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AN L^2-KÜNNETH FORMULA FOR TRACIAL ALGEBRAS

DAVID KYED

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ABSTRACT. We prove a Künneth formula computing the Connes–Shlyakhtenko L^2-Betti numbers of the algebraic tensor product of two tracial $*$-algebras in terms of the L^2-Betti numbers of the two original algebras. As an application, we construct examples of compact quantum groups with a non-vanishing first L^2-Betti number.

KEYWORDS: *L^2-Betti numbers, Künneth formula, quantum groups.*

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INTRODUCTION

The theory of L^2-invariants originates from the work of Atiyah [1] and was further developed by Cheeger and Gromov in [3] and later by Lück in [11], [12] and [13]. One of the pleasant features of Lück's approach is the algebraic setup which allows the usage of all the powerful tools of homological algebra; for instance the L^2-homology of a discrete group Γ can be written as

$$H_n^{(2)}(\Gamma) = \mathrm{Tor}_n^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), \mathbb{C}),$$

where $\mathscr{L}(\Gamma)$ denotes the group von Neumann algebra. The L^2-Betti numbers of Γ are then obtained by applying the extended dimension function to the L^2-homology; in symbols $\beta_n^{(2)}(\Gamma) = \dim_{\mathscr{L}(\Gamma)} H_n^{(2)}(\Gamma)$. These L^2-Betti numbers permit a Künneth formula ([3], 2.7); i.e. for two discrete groups Γ and Λ we have that

$$\beta_n^{(2)}(\Gamma \times \Lambda) = \sum_{k+l=n} \beta_k^{(2)}(\Gamma)\beta_l^{(2)}(\Lambda).$$

In the beginning of the present century, Connes and Shlyakhtenko [4] took the development of L^2-invariants a step further by defining L^2-homology and L^2-Betti numbers for any weakly dense $*$-subalgebra A in a tracial von Neumann algebra (M, τ); these are denoted $H_n^{(2)}(A, \tau)$ and $\beta_n^{(2)}(A, \tau)$ respectively. The

Connes–Shlyakhtenko L^2-Betti numbers generalize the classical L^2-Betti numbers for groups by means of the formula

$$\beta_n^{(2)}(\mathbb{C}\Gamma,\tau) = \beta_n^{(2)}(\Gamma),$$

where τ is the natural trace on the group von Neumann algebra $\mathscr{L}(\Gamma)$. The aim of the present note is to prove a Künneth formula for the Connes–Shlyakhtenko L^2-Betti numbers; i.e. to show that for weakly dense $*$-subalgebras A and B of tracial von Neumann algebras (M,τ) and (N,ρ) we have

$$\beta_n^{(2)}(A\odot B,\tau\otimes\rho) = \sum_{k+l=n}\beta_k^{(2)}(A,\tau)\beta_l^{(2)}(B,\rho).$$

0.1. NOTATION. Above, and in what follows, $\odot$ is used to denote algebraic tensor products which, unless specified otherwise, are assumed to be over the complex numbers. The symbol $\otimes$ will be reserved to denote the minimal tensor product of C^*-algebras, while $\overline{\otimes}$ will be used to denote the tensor product in the category of von Neumann algebras as well as the tensor product in the category of Hilbert spaces. Moreover, for any algebra A we denote by A^{op} the opposite algebra and by A^{ev} the enveloping algebra $A\odot A^{\mathrm{op}}$. For a von Neumann algebra M we let $M^{\overline{\mathrm{ev}}}$ denote the completed tensor product $M\overline{\otimes}M^{\mathrm{op}}$.

0.2. STRUCTURE. The rest of the paper is organized in the following way: Section 1 is devoted to prove some minor results concerning the extended dimension function. These will be used in the proof of the Künneth formula (Theorem 2.1) which is presented in Section 2. In the fourth and final section we show how the Künneth formula can be used to manufacture non-trivial compact quantum groups with a non-vanishing first L^2-Betti number.

1. A BIT OF DIMENSION THEORY

In this section we prove a few minor results related to Lück's generalized Murray–von Neumann dimension $\dim_M(-)$; this is a dimension function defined on the category of all (algebraic) modules over a finite von Neumann algebra M taking values in the interval $[0,\infty]$. For the definition and properties of this dimension function the reader is referred to Chapter 6 in [14]. All three results in this section are derived, without much effort, from the work of Lück, but since they are not made explicit in the literature we present them here for the convenience of the reader.

