{A}\}.$$

By (i) of Definition 4.1, $L_\infty(\nu) \subseteq M_{\mathfrak{A}}$. We claim that $\delta(af) = a\delta(f)$ for all $f \in \mathfrak{A}$ and $a \in M_{\mathfrak{A}}$. Indeed, given $a \in M_{\mathfrak{A}}$, there exists a ν-measurable partition $\{A_n\}_{n=1}^\infty$ of S such that $a\chi_{A_n} \in L_\infty(\nu)$ for all n. Using Corollary 4.4 and Proposition 4.9, it follows that

$$\chi_{A_n}\delta(af) = \delta(\chi_{A_n}af) = \chi_{A_n}a\delta(f)$$

for all $n \in \mathbb{N}$. This implies that $\delta(af) = a\delta(f)$.

In particular, if $\mathfrak{A} = L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$ for some closed standard subalgebra $\mathfrak{U}$ of $\mathcal{L}(X)$, then δ is $L_0(\nu)$-linear.

If $\mathfrak{U}$ is a closed standard subalgebra of $\mathcal{L}(X)$, then $\mathcal{T}_0$ denotes the topology in $L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$ of convergence in measure on sets of finite measure (see Section 2).

PROPOSITION 4.11. *Let $\mathfrak{U}$ be a closed standard subalgebra of $\mathcal{L}(X)$. If $\delta : L_0^{\mathrm{wo}}(\nu; \mathfrak{U}) \to L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$ is an $L_0(\nu)$-linear derivation, then δ is continuous with respect to $\mathcal{T}_0$.*

Proof. It follows from Lemma 2.4 that $L_0^{wo}(v; \mathfrak{U})$ is a complete metrizable topological vector space with respect to $\mathcal{T}_0$. Therefore, it is sufficient to show that δ is closed (see e.g. Theorem 2.15 of [16]). Accordingly, suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L_0^{wo}(v; \mathfrak{U})$ such that $f_n \overset{\mathcal{T}_0}{\to} 0$ and $\delta(f_n) \overset{\mathcal{T}_0}{\to} g$ as $n \to \infty$ for some $g \in L_0^{wo}(v; \mathfrak{U})$. Let $p, q \in \mathcal{I}(\mathcal{F}(X))$ be two rank one projections and note that $p, q \in \mathfrak{U}$. There exist $x_0, y_0 \in X$ and $x_0^*, y_0^* \in X^*$ satisfying $\langle x_0, x_0^* \rangle = \langle y_0, y_0^* \rangle = 1$, such that $px = \langle x, x_0^* \rangle x_0$ and $qx = \langle x, y_0^* \rangle y_0$, $x \in X$. Defining the rank one operator $v \in \mathcal{F}(X)$ by setting $vx = \langle x, y_0^* \rangle x_0$, $x \in X$, it follows that

(4.6) $$pTq = \langle Ty_0, x_0^* \rangle v, \quad T \in \mathcal{L}(X).$$

It follows from (4.6) that

$$((\mathbf{1} \otimes p)f_n(\mathbf{1} \otimes q))(s) = pf_n(s)q = a_n(s)v, \quad s \in S,$$

where $a_n \in L_0(v)$. Since $f_n \overset{\mathcal{T}_0}{\to} 0$ implies that $(\mathbf{1} \otimes p)f_n(\mathbf{1} \otimes q) \overset{\mathcal{T}_0}{\to} 0$, it is clear that $a_n \to 0$ as $n \to \infty$ in measure on sets of finite measure. Hence, $a_n \delta(v) \overset{\mathcal{T}_0}{\to} 0$ in $L_0^{wo}(v; \mathfrak{U})$. Furthermore, δ is L_0-linear and so

$$\delta((\mathbf{1} \otimes p)f_n(\mathbf{1} \otimes q)) = a_n \delta(v)$$

for all n. Consequently, $\delta((\mathbf{1} \otimes p)f_n(\mathbf{1} \otimes q)) \overset{\mathcal{T}_0}{\to} 0$ as $n \to \infty$. Since

$$\delta((\mathbf{1} \otimes p)f_n(\mathbf{1} \otimes q)) = \delta(\mathbf{1} \otimes p)f_n(\mathbf{1} \otimes q) + (\mathbf{1} \otimes p)\delta(f_n)(\mathbf{1} \otimes q) + (\mathbf{1} \otimes p)f_n\delta(\mathbf{1} \otimes q),$$

$\delta(\mathbf{1} \otimes p)f_n(\mathbf{1} \otimes q) \overset{\mathcal{T}_0}{\to} 0$ and $(\mathbf{1} \otimes p)f_n\delta(\mathbf{1} \otimes q) \overset{\mathcal{T}_0}{\to} 0$ as $n \to \infty$, it follows that

$$(\mathbf{1} \otimes p)\delta(f_n)(\mathbf{1} \otimes q) \overset{\mathcal{T}_0}{\to} 0, \quad n \to \infty.$$

On the other hand, $\delta(f_n) \overset{\mathcal{T}_0}{\to} g$ implies that

$$(\mathbf{1} \otimes p)\delta(f_n)(\mathbf{1} \otimes q) \overset{\mathcal{T}_0}{\to} (\mathbf{1} \otimes p)g(\mathbf{1} \otimes q), \quad n \to \infty$$

and so $(\mathbf{1} \otimes p)g(\mathbf{1} \otimes q) = 0$.

Since every finite rank projection in X may be written as a sum of finitely many rank one projections, it follows that $(\mathbf{1} \otimes p)g(\mathbf{1} \otimes p) = 0$ for all finite rank projections p in X. Consequently, via Lemma 4.6 it may be concluded that $g = 0$. This shows that δ is closed and hence, δ is $\mathcal{T}_0$-continuous. ∎

COROLLARY 4.12. *Assuming that X is infinite dimensional, let $\mathfrak{U}$ be a closed standard subalgebra of $\mathcal{L}(X)$. If $\delta : L_0^{wo}(v; \mathfrak{U}) \to L_0^{wo}(v; \mathfrak{U})$ is a derivation, then δ is continuous with respect to $\mathcal{T}_0$.*

Proof. It follows from Remark 4.10 that δ is L_0-linear and so the result is an immediate consequence of 4.11. ∎

PROPOSITION 4.13. *Let $\mathfrak{U}$ be a closed standard subalgebra of $\mathcal{L}(X)$. If $\delta : L_0^{wo}(v; \mathfrak{U}) \to L_0^{wo}(v; \mathfrak{U})$ is an $L_0(v)$-linear derivation, then there exists $w \in L_0(v; \mathcal{L}(X))$ such that $\delta(f) = wf - fw$ for all $f \in \mathfrak{U}$.*

Proof. It follows from Proposition 4.11 that δ is continuous with respect to $\mathcal{T}_0$. Consequently, by Proposition 2.8, there exists $0 \leqslant a \in L_0(\nu)$ such that

$$(4.7) \qquad \|\delta(f)(s)\|_{\mathcal{L}(X)} \leqslant a(s)\|f(s)\|_{\mathcal{L}(X)}, \quad \nu\text{-a.e. on } S,$$

for all $f \in L_0^{wo}(\nu; \mathfrak{U})$. Fix $x_0 \in X$ with $\|x_0\|_X = 1$ and let $x_0^* \in X^*$ be such that $\|x_0^*\|_{X^*} = 1$ and $\langle x_0, x_0^* \rangle = 1$. Define $F_0 \in L_0(\nu; X)$ by $F_0 = \mathbf{1} \otimes x_0$.

For $F \in L_0(\nu; X)$ and $s \in S$, define the rank one operator $g_F(s) \in \mathcal{F}(X) \subseteq \mathfrak{U}$ by setting

$$g_F(s)x = \langle x, x_0^* \rangle F(s), \quad x \in X.$$

It is clear that

$$(4.8) \qquad \|g_F(s)\|_{\mathcal{L}(X)} \leqslant \|F(s)\|_X.$$

Furthermore, if $x \in X$ and $x^* \in X^*$, then the function

$$s \longmapsto \langle g_F(s)x, x^* \rangle = \langle x, x_0^* \rangle \langle F(s), x^* \rangle, \quad s \in S,$$

is ν-measurable. Therefore, $g_F \in L_0^{wo}(\nu; \mathfrak{U})$. Identifying the elements of $L_0^{wo}(\nu; \mathfrak{U})$ with $L_0(\nu)$-linear operators on $L_0(\nu; X)$ (see Corollary 2.16), it is clear that $g_F F_0 = F$. Furthermore, the map $F \longmapsto g_F$, $F \in L_0(\nu; X)$, is linear from $L_0(\nu; X)$ into $L_0^{wo}(\nu; \mathfrak{U})$. It should be observed that

$$(4.9) \qquad g_{fF} = f g_F, \quad f \in L_0^{wo}(\nu; \mathfrak{U}), F \in L_0(\nu; X).$$

Indeed,

$$g_{fF}(s)x = \langle x, x_0^* \rangle f(s)F(s) = f(s)(\langle x, x_0^* \rangle F(s)) = f(s)g_F(s)x$$

for $x \in X$, $s \in S$, and so (4.9) holds. Similarly, it follows that

$$(4.10) \qquad g_{bF} = b g_F, \quad b \in L_0(\nu), F \in L_0(\nu; X),$$

that is, the map $F \longmapsto g_F$, $F \in L_0(\nu; X)$, is $L_0(\nu)$-linear.

