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PROPERTIES OF L_2-SOLUTIONS OF REFINEMENT EQUATIONS

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ABSTRACT. This paper is devoted to the study of L_2-solutions of the operator equations $f - B_M\mathfrak{F}a\mathfrak{F}^{-1}f = g$, where a is the operator of multiplication by a matrix $a \in L_\infty^{m\times m}(\mathbb{R}^s)$, $m, s \in \mathbb{N}$, $\mathfrak{F}$ denotes the Fourier transform, and B_M is the dilation operator $B_M f(x) := f(Mx)$, $x \in \mathbb{R}^s$, generated by a non-singular matrix $M \in \mathbb{R}^{m\times m}$. This class of equations contains discrete and continuous refinement equations widely used in wavelet analysis, signal processing, computer graphics and other fields of mathematics and in applications.

It is shown that the set of nontrivial solutions of the homogeneous equation is either empty or contains a subset isomorphic to a space $L_\infty(\mathbb{V}_M)$, where $\mathbb{V}_M$ is a Lebesgue measurable set with positive Lebesgue measure. It follows that the operator $I - B_M\mathfrak{F}a\mathfrak{F}^{-1}$ is Fredholm if and only if it is invertible. Moreover, if the dilation M satisfies some mild conditions, then $\ker(I - B_M\mathfrak{F}a\mathfrak{F}^{-1}) \subset \overline{\operatorname{im}(I - B_M\mathfrak{F}a\mathfrak{F}^{-1})}$.

KEYWORDS: *Refinement equation, L_2-solutions, Fredholm properties.*

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INTRODUCTION

Refinement equations play an important role in wavelet analysis and computational mathematics. Discrete homogeneous refinement equations are used to construct orthonormal wavelet bases in spaces $L_2^m(\mathbb{R}^s)$, $m \geqslant 1, s \geqslant 1$ [9], [14], [15], whereas discrete non-homogeneous equations arise while considering wavelets on compactly supported subsets of $\mathbb{R}^s$ [15], [37], [39] and also in signal processing to obtain multi-channel filters with good localization properties in time and frequency domains [41]. On the other hand, continuous refinement equations have important applications in non-stationary subdivision processes [11], multiresolution analysis and wavelets [5], and invariant densities for model sets and quasicrystals [3]. Therefore, solvability and properties of the solutions of refinement equations have attracted a wide audience.

However, despite a steady growth of results available in literature, many issues remain open. This can probably be attributed to the widespread use of specific methods to establish solvability. For example, two approaches are normally employed to show that a homogeneous refinement equation has a non-trivial solution in some normed space. One approach uses distributional solutions, and the other relies upon the convergence of certain approximation methods. Thus it is relatively simple to find sufficient conditions for the existence of distributional solutions of a discrete or continuous homogeneous refinement equation. Moreover, it is also possible to obtain effective representations for such solutions [4], [11], [16], [43]. The next step is more demanding — viz. to show that the distributional solution obtained belongs to a desired Hilbert or Banach space, which uses different tools. Among the most effective are joint spectral radii of auxiliary matrices [10], [17] and auxiliary operators [6], [7], [8], [44]. Another approach exploits the convergence of various approximation methods in the normed space of interest [14], [28], [31], [36], [45]. Of course, such methods depend upon the approximation method adopted and the initial approximation chosen.

Let us note that the above classification is somewhat arbitrary and in many cases the method can be assigned to either of these two groups. Indeed, although each of these two strategies provides additional information about the possible solution of refinement equations, neither discloses the whole picture — certain peculiarities of the initial problem remain unexplored. The solvability of refinement equations in the spaces L_1 and L_2 is the most completely studied, but even then the results obtained are mainly concerned with refinement equations generated by polynomials or by functions with compact support — and aimed at solutions with compact supports. Although such solutions play an important role in applications, they are not the only solutions. The Haar refinement equation is one of the simplest examples, but in the space $L_2(\mathbb{R})$ it possesses a very rich solution set in addition to the well-known and widely-used Haar function [38].

