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ORBIT REFLEXIVITY OF COMPOSITIONS ON WEIGHTED HARDY–HILBERT SPACES

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ABSTRACT. We establish orbit reflexivity of composition operators and their adjoints acting on a special family of weighted Hardy–Hilbert spaces.

KEYWORDS: *Composition operators, weighted Hardy–Hilbert spaces, reproducing kernels, orbit reflexivity, multiplier space.*

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1. INTRODUCTION

We start with a review on weighted Hardy–Hilbert spaces and on orbit reflexivity. After that, in the next section, we prove the two main results of this paper, Theorem 2.8 and Theorem 2.9, which assert that every composition operator and every adjoint of a composition operator is orbit reflexive.

1.1. WEIGHTED HARDY–HILBERT SPACES. For a sequence of positive numbers $\beta = \{\beta(j)\}_{j=0}^{\infty}$ with $\beta(0) = 1$, we follow Allen Shields [7] in defining the weighted Hardy–Hilbert space, denoted $H^2(\beta)$, to be the set of formal power series $f = \sum\limits_{j=0}^{\infty} a_j z^j$ whose coefficients are square-summable when weighted by β's; i.e.,

$$H^2(\beta) = \Big\{ f = \sum_{j=0}^{\infty} a_j z^j : \|f\|_\beta^2 = \sum_{j=0}^{\infty} |a_j|^2 \beta(j)^2 < \infty \Big\},$$

with the inner product given by

$$\Big\langle \sum_{j=0}^{\infty} a_j z^j, \sum_{j=0}^{\infty} c_j z^j \Big\rangle = \sum_{j=0}^{\infty} a_j \bar{c}_j \beta(j)^2.$$

In order to ensure that the formal power series $\sum_{j=0}^{\infty} a_j z^j$ actually represents an analytic function on the unit disk, we will assume that

$$\lim_{j\to\infty} \frac{\beta(j+1)}{\beta(j)} = 1. \tag{1.1}$$

It is immediate from Proposition 15 and Theorem 10 in [7] that this condition implies that the formal power series in $H^2(\beta)$ represent functions which are analytic in the unit disk.

Let $\mathbb{D}$ be the open unit disk in the complex plane. For $\omega \in \mathbb{D}$, the linear functional of evaluation at ω is continuous, so, by the Riesz representation theorem, there is a function K_ω in $H^2(\beta)$ such that $f(\omega) = \langle f, K_\omega \rangle$ for all $f \in H^2(\beta)$. The function K_ω is called the reproducing kernel. The reproducing kernel is analytic on the open unit disk and can be written using the generating function $k(z) = \sum_{j=0}^{\infty} \frac{z^j}{\beta(j)^2}$ as $K_\omega(z) = k(\overline{\omega} z)$. Moreover, we also have $\|K_\omega\|^2 = k(|\omega|^2)$.

Let φ be a map from $\mathbb{D}$ to itself. We define C_φ, the composition operator induced by φ, acting on $H^2(\beta)$, by

$$(C_\varphi f)(z) = f(\varphi(z)) \quad \forall z \in \mathbb{D} \text{ and } f \in H^2(\beta).$$

The action of the adjoint of the composition operators C_φ^* on the reproducing kernels is easily seen to be given by $C_\varphi^* K_\omega = K_{\varphi(\omega)}$.

It is well known that, on the classical Hardy–Hilbert space where $\beta(j) \equiv 1$, every composition operator is a bounded operator. On a general $H^2(\beta)$ space not every composition operator is bounded. We will restrict our attention to spaces on which all composition operators are bounded. Our next assumption on the sequence of β's is that they are decreasing; i.e., they satisfy

$$\beta(0) \geqslant \beta(1) \geqslant \beta(2) \geqslant \cdots . \tag{1.2}$$

We have the following result from [1], page 119: on such a space, if φ satisfies $\varphi(0) = 0$, then C_φ is bounded on $H^2(\beta)$ and $\|C_\varphi\| = 1$. We also note that this condition automatically guarantees the following:

$$k(1) = \sum_{j=0}^{\infty} \frac{1}{\beta(j)^2} = \infty, \tag{1.3}$$

which, in addition to being important by itself (see Proposition 2.5), also guarantees that the simple relation $C_\varphi^{-1} = C_{\varphi^{-1}}$ is valid whenever φ is invertible (Theorems 1.6 and 2.15 of [1]).