Define the linear operator $T : L_0(\nu; X) \to L_0(\nu; X)$ by setting

$$(4.11) \qquad TF = \delta(g_F)F_0, \quad F \in L_0(\nu; X).$$

Since δ is L_0-linear, it follows, in particular, from (4.10) that T is also L_0-linear. Furthermore, it follows from (4.7) and (4.8) that

$$\|(TF)(s)\|_X = \|\delta(g_F)(s)x_0\|_X \leqslant \|\delta(g_F)(s)\|_{\mathcal{L}(X)}$$
$$\leqslant a(s)\|g_F(s)\|_{\mathcal{L}(X)} \leqslant a(s)\|F(s)\|_X, \quad s \in S,$$

and so T is continuous with respect to the topology $\mathcal{T}_0$ (see Proposition 2.8). Therefore, it follows from Proposition 2.15 that there exists $w \in L_0^{wo}(\nu; \mathcal{L}(X))$ such that

$$(4.12) \qquad TF = wF, \quad F \in L_0(\nu; X).$$

Finally, it will be shown that $\delta(f) = [w, f]$ for all $f \in L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$. If $f \in L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$ and $F \in L_0(\nu; X)$, then it follows from (4.11), (4.12) and (4.9) that

$$[w, f]F = wfF - fwF = \delta(g_{fF})F_0 - f\delta(g_F)F_0 = \delta(fg_F)F_0 - f\delta(g_F)F_0$$
$$= \delta(f)g_F F_0 + f\delta(g_F)F_0 - f\delta(g_F)F_0 = \delta(f)g_F F_0 = \delta(f)F.$$

Since this holds for all $F \in L_0(\nu; X)$, we may conclude that $\delta(f) = [w, f]$. The proof is complete. ∎

COROLLARY 4.14. *Suppose that X is infinite dimensional and let $\mathfrak{U}$ be a closed standard subalgebra of $\mathcal{L}(X)$. If $\delta : L_0^{\mathrm{wo}}(\nu; \mathfrak{U}) \to L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$ is a derivation, then there exists $w \in L_0(\nu; \mathcal{L}(X))$ such that $\delta(f) = wf - fw$ for all $f \in \mathfrak{A}$.*

COROLLARY 4.15. *Let X be a separable Banach space and let $\mathcal{A}$ be the algebra $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ or $L_0^{\mathrm{wo}}(\nu; \mathcal{K}(X))$. If $\delta : \mathcal{A} \to \mathcal{A}$ is an $L_0(\nu)$-linear derivation, then there exists $w \in L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ such that $\delta(f) = wf - fw$ for all $f \in \mathcal{A}$.*

COROLLARY 4.16. *Let X be an infinite dimensional separable Banach space and let $\mathcal{A}$ be the algebra $L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ or $L_0^{\mathrm{wo}}(\nu; \mathcal{K}(X))$. If $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation, then there exists $w \in L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ such that $\delta(f) = wf - fw$ for all $f \in \mathcal{A}$.*

THEOREM 4.17. *Let X be a separable Banach space and (S, Σ, ν) be a σ-finite measure space. Suppose that $\mathfrak{U} \subseteq \mathcal{L}(X)$ is a closed standard algebra and that $\mathfrak{A} \subseteq L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ is a $\mathfrak{U}$-admissible subalgebra. If $\delta : \mathfrak{A} \to \mathfrak{A}$ is an $L_\infty(\nu)$-linear derivation, then there exists $w \in L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ such that $\delta(f) = wf - fw$ for all $f \in \mathfrak{A}$.*

Proof. It follows from condition (iii) of Definition 4.1 and Proposition 2.9 that δ extends uniquely to an $L_0(\nu)$-linear map $\widehat{\delta} : L_0^{\mathrm{wo}}(\nu; \mathfrak{U}) \to L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$. Recall from Proposition 2.9 that $\chi_A \widehat{\delta}(f) = \delta(\chi_A f)$ for all $f \in L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$ and all ν-measurable sets A satisfying $\chi_A f \in \mathfrak{A}$. We claim that $\widehat{\delta}$ is a derivation. Indeed, suppose that $f, g \in L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$. It follows from condition (iii) on the algebra $\mathfrak{A}$ that there exists a measurable partition $\{A_n\}_{n=1}^\infty$ such that $\chi_{A_n} f, \chi_{A_n} g \in \mathfrak{A}$ for all n. Using that $\widehat{\delta}$ is $L_0(\nu)$-linear, we find that

$$\chi_{A_n} \widehat{\delta}(fg) = \widehat{\delta}(\chi_{A_n} fg) = \delta(\chi_{A_n} f \chi_{A_n} g) = \delta(\chi_{A_n} f)\chi_{A_n} g + \chi_{A_n} f\delta(\chi_{A_n} g)$$
$$= \chi_{A_n} \widehat{\delta}(f)g + \chi_{A_n} f\widehat{\delta}(g)$$

for all $n \in \mathbb{N}$. Hence, $\widehat{\delta}(fg) = \widehat{\delta}(f)g + f\widehat{\delta}(g)$, which shows that $\widehat{\delta}$ is a derivation in $L_0^{\mathrm{wo}}(\nu; \mathfrak{U})$. Consequently, the result of the theorem follows immediately from Proposition 4.13. ∎

COROLLARY 4.18. *Let X be an infinite dimensional separable Banach space and (S, Σ, ν) be a σ-finite measure space. Suppose that $\mathfrak{U} \subseteq \mathcal{L}(X)$ is a closed standard algebra and that $\mathfrak{A} \subseteq L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ is a $\mathfrak{U}$-admissible subalgebra. If $\delta : \mathfrak{A} \to \mathfrak{A}$ is a derivation, then there exists $w \in L_0^{\mathrm{wo}}(\nu; \mathcal{L}(X))$ such that $\delta(f) = wf - fw$ for all $f \in \mathfrak{A}$.*

5. APPLICATIONS TO ALGEBRAS OF MEASURABLE OPERATORS

In the present section, we shall apply the results of the previous section to algebras of measurable operators with respect to certain von Neumann algebras. For details on von Neumann algebra theory, the reader is referred to e.g. [10], [11], [17] or [21]. General facts concerning measurable operators may be found in [20], [23] and [24] (see also Chapter IX of [22]). For the convenience of the reader, some of the basic definitions are recalled.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H. The set of all self-adjoint projections in $\mathcal{M}$ is denoted by $P(\mathcal{M})$. The commutant of $\mathcal{M}$ is denoted by $\mathcal{M}'$. A linear operator $x : \mathfrak{D}(x) \to H$, where the domain $\mathfrak{D}(x)$ of x is a linear subspace of H, is said to be affiliated with $\mathcal{M}$ if $yx \subseteq xy$ for all $y \in \mathcal{M}'$ (which is denoted by $x\eta\mathcal{M}$). A linear operator $x : \mathfrak{D}(x) \to H$ is termed measurable with respect to $\mathcal{M}$ if x is closed, densely defined, affiliated with $\mathcal{M}$ and there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in $P(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subseteq \mathfrak{D}(x)$ and $p_n^{\perp}$ is a finite projection (with respect to $\mathcal{M}$) for all n. It should be noted that the condition $p_n(H) \subseteq \mathfrak{D}(x)$ implies that $xp_n \in \mathcal{M}$. The collection of all measurable operators with respect to $\mathcal{M}$ is denoted by $S(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products (denoted simply by $x + y$ and xy for all $x, y \in S(\mathcal{M})$). Furthermore, a closed densely defined linear operator $x : \mathfrak{D}(x) \to H$ is called locally measurable (with respect to $\mathcal{M}$) if $x\eta\mathcal{M}$ and there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in $P(\mathcal{Z}(\mathcal{M}))$ such that $p_n \uparrow \mathbf{1}$ and $xp_n \in S(\mathcal{M})$ for all n. The collection of all locally measurable operators is denoted by $LS(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products. Evidently, $S(\mathcal{M})$ is a $*$-subalgebra of $LS(\mathcal{M})$. For details we refer e.g. to [24].

If the von Neumann $\mathcal{M}$ is semi-finite, equipped with a semi-finite faithful normal trace τ, then an operator $x \in S(\mathcal{M})$ is called τ-measurable if there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in $P(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subseteq \mathfrak{D}(x)$ and $\tau(p_n^{\perp}) < \infty$ for all n. The collection $S(\tau)$ of all τ-measurable operators is a unital $*$-subalgebra of $S(\mathcal{M})$. For details see e.g. [23]. An operator $x \in S(\tau)$ is said to τ-compact if for every $\varepsilon > 0$ there exists a projection $p \in P(\mathcal{M})$ such that $\|xp\|_{B(H)} < \varepsilon$ and $\tau(p^{\perp}) < \infty$. The collection $S_0(\tau)$ of all τ-compact operators is a two-sided ideal in $S(\tau)$.