Throughout this section, M will denote a finite von Neumann algebra with a specified normal, faithful, tracial state τ and all calculations of Murray–von Neumann dimensions of M-modules are implicitly assumed to be with respect to the trace τ. To fix notation, we recall that if X is a submodule of an M-module Y

the algebraic closure of X (relative to Y) is defined as

$$\overline{X}^{\mathrm{alg}} = \bigcap_{\varphi \in \mathrm{Hom}(Y,M),\ X \subseteq \ker(\varphi)} \ker(\varphi).$$

Moreover, the projective part $P(X)$ of a module X is defined as $X/\overline{\{0\}}^{\mathrm{alg}}$ and if X is finitely generated, $P(X)$ is in fact a finitely generated projective module. In the sequel, we will denote by π_X the natural surjection of X onto $P(X)$. As is easily seen, $P(-)$ becomes an endo-functor on the category of M-modules.

LEMMA 1.1. *For a homomorphism $f\colon X \to Y$ between finitely generated M-modules we have* $\dim_M \mathrm{Im}(f) = \dim_M \mathrm{Im}(P(f))$.

Proof. Since $\mathrm{Im}(P(f)) = \mathrm{Im}(\pi_Y \circ f) = \pi_Y(\mathrm{Im}(f))$ we have a short exact sequence

$$0 \longrightarrow \ker(\pi_Y|_{\mathrm{Im}(f)}) \xrightarrow{\subseteq} \mathrm{Im}(f) \xrightarrow{\pi_Y} \mathrm{Im}(P(f)) \longrightarrow 0. \tag{1.1}$$

The kernel $\ker(\pi_Y|_{\mathrm{Im}(f)})$ is contained in the zero-dimensional M-module $\ker(\pi_Y)$ and is therefore itself zero dimensional. Applying additivity of $\dim_M(-)$ to the short exact sequence (1.1) we therefore get, as desired, the following:

$$\dim_M \mathrm{Im}(P(f)) = \dim_M \mathrm{Im}(f) - \dim_M \ker(\pi_Y|_{\mathrm{Im}(f)}) = \dim_M \mathrm{Im}(f). \quad ∎$$

LEMMA 1.2. *Let $f\colon P \to Q$ be a homomorphism of finitely generated projective M-modules and consider the continuous extension $f^{(2)}\colon L^2(P) \longrightarrow L^2(Q)$ of f between the Hilbert M-module completions of P and Q. Then*

$$\dim_M \mathrm{Im}(f) = \dim_M \overline{\mathrm{Im}(f^{(2)})},$$

where the closure on the right hand side is with respect to the Hilbert space norm.

More details about the notion of L^2-completion of projective modules can be found in [11].

Proof. By Theorem 6.24 of [14], the completion-functor $L^2(-)$, from the category of finitely generated projective M-modules to the category of finitely generated Hilbert M-modules, is weakly exact and dimension preserving with weakly exact and dimension preserving inverse. Applying this to the weakly exact sequence

$$P \xrightarrow{f} \overline{\mathrm{Im}(f)}^{\mathrm{alg}} \longrightarrow 0$$

we get the following, where the last identity follows from Theorem 6.7 of [14]:

$$\dim_M \overline{\mathrm{Im}(f^{(2)})} = \dim_M L^2(\overline{\mathrm{Im}(f)}^{\mathrm{alg}}) = \dim_M \overline{\mathrm{Im}(f)}^{\mathrm{alg}} = \dim_M \mathrm{Im}(f). \quad ∎$$

The above two lemmas are included in order to prove the following result which will be essential in the proof of the Künneth formula. The claim in Lemma 1.3 is nested in the proof of Theorem 6.54 in [14], but in order to clarify the proof of Theorem 2.1 we have extracted the result as a separate lemma and included a proof.

LEMMA 1.3. *Let $F = (F_*, f_*)$ and $G = (G_*, g_*)$ be chain complexes consisting of finitely generated projective M-modules and consider a morphism of complexes $\varphi\colon F \to G$. Denote by $L^2(F) = (L^2(F_*), f_*^{(2)})$ and $L^2(G) = (L^2(G_*), g_*^{(2)})$ the completions of F and G into Hilbert M-chain complexes and by $\varphi^{(2)}\colon L^2(F) \to L^2(G)$ the morphism induced by φ. From these data we obtain three induced morphisms on the level of homology:*

$$H_n(\varphi)\colon H_n(F) \longrightarrow H_n(G) \stackrel{\text{def}}{=} \frac{\ker(g_n)}{\operatorname{Im}(g_{n+1})};$$

$$\overline{H}_n(\varphi)\colon \overline{H}_n(F) \longrightarrow \overline{H}_n(G) \stackrel{\text{def}}{=} \frac{\ker(g_n)}{\overline{\operatorname{Im}(g_{n+1})}^{\text{alg}}};$$

$$H_n^{(2)}(\varphi^{(2)})\colon H_n^{(2)}(L^2(F)) \longrightarrow H_n^{(2)}(L^2(G)) \stackrel{\text{def}}{=} \frac{\ker(g_n^{(2)})}{\overline{\operatorname{Im}(g_{n+1}^{(2)})}}.$$

The claim now is that $\dim_M \operatorname{Im}(H_n(\varphi)) = \dim_M \overline{\operatorname{Im}(H_n^{(2)}(\varphi^{(2)}))}$.

