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IDEALS AND STRUCTURE OF OPERATOR ALGEBRAS

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ABSTRACT. We continue the study of r-ideals, l-ideals, and HSA's in operator algebras. Some applications are made to the structure of operator algebras, including Wedderburn type theorems for a class of operator algebras. We also consider the one-sided M-ideal structure of certain tensor products of operator algebras.

KEYWORDS: *Nonselfadjoint operator algebra, one-sided ideals, hereditary subalgebra, matrix units, minimal ideals, annihilator algebra, Haagerup tensor product.*

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1. INTRODUCTION

An *operator algebra* (respectively *operator space*) is a norm closed algebra (respectively vector space) of operators on a Hilbert space. The present paper is a continuation of a program (see e.g. [7], [8], [9], [15], [25], [38]) studying the structure of operator algebras and operator spaces using "one-sided ideals". We shall have nothing to say about general one-sided ideals in an operator algebra A, indeed not much is known about general closed ideals in some of the simplest classical function algebras. However there is a tractable and often interesting class of one-sided ideals in A, which corresponds to a kind of "noncommutative topology" for A, and to a noncommutative variant of the theory of peak sets and peak interpolation for function algebras (see [25], [9]). These are the r-*ideals*, namely the right ideals of A possessing a left contractive approximate identity (cai). These ideals are in bijective correspondence with the l-*ideals* (left ideals with right cai), and with the *hereditary subalgebras (*HSA*'s)* of A, defined below. Much of our paper is a further development of the general properties and behaviors of these objects, with applications to the structure of operator algebras. Thus in Section 2 we record many new general facts about one-sided ideals and HSA's, as well as some other preliminary results. Section 3 mainly concerns the existence of nontrivial r-ideals, and of maximal r-ideals. There are some interesting connections here with the remaining open problems from [25], [9], [8].

In Section 4 we apply some of these ideas to study a class of operator algebras which we have not seen in the literature, which we call 1-*matricial algebras*. One of our original motivations for introducing these algebras, is that they might perhaps lead to some insight into important open questions about operator algebraic amenability and related notions, for example because their second duals are also easy to work with. This class of algebras is large enough to display some interesting behaviors, e.g. some 1-matricial algebras are right ideals in their biduals, and others are not. Amongst other things we prove Wedderburn type structure theorems for operator algebras, using 1-matricial algebras as the building blocks. A key ingredient, as one might expect given the history of Wedderburn type decompositions, is played by minimal r-ideals. In Section 5 we continue this theme by deriving some new characterizations of C^*-algebras consisting of compact operators.

In the final section we turn to the ideal structure of the Haagerup tensor product of operator algebras. We also discuss some connections with the study of operator spaces which are one-sided M- or L-ideals in their biduals (initiated in [38]). Our results provide natural examples of operator spaces which are right but not left ideals (or M-ideals) in their second dual. Their duals are left but not right L-summands in their biduals.

Turning to notation, we reserve the letters H and K for Hilbert spaces. We will use basic concepts from operator space theory, and from the operator space approach to operator algebras, which may be found e.g. in [11], [35].

A *topologically simple* algebra has no closed ideals. A *semiprime* algebra is one in which $J^2 = (0)$ implies $J = (0)$, for closed ideals J. For subsets J, I of an algebra A the *left* (respectively *right*) *annihilators* are $L(J) = \{a \in A : aJ = (0)\}$ and $R(I) = \{a \in A : Ia = (0)\}$. We recall that a Banach algebra A is a *right* (respectively *left*) *annihilator algebra* if $R(I) \neq (0)$ (respectively $L(J) \neq (0)$) for any proper closed left ideal I (respectively right ideal J) of A. An *annihilator algebra* is both a left and right annihilator algebra. In [27], Kaplansky studied a class of algebras which he called *dual*, these satisfy $R(L(J)) = J$ and $L(R(I)) = I$ for J, I as in the last lines. It is known that a C^*-algebra is an annihilator algebra if and only if it is dual in Kaplansky's sense, and these are precisely the c_0-direct sums of "elementary C^*-algebras", where the latter term refers to the space of compact operators on some Hilbert space. This class of algebras has very many diverse characterizations, some of which may be found in Kaplansky's works or Exercise 4.7.20 of [17], and which we shall use freely. For example, these are also the C^*-algebras A which are ideals in their bidual [24], or equivalently, $A^{**} = M(A)$. We shall usually not use the word "dual", to avoid any confusion with Banach space duality, and instead will call these *annihilator C^*-algebras*, or *C^*-algebras consisting of compact operators*. In Section 4, we will be more interested in several properties weaker than being an annihilator algebra: *modular annihilator, compact, dense socle*. See Chapter 8 of [32] for the definitions, and a thorough discussion of these properties and their connections to each other.