Now, we specialize to the case where H is a separable Hilbert space and $\mathcal{M} = L_{\infty}(\nu)\overline{\otimes}B(H)$, the von Neumann algebra tensor product of $L_{\infty}(\nu)$ and $B(H)$ acting on the Hilbert space tensor product $L_2(\nu)\overline{\otimes}H$. Here, (S, Σ, ν) is a σ-finite measure space and $L_{\infty}(\nu)$ is considered as a von Neumann algebra on the Hilbert space $L_2(\nu)$, acting via multiplication. The von Neumann algebra $B(H)$ is equipped with its standard trace tr and $L_{\infty}(\nu)$ is equipped with the trace ν, given by $\nu(f) = \int_S f d\nu$, $0 \leqslant f \in L_{\infty}(\nu)$. Let $\tau = \nu \otimes \text{tr}$ be the tensor product trace on $\mathcal{M}$, that is, τ is the unique semi-finite normal faithful trace τ satisfying $\tau(f \otimes x) = \nu(f)\text{tr}(x)$ for all $0 \leqslant f \in L_{\infty}(\nu)$ and $0 \leqslant x \in B(H)$. Identifying

the tensor product $L_2(\nu)\overline{\otimes}H$ with the Bochner space $L_2(\nu; H)$, the von Neumann algebra $L_\infty(\nu)\overline{\otimes}B(H)$ may be identified with the space $L_\infty^{wo}(\nu; B(H))$, where the function $f \in L_\infty^{wo}(\nu; B(H))$ corresponds to the operator x_f on $L_2(\nu; H)$ given by $(x_f F)(s) = f(s)F(s)$, ν-a.e. on S for all $F \in L_2(\nu; H)$ (see e.g. [21]). It should be observed that if $f \in L_\infty^{wo}(\nu; B(H))$, then $f \geqslant 0$ (that is, $x_f \geqslant 0$) if and only if $f(s) \geqslant 0$ in $B(H)$ ν-a.e. on S. Using this observation, it is not difficult to show that a sequence $\{f_n\}_{n=1}^\infty$ in $L_\infty^{wo}(\nu; B(H))$ satisfies $f_n \downarrow_n 0$ (in the von Neumann algebra $L_\infty^{wo}(\nu; B(H))$) if and only if $f_n(s) \downarrow_n 0$ (in $B(H)$) ν-a.e. on S. The (tensor product) trace on $L_\infty^{wo}(\nu; B(H))$ is given by

$$(5.1) \qquad \tau(f) = \int_S \mathrm{tr}(f(s))\mathrm{d}\nu(s), \quad 0 \leqslant f \in L_\infty^{wo}(\nu; B(H)).$$

Indeed, using the preceding observations, it is not difficult to show that (5.1) defines a semi-finite normal faithful trace τ on $L_\infty^{wo}(\nu; B(H))$ satisfying $\tau(f \otimes x) = \nu(f)\mathrm{tr}(x)$, $0 \leqslant f \in L_\infty(\nu)$, $0 \leqslant x \in B(H)$.

The following observation will be used.

LEMMA 5.1. *A projection $p \in \mathcal{M} \cong L_\infty^{wo}(\nu; B(H))$ is finite if and only if $p(s)$ is a finite projection in $B(H)$ for ν-almost all $s \in S$.*

Proof. First assume that $p(s)$ is a finite projection in $B(H)$ for ν-almost all $s \in S$ and suppose that $q \in P(L_\infty^{wo}(\nu; B(H)))$ is such that $q \leqslant p$ and $q \sim p$. Since $q \sim p$, there exists a partial isometry $v \in L_\infty^{wo}(\nu; B(H))$ such that $v^*v = p$ and $vv^* = q$. This implies that $v(s)^*v(s) = p(s)$ and $v(s)v(s)^* = q(s)$ ν-a.e. on S. Since $p(s)$ is finite ν-a.e. , it follows that $q(s) = p(s)$ ν-a.e. on S and hence, $p = q$. This shows that the projection p is finite in $L_\infty^{wo}(\nu; B(H))$.

Assume now that $p \in P(L_\infty^{wo}(\nu; B(H)))$ is finite and let $H(s) = p(s)(H)$, $s \in S$. Setting $S_\infty = \{s \in S : \dim H(s) = \infty\}$, suppose that $\nu(S_\infty) > 0$ and define $p_1 \in P(L_\infty^{wo}(\nu; B(H)))$ by $p_1 = \chi_{S_\infty} p$. It follows from Proposition II.1.9 and Proposition II.1.1 of [10], that there exist Bochner ν-measurable functions $\eta_n : S_\infty \to H$, $n \in \mathbb{N}$, such that $\{\eta_n(s)\}_{n=1}^\infty$ is an orthonormal basis in $H(s)$ for all $s \in S_\infty$. Defining $q \in P(L_\infty^{wo}(\nu; B(H)))$ by setting

$$q(s)\xi = \sum_{n=1}^\infty \langle \xi, \eta_{2n}(s)\rangle \eta_{2n}(s), \quad s \in S_\infty$$

and $q(s) = 0$ if $s \in S\backslash S_\infty$, it is easily verified that $q < p_1$ and $q \sim p_1$. Hence, p_1 is not finite. Since $p_1 \leqslant p$, this is a contradiction. Therefore, we may conclude that $\nu(S_\infty) = 0$, that is, $p(s)$ is finite for ν-almost all $s \in S$. The proof is complete. ∎

We will show next that the algebra $L_0^{wo}(\nu; B(H))$ may be identified with the algebra $LS(\mathcal{M})$. Let $f \in L_0^{wo}(\nu; B(H))$ be given. It should be recalled that, by Lemma 2.10, $fF \in L_0^{wo}(\nu; H)$ for all $F \in L_2(\nu; H)$. Define the linear subspace $\mathfrak{D}(x_f)$ of $L_2(\nu; H)$ by setting

$$\mathfrak{D}(x_f) = \{F \in L_2(\nu; H) : fF \in L_2(\nu; H)\}$$

and define the linear operator $x_f : \mathfrak{D}(x_f) \to L_2(\nu; H)$ by

$$(5.2) \qquad\qquad x_f F = f F, \quad F \in \mathfrak{D}(x_f).$$

LEMMA 5.2. *If $f \in L_0^{wo}(\nu; B(H))$, then the operator x_f, defined by (5.2), is locally measurable with respect to $\mathcal{M}$.*

Proof. Define the subsets A_n of S by setting

$$(5.3) \qquad\qquad A_n = \{s \in S : \|f(s)\|_{B(H)} \leqslant n\}, \quad n \in \mathbb{N},$$

which are ν-measurable by Lemma 2.1. It is clear that

$$\chi_{A_n} L_2(\nu; H) = \{\chi_{A_n} F : F \in L_2(\nu; H)\} \subseteq \mathfrak{D}(x_f)$$

and it follows from the dominated convergence theorem that

$$(5.4) \qquad\qquad \mathcal{D} = \bigcup_{n=1}^{\infty} \chi_{A_n} L_2(\nu; H)$$

is dense in $L_2(\nu; H)$. This shows that $\mathfrak{D}(x_f)$ is dense in $L_2(\nu; H)$. Suppose that $\{F_n\}_{n=1}^{\infty}$ is a sequence in $\mathfrak{D}(x_f)$ and that $F, G \in L_2(\nu; H)$ such that $F_n \to F$ and $x_f F_n \to G$ in $L_2(\nu; H)$. Passing to a subsequence, it may be assumed that $F_n(s) \to F(s)$ and $f(s)F_n(s) \to G(s)$ ν-a.e. on S. Therefore, $f(s)F(s) = G(s)$ ν-a.e. and hence, $F \in \mathfrak{D}(x_f)$ and $x_f F = G$. This shows that x_f is closed.

Using that $\mathcal{M}' = L_\infty(\nu) \otimes \mathbb{C}\mathbf{1}$, it is easy to verify that x_f is affiliated with $\mathcal{M}$. Finally, defining the projections $\{p_n\}_{n=1}^{\infty}$ in $P(\mathcal{Z}(\mathcal{M}))$ by $p_n = \chi_{A_n} \otimes \mathbf{1}$, it is evident that $p_n \uparrow \mathbf{1}$ and $x_f p_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. Consequently, $x_f \in LS(\mathcal{M})$. ■

If $f \in L_0^{wo}(\nu; B(H))$ and the sets $A_n \subseteq S$ are defined by (5.3), then it follows easily that the subspace $\mathcal{D}$, defined by (5.4), is a core for the operator x_f, that is, x_f is the closure of its restriction to $\mathcal{D}$. Using this observation, the first part of the following proposition follows via a standard argument.