Proof. We first note that the homology modules $H_n(F)$ and $H_n(G)$ appearing in Lemma 1.3 are finitely generated so that Lemma 1.1 and Lemma 1.2 apply; this is due to the fact that M is a semihereditary ring and therefore ([11], 0.2) its category of finitely presented modules is abelian. From Lemma 6.52 of [14] we get an isomorphism $\overline{H}_n(F) \simeq P(H_n(F))$ under which $\overline{H}_n(\varphi)$ corresponds to $PH_n(\varphi)$ and an isomorphism $L^2(PH_n(F)) \simeq H_n^{(2)}(L^2(F))$ under which $(PH_n(\varphi))^{(2)}$ corresponds to $H_n^{(2)}(\varphi^{(2)})$. Hence

$$\begin{aligned}
\dim_M \operatorname{Im}(H_n(\varphi)) &= \dim_M \operatorname{Im}(PH_n(\varphi)) \quad \text{(by Lemma 1.1)}\\
&= \dim_M \overline{\operatorname{Im}((PH_n(\varphi))^{(2)})} \quad \text{(by Lemma 1.2)}\\
&= \dim_M \overline{\operatorname{Im}(H_n^{(2)}(\varphi^{(2)}))}. \quad \blacksquare
\end{aligned}$$

2. THE KÜNNETH FORMULA

Let (M, τ) and (N, ρ) be tracial von Neumann algebras and let $A \subseteq M$ and $B \subseteq N$ be weakly dense $*$-subalgebras. Then the algebraic tensor product $A \odot B$ is a weakly dense $*$-subalgebra in the tracial von Neumann algebra $(M\overline{\otimes}N, \tau \otimes \rho)$, and our aim now is to prove the following Künneth formula for the Connes–Shlyakhtenko L^2-Betti numbers introduced in [4].

THEOREM 2.1. *For every $n \geqslant 0$ we have*

$$\beta_n^{(2)}(A \odot B, \tau \otimes \rho) = \sum_{k+l=n} \beta_k^{(2)}(A,\tau)\beta_l^{(2)}(B,\rho),$$

where $\beta_*^{(2)}(-,-)$ are the Connes–Shlyakhtenko L^2-Betti numbers of the tracial $*$-algebra in question.

Note that the L^2-Betti numbers might be infinite and the Künneth formula is therefore to be understood with respect to the standard rules for addition and multiplication in $[0,\infty]$.

Proof. Let $F = (F_*, f_*)$ and $G = (G_*, g_*)$ denote the bar-resolutions ([10], 1.1.12) of A and B respectively. Consider their tensor product $F \odot G = E$ which in degree n has the module

$$E_n = \bigoplus_{k+l=n} F_k \odot G_l,$$

and whose n-th differential $e_n \colon E_n \to E_{n-1}$ is given by the formula

$$e_n(x \otimes y) = f_k(x) \otimes y + (-1)^k x \otimes g_l(y),$$

for a homogeneous element $x \otimes y \in F_k \odot G_l$. Since both F and G are acyclic the same is true for E (see e.g. 2.7.3 of [16]) and E therefore constitutes a resolution of $A \odot B$ in the category of $A \odot B$-bimodules — a category we will freely identify with the category of left modules over $(A \odot B)^{\mathrm{ev}} = (A \odot B) \odot (A \odot B)^{\mathrm{op}}$. As proven in Lemma 2.2 of [4], the bar resolution F can be written as an inductive limit of a family of subcomplexes $(F_{*,i}, f_{*,i})_{i \in I}$ where each $F_i = (F_{*,i}, f_{*,i})$ is a complex of finite length consisting of finitely generated free A^{ev}-modules. If we denote the n-th homology of the complex

$$F_i^{\mathrm{vN}} \stackrel{\mathrm{def}}{=} (M \overline{\otimes} M^{\mathrm{op}} \odot_{A \odot A^{\mathrm{op}}} F_{*,i}, 1 \otimes f_{*,i})$$