The aim of this paper is to study L_2-solutions of homogeneous and non-homogeneous refinement equations, but our approach is not based on a distributional solution or approximation method. Moreover, different types of refinement equations are considered from a unified point of view. This allows us to describe common properties of such equations and their solutions. Thus the properties established here do not depend on whether a refinement equation is discrete or continuous, scalar or vector — or whether univariate or multivariate case is considered. They rely upon properties of the dilation matrix; and if this matrix satisfies certain quite general conditions, then the solutions of the corresponding refinement equation have distinctive features. For example, for any homogeneous refinement equation, the set of non-trivial L_2-solutions is either empty or contains a set isomorphic to a space $L_\infty(\mathbb{V}_M)$ with a set $\mathbb{V}_M$ having a positive Lebesgue measure, and the corresponding refinement operator is Fredholm if and only if it is invertible. Thus if an homogeneous refinement equation has a non-trivial solution, the set of its solutions is quite large. Note that for some discrete refinement equations with special dilations, the non-uniqueness of L_2-solutions has already been mentioned [12], [16], [20], [38], [40], [46]. Furthermore, in the present paper, a description of the structure of the kernel space for refinement operators is given. Necessary and certain sufficient conditions of the L_2-solvability of homogeneous and non-homogeneous equations are also obtained. Some conditions provide the solvability of the refinement equation for any right-hand side from the space $L_2^m(\mathbb{R}^s)$, while others deal with a more delicate situation when the corresponding equation is solvable under additional conditions — e.g. if an associated operator is normally solvable and the right-hand side g of the non-homogeneous equation

$$f - B_M \mathfrak{F} a \mathfrak{F}^{-1} f = g \tag{0.1}$$

is a solution of the corresponding homogeneous equation, then equation (0.1) is solvable.

It is also worth noting that the refinement operators are closely connected with the weighted shift operators, and there is vast literature where such operators have been studied. In particular, Fredholmness of operators from different algebras of weighted shift operators and properties of solutions of the corresponding equations are presented in [1], [2]. Nevertheless, it seems that properties of the refinement operators established here, have not been observed for other classes of weighted shift operators.

1. REFINEMENT OPERATORS AND REFINEMENT EQUATIONS

Let s and m be positive integers, and let $\mathfrak{F}$ and $\mathfrak{F}^{-1}$, respectively, denote the direct and inverse Fourier transforms, i.e.

$$(\mathfrak{F}f)(t) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} \mathrm{e}^{-\mathrm{i}(y,t)} f(y)\,\mathrm{d}y, \quad t \in \mathbb{R}^s$$

$$(\mathfrak{F}^{-1}f)(t) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} \mathrm{e}^{\mathrm{i}(y,t)} f(y)\,\mathrm{d}y, \quad t \in \mathbb{R}^s,$$

where

$$(y,t) := \sum_{k=1}^{s} y_k t_k,$$

is the scalar product of vectors $y = (y_1, y_2, \ldots, y_s)$ and $t = (t_1, t_2, \ldots, t_s)$ of $\mathbb{R}^s$.

Let $L_2^m(\mathbb{R}^s)$ denote the set of vector-functions $f = (f_1, f_2, \ldots, f_m)$ with entries from the space $L_2(\mathbb{R}^s)$ and the norm

$$\|f\|_2 = \Big(\sum_{j=1}^{m} \|f_j\|^2_{L_2(\mathbb{R}^s)} \Big)^{1/2}.$$

It is well known that if $a \in L_\infty^{m\times m}(\mathbb{R}^s)$, then the convolution operator $C(a) := \mathfrak{F}a\mathfrak{F}^{-1}$ is bounded on the space $L_2^m(\mathbb{R}^s)$ and

$$\|\mathfrak{F}a\mathfrak{F}^{-1}\|_2 = \|a\|_\infty,$$

where

$$\|a\|_\infty = \max_{1\leqslant j\leqslant m} \operatorname*{ess\,sup}_{x\in\mathbb{R}^s} \sqrt{\lambda_j(x)}, \tag{1.1}$$

and $\lambda_j(x), j = 1,2,\dots,m$ are the eigenvalues of the matrix a^*a. Note that for any $m \geqslant 1$ we use the same symbol $\mathfrak{F}$ to denote the Fourier transform on space $L_2^m(\mathbb{R}^s)$. If M is a non-singular $s\times s$ matrix of real numbers, then one can consider an operator $R_a^M : L_2^m(\mathbb{R}^s) \to L_2^m(\mathbb{R}^s)$ defined by

$$R_a^M := B_M\mathfrak{F}a\mathfrak{F}^{-1}, \tag{1.2}$$

where

$$B_M f(x) := f(Mx), \quad x \in \mathbb{R}^s.$$

The operator R_a^M is called *the refinement operator generated by the matrix function a and matrix M*, or simply the *refinement operator*. The matrix function a is called the *symbol of the operator* R_a^M. In wavelet literature it is usually assumed that $M \in \mathbb{Z}^{s\times s}$ and

$$\lim_{n\to\infty} M^{-n} = 0. \tag{1.3}$$

Such matrices are called *dilation matrices*. For convenience, we will also use the term "dilation" while speaking about the matrix M but now this term is attributed not only to integer matrices with the property (1.3) but to any non-singular matrix of real numbers.