PROPOSITION 5.3. *The map $f \longmapsto x_f$, $f \in L_0^{wo}(\nu; B(H))$, is a unital $*$-isomorphism from $L_0^{wo}(\nu; B(H))$ onto $LS(\mathcal{M})$.*

Proof. It remains to be shown that for every $x \in LS(\mathcal{M})$ there exists $f \in L_0^{wo}(\nu; B(H))$ such that $x = x_f$. To this end, first assume that $x \in S(\mathcal{M})$. By definition, there exists a sequence $\{p_n\}_{n=1}^{\infty}$ such that $p_n \uparrow \mathbf{1}$, $xp_n \in \mathcal{M}$ and $p_n^\perp$ is finite for all n. As observed in Lemma 5.1, this implies that $p_n^\perp(s)$ is a finite projection in $B(H)$ ν-a.e., that is, $\mathrm{tr}(p_n^\perp(s)) \in \mathbb{N} \cup \{0\}$, for ν-almost all $s \in S$. Since $p_n^\perp \downarrow 0$ in $\mathcal{M}$ implies that $p_n^\perp(s) \downarrow 0$ in $B(H)$ ν-a.e. on S (see the observations preceding (5.1)) and $\mathrm{tr}(p_n^\perp(s)) \in \mathbb{N} \cup \{0\}$, it follows that for ($\nu$-almost) every $s \in S$ there exists $n_s \in \mathbb{N}$ such that $p_n(s) = \mathbf{1}$ for all $n \geqslant n_s$. Consequently, if the ν-measurable sets A_n are defined by setting

$$A_n = \{s \in S : p_n(s) = \mathbf{1}\},$$

then $A_n \uparrow S$. Note that $p_n(\chi_{A_n} \otimes \mathbf{1}) = \chi_{A_n} \otimes \mathbf{1}$. For each $n \in \mathbb{N}$, there exists a unique $f_n \in L_\infty^{\mathrm{wo}}(\nu; B(H))$ such that $xp_n = x_{f_n}$. If $m \leqslant n$, then

$$x_{\chi_{A_m}f_m} = x_{f_m}(\chi_{A_m} \otimes \mathbf{1}) = xp_m(\chi_{A_m} \otimes \mathbf{1}) = x(\chi_{A_m} \otimes \mathbf{1})$$
$$= x(\chi_{A_n} \otimes \mathbf{1})(\chi_{A_m} \otimes \mathbf{1}) = xp_n(\chi_{A_n} \otimes \mathbf{1})(\chi_{A_m} \otimes \mathbf{1}) = x_{f_n}(\chi_{A_m} \otimes \mathbf{1}) = x_{\chi_{A_m}f_n}$$

and so $\chi_{A_m}f_m = \chi_{A_m}f_n$. Therefore, there exists a unique $f \in L_0^{\mathrm{wo}}(\nu; B(H))$ such that $\chi_{A_n}f = \chi_{A_n}f_n$ for all n. We claim that $x = x_f$. Indeed,

$$x(\chi_{A_n} \otimes \mathbf{1}) = xp_n(\chi_{A_n} \otimes \mathbf{1}) = x_{f_n}(\chi_{A_n} \otimes \mathbf{1}) = x_f(\chi_{A_n} \otimes \mathbf{1})$$

for all n, from which it easily follows that $x = x_f$.

Assume now that $x \in LS(\mathcal{M})$. There exists a sequence $\{q_n\}_{n=1}^\infty$ of mutually orthogonal central projections in $\mathcal{M}$ such that $\sum_{n=1}^\infty q_n = \mathbf{1}$ and $xq_n \in S(\mathcal{M})$ for all n. Each q_n is of the form $q_n = \chi_{B_n} \otimes \mathbf{1}$, where B_n is a ν-measurable set. Since $q_nq_m = 0$ whenever $n \neq m$, it follows that the sets $\{B_n\}_{n=1}^\infty$ are mutually disjoint (modulo ν-null sets). Furthermore, it is clear that $\bigcup_n B_n = S$. From the first part of the proof it follows that for every n there exists a unique $f_n \in L_0^{\mathrm{wo}}(\nu; B(H))$ such that $xq_n = x_{f_n}$. A moment's reflection shows that $f_n = \chi_{B_n}f_n$ for all n and so there exists a unique $f \in L_0^{\mathrm{wo}}(\nu; B(H))$ such that $f_n = \chi_{B_n}f$ for all $n \in \mathbb{N}$. Since $x(\chi_{B_n} \otimes \mathbf{1}) = x_f(\chi_{B_n} \otimes \mathbf{1})$ for all n, it is now clear that $x = x_f$, which completes the proof of the proposition. ∎

REMARK 5.4. In Definition 3.1 of [24], Yeadon introduces the *topology of convergence locally in measure* in the algebra $LS(\mathcal{M})$ for a general von Neumann algebra $\mathcal{M}$. If $\mathcal{M} = L_\infty(\nu) \overline{\otimes} B(H)$ and the algebra $LS(\mathcal{M})$ is identified with the algebra $L_0^{\mathrm{wo}}(\nu; B(H))$ via the above proposition, then this topology in $LS(\mathcal{M})$ corresponds to the topology of convergence in measure on sets of finite measure in $L_0^{\mathrm{wo}}(\nu; B(H))$ (as defined in Section 2 of the present paper). To verify this claim, it should be observed that $\mathcal{Z}(\mathcal{M}) \cong L_\infty(\nu)$ and that the *dimension function* $D : P(\mathcal{M}) \to M(\nu)^+$ (in the sense of Segal, ([20], Definition 1.4)) is given by $(Dp)(s) = \mathrm{tr}(p(s)), s \in S$. Here, $M(\nu)^+$ denotes the collection of all (equivalence classes of) ν-measurable functions with values in $[0, \infty]$. The details are left to the reader.

If $f \in L_0^{\mathrm{wo}}(\nu; B(H))$ is self-adjoint, that is, $f(s)^* = f(s)$ ν-a.e. on S, and if ϕ is a bounded Borel function on $\mathbb{R}$, then $\phi(f) : s \longmapsto \phi(f(s)), s \in S$, is a bounded function from S into $B(H)$. On the other hand, the operator $x_f \in LS(\mathcal{M})$ is self-adjoint and so the operator $\phi(x_f) \in \mathcal{M}$ is defined via the functional calculus of x_f. The next lemma describes the relationship between $\phi(f)$ and $\phi(x_f)$. The $*$-algebra of all bounded Borel functions on $\mathbb{R}$ is denoted by $B_\mathrm{b}(\mathbb{R})$.

LEMMA 5.5. *If* $f \in L_0^{\mathrm{wo}}(\nu; B(H))$ *is self-adjoint and* $\phi \in B_\mathrm{b}(\mathbb{R})$, *then* $\phi(f) \in L_\infty^{\mathrm{wo}}(\nu; B(H))$ *and* $x_{\phi(f)} = \phi(x_f)$.

Proof. Let $f \in L_0^{wo}(\nu; B(H))$ be fixed and self-adjoint. Define $\mathcal{U} \subseteq B_b(\mathbb{R})$ by setting

$$\mathcal{U} = \{\phi \in B_b(\mathbb{R}) : \phi(f) \in L_\infty^{wo}(\nu; B(H)) \text{ and } x_{\phi(f)} = \phi(x_f)\}.$$

It is easily verified that $\mathcal{U}$ is a unital $*$-subalgebra of $B_b(\mathbb{R})$. If $\{\phi_n\}_{n=1}^\infty$ is a sequence in $\mathcal{U}$ and if $\phi \in B_b(\mathbb{R})$ is such that $\phi_n \to \phi$ uniformly on $\mathbb{R}$, then it follows from the properties of the functional calculus that $\phi_n(f) \to \phi(f)$ uniformly on S. Hence, $\phi(f) \in L_\infty^{wo}(\nu; B(H))$ and $\|x_{\phi_n(f)} - x_{\phi(f)}\|_{\mathcal{M}} \to 0$ as $n \to \infty$. Similarly, $\|\phi_n(x_f) - \phi(x_f)\|_{\mathcal{M}} \to 0$ and so $x_{\phi(f)} = \phi(x_f)$. This shows that $\mathcal{U}$ is uniformly closed in $B_b(\mathbb{R})$. If $\lambda \in \mathbb{C} \backslash \mathbb{R}$ and the function $r_\lambda \in B_b(\mathbb{R})$ is defined by setting $r_\lambda(t) = (\lambda - t)^{-1}$, $t \in \mathbb{R}$, then a straightforward calculation shows that $x_{r_\lambda(f)} = r_\lambda(x_f) = (\lambda\mathbf{1} - x_f)^{-1}$. Consequently, $r_\lambda \in \mathcal{U}$ for all $\lambda \in \mathbb{C} \backslash \mathbb{R}$. The Stone–Weierstrass theorem now implies that $C_0(\mathbb{R}) \subseteq \mathcal{U}$.

Suppose that the sequence $\{\phi_n\}_{n=1}^\infty$ in $\mathcal{U}^+$ and $\phi \in B_b(\mathbb{R})^+$ are such that $0 \leqslant \phi_n(t) \uparrow_n \phi(t)$, $t \in \mathbb{R}$. From the properties of the functional calculus it follows that $\phi_n(x_f) \uparrow_n \phi(x_f)$ in $\mathcal{M}$. On the other hand,

$$\phi_n(f(s)) \uparrow_n \phi(f(s)), \quad s \in S,$$

in $B(H)$. Hence, $\phi(f) \in L_\infty^{wo}(S; B(H))$ and $\phi_n(f) \uparrow \phi(f)$ in $L_\infty^{wo}(S; B(H))$, which implies that $x_{\phi_n(f)} \uparrow_n x_{\phi(f)}$ in $\mathcal{M}$. This shows that $\phi \in \mathcal{U}^+$. From this observation it follows that $\mathcal{U}^+$ is closed in $B_b(\mathbb{R})^+$ under monotone pointwise convergence of sequences. Together with the inclusion $C_0(\mathbb{R}) \subseteq \mathcal{U}$, this suffices to conclude that $\mathcal{U} = B_b(\mathbb{R})$, which completes the proof of the lemma. ∎

In the sequel, we shall frequently identify a function $f \in L_0^{wo}(S; B(H))$ with the corresponding operator $x_f \in LS(\mathcal{M})$. An immediate consequence of Lemma 5.5 is the following.