by $H_n(F_i^{\mathrm{vN}})$ and by $H_n(\varphi_{i_2 i_1}^{\mathrm{vN}}) \colon H_n(F_{i_1}^{\mathrm{vN}}) \to H_n(F_{i_2}^{\mathrm{vN}})$ the map induced by the inclusion $\varphi_{i_2 i_1} \colon F_{i_1} \hookrightarrow F_{i_2}$ whenever $i_2 \geqslant i_1$, then the L^2-homology $H_n^{(2)}(A,\tau)$ can be calculated as the inductive limit $\varinjlim(H_n(F_i^{\mathrm{vN}}), H_n(\varphi_{i_2 i_1}^{\mathrm{vN}}))$. Since each $F_{n,i}$ is finitely generated, it follows from Theorem 6.13 of [14] that

$$\begin{aligned} \beta_n^{(2)}(A,\tau) &= \sup_{i_1} \inf_{i_2 \geqslant i_1} \dim_{M \overline{\otimes} M^{\mathrm{op}}} \mathrm{Im}(H_n(\varphi_{i_2 i_1}^{\mathrm{vN}})) \\ &= \sup_{i_1} \inf_{i_2 \geqslant i_1} \dim_{M \overline{\otimes} M^{\mathrm{op}}} \overline{\mathrm{Im}(H_n^{(2)}(\varphi_{i_2 i_1}^{(2)}))}, \end{aligned} \tag{2.1}$$

where the last equality follows from Lemma 1.3. In a similar manner we obtain the L^2-homology $H_n^{(2)}(B,\rho)$ as the inductive limit $\varinjlim(H_n(G_j^{\mathrm{vN}}), H_n(\psi_{j_2 j_1}^{\mathrm{vN}}))$ arising from a family of finite length subcomplexes $(G_{*,j}, g_{*,j})_{j \in J}$, each consisting of finitely generated free B^{ev}-modules, and hence

$$\begin{aligned}\beta_n^{(2)}(B,\rho) &= \sup_{j_1} \inf_{j_2 \geqslant j_1} \dim_{N \overline{\otimes} N^{\mathrm{op}}} \mathrm{Im}(H_n(\psi_{j_2 j_1}^{\mathrm{vN}})) \\ &= \sup_{j_1} \inf_{j_2 \geqslant j_1} \dim_{N \overline{\otimes} N^{\mathrm{op}}} \overline{\mathrm{Im}(H_n^{(2)}(\psi_{j_2 j_1}^{(2)}))}. \end{aligned} \tag{2.2}$$

The two families $(F_i)_{i \in I}$ and $(G_j)_{j \in J}$ define a directed family of subcomplexes $(F_i \odot G_j)_{(i,j) \in I \times J}$ of E ($I \times J$ is ordered by setting $(i_1, j_1) \leqslant (i_2, j_2)$ if and only if $i_1 \leqslant i_2$ and $j_1 \leqslant j_2$) which has E as its inductive limit. We now put $E_{i,j} = F_i \odot G_j$ and denote by $H_n(E_{i,j}^{\mathrm{vN}})$ the n-th homology of the induced complex

$$E_{i,j}^{\mathrm{vN}} \overset{\mathrm{def}}{=} ((M \overline{\otimes} N) \overline{\otimes} (M \overline{\otimes} N)^{\mathrm{op}} \odot_{(A \odot B)^{\mathrm{ev}}} E_{*,i,j}, 1 \otimes e_{*,i,j}).$$

For $(i_2, j_2) \geqslant (i_1, j_1)$, the inclusion $\varphi_{i_2 i_1} \otimes \psi_{j_2 j_1}$ induces a map

$$H_n((\varphi_{i_2 i_1} \otimes \psi_{j_2 j_1})^{\mathrm{vN}}) \colon H_n(E_{i_1, j_1}^{\mathrm{vN}}) \longrightarrow H_n(E_{i_2, j_2}^{\mathrm{vN}}),$$

and just as above we get

$$H_n^{(2)}(A \odot B, \tau \otimes \rho) = \varinjlim (H_n(E_{i,j}^{\mathrm{vN}}), H_n((\varphi_{i_2 i_1} \otimes \psi_{j_2 j_1})^{\mathrm{vN}})).$$

Denoting the completed tensor product $(M \overline{\otimes} N) \overline{\otimes} (M \overline{\otimes} N)^{\mathrm{op}}$ by $(M \overline{\otimes} N)^{\overline{\mathrm{ev}}}$ we get

$$\begin{aligned}\beta_n^{(2)}(A \odot B, \tau \otimes \rho) &= \sup_{(i_1, j_1)} \inf_{(i_2, j_2) \geqslant (i_1, j_1)} \dim_{(M \overline{\otimes} N)^{\overline{\mathrm{ev}}}} \mathrm{Im}(H_n((\varphi_{i_2 i_1} \otimes \psi_{j_2 j_1})^{\mathrm{vN}})) \\ &= \sup_{(i_1, j_1)} \inf_{(i_2, j_2) \geqslant (i_1, j_1)} \dim_{(M \overline{\otimes} N)^{\overline{\mathrm{ev}}}} \overline{\mathrm{Im}(H_n^{(2)}((\varphi_{i_2 i_1} \otimes \psi_{j_2 j_1})^{(2)}))}. \end{aligned} \tag{2.3}$$