In this work we study Fredholm properties of the operator $I - R_a^M$ and L_2-solutions of the corresponding homogeneous and non-homogeneous equations

$$f - R_a^M f = 0; \tag{1.4}$$

$$f - R_a^M f = g, \quad g \in L_2^m(\mathbb{R}^s). \tag{1.5}$$

There are various papers where the solvability of equations (1.4) and (1.5) has been studied under different assumptions on the symbol a and dilation matrix M, and some results are discussed below. These investigations usually rely on whether the refinement equation generated by a and M is discrete or continuous, whether it is univariate or multivariate, and whether this is an equation for a vector or scalar unknown function.

Throughout this paper, $I - R_a^M$ will be also referred as the refinement operator. Moreover, although the null function $f_0(x) = 0$ almost everywhere on $\mathbb{R}^s$ is always a solution of the equation (1.4), for convenience this *equation* is called *solvable* if it has a non-trivial solution.

EXAMPLE 1.1. Let $m = s = 1$, $M = 2$ and let a be a trigonometric polynomial

$$a(t) = \sum_{k=0}^{N} a_k t^k, \quad t = \mathrm{e}^{\mathrm{i}x}, \quad x \in \mathbb{R}.$$

The corresponding discrete homogeneous refinement equation has the form

$$f(x) = \sum_{k=0}^{N} a_k f(2x - k). \tag{1.6}$$

The study of the solvability of equation (1.6) in different spaces has a long history [4], [10], [14], [15], [16], [17], [31], [36], [44]. In particular, if

$$\sum_{k=0}^{N} a_k = 2, \tag{1.7}$$

then equation (1.6) has the distributional solution

$$f_0 = \mathfrak{F}^{-1}\Big(\prod_{j=1}^{\infty} a(2^{-j}\cdot)\Big). \tag{1.8}$$

Note that both condition (1.7) and solution (1.8) are especially valuable for the refinement equations considered in the space $L_1(\mathbb{R})$. Thus if equation (1.6) is solvable in $L_1(\mathbb{R})$ under condition (1.7), then the space of its solutions $\ker_{L_1(\mathbb{R})}(I - R_a^2) = \operatorname{span}\{f_0\}$. On the other hand, if equation (1.6) is solvable, then the coefficients $a_k, k = 0, 1, \ldots, N$ satisfy the condition

$$\sum_{k=0}^{N} a_k = 2^n$$

for certain $n \in \mathbb{N}$, [16]. However, for the L_2-solvability the condition (1.7) is not as important as for the L_1 case. Nevertheless, this condition is often used to study the solvability of (1.6) not only in the space L_1 but also in other normed spaces.

For the space $L_2(0, N)$, the solvability of the associate non-homogeneous equation has been studied in [29], [42], and [29] also discusses the multivariate analogue of equation (1.6). Distributional solutions of discrete non-homogeneous equations were investigated in [24], [25], [30], [42]. Moreover, for compactly supported right-hand sides certain conditions of L_p-solvability of discrete non-homogeneous equations are presented in [42].

EXAMPLE 1.2. Let $m \geqslant 1, s \geqslant 1$ and $c \in L_1^{m\times m}(\mathbb{R}^s)$ and let M be an $s \times s$ matrix of real numbers. We define the matrix function a by

$$a := (2\pi)^{s/2}\mathfrak{F}^{-1}(c).$$

Then by the convolution theorem, R_a^M is the continuous refinement operator

$$R_a^M f(x) = \int_{\mathbb{R}^s} c(Mx - y) f(y) \mathrm{d}y.$$

For $m = 1, s = 1$, the solvability of the corresponding refinement equations and spectral properties of the operator R_a^M have been studied in [27], [32], [34], [35], [43], and the multivariate case has also been investigated [33].

There are also papers where continuous and discrete refinement equations are treated simultaneously. Distributional solutions of homogeneous equations are considered in [11], and non-homogeneous refinement equations in [29], where the multivariate case is studied as well.

2. NECESSARY CONDITIONS OF SOLVABILITY

Let us recall notions associated with Fredholm operators. For any normed space X and bounded linear operator A on X, let $\operatorname{im}_X A$ and $\ker_X A$ denote the range and null space of the operator A on X, respectively. The subscript X is usually omitted if that does not cause confusion. The *operator* $A : X \mapsto X$ is called *Fredholm* if it is normally solvable and the dimensions of the subspaces $\ker A$ and $\ker A^*$ are finite. Recall that the normal solvability of the operator A is equivalent to the property that the range $\operatorname{im}_X A$ of A is a closed subspace of X — i.e. an operator $A : X \mapsto X$ is normally solvable if and only if $\overline{\operatorname{im}_X A} = \operatorname{im}_X A$, where $\overline{Y}$ denotes the closure of the subset Y in X, [26].