A normed algebra is *unital* if it has an identity of norm 1; any operator algebra A has a (unique) operator algebra unitization A^1. A *bai* (respectively *cai*) is a bounded (respectively contractive) two-sided approximate identity. The existence of a bai implies of course that A is *left* and *right essential*, by which we mean that the left or right multiplication by an element in A induces a bicontinuous injection of A in $B(A)$. An operator algebra A is Arens regular, and A has a right bai (respectively right cai) if and only if A^{**} has a right identity (respectively right identity of norm 1) — see e.g. Proposition 5.1.8 of [32], Section 2.5 of [11]. An algebra is *approximately unital* if it has a cai. For an approximately unital operator algebra A, we denote the left, right, and two-sided, multiplier algebras by $LM(A), RM(A), M(A)$, usually viewed as subalgebras of A^{**} (see Section 2.5 of [11]). If A is an operator algebra, or operator space containing the identity operator, then we write $\Delta(A)$ for the *diagonal* $A \cap A^*$, and write A_{sa} for the selfadjoint part of $\Delta(A)$.

The one-sided approximate identity in any r-ideal (respectively l-ideal) J in an operator algebra A, converges weak* to the *support projection* p of J in A^{**}, and $J^{\perp\perp} = pA^{**}$ (respectively $J^{\perp\perp} = A^{**}p$). Indeed, we recall from [25], [9] that r-ideals are precisely those right ideals of the form $pA^{**} \cap A$ for a projection p in A^{**} which is *open*. By the latter term we mean that there is a net in $pA^{**}p \cap A$ converging weak* to p; there are several other equivalent characterizations in [9]. Similarly for l-ideals. A *hereditary subalgebra (*HSA*)* of A is an approximately unital subalgebra D of A such that $DAD \subset D$: these are also the subalgebras of the form $pA^{**}p \cap A$ for an open projection p in A^{**}.

For subspaces V_k of a vector space, we write $\sum_k V_k$ for the vector subspace of finite sums of elements in V_k, for all k. A *projection* in an operator algebra A is always an orthogonal projection. A projection in A is called $*$*-minimal* if it dominates no nontrivial projection in A. We use this nonstandard notation to distinguish it from the next concept: a projection or idempotent e in A is called *algebraically minimal* if $eAe = \mathbb{C}\, e$. Clearly an algebraically minimal projection is $*$-minimal. In certain algebras the converse is true too, but this is not common. In a semiprime Banach algebra the minimal left ideals are all closed, and all of the form Ae for an algebraically minimal idempotent. Indeed, if e is an idempotent then Ae is a minimal left ideal if and only if e is algebraically minimal [32]. Similarly for right ideals. Idempotents e, f are *mutually orthogonal* if $ef = fe = 0$. Any A-modules appearing in our paper will be operator spaces too; hence for morphisms we use $CB(X, Y)_A$, the *completely bounded* right A-module maps.