For each $(i,j) \in I \times J$ we now have two finitely generated Hilbert chain complexes; namely the Hilbert $(M \overline{\otimes} N)^{\overline{\mathrm{ev}}}$-chain complex $L^2(E_{i,j})$ and the tensor product of Hilbert chain complexes $L^2(F_i) \overline{\otimes} L^2(F_j)$ (see e.g [14]) which becomes a Hilbert chain complex for the von Neumann algebra $M^{\overline{\mathrm{ev}}} \overline{\otimes} N^{\overline{\mathrm{ev}}}$. The two von Neumann algebras in question are $*$-isomorphic in a trace-preserving way via the map

$$(M \overline{\otimes} M^{\mathrm{op}}) \overline{\otimes} (N \overline{\otimes} N^{\mathrm{op}}) \xrightarrow{\alpha} (M \overline{\otimes} N) \overline{\otimes} (M \overline{\otimes} N)^{\mathrm{op}},$$

given by $a \otimes c^{\mathrm{op}} \otimes b \otimes d^{\mathrm{op}} \mapsto a \otimes b \otimes (c \otimes d)^{\mathrm{op}}$. Through the isomorphism α we may therefore consider $L^2(E_{i,j})$ as a Hilbert $M^{\overline{\mathrm{ev}}} \overline{\otimes} N^{\overline{\mathrm{ev}}}$-chain complex; when doing so we write it as ${}_{\alpha}L^2(E_{i,j})$. The Hilbert chain complex ${}_{\alpha}L^2(E_{i,j})$ is nothing but the tensor product complex $L^2(F_i) \overline{\otimes} L^2(G_j)$ and by Lemma 1.22 of [14] this identification gives rise to an $M^{\overline{\mathrm{ev}}} \overline{\otimes} N^{\overline{\mathrm{ev}}}$-isomorphism on the level of L^2-homology:

$${}_{\alpha}H_n^{(2)}(L^2(E_{i,j})) = H_n^{(2)}({}_{\alpha}L^2(E_{i,j})) \xrightarrow{\sim} \bigoplus_{k+l=n} H_k^{(2)}(L^2(F_i)) \overline{\otimes} H_l^{(2)}(L^2(G_j)).$$

For each $(i_1, j_1) \leqslant (i_2, j_2)$ we therefore get a commutative diagram of Hilbert $M^{\overline{\mathrm{ev}}}\overline{\otimes}N^{\overline{\mathrm{ev}}}$-modules

$$\begin{tikzcd}
{}_\alpha H_n^{(2)}(L^2(E_{i_1,j_1})) \arrow[r, "\sim"] \arrow[d, "{}_\alpha H_n^{(2)}((\varphi_{i_2 i_1}\otimes\psi_{j_2 j_1})^{(2)})"'] & \bigoplus_{k+l=n} H_k^{(2)}(L^2(F_{i_1}))\overline{\otimes} H_l^{(2)}(L^2(G_{j_1})) \arrow[d, "\bigoplus_{k+l=n} H_k^{(2)}(\varphi^{(2)}_{i_2 i_1})\otimes H_l^{(2)}(\psi^{(2)}_{j_2 j_1})"] \\
{}_\alpha H_n^{(2)}(L^2(E_{i_2,j_2})) \arrow[r, "\sim"] & \bigoplus_{k+l=n} H_k^{(2)}(L^2(F_{i_2}))\overline{\otimes} H_l^{(2)}(L^2(G_{j_2})).
\end{tikzcd}$$

For any finitely generated Hilbert $(M\overline{\otimes}N)^{\overline{\mathrm{ev}}}$-module X we have

$$\dim_{(M\overline{\otimes}N)^{\overline{\mathrm{ev}}}} X = \dim_{M^{\overline{\mathrm{ev}}}\overline{\otimes}N^{\overline{\mathrm{ev}}}} {}_\alpha X,$$