LEMMA 2.1. *Let $a \in L_\infty^{m\times m}(\mathbb{R}^s)$ and let $\widehat{M} := (M^T)^{-1}$, where M^T denotes the matrix transpose to M. Equation* (1.4) *is solvable in space $L_2^m(\mathbb{R}^s)$ if and only if the equation*

$$B_{M^T} f - \frac{a}{|\det M|} f = 0 \tag{2.1}$$

is solvable in $L_2^m(\mathbb{R}^s)$, and

$$\begin{aligned} \dim \ker_{L_2^m(\mathbb{R}^s)} (I - R_a^M) &= \dim \ker_{L_2^m(\mathbb{R}^s)} \left(B_{M^T} - \frac{a}{|\det M|} I \right) \\ &= \dim \ker_{L_2^m(\mathbb{R}^s)} \left(I - \frac{a}{|\det M|} B_{\widehat{M}} \right). \end{aligned} \tag{2.2}$$

Proof. It is easily seen that

$$\mathfrak{F}^{-1} B_{M^{-1}} = |\det M| \cdot B_{M^T} \mathfrak{F}^{-1}. \tag{2.3}$$

Since $\mathfrak{F}$ is a unitary operator on the space $L_2^m(\mathbb{R}^s)$, the refinement operator R_a^M is unitary equivalent to the operator

$$C_M(a) := \frac{a}{|\det M|} B_{\widehat{M}}. \tag{2.4}$$

Therefore, the spectra of the operators R_a^M and $C_M(a)$ coincide, and relations (2.1), (2.2) follow. ∎

Let $a \in L_\infty^{m\times m}(\mathbb{R}^s)$. For a non-singular matrix $M \in \mathbb{R}^{s\times s}$, let us consider the non-negative number

$$\Delta_M(a) := \lim_{n\to\infty} \Big\| \prod_{j=0}^{n-1} a((M^T)^j \cdot) \Big\|_\infty^{1/n}.$$

Note that the quantity $\Delta_M(a)$ is connected with the spectral radius of the operator R_a^M. For some classes of symbols a and dilations M, the number $\Delta_M(a)$ is evaluated in [18], [19], [21], [23].

THEOREM 2.2. *Let the matrices M and a be as above, and $\lambda_j, j = 1,2,\dots,m$ denote the eigenvalues of the matrix a^*a. If equation* (1.4) *has a non-trivial solution, then*

$$\sqrt{|\det M|} \leqslant \Delta_M(a). \tag{2.5}$$

If, in addition,

$$\min_{1\leqslant j\leqslant m} \operatorname*{ess\,inf}_{x\in\mathbb{R}^s} \sqrt{\lambda_j(x)} > 0, \tag{2.6}$$

then

$$\sqrt{|\det M|} \in [\Delta_M^{-1}(a^{-1}), \Delta_M(a)]. \tag{2.7}$$

Proof. Since the refinement operator R_a^M and the operator $C_M(a)$ of (2.4) are unitarily equivalent the spectral radii $\rho(R_a^M)$ and $\rho(C_M(a))$ are equal. However, the spectral radius of the operator $C_M(a)$ can be calculated by the formula [1], [2],

$$\rho(C_M(a)) = \frac{\Delta_M(a)}{\sqrt{|\det M|}},$$

that implies relation (2.5). On the other hand, if condition (2.6) is also satisfied, the operator of multiplication by a is invertible, and repeating the previous arguments for the operator $a^{-1}|\det M| \cdot B_{\widehat{M}^{-1}}$ leads to inclusion (2.7). ∎

COROLLARY 2.3. *Let $a \in L_\infty^{m\times m}(\mathbb{R}^s)$. If*

$$\sqrt{|\det M|} \notin [\Delta_M^{-1}(a^{-1}), \Delta_M(a)], \tag{2.8}$$

then the homogeneous refinement equation (1.4) *has the trivial solution only; whereas the non-homogeneous equation* (1.5) *is solvable for any right-hand side $g \in L_2(\mathbb{R}^s)$, and its solution is unique.*

Note that if the operator of multiplication by the matrix a is not invertible, then we set $\Delta_M^{-1}(a^{-1}) := 0$.

Consider now refinement equation (1.6) with a symbol a under the condition (1.7). Our analysis [18] shows that for such symbols

$$\Delta_2(a) \geqslant 2,$$

so condition (2.5) of Theorem 2.2 is always satisfied. However, for invertible symbols a, additional effort to verify condition (2.7) is needed. In fact, the author is not sure whether this has ever been done before. Instead, one usually considers symbols that satisfy condition (2.5) and vanish somewhere in $[0,2\pi]$. In wavelet theory, the last requirement is often satisfied by imposing an additional condition on the Fourier coefficients of the symbol a, viz.

$$\sum_j a_{2j+1} = \sum_j a_{2j} = 1,$$

[17]. For the continuous refinement equation

$$f(x) = \alpha \int_{\mathbb{R}} c(\alpha x - y) f(y) \mathrm{d}y,$$

with a dilation $\alpha \neq 0$, one often considers kernels $c \in L_1(\mathbb{R})$ that satisfy the condition [27], [32]

$$\int_{\mathbb{R}} c(y) \mathrm{d}y = 1.$$

In this case $\Delta_\alpha(a) = \sqrt{\alpha}$, see[21], so the inequality (2.5) is obviously satisfied.