We will also need in a couple of places some concepts from the theory of operator space multipliers and one-sided M-ideals, which can be found e.g. in Chapter 4 of [11], [8], or the first few pages of [15]. We write $\mathcal{M}_{\mathrm{l}}(X)$ and $\mathcal{M}_{\mathrm{r}}(X)$ for the unital operator algebras of *left and right operator space multipliers* of an operator space X. See Chapter 4 of [11] for the definition of these. The diagonal of these two operator algebras are the C^*-algebras $\mathcal{A}_{\mathrm{l}}(X)$ and $\mathcal{A}_{\mathrm{r}}(X)$ of *left* *and right adjointable multipliers*. The operator space centralizer algebra $Z(X)$ is $\mathcal{A}_{\mathrm{l}}(X) \cap \mathcal{A}_{\mathrm{r}}(X)$. The projections in these three C^*-algebras are called respectively *left, right,* and *complete, M-projections* on X. A subspace J of X is called, respectively, a *right, left, or complete, M-ideal* in X, if $J^{\perp\perp}$ is, respectively, the range of a left, right, or complete, M-projection on X^{**}. The right (respectively left, complete) M-ideals in an approximately unital operator algebra are precisely the r-ideals (respectively l-ideals, closed ideals with cai).

If X is an operator space then $C_\infty(X)$ (respectively $R_\infty(X)$) denotes the operator space of (countably) infinite columns (respectively rows) with entries in X, formed by taking the closure of those columns (respectively rows) with only finitely many nonzero entries. Also, if E_k is a subspace of X for each k, then the "column sum" $\bigoplus_k^{\mathrm{c}} E_k$ consists of the tuples $(x_k) \in C_\infty(X)$ with $x_k \in E_k$ for each k. Similarly for the "row sum" $\bigoplus_k^{\mathrm{r}} E_k$.

2. HSA'S AND r-IDEALS

In this section we collect several new results about r-ideals and HSA's. The first is a generalization of Proposition 6.4 in [8].

PROPOSITION 2.1. *Let A be an operator algebra with right* cai (e_t). *Then we have $\mathcal{M}_{\mathrm{r}}(A) = CB_A(A) = RM(\{a \in A : e_t a \to a\})$. Also, the left M-ideals of A are the* l*-ideals in A. The left M-summands of A are precisely the left ideals of form Ae, for a projection e in the multiplier algebra of $\{a \in A : e_t a \to a\}$; and also coincide with the ranges of completely contractive idempotent left A-module maps on A.*

Proof. That $\mathcal{M}_{\mathrm{r}}(A) = CB_A(A) = RM(\{a \in A : e_t a \to a\})$ follows from Theorem 6.1 of [7]. We are using the quite nontrivial fact that the hypothesis of Theorem 6.1 (5) in [7] is removable, which was established in [9]. This, together with Proposition 5.1 in [9], proves the summand assertion. It follows that if A has a right identity of norm 1, then the right M-projections correspond bijectively to the projections in A. Using this, the proof of the left M-ideal assertion is just as in Proposition 6.4 of [8]. ∎

REMARK 2.2. It is not hard to see that the last result is not true for general operator algebras. Indeed, right M-summands or right M-ideals of operator algebras with a left cai, will be right ideals, but they need not have a left cai.

LEMMA 2.3. *An* r*-ideal which has a left identity has a left identity of norm* 1*.*

The proof follows from Corollary 4.7 of [7].

PROPOSITION 2.4. *If A is an operator algebra with right* cai*, and if p is a projection in A^{**} such that $pA \subset A$, then $Ap \subset A$.*

The proof is as in Proposition 5.1 of [9].

PROPOSITION 2.5. *Semiprimeness and topological simplicity pass to (approximately unital)* HSA*'s of operator algebras.*

Proof. Suppose that D is an HSA in an operator algebra A. If A is semiprime and J is an ideal in D with $J^2 = (0)$, then since D is approximately unital we have

$$JAJ \subset JDADJ \subset JDJ \subset J^2 = (0).$$

Thus AJA is a nil ideal in A, hence is zero, so that $J \subset DJD \subset AJA = (0)$.