simply because α is a trace-preserving $*$-isomorphism. We therefore get

$$\begin{aligned}
&\dim_{(M\overline{\otimes}N)^{\overline{\mathrm{ev}}}} \overline{\mathrm{Im}(H_n^{(2)}((\varphi_{i_2 i_1}\otimes\psi_{j_2 j_1})^{(2)}))} \\
&\quad = \dim_{M^{\overline{\mathrm{ev}}}\overline{\otimes}N^{\overline{\mathrm{ev}}}} \overline{\mathrm{Im}({}_\alpha H_n^{(2)}((\varphi_{i_2 i_1}\otimes\psi_{j_2 j_1})^{(2)}))} \\
&\quad = \dim_{M^{\overline{\mathrm{ev}}}\overline{\otimes}N^{\overline{\mathrm{ev}}}} \overline{\mathrm{Im}\Big(\bigoplus_{k+l=n} H_k^{(2)}(\varphi^{(2)}_{i_2 i_1})\otimes H_l^{(2)}(\psi^{(2)}_{j_2 j_1})\Big)} \\
&\quad = \dim_{M^{\overline{\mathrm{ev}}}\overline{\otimes}N^{\overline{\mathrm{ev}}}} \bigoplus_{k+l=n} \overline{\mathrm{Im}(H_k^{(2)}(\varphi^{(2)}_{i_2 i_1}))}\overline{\otimes}\, \overline{\mathrm{Im}(H_l^{(2)}(\psi^{(2)}_{j_2 j_1}))} \\
&\quad = \sum_{k+l=n} \dim_{M^{\overline{\mathrm{ev}}}} (\overline{\mathrm{Im}(H_k^{(2)}(\varphi^{(2)}_{i_2 i_1}))}) \dim_{N^{\overline{\mathrm{ev}}}} (\overline{\mathrm{Im}(H_l^{(2)}(\psi^{(2)}_{j_2 j_1}))}),
\end{aligned} \tag{2.4}$$

where the last equality follows from Theorem 1.12 of [14]. Combining all the formulas obtained so far, the desired identity follows:

$$\begin{aligned}
&\beta_n^{(2)}(A\odot B, \tau\otimes\rho) \\
&\overset{(2.3)}{=} \sup_{(i_1,j_1)} \inf_{(i_2,j_2)\geqslant(i_1,j_1)} \dim_{(M\overline{\otimes}N)^{\overline{\mathrm{ev}}}} \overline{\mathrm{Im}(H_n^{(2)}((\varphi_{i_2 i_1}\otimes\psi_{j_2 j_1})^{(2)}))} \\
&\overset{(2.4)}{=} \sup_{(i_1,j_1)} \inf_{(i_2,j_2)\geqslant(i_1,j_1)} \sum_{k+l=n} \dim_{M^{\overline{\mathrm{ev}}}} (\overline{\mathrm{Im}(H_k^{(2)}(\varphi^{(2)}_{i_2 i_1}))}) \dim_{N^{\overline{\mathrm{ev}}}} (\overline{\mathrm{Im}(H_l^{(2)}(\psi^{(2)}_{j_2 j_1}))}) \\
&= \sum_{k+l=n} \Big(\sup_{i_1}\inf_{i_2\geqslant i_1} \dim_{M^{\overline{\mathrm{ev}}}} \overline{\mathrm{Im}(H_k^{(2)}(\varphi^{(2)}_{i_2 i_1}))}\Big)\Big(\sup_{j_1}\inf_{j_2\geqslant j_1} \dim_{N^{\overline{\mathrm{ev}}}} \overline{\mathrm{Im}(H_l^{(2)}(\psi^{(2)}_{j_2 j_1}))}\Big) \\
&\overset{(2.1),(2.2)}{=} \sum_{k+l=n} \beta_k^{(2)}(A,\tau)\beta_l^{(2)}(B,\rho). \quad \blacksquare
\end{aligned}$$

The Künneth formula gives an alternative proof of the following fact which is usually derived from Theorem 2.4 in [4].

COROLLARY 2.2. *The n-th L^2-Betti number of the hyperfinite factor R is either zero or infinite.*

Proof. Denote by τ the unique trace on R and by ρ the normalized trace on $\mathbb{M}_2(\mathbb{C})$. Since R is hyperfinite it absorbs $\mathbb{M}_2(\mathbb{C})$ tensorially and since both R and $\mathbb{M}_2(\mathbb{C})$ are factors any isomorphism $R \simeq R \odot \mathbb{M}_2(\mathbb{C})$ is bound to preserve the traces. Moreover, it follows from Proposition 2.9 of [4] that

$$\beta_k^{(2)}(\mathbb{M}_2(\mathbb{C}),\rho) = \begin{cases} \frac{1}{4} & \text{if } k=0, \\ 0 & \text{if } k \geqslant 1. \end{cases}$$

Hence

$$\beta_n^{(2)}(R,\tau) = \beta_n^{(2)}(R \odot \mathbb{M}_2(\mathbb{C}), \tau \otimes \rho) = \sum_{k+l=n} \beta_k^{(2)}(R,\tau)\beta_l^{(2)}(\mathbb{M}_2(\mathbb{C}),\rho) = \beta_n^{(2)}(R,\tau)\frac{1}{4},$$

which forces $\beta_n^{(2)}(R,\tau)$ to be either zero or infinite. ∎

It is, to the best of the author's knowledge, still not known what the L^2-Betti numbers of the hyperfinite factor are, except in degree zero where it follows from Corollary 2.8 of [4] that $\beta_0^{(2)}(R,\tau) = 0$. However, having in mind the well known analogy between hyperfiniteness and amenability, it is of course natural to expect that also the higher L^2-Betti numbers of R vanish.