COROLLARY 5.6. *If $f \in L_0^{wo}(S; B(H))$ is self-adjoint, then the spectral measure e^f of f (as a self-adjoint operator in $LS(\mathcal{M})$) is given by*

$$e^f(B)(s) = e^{f(s)}(B), \quad s \in S,$$

for every Borel set $B \subseteq \mathbb{R}$. In particular, for every Borel set $B \subseteq \mathbb{R}$, the function $s \longmapsto e^{f(s)}(B)$, $s \in S$, is ν-measurable.

To apply the results of Section 4 in the present setting, the following observations are relevant.

PROPOSITION 5.7. (i) *The algebras $LS(\mathcal{M}), S(\mathcal{M})$ and $S(\tau)$ are $B(H)$-admissible.* (ii) *The algebra $S_0(\tau)$ is $K(H)$-admissible.*

Proof. (i) It is evident that $LS(\mathcal{M}) \cong L_0^{wo}(\nu; B(H))$ is $B(H)$-admissible. It is clear that $S(\mathcal{M})$ and $S(\tau)$ satisfy condition (i) of Definition 4.1. Since

$$L_\infty^{wo}(\nu; B(H)) \cong \mathcal{M} \subseteq S(\tau) \subseteq S(\mathcal{M}),$$

it is also clear that both $S(\mathcal{M})$ and $S(\tau)$ satisfy condition (ii). If $f \in L_0^{\mathrm{wo}}(\nu; B(H))$, then there exists a ν-measurable partition $\{A_n\}_{n=1}^{\infty}$ of S such that $\chi_{A_n} f \in L_{\infty}^{\mathrm{wo}}(\nu; B(H))$, that is, $\chi_{A_n} f \in \mathcal{M}$ for all n. Hence, $S(\mathcal{M})$ and $S(\tau)$ also satisfy condition (iii) of Definition 4.1.

(ii) It should be observed first that $S_0(\tau) \subseteq L_0^{\mathrm{wo}}(\nu; K(H))$. Indeed, if $f \in S_0(\tau)$, then there exists for every $n \in \mathbb{N}$ a projection $p_n \in P(\mathcal{M})$ satisfying $\|fp_n\|_{\infty} \leqslant 1/n$ and $\tau(p_n^{\perp}) < \infty$. Using (5.1), it follows that there exists a ν-measurable set $A \subseteq S$ satisfying $\nu(S \setminus A) = 0$, such that $\|f(s)p_n(s)\|_{B(H)} \leqslant 1/n$ and $\mathrm{tr}(p_n(s)^{\perp}) < \infty$ for all $s \in A$. Consequently, $f(s) \in K(H)$ for all $s \in A$ and so $f \in L_0^{\mathrm{wo}}(\nu; K(H))$. Since $S_0(\tau)$ is a two-sided ideal in $S(\tau)$, it is clear that $S_0(\tau)$ is an $L_{\infty}(\nu)$-module. To show that $S_0(\tau)$ satisfies condition (ii) of Definition 4.1, suppose that $f \in L_{\infty}^{\mathrm{wo}}(\nu; B(H))$ is such that for every $\varepsilon > 0$ there exists $p \in \mathcal{I}(F(H))$ satisfying $\|f(\mathbf{1} \otimes p^{\perp})\|_{\infty} \leqslant \varepsilon$ and let the ν-measurable set $A \subseteq S$ with $\nu(A) < \infty$ be given. We have to show that $\chi_A f \in S_0(\tau)$. If $\varepsilon > 0$, then there exists $p \in \mathcal{I}(F(H))$ such that $\|f(\mathbf{1} \otimes p^{\perp})\|_{\infty} \leqslant \varepsilon$. Replacing p by the orthogonal projection in H onto $\mathrm{Ker}(p)^{\perp}$, it is easily seen that the projection p may be assumed to be self-adjoint, that is, $p \in P(F(H))$. Defining the projection $q \in P(\mathcal{M})$ by $q = \chi_A \otimes p$, it is clear that $\tau(q) = \nu(A)\mathrm{tr}(p) < \infty$. Furthermore, $q^{\perp} = \chi_A \otimes p^{\perp} + \chi_{A^c} \otimes \mathbf{1}$ and so

$$\|(\chi_A f)q^{\perp}\|_{\infty} = \|(\chi_A f)(\mathbf{1} \otimes p^{\perp})\|_{\infty} \leqslant \varepsilon.$$

This shows that $\chi_A f \in S_0(\tau)$.

Finally, it will be shown that $S_0(\tau)$ satisfies condition (iii) of Definition 4.1 with respect to $\mathfrak{U} = K(H)$. Since the measure ν is assumed to be σ-finite, it suffices to consider the case that $\nu(S) < \infty$. Given $f \in L_0^{\mathrm{wo}}(\nu; K(H))$, define the sets $B_{k,n}$ $(k, n \in \mathbb{N})$ by setting

$$B_{k,n} = \{s \in S : \mathrm{tr}(e^{|f(s)|}(1/k, \infty)) > n\}.$$

It follows from Corollary 5.6 that these sets are ν-measurable. Furthermore, since $f(s) \in K(H)$, it is clear that $\mathrm{tr}(e^{|f(s)|}(1/k, \infty)) \in \mathbb{N}$ for all $k \in \mathbb{N}$ and $s \in S$. Consequently, $B_{k,n} \downarrow_n \varnothing$ for all k and so for each k there exists $n_k \in \mathbb{N}$ such that $\nu(B_{k,n_k}) \leqslant 2^{-k}$. Defining

$$A_l = \bigcap_{k=l}^{\infty} B_{k,n_k}^c, \quad l \in \mathbb{N},$$

it is clear that $\nu(A_l^c) \leqslant 2^{-l+1}$ and so $A_l \uparrow S$. Furthermore, if $s \in A_l$, then $\mathrm{tr}(e^{|f(s)|}(1/k, \infty)) \leqslant n_k$ for all $k \geqslant l$. We claim that $\chi_{A_l} f \in S_0(\tau)$ for all $l \in \mathbb{N}$. Indeed, given l and $\varepsilon > 0$ let $k \in \mathbb{N}$ be such that $k \geqslant l$ and $1/k \leqslant \varepsilon$. The projection $p = \chi_{A_l} e^{|f|}(\varepsilon, \infty)$ satisfies

$$\tau(p) = \int\limits_{A_l} \mathrm{tr}(e^{|f(s)|}(\varepsilon, \infty)) \mathrm{d}\nu(s) \leqslant \int\limits_{A_l} \mathrm{tr}(e^{|f(s)|}(1/k, \infty)) \mathrm{d}\nu(s) \leqslant n_k \nu(A_l) < \infty$$

and

$$\|(\chi_{A_l} f)p^\perp\|_\infty = \|(\chi_{A_l} f)e^{|f|}[0,\varepsilon]\|_\infty \leqslant \varepsilon.$$

Hence, $\chi_{A_l} f \in S_0(\tau)$ for all l. This suffices to show that $S_0(\tau)$ satisfies condition (iii) of Definition 4.1 with respect to $\mathfrak{U} = K(H)$. ∎

A combination of Theorem 4.17, Corollary 4.18 and Proposition 5.7 immediately yield the following results.

COROLLARY 5.8. *Let H be a separable Hilbert space, (S, Σ, ν) be a σ-finite measure space and $\mathcal{M} = L_\infty(\nu)\overline{\otimes}B(H)$. If $\mathfrak{U}$ is one of the algebras $LS(\mathcal{M})$, $S(\mathcal{M})$, $S(\tau)$ or $S_0(\tau)$ and if δ is a $\mathcal{Z}(\mathcal{M})$-linear derivation in $\mathfrak{U}$, then there exists $w \in LS(\mathcal{M})$ such that $\delta(x) = wx - xw$, $x \in \mathfrak{U}$.*

COROLLARY 5.9. *Let H be a separable infinite dimensional Hilbert space, (S, Σ, ν) be a σ-finite measure space and $\mathcal{M} = L_\infty(\nu)\overline{\otimes}B(H)$. If $\mathfrak{U}$ is one of the algebras $LS(\mathcal{M})$, $S(\mathcal{M})$, $S(\tau)$ or $S_0(\tau)$ and if δ is a derivation in $\mathfrak{U}$, then there exists $w \in LS(\mathcal{M})$ such that $\delta(x) = wx - xw$, $x \in \mathfrak{U}$.*

If $\mathfrak{U}$ is equal to either $S(\mathcal{M})$ or $S(\tau)$, then it actually follows that the element w in Corollary 5.8 may be chosen from the algebra $\mathfrak{U}$ itself (that is, δ is inner). This will follow from the general result of Proposition 5.17, which is of interest in its own right. The proof of this proposition is divided into a number of lemmas.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H and suppose that $\mathcal{A}$ is a $*$-subalgebra of $LS(\mathcal{M})$. If $w \in LS(\mathcal{M})$ is such that $[w, x] \in \mathcal{A}$ for all $x \in \mathcal{A}$, then the map $\delta_w : \mathcal{A} \to \mathcal{A}$, defined by setting $\delta_w(x) = [w, x]$, $x \in \mathcal{A}$, is a derivation in $\mathcal{A}$. In general, δ_w need not be inner in $\mathcal{A}$. It will be shown that if $\mathcal{A}$ is *absolutely solid* in $LS(\mathcal{M})$ (that is, if $x \in LS(\mathcal{M})$ and $y \in \mathcal{A}$ are such that $|x| \leqslant |y|$, then $x \in \mathcal{A}$) and if $\mathcal{M} \subseteq \mathcal{A}$, then such a derivation δ_w is always inner in $\mathcal{A}$, that is, there exists $w_1 \in \mathcal{A}$ such that $\delta_w = \delta_{w_1}$.