We say that $H^2(\beta)$ is disk-automorphism invariant if all disk-automorphisms induce bounded composition operators. Our final requirement on the sequence of the β's is that the space $H^2(\beta)$ is disk-automorphism invariant. There are several sufficient conditions for a space to be disk-automorphism invariant which, for example, can be found in [1], page 119. The nested family S_α, $\alpha \leqslant 0$

corresponding to $\beta(j) = (j+1)^\alpha$, is an example of a family which satisfies all of our conditions, and can be considered as a prototype.

We are now in position to show that condition (1.2) and disk-automorphism invariance guarantee the boundness of composition operators.

PROPOSITION 1.1. *Let the sequence of β's be decreasing and such that the space $H^2(\beta)$ is disk-automorphism invariant. Then every composition operator on $H^2(\beta)$ is bounded.*

Proof. Let φ be any analytic function from the unit disk into itself. Let $\varphi_0 = \psi \circ \varphi$, where

$$\psi(z) = \frac{\varphi(0) - z}{1 - \overline{\varphi(0)}z}.$$

We have $\varphi_0(0) = 0$. Since $\psi^{-1} = \psi$ is an automorphism, we also have

$$C_\varphi = C_{\varphi_0} C_{\psi^{-1}} = C_{\varphi_0} C_\psi.$$

Thus

$$\|C_\varphi\| \leqslant \|C_{\varphi_0}\| \cdot \|C_\psi\| = 1 \cdot \|C_\psi\| = \|C_\psi\| < \infty,$$

where in the last step we have used the fact that $H^2(\beta)$ is disk-automorphism invariant. ∎

We denote the n-th iteration of φ by $\varphi^{(n)}$; i.e.,

$$\varphi^{(n)} = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}},$$

with the corresponding composition operator

$$C_{\varphi^{(n)}} = C_\varphi^n.$$

The definition and the existence of the Denjoy–Wolff point is included in the next theorem (see, for example, [1], [6]):

THEOREM 1.2 (Denjoy–Wolff Theorem). *Let φ be an analytic self-map of the unit disk other than an elliptic disk-automorphism. Then φ has a unique fixed point in $\overline{\mathbb{D}}$ (called the Denjoy–Wolff point) such that $|\varphi'(a)| \leqslant 1$, and, if $a \in \partial\mathbb{D}$, then $0 < \varphi'(a) \leqslant 1$. Moreover, $\varphi^{(n)} \to a$ uniformly on compact subsets of $\mathbb{D}$.*

PROPOSITION 1.3. *Let the sequence of β's satisfy* (1.2) *and be such that the space $H^2(\beta)$ is disk-automorphism invariant. Assume further that φ, an analytic self-map of the unit disk, has its Denjoy–Wolff point a in $\mathbb{D}$. Then C_φ is power bounded.*

Proof. The proof is similar to the proof of Proposition 1.1. Define the function $\varphi_0 = \psi \circ \varphi \circ \psi^{-1}$, where ψ is the following automorphism:

$$\psi(z) = \frac{a - z}{1 - \bar{a}z}.$$

We have $\varphi_0(0) = 0$, as a is a fixed point of φ. Furthermore,

$$\varphi_0 = \psi \circ \varphi \circ \psi^{-1} \implies \varphi = \psi^{-1} \circ \varphi_0 \circ \psi \implies \varphi^{(n)} = \psi^{-1} \circ \varphi_0^{(n)} \circ \psi.$$

Notice that we have $\varphi_0^{(n)}(0) = 0$ for all n. Using the relation $C_\psi C_\varphi = C_{\varphi\circ\psi}$, we obtain $C_{\varphi^{(n)}} = C_\psi C_{\varphi_0^{(n)}} C_{\psi^{-1}}$. Thus

$$\|C_\varphi^n\| = \|C_{\varphi^{(n)}}\| = \|C_\psi C_{\varphi_0^{(n)}} C_{\psi^{-1}}\| \leqslant \|C_\psi\| \cdot \|C_{\varphi_0^{(n)}}\| \cdot \|C_{\psi^{-1}}\| = \|C_\psi\| \cdot 1 \cdot \|C_{\psi^{-1}}\| < \infty,$$

where in the last step we have used the fact that $H^2(\beta)$ is disk-automorphism invariant. ∎

Finally, we need to know the multiplier set of $H^2(\beta)$, which we denote by $H^\infty(\beta)$. The following is an easy corollary of Problem 68 in [4] and Lemma 5 in [8].