3. AN APPLICATION TOWARDS QUANTUM GROUPS

We take as our starting point Woronowicz's definition [17] of a compact quantum group. Thus, a compact quantum group $\mathbb{G}$ consists of a (not necessarily commutative) unital, separable C^*-algebra $C(\mathbb{G})$ together with a coassociative, unital $*$-homomorphism $\Delta_{\mathbb{G}}\colon C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$, called the comultiplication, which furthermore has to satisfy a certain non-degeneracy condition. We remind the reader that such a C^*-algebraic quantum group automatically gives rise to a purely algebraic quantum group (i.e. a Hopf $*$-algebra [5]), whose underlying algebra will be denoted $\mathrm{Pol}(\mathbb{G})$, as well as a von Neumann algebraic quantum group, in the sense of [6], whose underlying algebra will be denoted $L^\infty(\mathbb{G})$.

EXAMPLE 3.1. The canonical example of a compact quantum group, on which the general definition is modeled, is obtained by considering a compact, second countable, Hausdorff topological group G and its commutative C^*-algebra $C(G)$ of continuous, complex valued functions. In this case the von Neumann algebra becomes $L^\infty(G,\mu)$, where μ denotes the Haar probability measure, and the associated Hopf $*$-algebra becomes the algebra generated by matrix coefficients arising from the irreducible representations of G. Moreover, every compact quantum group whose underlying C^*-algebra is commutative is of this form.

Recall that the C^*-algebra $C(\mathbb{G})$ of a compact quantum group $\mathbb{G}$ comes with a distinguished state $h_{\mathbb{G}}$, called the Haar state, which plays the role corresponding to the Haar measure on a genuine, compact group. If $\mathbb{G}$ and $\mathbb{H}$ are two compact quantum groups they give rise to a third quantum group, denoted $\mathbb{G} \times \mathbb{H}$, whose underlying C^*-algebra is $C(\mathbb{G}) \otimes C(\mathbb{H})$ and whose comultiplication is given by $\Delta_{\mathbb{G}\times\mathbb{H}} = (\mathrm{id} \otimes \sigma \otimes \mathrm{id})\Delta_{\mathbb{G}} \otimes \Delta_{\mathbb{H}}$; here $\sigma\colon C(\mathbb{G}) \otimes C(\mathbb{H}) \to C(\mathbb{H}) \otimes C(\mathbb{G})$ denotes the flip-isomorphism. The Haar state $h_{\mathbb{G}\times\mathbb{H}}$ is given by $h_{\mathbb{G}} \otimes h_{\mathbb{H}}$ and $\mathrm{Pol}(\mathbb{G} \times \mathbb{H}) = \mathrm{Pol}(\mathbb{G}) \odot \mathrm{Pol}(\mathbb{H})$. See [15] for more details.

In [7] and [8] the notion of L^2-invariants was studied in the setting of compact quantum groups of Kac type; i.e. those quantum groups for which the Haar state is a trace. If $\mathbb{G}$ is such a quantum group its n-th L^2-homology $H_n^{(2)}(\mathbb{G})$ is defined as $\mathrm{Tor}_n^{\mathrm{Pol}(\mathbb{G})}(L^\infty(\mathbb{G}), \mathbb{C})$ and the n-th L^2-Betti number $\beta_n^{(2)}(\mathbb{G})$ is defined by applying Lück's generalized Murray–von Neumann dimension $\dim_{L^\infty(\mathbb{G})}(-)$ to the $L^\infty(\mathbb{G})$-module $H_n^{(2)}(\mathbb{G})$. As a consequence of Theorem 2.1 we also obtain a Künneth formula for these quantum group L^2-Betti numbers.