The following theorem will be used (see Theorem 6.2.7 of [11]).

THEOREM 5.10. *If $e, f \in P(\mathcal{M})$, then there exist unique mutually orthogonal central projection p_0 and q_0 which are maximal with respect to the properties:*
   (i) $q_0 e \sim q_0 f$;
   (ii) $pe \prec pf$ whenever $p \in P(\mathcal{Z}(\mathcal{M}))$ satisfies $p \leqslant p_0$;
   (iii) $pf \prec pe$ whenever $p \in P(\mathcal{Z}(\mathcal{M}))$ satisfies $p \leqslant \mathbf{1} - p_0 - q_0$.

For $e, f \in P(\mathcal{M})$ define $z(e, f) \in P(\mathcal{Z}(\mathcal{M}))$ by setting $z(e, f) = p_0 + q_0$. It should be observed that $z(e, f)$ is also given by

(5.5) $z(e, f) = \bigvee\{p \in P(\mathcal{Z}(\mathcal{M})) : pe \precsim pf\}.$

Moreover, $z(f, e) = \mathbf{1} - p_0$. This implies, in particular, that

$$z(f, e) \geqslant \mathbf{1} - z(e, f)$$

and so

(5.6) $z(e, f) \vee z(f, e) = \mathbf{1}.$

Let $a \in LS_h(\mathcal{M})$ with spectral measure e^a. For $n \in \mathbb{Z}$, define the central projections p_n^a and q_n^a by setting

$$p_n^a = z(e^a(-\infty, n], e^a(n+1, \infty)), \quad q_n^a = z(e^a(n+1, \infty), e^a(-\infty, n]).$$

In the proof of Lemma 5.12 the following observation will be used.

LEMMA 5.11. *If $\{e_n\}_{n=1}^{\infty}$ is a sequence of finite projections in $\mathcal{M}$ such that $e_n \downarrow 0$ and if $p \in P(\mathcal{M})$ is such that $p \precsim e_n$ for all n, then $p = 0$.*

Proof. Since e_1 is finite and $p \precsim e_1$, it follows that p is finite and so the projection $e = e_1 \vee p$ is also finite. The reduced von Neumann algebra $e\mathcal{M}e$ is finite, $e_n \downarrow 0$ in $e\mathcal{M}e$ and $p \precsim e_n$ in $e\mathcal{M}e$ for all n. Let $T : e\mathcal{M}e \to Z(e\mathcal{M}e)$ be the center-valued trace (see Theorem 8.2.8 of [11]). It follows from $e_n \downarrow 0$ that $T(e_n) \downarrow 0$, and $p \precsim e_n$ implies that $T(p) \leqslant T(e_n)$ for all n. Hence, $T(p) = 0$ and so $p = 0$, which proves the lemma. ∎

LEMMA 5.12. *The central projections $\{p_n^a\}_{n \in \mathbb{Z}}$ and $\{q_n^a\}_{n \in \mathbb{Z}}$ satisfy:*
  (i) $p_n^a \vee q_n^a = \mathbf{1}$ *for all $n \in \mathbb{Z}$;*
  (ii) $p_n^a \downarrow_{n \in \mathbb{Z}}$ *and* $q_n^a \uparrow_{n \in \mathbb{Z}}$*;*
  (iii) $\bigwedge_{n \in \mathbb{Z}} p_n^a = 0$ *and* $\bigwedge_{n \in \mathbb{Z}} q_n^a = 0$.

Proof. (i) This follows immediately from (5.6).
  (ii) If $n \in \mathbb{Z}$, then

$$p_{n+1}^a e^a(-\infty, n] \leqslant p_{n+1}^a e^a(-\infty, n+1] \precsim p_{n+1}^a e^a(n+2, \infty) \leqslant p_{n+1}^a e^a(n+1, \infty)$$

and so it follows from (5.5) that $p_{n+1}^a \leqslant p_n^a$. Similarly,

$$q_n^a e^a(n+2, \infty) \leqslant q_n^a e^a(n+1, \infty) \precsim q_n^a e^a(-\infty, n] \leqslant q_n^a e^a(-\infty, n+1]$$

and hence, $q_n^a \leqslant q_{n+1}^a$.
  (iii) Since $a \in LS(\mathcal{M})$, there exists a sequence $\{z_k\}_{k=1}^{\infty}$ in $P(\mathcal{Z}(\mathcal{M}))$ such that $z_k \uparrow \mathbf{1}$ and $az_k \in S(\mathcal{M})$ for all $k \in \mathbb{N}$. Defining $p_\infty \in P(\mathcal{Z}(\mathcal{M}))$ by $p_\infty = \bigwedge_{n \in \mathbb{Z}} p_n^a$, it follows that

(5.7) $$p_\infty e^a(-\infty, n] \precsim p_\infty e^a(n+1, \infty), \quad n \in \mathbb{Z}.$$

([11], Proposition 6.2.3). It follows from (5.7) that for each $k \in \mathbb{N}$,

$$p_\infty z_k e^{az_k}(-\infty, n] \precsim p_\infty z_k e^{az_k}(n+1, \infty), \quad n \in \mathbb{Z}.$$

Let $k \in \mathbb{N}$ be fixed for the moment. Since $az_k \in S(\mathcal{M})$, there exists $n_0 \in \mathbb{N}$ such that $e^{az_k}(n_0+1, \infty)$ is a finite projection. Consequently, if $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ satisfies $n \geqslant \max(m, n_0)$, then

$$p_\infty z_k e^{az_k}(-\infty, m] \leqslant p_\infty z_k e^{az_k}(-\infty, n] \precsim p_\infty z_k e^{az_k}(n+1, \infty)$$

and so it follows from Lemma 5.11 that $p_\infty z_k e^{az_k}(-\infty, m] = 0$. Letting $m \uparrow \infty$, it follows that $p_\infty z_k = 0$ for all $k \in \mathbb{N}$. Since $z_k \uparrow \mathbf{1}$, this implies that $p_\infty = 0$. This shows that $p_n^a \downarrow_{n \in \mathbb{Z}} 0$.

Similarly, if $q_{-\infty} = \bigwedge\limits_{n \in \mathbb{Z}} q_n^a$, then

$$q_{-\infty} e^a (n+1, \infty) \precsim q_{-\infty} e^a (-\infty, n], \quad n \in \mathbb{Z},$$

which implies that

$$q_{-\infty} z_k e^{a z_k} (n+1, \infty) \precsim q_{-\infty} z_k e^{a z_k} (-\infty, n], \quad n \in \mathbb{Z},$$

for all $k \in \mathbb{N}$. If $n < 0$, then $e^{a z_k} (-\infty, n] \leqslant e^{|a z_k|} [-n, \infty)$. Since $a z_k \in S(\mathcal{M})$, there exists $n_0 \in \mathbb{Z}$ such that $e^{|a z_k|} [-n, \infty)$ is a finite projection for all $n \leqslant n_0$. As above, it now follows that $q_{-\infty} z_k = 0$ for all $k \in \mathbb{N}$. Since $z_k \uparrow \mathbf{1}$, this implies that $q_{-\infty} = 0$. ∎

The next general observation concerning Boolean algebras will be needed.

LEMMA 5.13. *Let $\mathcal{B}$ be a complete Boolean algebra (with smallest element $\mathbf{0}$ and largest element $\mathbf{1}$).*

(i) If $\{A_n\}_{n \in \mathbb{Z}}$ and $\{B_n\}_{n \in \mathbb{Z}}$ are two sequences in $\mathcal{B}$ such that $A_n \downarrow_{n \in \mathbb{Z}}$, $B_n \uparrow_{n \in \mathbb{Z}}$, $\bigwedge\limits_{n \in \mathbb{Z}} A_n = 0$, $\bigwedge\limits_{n \in \mathbb{Z}} B_n = 0$ and $A_n \vee B_n = \mathbf{1}$ for all n, then $\bigvee\limits_{n \in \mathbb{Z}} (A_n \wedge B_{n+1}) = \mathbf{1}$.

(ii) If $\{A_j\}_{j \in J}$ is a countable family in $\mathcal{B}$ such that $\bigvee\limits_{j \in J} A_j = \mathbf{1}$, then there exists a disjoint family $\{B_j\}_{j \in J}$ in $\mathcal{B}$ such that $B_j \leqslant A_j$ for all $j \in J$ and $\bigvee\limits_{j \in J} B_j = \mathbf{1}$.