COROLLARY 1.4. *$H^\infty(\beta) = H^\infty(\mathbb{D})$ with equal norms.*

This corollary will be used in the proof of Proposition 2.6.

1.2. ORBIT REFLEXIVITY.

DEFINITION 1.5. An operator A is *orbit reflexive* if whenever an operator B leaves invariant all closed sets invariant under A, then B is in the strongly closed semigroup generated by A and the identity operator I.

Orbit reflexive operators have been studied extensively in [3], where the authors give sufficient conditions for an operator to be orbit reflexive. They also conjectured that there were operators that were not orbit reflexive. An example has recently been found by Grivaux and Roginskaya [2]. The operator in their example is obtained by a modified Read-type construction and it is quite complicated. Another example was also found very recently by Muller and Vršovský [5]. They gave a simple Hilbert space example, and also a second example, this time on a Banach space, of an operator which is reflexive but not orbit reflexive.

In [3] it is proven that any Hilbert space operator similar to a contraction is orbit reflexive. Thus, whenever φ has its Denjoy–Wolff point a in $\mathbb{D}$ then $C_\varphi : H^2(\beta) \to H^2(\beta)$ is orbit reflexive since it is similar to the contraction C_{φ_0} for $\varphi_0 = \psi \circ \varphi \circ \psi$, where $\psi(z) = \frac{a-z}{1-\bar{a}z}$. Nevertheless, we will prove orbit reflexivity of composition operators and their adjoints without reliance on any result from [3].

2. ORBIT REFLEXIVITY OF COMPOSITION OPERATORS

In this section we prove that every composition operator and every adjoint of a composition operator is orbit reflexive. We start by stating a proposition that we call "the standard argument", and then apply it to composition operators and their adjoints.

2.1. PRELIMINARIES. The following proposition is fundamental.

PROPOSITION 2.1. *Let I be any uncountable subset of $\mathbb{D}$. Then*

$$\bigvee_{\omega \in I} \{K_\omega\} = H^2(\beta).$$

Proof. Assume $f \in \Big(\bigvee_{\omega \in I} \{K_\omega\} \Big)^{\perp}$. Then $f(\omega) = \langle f, K_\omega \rangle = 0$ for all $\omega \in I$. Since I is uncountable it has an accumulation point in $\mathbb{D}$. Since f is analytic in $\mathbb{D}$, and has zeros with an accumulation point in $\mathbb{D}$, f must be the zero function. Thus $\Big(\bigvee_{\omega \in I} \{K_\omega\} \Big)^{\perp} = 0$, and hence $\bigvee_{\omega \in I} \{K_\omega\} = H^2(\beta)$. ∎

We are now ready to state and prove our standard argument.

PROPOSITION 2.2 (The standard argument). *Assume that A and B are operators on $H^2(\beta)$ such that A is not an iterate of B and, for every ω in $\mathbb{D}$, AK_ω is in the closure of the orbit of K_ω with respect to B. Then there exists a dense subset E of $\mathbb{D}$ ($\mathbb{D} \setminus E$ is countable) such that, for every ω in E, AK_ω is a limit in $H^2(\beta)$ of a subsequence of the sequence of iterates of B applied to K_ω.*

Proof. Let G be the set of ω's in $\mathbb{D}$ such that AK_ω is an iterate of B applied to K_ω. If G is uncountable, then there exists a nonnegative integer m such that $AK_\omega = B^m K_\omega$ for an uncountable set of ω's in G. Since, by Proposition 2.1, the closed linear span of an uncountable set of K_ω, ω in $\mathbb{D}$, is $H^2(\beta)$, we get that A is the m-th iterate of B, contrary to the assumption. Thus G is countable, so the set $E = \mathbb{D} \setminus G$ has the required property. ∎

In order to be able to apply Proposition 2.2 to composition operators we need some more preliminaries.

The following proposition is well known for $H^2(\mathbb{D})$.