In order to study the Fredholm properties of refinement operators, let us introduce certain notions concerning Lebesgue measurable sets of $\mathbb{R}^s$, where all relations for such sets are always understood to be modulo sets of Lebesgue measure zero. Let $M \in \mathbb{R}^{s\times s}$ be an invertible matrix. We say that $S \subset \mathbb{R}^s$ is a *quasi-wandering set for* M, or that the matrix M has a quasi-wandering set S, if S is a Lebesgue measurable set and for any $k, j \in \mathbb{Z}$, $k \neq j$ either $M^k S \cap M^j S = \emptyset$ or $M^k x = M^j x$ for all $x \in S$. Recall that in theory of dynamical systems, a set S is called a *wandering set for a matrix* M if $M^k S \cap M^j S = \emptyset$ for all $k, j \in \mathbb{Z}$, $k \neq j$. A quasi-wandering set is said to be complete if $\mathbb{R}^s = \bigcup_{k\in\mathbb{Z}} M^k S$. The class of matrices possessing complete quasi-wandering sets is large enough, for it contains the dilation matrices used in wavelet analysis, and it also includes a variety of other non-singular matrices. Some examples of matrices with complete quasi-wandering sets are given below.

Of course, if a matrix M has a complete quasi-wandering set S, then it is always possible to choose an index set A such that $M^k S \cap M^j S = \emptyset$ for all $k, j \in A$, $k \neq j$ and $\mathbb{R}^s = \bigcup_{k\in A} M^k S$. The cardinality of such a set A is called the *wandering index of the set* S *under the action* M, or simply the *wandering index of* S. The wandering index is needed below to describe properties of null spaces of refinement operators.

EXAMPLE 2.4. Let s be any positive integer and $M \in \mathbb{R}^{s\times s}$ be an expansive matrix — i.e. all eigenvalues of M have modulus greater than 1. From [13] the matrix M has a complete wandering set S, so $\mathbb{R}^s = \bigcup_{k\in\mathbb{Z}} M^k S$. As was already mentioned, in wavelet theory one usually considers expansive matrices $M \in \mathbb{Z}^s$ as dilation matrices.

The next two examples show that not expansive matrices can possess complete quasi-wandering sets as well.

EXAMPLE 2.5. Let $s = 2$, l be a positive integer, and let M be the rotation matrix

$$M = \begin{pmatrix} \cos\frac{\pi}{l} & -\sin\frac{\pi}{l} \\ \sin\frac{\pi}{l} & \cos\frac{\pi}{l} \end{pmatrix}.$$

The set

$$S = \left\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0 \quad \text{and} \quad 0 < \tan^{-1}\left(\frac{y}{x}\right) \leqslant \frac{\pi}{l}\right\}$$

is a complete quasi-wandering set for M and $\mathbb{R}^2 = \bigcup_{k=0}^{2l-1} M^k S$.

EXAMPLE 2.6. Let $s = 2$, and let M be the diagonal matrix

$$M = \begin{pmatrix} r & 0 \\ 0 & p \end{pmatrix}$$

where $0 < r < 1$ and $p \neq 0$. Then the set

$$S = \{(x,y) \in \mathbb{R}^2 : x \in (-r^{-1}, -1] \cup [1, r^{-1}) \text{ and } y \in \mathbb{R}\}$$

is a complete quasi-wandering set for M and $\mathbb{R}^2 = \bigcup_{k\in\mathbb{Z}} M^k S$.

THEOREM 2.7. *Let $a \in L_\infty^{m\times m}(\mathbb{R}^s)$, and let the matrix $M^T \in \mathbb{R}^{s\times s}$ have a complete quasi-wandering set. Then either*

$$\ker_{L_2^m(\mathbb{R}^s)}(I - R_a^M) = 0$$

or there is a subspace $\mathfrak{S} \subset \ker_{L_2^m(\mathbb{R}^s)}(I - R_a^M)$ and a set $\mathbb{V} \subset \mathbb{R}^s$, the Lebesgue measure of which is positive, such that $\mathfrak{S}$ is isomorphic to the space $L_\infty(\mathbb{V})$.