Proof. (i) It follows from $A_n \vee B_n = \mathbf{1}$ that $A_n \geqslant B_n^c$ and $B_n \geqslant A_n^c$ for all n. This implies, in particular, that $B_n \uparrow_{n \in \mathbb{Z}} \mathbf{1}$. Hence, for each $n \in \mathbb{Z}$,

$$B_n \vee \bigvee_{k=n}^{\infty} (B_{k+1} \setminus B_k) = \mathbf{1}$$

and so

$$\bigvee_{k=n}^{\infty} (B_{k+1} \setminus B_k) = B_n^c.$$

Consequently,

$$\bigvee_{n \in \mathbb{Z}} (B_{n+1} \setminus B_n) = \bigvee_{n \in \mathbb{Z}} B_n^c = \mathbf{1}.$$

It follows from $A_n \geqslant B_n^c$ that

$$A_n \wedge B_{n+1} \geqslant B_n^c \wedge B_{n+1} = B_{n+1} \setminus B_n$$

and hence,

$$\bigvee_{n \in \mathbb{Z}} (A_n \wedge B_{n+1}) = \mathbf{1}.$$

(ii) Without loss of generality, it may be assumed that $J = \mathbb{N}$. Defining $B_1 = A_1$ and

$$B_j = A_j \setminus (A_1 \vee \cdots \vee A_{j-1}), \quad j \geqslant 2,$$

it is clear that the sequence $\{B_j\}_{j \in \mathbb{N}}$ has the desired properties. ∎

A combination of Lemma 5.12 and Lemma 5.13 (applied to the complete Boolean algebra of central projections in $\mathcal{M}$) yields the following result.

COROLLARY 5.14. *If $a \in LS_h(\mathcal{M})$, then there exists a disjoint sequence $\{z_n\}_{n \in \mathbb{Z}}$ in $P(\mathcal{Z}(\mathcal{M}))$ such that $z_n \leqslant p_n^a q_{n+1}^a$ for all $n \in \mathbb{Z}$ and $\bigvee\limits_{n \in \mathbb{Z}} z_n = \mathbf{1}$.*

LEMMA 5.15. *If $a \in LS_h(\mathcal{M})$ and $n \in \mathbb{Z}$ satisfies $e^a(-\infty, n] \precsim e^a(n+1, \infty)$ and $e^a(n+2, \infty) \precsim e^a(-\infty, n+1]$, then there exists a partial isometry $v \in \mathcal{M}$ such that*

$$|av - va| + \mathbf{1} \geqslant |a - (n+1)\mathbf{1}|.$$

Proof. Let $u_1, u_2 \in \mathcal{M}$ be partial isometries such that

$$u_1^* u_1 = e^a(-\infty, n] \quad \text{and} \quad u_1 u_1^* \leqslant e^a(n+1, \infty),$$
$$u_2^* u_2 = e^a(n+2, \infty) \quad \text{and} \quad u_2 u_2^* \leqslant e^a(-\infty, n+1].$$

It should be observed that

$$u_1^* u_2 = (e^a(n+1, \infty) u_1)^* (e^a(-\infty, n+1] u_2) = u_1^* e^a(n+1, \infty) e^a(-\infty, n+1] u_2 = 0$$

and hence, $u_2^* u_1 = (u_1^* u_2)^* = 0$. Defining $v = u_1 + u_2$, this implies that

$$v^* v = u_1^* u_1 + u_2^* u_2 = e^a(-\infty, n] + e^a(n+2, \infty)$$

and so v is a partial isometry.

Since

$$|v^* av - v^* va|^2 = |v^*(av - va)|^2 = (av - va)^* vv^*(av - va)$$
$$\leqslant (av - va)^*(av - va) = |av - va|^2,$$

it follows that $|v^* av - v^* va| \leqslant |av - va|$. Therefore, it is sufficient to show that

(5.8) $$|v^* av - v^* va| + \mathbf{1} \geqslant |a - (n+1)\mathbf{1}|.$$

It should be observed that $v^* v$ is a projection which commutes with a and that the operator $v^* av - v^* va$ is self-adjoint.

Note that

$$u_1^* au_2 = (e^a(n+1, \infty) u_1)^* a(e^a(-\infty, n+1] u_2) = u_1^* e^a(n+1, \infty) ae^a(-\infty, n+1] u_2 = 0$$

and hence, $u_2^* au_1 = (u_1^* au_2)^* = 0$. This implies that

$$v^* av = u_1^* au_1 + u_2^* au_2$$

and so

(5.9) $$v^* av - v^* va = [u_1^* au_1 - a]e^a(-\infty, n] + [u_2^* au_2 - a]e^a(n+2, \infty).$$

Since

$$u_1^* au_1 = u_1^* ae^a(n+1, \infty) u_1 \geqslant (n+1) u_1^* e^a(n+1, \infty) u_1 = (n+1) u_1^* u_1 = (n+1) e^a(-\infty, n]$$

and

$$ae^a(-\infty, n] \leqslant ne^a(-\infty, n] \leqslant (n+1) e^a(-\infty, n],$$

it follows that

$$[u_1^* a u_1 - a] e^a(-\infty, n] \geqslant -[a - (n+1)\mathbf{1}] e^a(-\infty, n] \geqslant 0.$$

Similarly,

$$u_2^* a u_2 = u_2^* a e^a(-\infty, n+1] u_2 \leqslant (n+1) u_2^* u_2 = (n+1) e^a(n+2, \infty)$$

and

$$a e^a(n+2, \infty) \geqslant (n+2) e^a(n+2, \infty) \geqslant (n+1) e^a(n+2, \infty)$$

and so

$$-[u_2^* a u_2 - a] e^a(n+2, \infty) \geqslant [a - (n+1)\mathbf{1}] e^a(n+2, \infty) \geqslant 0.$$

The above estimates, in combination with (5.9), show that

(5.10) $$|v^* a v - v^* v a| \geqslant |a - (n+1)\mathbf{1}| (e^a(-\infty, n] + e^a(n+2, \infty)).$$

Observe next that

$$n e^a(n, n+2] \leqslant a e^a(n, n+2] \leqslant (n+2) e^a(n, n+2]$$

and so

$$-e^a(n, n+2] \leqslant [a - (n+1)\mathbf{1}] e^a(n, n+2] \leqslant e^a(n, n+2].$$

This implies that

(5.11) $$e^a(n, n+2] \geqslant |a - (n+1)\mathbf{1}| e^a(n, n+2].$$

Adding (5.11) and (5.10) yields

$$|v^* a v - v^* v a| + e^a(n, n+2] \geqslant |a - (n+1)\mathbf{1}|,$$

from which (5.8) follows. The proof is complete. ∎

LEMMA 5.16. *If $a \in LS_h(\mathcal{M})$, then there exist a partial isometry $v \in \mathcal{M}$ and an operator $a_1 \in LS_h(\mathcal{M})$ such that $a - a_1 \in \mathcal{Z}(LS(\mathcal{M}))_h$ and*

$$|av - va| + \mathbf{1} \geqslant |a_1|.$$

Proof. Let the sequence $\{z_n\}_{n \in \mathbb{Z}}$ in $P(\mathcal{Z}(\mathcal{M}))$ be as in Corollary 5.14. For each n, the operator $a z_n$ satisfies the conditions of Lemma 5.15 (with respect to the reduced von Neumann algebra $\mathcal{M} z_n$). Consequently, for each $n \in \mathbb{Z}$ there exists a partial isometry $v_n \in \mathcal{M} z_n$ such that

$$|(a z_n) v_n - v_n(a z_n)| + z_n \geqslant |a z_n - (n+1) z_n|.$$

Defining the partial isometry $v \in \mathcal{M}$ by setting $v = \sum_{n \in \mathbb{Z}} v_n z_n$, the operator $b \in \mathcal{Z}(LS(\mathcal{M}))_h$ by $b = \sum_{n \in \mathbb{Z}} (n+1) z_n$ and $a_1 = a - b$, it follows that $a - a_1 = b \in \mathcal{Z}(LS(\mathcal{M}))_h$ and

$$|av - va| + \mathbf{1} \geqslant |a_1|.$$

The proof of the lemma is complete. ∎

We are now in a position to prove the already advertised proposition.

PROPOSITION 5.17. *Let $\mathcal{A}$ be a $*$-subalgebra of $LS(\mathcal{M})$ such that $\mathcal{M} \subseteq \mathcal{A}$ and $\mathcal{A}$ is absolutely solid (that is, if $x \in LS(\mathcal{M})$ and $y \in \mathcal{A}$ satisfy $|x| \leqslant |y|$, then $x \in \mathcal{A}$). If $w \in LS(\mathcal{M})$ is such that $[w, x] \in \mathcal{A}$ for all $x \in \mathcal{A}$, then there exists $w_1 \in \mathcal{A}$ such that $[w, x] = [w_1, x]$ for all $x \in \mathcal{A}$.*

Proof. First observe that $[w^*, x] = -[w, x^*]^* \in \mathcal{A}$ for all $x \in \mathcal{A}$. Consequently, $\mathrm{Re}\,w$ and $\mathrm{Im}\,w$ satisfy the hypothesis of the proposition. Therefore, without loss of generality, it may be assumed that w is self-adjoint. It follows from Lemma 5.16 that there exists a partial isometry $v \in \mathcal{M}$ and an operator $w_1 \in LS(\mathcal{M})$ such that $w - w_1 \in \mathcal{Z}(LS_h(\mathcal{M}))$ and $|[w, v]| + \mathbf{1} \geqslant |w_1|$. Since $|[w, v]| + \mathbf{1} \in \mathcal{A}$, the assumptions on $\mathcal{A}$ imply that $w_1 \in \mathcal{A}$. Finally, since $w - w_1 \in \mathcal{Z}(LS_h(\mathcal{M}))$, it is also clear that $[w, x] = [w_1, x]$ for all $x \in \mathcal{A}$. ∎

Combining Proposition 5.17 with Corollary 5.8 and Corollary 5.9 we obtain the following results.