COROLLARY 3.2. *Let $\mathbb{G}$ and $\mathbb{H}$ be compact quantum groups of Kac type. Then, for every $n \geqslant 0$,*

$$\beta_n^{(2)}(\mathbb{G} \times \mathbb{H}) = \sum_{k+l=n} \beta_k^{(2)}(\mathbb{G})\beta_l^{(2)}(\mathbb{H}).$$

Proof. It was shown in Theorem 4.1 of [8] that $\beta_n^{(2)}(\mathbb{G})$ coincides with the n-th Connes–Shlyakhtenko L^2-Betti number $\beta_n^{(2)}(\mathrm{Pol}(\mathbb{G}), h_{\mathbb{G}})$, and since we have $\mathrm{Pol}(\mathbb{G} \times \mathbb{H}) = \mathrm{Pol}(\mathbb{G}) \odot \mathrm{Pol}(\mathbb{H})$ and $h_{\mathbb{G}\times\mathbb{H}} = h_{\mathbb{G}} \otimes h_{\mathbb{H}}$ (see e.g. [15]) the result follows from Theorem 2.1. ∎

Next we explain how the Künneth formula provides us with non-trivial examples of quantum groups with non-vanishing L^2-Betti numbers.

Any cocommutative quantum group is isomorphic to a quantum group of the form $(C^*_{\mathrm{red}}(\Gamma), \Delta_{\mathrm{red}})$ where Γ is a discrete group and $\Delta_{\mathrm{red}}(\gamma) = \gamma \otimes \gamma$. It follows from the Proposition 1.3 of [8] that the L^2-Betti numbers of such a quantum group coincide with the classical L^2-Betti numbers of the underlying group Γ. Considering, for instance, the case of the free group on two generators $\mathbb{F}_2$ with $\beta_1^{(2)}(\mathbb{F}_2) = 1$ we therefore get, in a somewhat trivial way, a compact quantum group with a non-vanishing first L^2-Betti number. Another trivial source of non-vanishing results is the class of finite quantum groups (i.e. those whose C^*-algebra is finite dimensional); for such a quantum group $\mathbb{G}$ the zeroth L^2-Betti number equals $\dim_{\mathbb{C}}(C(\mathbb{G}))^{-1}$ and all the higher L^2-Betti numbers vanish [8]. So far, these are the only known examples of quantum groups with non-vanishing L^2-Betti numbers and the following question is therefore natural.

QUESTION 3.3. What is an example of an infinite, non-cocommutative compact quantum group with a positive L^2-Betti number?

The Künneth formula provides an answer to this question. For this, let $\mathbb{G}$ be a finite, non-cocommutative quantum group and denote by N the dimension of $C(\mathbb{G})$ and let $\mathbb{H}$ be the compact quantum group arising from $\mathbb{F}_2$. We then have

$$\beta_p^{(2)}(\mathbb{G}) = \begin{cases} \frac{1}{N} & \text{when } p = 0, \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad \beta_p^{(2)}(\mathbb{H}) = \begin{cases} 1 & \text{when } p = 1, \\ 0 & \text{otherwise;} \end{cases}$$

and the Künneth formula therefore yields

$$\beta_1^{(2)}(\mathbb{G} \times \mathbb{H}) = \beta_0^{(2)}(\mathbb{G})\beta_1^{(2)}(\mathbb{H}) + \beta_1^{(2)}(\mathbb{G})\beta_0^{(2)}(\mathbb{H}) = \frac{1}{N}.$$

By construction, $C(\mathbb{G} \times \mathbb{H})$ has infinite linear dimension and since $\mathbb{G}$ is assumed non-cocommutative $\mathbb{G} \times \mathbb{H}$ becomes non-cocommutative.

REMARK 3.4. For the free group on k generators $\mathbb{F}_k$ the only non-vanishing L^2-Betti number is the first which has value $k-1$. Also, for each $n \in \mathbb{N}$ it is easy to produce a finite quantum group of dimension n; one may simply take a group G with n elements and consider the associated commutative quantum group $C(G)$. By copying the example from above we can therefore construct quantum groups with any prescribed positive, rational number as its first L^2-Betti number. Note, however, that if the group G is chosen (or forced) to be abelian the example becomes cocommutative.

In the opposite direction, the Künneth formula also gives rise to the following vanishing result.

COROLLARY 3.5. *Let $\mathbb{G}$ and $\mathbb{H}$ be compact quantum groups of Kac type and assume one of them to be infinite and coamenable. Then $\beta_n^{(2)}(\mathbb{G} \times \mathbb{H}) = 0$ for all $n \geqslant 0$.*

We remind the reader that a compact quantum group is called coamenable [2] if the counit $\varepsilon\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ extends to a character on the image of $C(\mathbb{G})$ under the GNS-representation arising from $h_{\mathbb{G}}$.

Proof. Corollary 6.2 in [7] together with Proposition 5.1.5 of [9] shows that all L^2-Betti numbers of an infinite, coamenable quantum group vanish and the claim now follows from Corollary 3.2. ∎

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DAVID KYED, MATHEMATISCHES INSTITUT, GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN, BUNSENSTRASSE 3-5, D-37073 GÖTTINGEN, GERMANY

E-mail address: kyed@uni-math.gwdg.de

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