Proof. Let $S_M \subset \mathbb{R}^s$ be a complete quasi-wandering set for the matrix M^T. If the homogeneous refinement equation (1.4) has a non-trivial solution, then by Lemma 2.1, there is a function $f_0 \in L_2^m(\mathbb{R}^s)$, $f_0 \neq 0$ such that

$$f_0(M^T x) = \frac{a(x)}{|\det M|} f_0(x), \quad x \in \mathbb{R}^s. \tag{2.9}$$

Moreover, since $f_0 \neq 0$, there is at least one $j_0 \in \mathbb{Z}$ and a $\sigma > 0$ such that the set

$$\mathbb{V}_M = \mathbb{V}_M^\sigma := \{x \in (M^T)^{j_0} S_M : |f_0(x)| \geqslant \sigma\}$$

has a positive Lebesgue measure. For any $q_0 \in L_\infty((M^T)^{j_0} S_M)$ such that the restriction of q_0 onto $\mathbb{V}_M$ is a non-zero element from $L_\infty(\mathbb{V}_M)$, we define its extension $\widetilde{q}_0$ on the whole space $\mathbb{R}^s$ by

$$\widetilde{q}_0(x) = \begin{cases} q_0((M^T)^{-j}x) & \text{if } x \in (M^T)^{j+j_0}\mathbb{V}_M,\ j \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\widetilde{q}_0$ is well-defined, belongs to the class $L_\infty(\mathbb{R}^s)$, and satisfies the equation

$$\widetilde{q}_0(x) = \widetilde{q}_0(M^T x) \quad \text{for all } x \in \mathbb{R}^s. \tag{2.10}$$

Multiplying equations (2.10) and (2.9), one obtains that $\widetilde{q}_0 f_0$ also is a solution of equation (2.1), and it is not hard to show that $\|\widetilde{q}_0 f_0\|_2 \neq 0$. Thus the space $\ker_{L_2^m(\mathbb{R}^s)}(B_{M^T} - (a/|\det M|)I)$ contains a subset isomorphic to $L_\infty(\mathbb{V}_M)$ and so is the space $\ker(I - R_a^M)$. ∎

The results of Theorem 2.7 allows us to characterise the Fredholmness of the operator $I - R_a^M$.

COROLLARY 2.8. *Let matrices a and M satisfy the conditions of Theorem* 2.7. *If the refinement operator $I - R_a^M$ is Fredholm, then:*

(i) $\dim \ker_{L_2^m(\mathbb{R}^s)}(I - R_a^M) = 0$;

(ii) $\dim \ker_{L_2^m(\mathbb{R}^s)}(I - (R_a^M)^*) = 0$.

Proof. If the operator $I - R_a^M$ is Fredholm, then $\dim \ker_{L_2^m(\mathbb{R}^s)}(I - R_a^M) < \infty$. However, if the dilation matrix M^T possesses a complete quasi-wandering set, then by Theorem 2.7 either $\dim \ker_{L_2^m(\mathbb{R}^s)}(I - R_a^M) = 0$ or $\dim \ker_{L_2^m(\mathbb{R}^s)}(I - R_a^M) = \infty$ and relation (i) follows. To establish relation (ii) let us rewrite the adjoint operator $(I - R_a^M)^*$ as

$$(I - R_a^M)^* = \frac{1}{|\det M|}\mathfrak{F}(B_{\widehat{M}} - a^* I)\mathfrak{F}^{-1} B_{M^{-1}}.$$

In the above transformation we used equation (2.3) with the matrix M instead of M^{-1}. Thus, the adjoint homogeneous equation

$$(I - R_a^M)^* \varphi = 0 \tag{2.11}$$

has a non-trivial L_2-solution if and only if the equation

$$B_{\widehat{M}} \varphi = a^* \varphi \tag{2.12}$$

has a non-trivial L_2-solution. If S is a complete quasi-wandering set for a matrix M^T, then it is also a complete quasi-wandering set for the matrix $\widehat{M} := (M^T)^{-1}$. Following the proof of Theorem 2.7, one can show that the assumption $\dim \ker_{L_2^m(\mathbb{R}^s)}(I - R_a^M)^* \neq 0$ implies $\dim \ker_{L_2^m(\mathbb{R}^s)}(I - R_a^M)^* = \infty$. The latter relation contradicts the assumption that $I - R_a^M$ is a Fredholm operator. ∎

Corollary 2.8 implies the following Fredholmness criterion for the operator $I - R_a^M$.

COROLLARY 2.9. *Let matrices a and M satisfy the conditions of Theorem* 2.7. *The refinement operator $I - R_a^M$ is Fredholm if and only if it is invertible.*

REMARK 2.10. There are similar results for some classes of weighted shift operators. Their proof is based on the fact that a C^*-algebra generated by the corresponding weighted shift operator, contains no non-trivial compact operators ([1], Theorem 8.3). However, our approach allows us to obtain additional information concerning the kernels of the operators under consideration.