PROPOSITION 2.3. *If $\{f_n\} \xrightarrow{\beta} f$ in $H^2(\beta)$, then $\{f_n\} \to f$ uniformly on compact subsets of $\mathbb{D}$.*

Proof. Let K be a compact subset of $\mathbb{D}$. For a fixed $\omega \in \mathbb{D}$, using the Cauchy-Schwartz inequality, we obtain

$$|f_n(\omega) - f(\omega)| = |(f_n - f)(\omega)| = |(f_n - f, K_\omega)| \leqslant \|f_n - f\|_\beta \cdot \|K_\omega\|_\beta. \tag{2.1}$$

We know that $\|K_\omega\|^2 = k(|\omega|^2)$, so if we take $M^2 = \sup_{\omega \in K} k(|\omega|^2) < \infty$ we obtain $\|K_\omega\| \leqslant M$ for all $\omega \in K$. Using this in equation (2.1) gives

$$|f_n(\omega) - f(\omega)| \leqslant M \|f_n - f\|_\beta \quad \forall\, \omega \in K,$$

which clearly implies the proposition. ∎

We need two more technical results about the reproducing kernel K_ω. We start with a simple proposition.

PROPOSITION 2.4. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers inside $\mathbb{D}$ such that $\lim\limits_{n\to\infty} a_n = a$, where $a \in \mathbb{D}$. Then $\{K_{a_n}\}_{n=1}^{\infty}$ converges in norm to K_a.*

Proof. We have

$$\begin{aligned}\|K_{a_n} - K_a\|_\beta^2 &= \langle K_{a_n} - K_a, K_{a_n} - K_a\rangle = \langle K_{a_n}, K_{a_n}\rangle - \langle K_{a_n}, K_a\rangle - \langle K_a, K_{a_n}\rangle + \langle K_a, K_a\rangle \\ &= \|K_{a_n}\|^2 - K_{a_n}(a) - K_a(a_n) + \|K_a\|^2 = k(|a_n|^2) - k(\overline{a}_n a) - k(\overline{a} a_n) \\ &\quad + k(|a|^2) \xrightarrow{n\to\infty} k(|a|^2) - k(|a|^2) - k(|a|^2) + k(|a|^2) = 0. \quad \blacksquare\end{aligned}$$

PROPOSITION 2.5. *Let $a \in \overline{\mathbb{D}}$. If A is a bounded operator that satisfies $AK_\omega = K_\omega(a)$ for all $\omega \in \mathbb{D}$, then $a \in \mathbb{D}$.*

Proof. Let us assume that $a \in \partial\mathbb{D}$, and deduce a contradiction. We will prove that

$$\sup_{\omega\in\mathbb{D}} \frac{\|AK_\omega\|}{\|K_\omega\|} = \infty,$$

contradicting the hypothesis that A is bounded.

Let $\omega_n = \left(1 - \frac{1}{n}\right)a \in \mathbb{D}$. Then

$$\frac{\|AK_\omega\|}{\|K_\omega\|} = \frac{K_\omega(a)}{\|K_\omega\|} = \frac{k(\overline{\omega} a)}{\sqrt{k(|\omega|^2)}}.$$

Since $a\overline{a} = |a|^2 = 1$ we have

$$\sup_{\omega\in\mathbb{D}} \frac{\|AK_\omega\|}{\|K_\omega\|} = \sup_{\omega\in\mathbb{D}} \frac{k(\overline{\omega} a)}{\sqrt{k(|\omega|^2)}} \geqslant \sup_n \frac{k(\overline{\omega}_n a)}{\sqrt{k(|\omega_n|^2)}} = \sup_n \frac{k\left(1 - \frac{1}{n}\right)}{\sqrt{k\left(\left(1 - \frac{1}{n}\right)^2\right)}}.$$

Finally, since we are also assuming that $k(1) = \infty$ (equation (1.3)), this last supremum is obviously infinite. $\blacksquare$

2.2. APPLICATION TO COMPOSITION OPERATORS.

For the proof that every composition operator and every adjoint of a composition operator is orbit reflexive, we need the following two propositions.

PROPOSITION 2.6. *Let φ be an analytic self-map of the unit disk with Denjoy–Wolff point $a \in \mathbb{D}$. Let the operator A be defined by the equations $Af = \langle f, K_a\rangle$ for all $f \in H^2(\beta)$. If Az belongs to the $H^2(\beta)$-closure of $\{\varphi^{(n)}\}_{n=0}^{\infty}$, then*

$$A = \lim_{n\to\infty} C_\varphi^n$$

in the strong operator topology.