If A is topologically simple and J is a closed ideal in D, then $AJA = (0)$ or $AJA = A$. In the first case, $J = DJD = (0)$. In the second case,

$$D = D^3 \subset DAJAD \subset DADJDAD \subset J.$$

So $D = J$, and so D is topologically simple. ∎

REMARK 2.6. We show elsewhere that semisimplicity passes to HSA's.

We now state a series of simple results about the diagonal of an operator algebra.

PROPOSITION 2.7. *For any operator algebra A, we have $\Delta(A) = \Delta(A^{**}) \cap A$.*

Proof. Suppose that A is a subalgebra of a C^*-algebra B. Clearly $\Delta(A) \subset \Delta(A^{**}) \cap A$. If $x \in \Delta(A^{**}) \cap A$, we may write $x = x_1 + \mathrm{i}x_2$ with x_k selfadjoint in $\Delta(A^{**})$. Then $x + x^* = 2x_1$, so that $x_1 \in B \cap A^{\perp\perp} = A$. Since x_1 is selfadjoint it is in $\Delta(A)$. Similarly for x_2, so that $x \in \Delta(A)$. ∎

COROLLARY 2.8. *Let A be an operator algebra. If A is an ideal in its bidual, then $\Delta(A)$ is an ideal in $\Delta(A^{**})$, and also in $\Delta(A)^{**}$. Thus $\Delta(A)$ is an annihilator C^*-algebra.*

Proof. Since $\Delta(A^{**})A\Delta(A^{**}) \subset A^{**}AA^{**} \subset A$, the first assertion follows. The second assertion follows from the first, and the third from the second. ∎

PROPOSITION 2.9. *If A is an* HSA *in an approximately unital operator algebra B, then $\Delta(A) = \Delta(B) \cap A$, and this is an* HSA *in $\Delta(B)$ if it is nonzero.*

Proof. If $x \in \Delta(B) \cap A$, write $x = x_1 + \mathrm{i}x_2$ with x_k selfadjoint. If $A = eB^{**}e \cap B$ then $x = ex_1e + \mathrm{i}ex_2e$. Thus $x_k = ex_ke \in eB^{**}e \cap B = A$, and so $x \in \Delta(A)$. Hence $\Delta(B) \cap A = \Delta(A)$. Clearly $\Delta(A)\Delta(B)\Delta(A) \subset \Delta(B) \cap (ABA) \subset \Delta(B) \cap A = \Delta(A)$, and so $\Delta(A)$ is an HSA in $\Delta(B)$ if it is nonzero. ∎

REMARK 2.10. (i) Idempotents e in an HSA D of A are algebraically minimal in A if and only if they are algebraically minimal in D, since $eAe = e^2Ae^2 \subset eDe \subset \mathbb{C}\,e$.

(ii) If A is an approximately unital operator algebra, and p is a projection in $M(A)$ then $\Delta(pAp) = p\Delta(A)p$. This follows from Proposition 2.9, or directly.

For any operator algebra A, the diagonal $\Delta(A)$ acts nondegenerately on A if and only if A has a positive cai, and if and only if $1_{\Delta(A)^{\perp\perp}} = 1_{A^{**}}$. The latter is equivalent to $1_{A^{**}} \in \Delta(A)^{\perp\perp}$. To see these last equivalences, note that a positive cai in A will converge weak* to both $1_{\Delta(A)^{\perp\perp}}$ and $1_{A^{**}}$, so they are equal. Conversely, if $1_{\Delta(A)^{\perp\perp}} = 1_{A^{**}}$, and if (e_t) is a cai for $\Delta(A)$, then $e_t a \to a$ weak*, hence weakly, for all $a \in A$. Thus $\Delta(A)A$ is weakly dense in A, hence norm dense by Mazur's theorem.