The operator

$$G_{a,M} := B_{M^T} - \frac{a}{|\det M|} I$$

plays a remarkable role in investigation of solvability of refinement equations. Let us study this operator in more detail. Consider the adjoint operator

$$G^*_{a,M} = \frac{1}{|\det M|}(B_{\widehat{M}} - a^* I)$$

for the operator $G_{a,M}$.

For any $f, \varphi \in L_2^m(\mathbb{R}^s)$, $f = (f_1, f_2, \ldots, f_m)$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)$, we define a product $\langle f, \varphi \rangle$ of f and φ by

$$\langle f, \varphi \rangle := \sum_{k=1}^{m} f_k \overline{\varphi}_k.$$

LEMMA 2.11. *Let matrices a and M satisfy the conditions of Theorem* 2.7. *If the matrix M^T has a complete quasi-wandering set S with infinite wandering index, then for any $f \in \ker G_{a,M}$ and for any $\varphi \in \ker G^*_{a,M}$ the function $\Psi_{f,\varphi} : \mathbb{R}^s \mapsto \mathbb{C}$ defined by*

$$\Psi_{f,\varphi} := \langle f(M^T \cdot), \varphi \rangle, \tag{2.13}$$

is equal to zero almost everywhere on $\mathbb{R}^s$.

Proof. Rewriting the homogeneous equations $G_{a,M} f = 0$ and $G^*_{a,M}\varphi = 0$ in the form

$$|\det M|\, f(M^T x) = a(x) f(x) \tag{2.14}$$

and

$$a^*(M^T x)\varphi(M^T x) = \varphi(x), \tag{2.15}$$

and considering the products $\langle \cdot, \cdot \rangle$ for the corresponding parts of (2.14), (2.15) one obtains

$$|\det M| \langle f(M^T \cdot), a^*(M^T \cdot)\varphi(M^T \cdot) \rangle = \langle af, \varphi \rangle.$$

Hence

$$|\det M| \langle a(M^T \cdot) f(M^T \cdot), \varphi(M^T \cdot) \rangle = \langle af, \varphi \rangle. \tag{2.16}$$

Let us recall that $a \in L_\infty^{m \times m}(\mathbb{R}^s)$ whereas both the elements f and φ belong to the space $L_2^m(\mathbb{R}^s)$. The function $\psi_{f,\varphi} : \mathbb{R}^s \mapsto \mathbb{C}$ defined by

$$\psi_{f,\varphi} := \langle af, \varphi \rangle,$$

therefore belongs to the space $L_1(\mathbb{R}^s)$ and satisfies the equation

$$\psi_{f,\varphi} = |\det M|\, \psi_{f,\varphi}(M^T \cdot) \tag{2.17}$$

almost everywhere on $\mathbb{R}^s$. Then for any $k \in \mathbb{Z}$, one obtains

$$\psi_{f,\varphi} = |\det M|^k\, \psi_{f,\varphi}((M^T)^k \cdot). \tag{2.18}$$

Using relations (2.17), (2.18) and the fact that the set S has infinite wandering index, the L_1-norm of the function $\psi_{f,\varphi}$ may be represented as

$$\begin{aligned} \|\psi_{f,\varphi}\|_{L_1(\mathbb{R}^s)} &= \int_{\mathbb{R}^s} |\psi_{f,\varphi}(x)|\, \mathrm{d}x = \sum_{k\in\mathbb{Z}} \int_{(M^T)^k S} |\psi_{f,\varphi}(x)|\, \mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} |\det M|^k \int_S |\psi_{f,\varphi}((M^T)^k x)|\, \mathrm{d}x = \sum_{k\in\mathbb{Z}} \int_S |\psi_{f,\varphi}(x)|\, \mathrm{d}x. \end{aligned}$$

Thus if

$$\int_S |\psi_{f,\varphi}(x)|\, \mathrm{d}x \neq 0$$

then $\psi_{f,\varphi} \notin L_1(\mathbb{R}^s)$. Hence $\psi_{f,\varphi} = 0$ almost everywhere on S, and using the equation (2.18) once more it follows that $\langle af, \varphi\rangle = 0$ almost everywhere on $\mathbb{R}^s$. The application of the equation (2.14) leads to the equality

$$\langle B_{M^T} f, \varphi\rangle = \frac{1}{|\det M|} \langle af, \varphi\rangle = 0,$$

and the proof is complete. ∎

For any set $X \subset L_2^m(\mathbb{R}^s)$, let $X^\perp$ denote the orthogonal complement of X in $L_2^m(\mathbb{R}^s)$.