COROLLARY 5.18. *Let H be a separable Hilbert space, (S, Σ, ν) be a σ-finite measure space and $\mathcal{M} = L_\infty(\nu)\overline{\otimes}B(H)$. Any $\mathcal{Z}(\mathcal{M})$-linear derivation on each of the algebras $LS(\mathcal{M})$, $S(\mathcal{M})$ or $S(\tau)$ is inner.*

Proof. For the algebra $LS(\mathcal{M})$, this is already stated in Corollary 5.8. Since the algebras $S(\mathcal{M})$ and $S(\tau)$ are solid in $LS(\mathcal{M})$ and $\mathcal{M}$ is contained in both algebras, the result follows from Corollary 5.8 in combination with Proposition 5.17. ∎

COROLLARY 5.19. *Let H be a separable infinite dimensional Hilbert space, (S, Σ, ν) be a σ-finite measure space and $\mathcal{M} = L_\infty(\nu)\overline{\otimes}B(H)$. Any derivation on each of the algebras $LS(\mathcal{M})$, $S(\mathcal{M})$ or $S(\tau)$ is inner.*

It should be noted that a derivation on $S_0(\tau)$ need not be inner. Indeed, assuming that H is infinite dimensional, let $p \in P(B(H))$ be such that $\mathrm{tr}(p) = \mathrm{tr}(\mathbf{1} - p) = \infty$ and define $w \in P(\mathcal{M})$ by $w = \chi_S \otimes p$. It is easily verified that the derivation δ in $S_0(\tau)$, given by $\delta(f) = wf - fw$, $f \in S_0(\tau)$, is not inner.

REMARK 5.20. It should be observed that if $\mathcal{M} = L_\infty(\nu)\overline{\otimes}B(H)$, where $S = [0, 1]$ equipped with Lebesgue measure ν and H is an infinite dimensional separable Hilbert space, then

$$\mathcal{M} \subsetneq S(\tau) \subsetneq S(\mathcal{M}) \subsetneq LS(\mathcal{M}).$$

To verify this claim, we make the following observations.

(i) Let $p_0 \in B(H)$ be a rank one projection and $a \in L_0(\nu)$. Define $f \in L_0^{\mathrm{wo}}(\nu; B(H)) = LS(\mathcal{M})$ by setting $f = a \otimes p_0$. Since $\tau(\mathbf{1} \otimes p_0) = 1$ and $f(\mathbf{1} \otimes p_0)^\perp = 0$, it is clear that $f \in S_0(\tau) \subseteq S(\tau)$. If $a \notin L_\infty(\nu)$, then $f \notin \mathcal{M}$. Hence, $\mathcal{M} \subsetneq S(\tau)$.

(ii) Let $f \in L_0^{\mathrm{wo}}(\nu; B(H))$ be such that $f(s) \in K(H)$ ν-a.e. on S and let $e^{|f|}$ be the spectral measure of $|f|$ (see Corollary 5.6). Defining $p_n = e^{|f|}(n, \infty)$, $n \in \mathbb{N}$, it follows that $p_n(s) = e^{|f(s)|}(n, \infty)$ is a finite rank projection for ν-almost all $s \in S$

and hence, by Lemma 5.1, p_n is a finite projection in $\mathcal{M}$. Since $f p_n^\perp \in \mathcal{M}$ for all n and $p_n^\perp \uparrow \mathbf{1}$ in $\mathcal{M}$, it follows that $f \in S(\mathcal{M})$.

For $n \in \mathbb{N}$, define the intervals I_n by $I_n = (2^{-n}, 2^{-n+1}]$ and let $q_n \in P(B(H))$ be such that $\mathrm{tr}(q_n) = 2^n$. Let $f \in L_0^{\mathrm{wo}}(\nu; B(H))^+$ be defined by setting

$$f = \sum_{n=1}^\infty n \chi_{I_n} \otimes q_n,$$

as a pointwise convergent series. If $0 \leqslant \lambda \in \mathbb{R}$ and $s \in I_n$, then $e^{f(s)}(\lambda, \infty) = e^{n q_n}(\lambda, \infty)$, which is equal to q_n if $0 \leqslant \lambda < n$ and equal to 0 if $\lambda \geqslant n$. Consequently,

$$\tau(e^f(\lambda, \infty)) = \sum_{n > \lambda} \mathrm{tr}(q_n) \nu(I_n) = \infty$$

for all $\lambda \geqslant 0$. This shows that $f \notin S(\tau)$ and hence, $S(\tau) \subsetneq S(\mathcal{M})$.

It should be observed that the function $g \in L_\infty^{\mathrm{wo}}(\nu; B(H))^+$, given by $g = \sum_{n=1}^\infty \chi_{I_n} \otimes q_n$, takes its values in $F(H)$ but, $\tau(e^g(\lambda, \infty)) = \infty$ whenever $0 \leqslant \lambda < 1$ and so $g \notin S_0(\tau)$.

(iii) If $a \in L_0(\nu)$, then it is clear that $a \otimes \mathbf{1} \in LS(\mathcal{M})$. We claim that $a \otimes \mathbf{1} \in S(\mathcal{M})$ only if $a \in L_\infty(\nu)$. Indeed, if $a \otimes \mathbf{1} \in S(\mathcal{M})$, then there exists a projection $p \in P(\mathcal{M})$ such that $(a \otimes \mathbf{1})p \in L_\infty^{\mathrm{wo}}(\nu; B(H))$ and $p^\perp$ is finite (that is, $p(s)^\perp$ is finite rank ν-a.e.). Define the ν-measurable set $A \subseteq S$ by setting

$$A = \{s \in S : |a(s)| > \|(a \otimes \mathbf{1})p\|_\infty\}.$$

Since $|a(s)| \|p(s)\|_{B(H)} \leqslant \|(a \otimes \mathbf{1})p\|_\infty$ ν-a.e. on S, it follows that $p(s) = 0$ ν-a.e. on A, that is, $p(s)^\perp = \mathbf{1}$ ν-a.e. on A. Consequently, $\nu(S) = 0$ and hence, $a \in L_\infty(\nu)$. It is now clear that $S(\mathcal{M}) \subsetneq LS(\mathcal{M})$.

REMARK 5.21. As follows from Corollary 5.18, any derivation δ in $LS(\mathcal{M})$, $S(\mathcal{M})$ or $S(\tau)$, where $\mathcal{M} = L_\infty(\nu) \overline{\otimes} B(H)$, is inner and hence, δ is $\mathcal{Z}$-linear. The main ingredient to obtain this $\mathcal{Z}$-linearity has been Proposition 4.9 (and Remark 4.10). In the setting of von Neumann algebra, the $\mathcal{Z}$-linearity of derivations also follows from the following proposition, which is of interest in its own right. The proof of this proposition follows the same lines as the proof of Proposition 4.9, although there are some non-trivial other ingredients. However, in view of the length of the present paper, a detailed proof of this proposition will appear elsewhere.

PROPOSITION 5.22. *Suppose that $\mathcal{M}$ is properly infinite, that is, every non-zero projection in $P(\mathcal{Z}(\mathcal{M}))$ is infinite (with respect to $\mathcal{M}$). Let $\mathcal{A} \subseteq LS(\mathcal{M})$ be a $*$-subalgebra such that $\mathcal{M} \subseteq \mathcal{A}$. If $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation, then δ is $\mathcal{Z}$-linear.*

A typical example of a von Neumann algebra $\mathcal{M}$ which satisfies the conditions of the previous proposition is $\mathcal{M} = L_\infty(\nu) \overline{\otimes} \mathcal{N}$, where $\mathcal{N}$ is a factor for which $\mathbf{1}_\mathcal{N}$ is infinite. Indeed, if $0 \neq p \in \mathcal{Z}(\mathcal{M})$, then $p = \chi_A \otimes \mathbf{1}_\mathcal{N}$ for some projection $\chi_A \in L_\infty(\nu)$. Since $\mathbf{1}_\mathcal{N}$ is infinite, there exists $q \in P(\mathcal{N})$ such that $q \neq \mathbf{1}_\mathcal{N}$ and

$q \sim \mathbf{1}_{\mathcal{N}}$. This implies that $\chi_A \otimes q < p$ and $\chi_A \otimes q \sim p$. Hence, p is an infinite projection in $\mathcal{M}$.

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A.F. BER, CHIEF SOFTWARE DEVELOPER, ISV "SOLUTIONS", TASHKENT, UZBEK-ISTAN
E-mail address: ber@ucd.uz

B. DE PAGTER, DELFT INSTITUTE OF APPLIED MATHEMATICS, FACULTY EEMCS, DELFT UNIVERSITY OF TECHNOLOGY, P.O. BOX 5031, 2600 GA DELFT, THE NETHER-LANDS
E-mail address: b.depagter@tudelft.nl

F.A. SUKOCHEV, SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW SOUTH WALES, KENSINGTON, NSW 2052, AUSTRALIA
E-mail address: f.sukochev@unsw.edu.au