PROPOSITION 2.11. *If A is an operator algebra such that $\Delta(A)$ acts nondegenerately on A, then $M(\Delta(A)) = \Delta(M(A))$.*

Proof. For any approximately unital operator algebra A, viewing multipliers of A as elements of A^{**}, and multipliers of $\Delta(A)$ in $\Delta(A)^{**} = \Delta(A)^{\perp\perp} \subset A^{**}$, we have $\Delta(M(A)) \cap \Delta(A)^{\perp\perp} \subset M(\Delta(A))$. For if $T \in \Delta(M(A))$ and if $a \in \Delta(A)$, then to see that $Ta \in \Delta(A)$, we may assume by linearity that T and a are selfadjoint. Then $Ta \in A$, but also $(Ta)^* = aT \in A$, so that $Ta \in \Delta(A)$. Suppose that A and $\Delta(A)$ share a common positive cai (e_t). Then $Te_t \in \Delta(A)$, and so $T \in \Delta(A)^{\perp\perp}$. Thus $\Delta(M(A)) \subset M(\Delta(A))$.

For $T \in M(\Delta(A)), a \in A$, we have $Ta = \lim_t Te_t a \in \Delta(A)A \subset A$. Similarly $aT \in A$, and so the C^*-algebra $M(\Delta(A))$ is a unital subalgebra of $M(A)$. So $M(\Delta(A)) \subset \Delta(M(A))$. ∎

We define a notion based on the noncommutative topology of operator algebras [9], [25]. We will say that an operator algebra A is *nc-discrete* if it satisfies the equivalent conditions in the next result.

PROPOSITION 2.12. *For an approximately unital operator algebra A the following are equivalent:*

(i) *Every open projection e in A^{**} is also closed (in the sense that $1 - e$ is open).*

(ii) *The open projections in A^{**} are exactly the projections in $M(A)$.*

(iii) *Every* r*-ideal J of A is of the form eA for a projection $e \in M(A)$.*

(iv) *The left annihilator of every nontrivial* r*-ideal of A is a nontrivial* l*-ideal.*

(v) *Every* HSA *of A is of the form eAe for a projection $e \in M(A)$.*

If any of these hold then $\Delta(A)$ is an annihilator C^*-algebra.

Proof. If a projection p in A is both open and closed, then $p \in M(B)$ for any C^*-cover B of A by Theorem 3.12.9 of [34] and Theorem 2.4 of [9]. Hence $pA \subset A^{\perp\perp} \cap B = A$. Similarly $Ap \subset A$, so that $p \in M(A)$. So (i) implies (ii), and the converse follows from the fact that any projection in $M(A)$ is open [9]. The equivalence of (i) and (ii) with (iii) and (v) follows from basic correspondences from [9], [25]. That (iii) implies (iv) follows from the fact that here $L(eA) = Ae^{\perp}$, an r-ideal. Conversely, if (iv) holds, and $e \neq 1$ is an open projection, then we claim that there is a nonzero open projection f with $f \leqslant e^{\perp}$. Indeed choose such f to be the support projection of $L(J)$ where $J = A^{**}e \cap A$. If $f = e^{\perp}$ then (i) follows, so assume that $e + f \neq 1$. Since $e + f$ is open, by the claim there exists a nonzero open projection p with $p \leqslant (e+f)^{\perp}$. Then the HSA $D = pA^{**}p \cap A \subset L(J) \cap R(L(J))$, so that $D = D^2 = (0)$. Thus $p = 0$, a contradiction.

If (i)–(v) hold, then every open projection in $\Delta(A)^{**}$ is in $\Delta(M(A))$, hence in $M(\Delta(A))$ by the argument in Proposition 2.11. Thus every left ideal J of $\Delta(A)$ is of the form $e\Delta(A)$ for a projection $e \in M(\Delta(A))$, hence has a nonzero right annihilator. Thus $\Delta(A)$ is an annihilator C^*-algebra. ∎

REMARK 2.13. (i) Every finite dimensional unital operator algebra is obviously nc-discrete. We will see more examples later.

(ii) Of course the "other-handed" variants of (iii) and (iv) in the Proposition 2.12 are also equivalent to the others, by symmetry. For a projection $e \in M(A)$, the l-ideal and HSA corresponding to the r-ideal eA, are Ae and eAe respectively, by the theory in [9].