COROLLARY 2.12. *If matrices a and M satisfy the conditions of Lemma* 2.11, *then*

$$B_{M^T}(\ker G_{a,M}) \subset (\ker G^*_{a,M})^\perp. \tag{2.19}$$

Proof. For any $f \in \ker G_{a,M}$ and for any $\varphi \in \ker G^*_{a,M}$, the function $\Psi_{f,\varphi}(x) = 0$ almost everywhere on $\mathbb{R}^s$, so the inner product

$$(B_{M^T} f, \varphi) = \int_{\mathbb{R}^s} \Psi_{f,\varphi}(x)\, \mathrm{d}x = 0. \tag{2.20}$$

This finishes the proof. ∎

THEOREM 2.13. *If matrices a and M satisfy the conditions of Lemma* 2.11, *then*

$$\ker (I - R_a^M) \subset \overline{\mathrm{im}\, (I - R_a^M)}. \tag{2.21}$$

Proof. Let $\widetilde{f}$ and $\widetilde{\varphi}$ be solutions of the equations (1.4) and (2.11), respectively. Then $f = \mathfrak{F}^{-1}\widetilde{f}$ and $\varphi = \mathfrak{F}^{-1}B_{M^{-1}}\widetilde{\varphi}$ are solutions of the homogeneous equations (2.1) and (2.12), respectively. From (2.3) and (2.20)

$$(\widetilde{f}, \widetilde{\varphi}) = (\mathfrak{F}f, B_M\mathfrak{F}\varphi) = (B_M^*\mathfrak{F}f, \mathfrak{F}\varphi) = \frac{1}{|\det M|}(B_{M^{-1}}\mathfrak{F}f, \mathfrak{F}\varphi)$$
$$= \frac{1}{|\det M|}(\mathfrak{F}^{-1}B_{M^{-1}}\mathfrak{F}f, \varphi) = (B_{M^T}f, \varphi) = 0,$$

hence

$$\ker(I - R_a^M) \subset (\ker(I - R_a^M)^*)^\perp, \tag{2.22}$$

and since $(\ker A^*)^\perp = \overline{\operatorname{im} A}$ for any linear operator A [26], the inclusion (2.21) follows. ∎

Immediate application of this result leads to a solvability condition of non-homogeneous refinement equations.

COROLLARY 2.14. *Let matrices a and M satisfy the conditions of Lemma* 2.11. *If the operator* $I - R_a^M : L_2^m(\mathbb{R}^s) \mapsto L_2^m(\mathbb{R}^s)$ *is normally solvable, then for any* $g \in \ker(I - R_a^M)$ *the non-homogeneous refinement equation* (1.5) *is solvable.*

Note that some methods for construction of wavelet bases on interval use non-homogeneous equations with right-hand sides obtained from the solutions of homogeneous refinement equations [39]. Thus the solvability conditions of non-homogeneous equations presented in Corollary 2.14 can have a straightforward application in this case.

On the other hand, the normal solvability of the operator $I - R_a^M$ required in Corollary 2.14, is not particularly well studied. Thus for operators with polynomial symbols, conditions of normal solvability can be formulated in terms of sequences of singular values of matrices arising in Galerkin approximations of some auxiliary operators ([22], Theorem 6.3). However, it is desirable to have more practical and more general results related to this issue.

In conclusion, we would like to emphasize that results of this paper are valid for refinement equations of any kind: scalar or vector, univariate or multivariate, discrete or continuous, or even for combination of discrete and continuous equations. It is interesting to compare them with known results for L_1-solvability. For definiteness, consider a discrete refinement equation with a polynomial symbol on $\mathbb{R}$. The most obvious difference is the number of solutions for the homogeneous equation. Thus if a discrete homogeneous equation has a compactly supported solution then $\dim\ker_{L_1(\mathbb{R})}(I - R_a^M) = 1$, whereas in similar circumstances $\dim\ker_{L_2(\mathbb{R})}(I - R_a^M) = \infty$. The spectrum structure of the operator R_a^M considered on $L_2(\mathbb{R})$ and $L_1(-K, K)$, where $(-K, K)$ is an interval of $\mathbb{R}$, is also different. In the $L_1(-K, K)$-case, the spectrum of the operator R_a^M consists of disjoint points of the real line $\mathbb{R}$ with zero as the only possible accumulation point, whereas for the $L_2(\mathbb{R})$-case the spectrum of the same operator can include intervals as well [38]. It would probably be more appropriate to make comparison with the spectra of the operator R_a^M considered on the space $L_1(\mathbb{R})$ but the author is not aware of the corresponding results. Nevertheless it is evident that the spectral properties of this operator are very different for different normed spaces. Indeed, although solutions with compact support play an extremely important role in the solvability of refinement equations, their choice as a starting point for the study of solvability in other normed spaces can lead to a loss of useful information.

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