Proof. If $a = Az = \varphi^{(k)}$ for some positive integer k, we have $\varphi = a$ (since every analytic function which is not constant is an open map). This would imply $A = \langle \cdot, K_a\rangle = C_a = C_\varphi$, and so the result follows. So, without loss of generality, assume that there is a subsequence $\{m_j\}_{j=0}^{\infty}$ of the sequence of nonnegative integers such that

$$Az = \lim_{j\to\infty} C_{\varphi}^{m_j} z. \tag{2.2}$$

For a fixed $n \geqslant 1$ we have

$$\begin{aligned}
\|(C_{\varphi}^{m_j}-A)z^n\| &= \|(\varphi^{(m_j)})^n - a^n\| = \|(\varphi^{(m_j)}-a)((\varphi^{(m_j)})^{n-1}+a(\varphi^{(m_j)})^{n-1}+\cdots+a^{n-1})\| \\
&\overset{(\heartsuit)}{\leqslant} \|\varphi^{(m_j)} - a\| \cdot \|(\varphi^{(m_j)})^{n-1} + \cdots + a^{n-1}\|_{H^\infty(\mathbb{D})} \\
&\leqslant \|\varphi^{(m_j)} - a\| \cdot (\|(\varphi^{(m_j)})^{n-1}\|_{H^\infty} + \cdots + \|a^{n-1}\|_{H^\infty}) \\
&\leqslant n \cdot \|\varphi^{(m_j)} - a\| = n \cdot \|(C_{\varphi}^{m_j} - A)z\| \overset{j\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where we applied Corollary 1.4 in the step ($\heartsuit$), and used equation (2.2) in the last step. Hence we proved that

$$A(z^n) = \lim_{j\to\infty} C_{\varphi}^{m_j}(z^n) \quad \forall\, n \geqslant 1. \tag{2.3}$$

We also have

$$A1 = \langle 1, K_a\rangle = 1 = \lim_{j\to\infty} C_{\varphi}^{m_j} 1. \tag{2.4}$$

From equations (2.3) and (2.4) it follows that $\{C_{\varphi}^{m_j}\}$ converges to A on the basis elements $\{z^n\}_{n=0}^{\infty}$. So, $A = \lim_{j\to\infty} C_{\varphi}^{m_j}$ on a dense set. Since φ has Denjoy–Wolff point $a \in \mathbb{D}$, we know by Theorem 1.3 that C_φ is power bounded. Hence, we conclude that $A = \lim_{j\to\infty} C_{\varphi}^{m_j}$ in the strong operator topology.

We now claim that $A = \lim_{n\to\infty} C_{\varphi}^{n}$ in the strong operator topology. Let $f \in H^2(\beta)$ and $\varepsilon > 0$. Let M be such that $\|C_\varphi^n\| \leqslant M$ for all $n \in \mathbb{N} \cup \{0\}$. Choose j large enough so that

$$\|C_\varphi^{m_j} f - Af\| < \frac{\varepsilon}{M}.$$

For each $n > m_j$, we have

$$\|C_\varphi^n f - Af\| = \|C_\varphi^{n-m_j}(C_\varphi^{m_j} f - Af)\| \leqslant M\|C_\varphi^{m_j} f - Af\| < \varepsilon,$$

the key here being that composition operators leave constant functions alone. ∎

For the adjoint of a composition operator we have the following similar result.

PROPOSITION 2.7. *Let φ be an analytic self-map of the unit disk with Denjoy–Wolff point $a \in \mathbb{D}$. Let the operator A be defined by $Af = \langle f, 1\rangle K_a$ for all $f \in H^2(\beta)$. Then $A = \lim_{n\to\infty} (C_\varphi^*)^n$ in the strong operator topology.*

Proof. Using the definition of the operator and the Denjoy–Wolff Theorem together with Proposition 2.4, for each $\omega \in \mathbb{D}$ we have the following:

$$\|((C_\varphi^*)^n - A)K_\omega\|_\beta^2 = \|K_{\varphi^{(n)}(\omega)} - K_a\|^2 \overset{n\to\infty}{\longrightarrow} 0.$$

Hence, for every $\omega \in \mathbb{D}$,

$$AK_\omega = \lim_{n\to\infty}(C_\varphi^*)^n K_\omega.$$

Also, for any finite combination of K_ω's, we have

$$\begin{aligned}\Big\|((C_\varphi^*)^n - A)\Big(\sum_{j=0}^N b_j K_{\omega_j}\Big)\Big\| &= \Big\|\sum_{j=0}^N b_j((C_\varphi^*)^n - A)K_{\omega_j}\Big\|\\ &\leqslant \sum_{j=0}^N |b_j|\|((C_\varphi^*)^n - A)K_{\omega_j}\| \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

By Proposition 2.1, we know that $\bigvee_{\omega\in\mathbb{D}}\{K_\omega\} = H^2(\beta)$. We also note that, since $a \in \mathbb{D}$, we have by Proposition 1.3 that C_φ is power bounded, and hence so is C_φ^*. Thus $\{(C_\varphi^*)^n\}_{n=1}^\infty$ is a bounded sequence of operators that converges strongly to A on a dense set. We conclude that $A = \lim_{n\to\infty}(C_\varphi^*)^n$ in the strong topology. ∎

We are finally ready to state and prove the two main results of this paper.