We will say that an operator algebra A is *Δ-dual* if $\Delta(A)$ is a dual C^*-algebra in the sense of Kaplansky, and $\Delta(A)$ acts nondegenerately on A.

We briefly discuss the connections between the "Δ-dual" and the "nc-discrete" properties. The flow is essentially one-way between these properties. Being Δ-dual certainly is far from implying nc-discrete. For example the disk algebra $A(\mathbb{D})$ has no nontrivial projections and is Δ-dual; but it has many nontrivial approximately unital ideals (such as $\{f \in A(\mathbb{D}) : f(1) = 0\}$), thus is not nc-discrete. However nc-discrete implies Δ-dual under reasonable conditions. For future use we record some necessary and sufficient conditions for a nc-discrete algebra to be Δ-dual.

COROLLARY 2.14. *Let A be an approximately unital operator algebra which is* nc-*discrete. The following are equivalent:*

(i) *A is Δ-dual.*
(ii) *$\Delta(A)$ acts nondegenerately on A.*
(iii) *Every nonzero projection in $M(A)$ dominates a nonzero positive element in A.*
(iv) *If p is a nonzero projection in $M(A)$, then there exists $a \in A$ with pap selfadjoint.*
(v) $1_{A^{**}} \in \Delta(A)^{\perp\perp}$.

Proof. We said in Proposition 2.12 that $\Delta(A)$ is an annihilator C^*-algebra. Thus (i) $\Leftrightarrow$ (ii); and (ii) $\Leftrightarrow$ (v) by the arguments above the statement of Proposition 2.11. Clearly (iii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (iii) As in the last line $p \in M(\Delta(A)) = \Delta(A)^{\perp\perp}$, hence dominates a projection in $\Delta(A)$.

(iv) $\Rightarrow$ (ii) Since $\Delta(A)$ is a c_0-sum of C^*-algebras of compact operators, every projection p in $\Delta(A)^{**} = M(\Delta(A))$ is open in A^{**}, hence lies in $M(A)$ by Proposition 2.12(ii). In particular, e_Δ, the identity of $\Delta(A)^{**}$ is in $M(A)$. Hence $e_\Delta = 1$ (since $(1-e_\Delta)A(1-e_\Delta)$ cannot contain any nonzero (selfadjoint) element in $\Delta(A)$). So A has a positive cai. ∎

The nc-discrete algebras are reminiscent of Kaplansky's "dual algebras", in that the "left (respectively right) annihilator" operation is a lattice anti-isomorphism between the lattices of one-sided M-ideals of A:

COROLLARY 2.15. *Let A be an* nc*-discrete approximately unital operator algebra. If J is an* r*-ideal (respectively* l*-ideal) in A, then the left annihilator $L(J)$ (respectively right annihilator $R(J)$) equals $Ae^{\perp}$ (respectively $e^{\perp}A$) where e is the support projection of J. This annihilator is an* l*-ideal (respectively* r*-ideal), and $R(L(J)) = J$ (respectively $L(R(J)) = J$). Also, the intersection, and the closure of the sum, of any family of* r*-ideals (respectively* l*-ideals) in A is again an* r*-ideal (respectively* l*-ideal).*

Proof. The first statements are evident. The last statement is always true for the closure of the sum [15]. An intersection $\bigcap_i J_i$ of r-ideals J_i is an r-ideal, since it equals $R(\overline{\sum_i L(J_i)})$, and $\overline{\sum_i L(J_i)}$ is an l-ideal. ∎

PROPOSITION 2.16. *If A is an operator algebra, which is an* l*-ideal in its bidual, then every projection $e \in A^{**}$ is open, and the* r*-ideals (respectively* l*-ideals) in A are precisely the ideals eA (respectively Ae) for projections $e \in A^{**}$.*