THEOREM 2.8. *Every composition operator is orbit reflexive.*

Proof. Let a be the Denjoy–Wolff point of φ. Then the sequence of iterates of C_φ applied to K_ω converges pointwise on $\mathbb{D}$ to $K_\omega(a)$. Thus, if A is an operator on $H^2(\beta)$ such that Af is in the closure of the orbit of C_φ applied to f for every f in $H^2(\beta)$, and is not an iterate of C_φ, we get by the standard argument (Proposition 2.2), and the fact that convergence in $H^2(\beta)$ implies pointwise convergence in $\mathbb{D}$ (Proposition 2.3), that $AK_\omega = K_\omega(a)$, for a dense set of ω's in $\mathbb{D}$. Since both sides of this equality are continuous functions (of ω) from $\mathbb{D}$ to $H^2(\beta)$ (the right hand side is even continuous from $\mathbb{D}$ to $\mathbb{C}$), the equality holds for all ω in $\mathbb{D}$. Thus, by Proposition 2.5, a is in $\mathbb{D}$, so $AK_\omega = C_a K_\omega$ for all ω in $\mathbb{D}$. Consequently $A = C_a$, and therefore, since we also know that Az is in the closure of the orbit of C_φ applied to the function z, we get from Proposition 2.6 that A is in the strong closure of the set of iterates of C_φ. ∎

We have a similar result for the adjoint of a composition operator.

THEOREM 2.9. *Every adjoint of a composition operator is orbit reflexive.*

Proof. Let a be the Denjoy–Wolff point of φ. For every fixed z in $\mathbb{D}$, the function $K_\omega(z)$ of the variable ω is continuous on the closed unit disc. Therefore the sequence of iterates of C_φ^* applied to K_ω converges pointwise on $\mathbb{D}$ to K_a. Now, if a is in the unit circle, then the function K_a is not in $H^2(\beta)$. Therefore, since convergence in $H^2(\beta)$ implies pointwise convergence in $\mathbb{D}$ (Proposition 2.3), in this case the above sequence has no convergent subsequence in $H^2(\beta)$, for every ω in $\mathbb{D}$. Thus, if A is an operator on $H^2(\beta)$ such that Af belongs to the closure in $H^2(\beta)$ of the orbit of f by C_φ^*, for every f in $H^2(\beta)$, then for $f = K_\omega$, this orbit has no convergent subsequence in $H^2(\beta)$. Since this holds for every ω in $\mathbb{D}$, the standard argument (Proposition 2.2) shows that in this case there exists a nonnegative integer m such that A is the m-th iterate of C_φ^*. This proves the assertion for a in the unit circle.

Assume next that a is in $\mathbb{D}$. Since the function mapping ω to K_ω is continuous from $\mathbb{D}$ to $H^2(\beta)$, it follows that for every ω in $\mathbb{D}$, the the sequence of iterates of C_φ^* applied to K_ω converges in $H^2(\beta)$ to $K_a = C_a^* K_\omega$. Since this holds for every ω in $\mathbb{D}$ and C_φ^* is power bounded, we conclude that the sequence of iterates of C_φ^* converges strongly to the operator C_a^*. Thus, A satisfies the above condition, and is different from an iterate of C_φ^*. We get, by the standard argument (Proposition 2.2), that $AK_\omega = C_a^* K_\omega$, for all ω in $\mathbb{D}$ except perhaps for a countable set of ω's. Therefore $A = C_a^*$, which we know, by Proposition 2.7, is the strong limit of the iterates C_φ^*. This concludes the proof. ∎

REMARK 2.10. The results of Theorem 2.8 and 2.9 can easily be generalized to finite direct sums instead of one operator; i.e., every finite direct sum of composition operators, every finite direct sum of adjoint of composition operators, and every finite direct sum of composition operators and the adjoints of composition operators is orbit reflexive.

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