If in addition A is approximately unital, then A is nc*-discrete, and every projection in A^{**} is in $M(A)$.*

Proof. By Proposition 2.4, any projection in A^{**} is in the idealizer of A in A^{**}, and is open by an argument similar to that on p. 336 of [9]. Also, eA is an r-ideal, and Ae is an l-ideal, by Theorem 2.4 in [9]. The last statement is now easy. ∎

If in the first paragraph of the statement of the last result, A also happens to be semiprime, then it is automatically approximately unital, hence the second paragraph of the statement holds automatically. This follows from:

PROPOSITION 2.17. *A semiprime Arens regular Banach algebra A which is a left ideal in its bidual, has a* bai *(respectively* cai*) if it has a right* bai *(respectively right* cai*).*

Proof. Suppose that (e_t) is a right bai for A, with $e_t \to e \in A^{**}$ weak*. If we can show that e is an identity for A^{**}, then A has a bai. Let $J = \{ey - y : y \in A\} \subset A$. Then $AJ = (0)$, so that $J^2 = (0)$, and $JA \subset J$. Thus $J = (0)$, so that $ey = y$ for all $y \in A$. Hence A^{**} has an identity. Similarly in the cai case. ∎

The next result generalizes part of Proposition 2.1. Although we will not really use it in the sequel, it is of independent interest, extending a result of Lin on C^*-modules [30]. See Theorem 2.3 of [10] for the "weak* version" of the result. We refer to [6] for most of the notation used in the following statement and proof.

THEOREM 2.18. *If Y is a right rigged module in the sense of* [6] *over an approximately unital operator algebra A, then $\mathcal{M}_l(Y) = CB(Y)_A = LM(\mathbb{K}(Y)_A)$ completely isometrically isomorphically.*

Proof. By facts from the theory of multipliers of an operator space (see e.g. Chapter 4 of [11]), the "identity map" is a completely contractive homomorphism $\mathcal{M}_1(Y) \to CB(Y)$, which maps into $CB(Y)_A$. Since by [13], p. 34, $CB(Y)_A$ is an operator algebra, and Y is a left operator $CB(Y)_A$-module (with the canonical action), then by the aforementioned theory there exist a completely contractive homomorphism $\pi : CB(Y)_A \to \mathcal{M}_1(Y)$ with $\pi(T)(y) = T(y)$ for all $y \in Y, T \in CB(Y)_A$. That is, $\pi(T) = T$. Thus $CB(Y)_A = \mathcal{M}_1(Y)$. Finally, $\mathbb{K}(Y)_A$ is an essential left ideal in $CB(Y)_A$: it is easy to see that the left regular representation of $CB(Y)_A$ on $\mathbb{K}(Y)_A$ is completely isometric. Thus $CB(Y)_A \subset LM(\mathbb{K}(Y)_A)$ completely isometrically. However Y is a left operator module over $\mathbb{K}(Y)_A$, hence also over $LM(\mathbb{K}(Y)_A)$ (see Theorem 3.6 (5) in [6] and 3.1.11 in [11]), and so every $T \in LM(\mathbb{K}(Y)_A)$ corresponds to a map in $CB(Y)_A$. ∎

The following follows either from Proposition 2.1, or from Theorem 2.18 (since all r-ideals J in an approximately unital operator algebra A are right rigged modules over A).

COROLLARY 2.19. *If J is an* r*-ideal in an approximately unital operator algebra A then $\mathcal{M}_1(J) = CB(J)_A$. In particular, if e is a projection in A then $\mathcal{M}_1(eA) = CB(eA)_A = eAe$.*

3. EXISTENCE OF r-IDEALS

In Lemma 6.8 of [9] it is shown that if A is a unital operator algebra, and $x \in \mathrm{Ball}(A)$, then $J = \overline{(1-x)A}$ is an r-ideal. These ideals, which have also been studied by G. Willis in a Banach algebra context (see e.g. [41]), for us correspond to the noncommutative variant of *peak sets* from the theory of function algebras (see Proposition 6.7 of [9] for the correspondence). Here and in the following we write 1 for the identity in A^1 if A is a nonunital algebra. The following enlarges this class of examples:

PROPOSITION 3.1. *If A is an approximately unital operator algebra, which is an ideal in an operator algebra B, then $\overline{(1-x)A}$ is an* r*-ideal in A for all $x \in \mathrm{Ball}(B)$.*

Proof. We may assume that B is unital and closed. Then $\overline{(1-x)B}$ is an r-ideal in B. Also, A is a two-sided M-ideal in B. By Proposition 5.30(ii) in [15], $A \cap \overline{(1-x)B}$ is an r-ideal in B, so has a left cai. Clearly $\overline{(1-x)A} \subset A \cap \overline{(1-x)B}$. Conversely, if $a \in A$ is in $\overline{(1-x)B}$, then since the latter has left cai we have that $a \in \overline{(1-x)BA} \subset \overline{(1-x)A}$. Thus $\overline{(1-x)A}$ is an r-ideal. ∎

REMARK 3.2. This is false if A has only a right cai, e.g. C_2. We do not know if it is true if A has a left cai, or is a left ideal in B, or both. Note that if A has a left identity e of norm 1 then $\overline{(e-x)A}$ is an r-ideal in A for any $x \in \mathrm{Ball}(A)$; in this case $\overline{(e-x)Ae} = \overline{(e-xe)Ae}$ is an r-ideal in Ae, so has a left cai which serves as a left cai for $\overline{(e-x)A} = \overline{(e-xe)A}$.

We recall that a right ideal J of a normed algebra A is *regular* if there exists $y \in A$ such that $(1-y)A \subset J$. We shall say that J is *1-regular* if this can be done with $\|y\| \leqslant 1$ and $y \neq 1$. We do not know if every r-ideal in a unital operator algebra is 1-regular, this is related to a major open question in [25], [9] concerning the noncommutative variant of the notion of peak sets from the theory of function algebras. That is, if A is a unital operator algebra, then is every closed projection (i.e. the "perp" of an open projection) in A^{**} a p-projection in the sense of [25]? We recall that p-projections are a noncommutative generalization of the p-sets, and peak sets, from the theory of function algebras. By Proposition 6.7 of [9] and other principles from [25], [9], this open question is equivalent to whether every r-ideal in a unital operator algebra A is the closure of a union of (nested, if one wishes) right ideals of the form $\overline{(1-a)A}$ for elements $a \in \mathrm{Ball}(A)$? This is true if A is a unital function algebra (uniform algebra), by results of Glicksberg and others [25]. There are many reformulations of this question in [9].

PROPOSITION 3.3. *If A is a unital operator algebra, then every* r*-ideal in A whose support projection is the complement of a p-projection, is* 1*-regular. If every closed projection in A^{**} is a p-projection (that is, if the major open problem from* [25], [9] *mentioned above has an affirmative solution), then every* r*-ideal in A is* 1*-regular.*

Proof. The first statement follows easily from Theorem 6.1 of [9]. The second follows from the first and the correspondence between r-ideals and closed projections [25]. ∎

We will say that a one-sided ideal in an algebra A is nontrivial, if it is not A or (0). From Proposition 3.1, one would usually expect there to be many nontrivial r-ideals in an approximately unital operator algebra. For example, one can show that in every function algebra there are plenty of nontrivial r-ideals. However consider for example the semisimple commutative two dimensional operator algebra $A = \mathrm{Span}(\{I_2, E_{11} + E_{12}\})$ in M_2. This algebra has only two proper ideals, and neither is an r-ideal. The following results give criteria for the existence of nontrivial r-ideals, and make it apparent that the operator algebras which contain nontrivial r-ideals are far from being a small or uninteresting class. We have not seen item (ii) of the next result in the literature, but probably it is well known in some quarters.

THEOREM 3.4. *If A is a Banach algebra, and $x \in \mathrm{Ball}(A)$, then*

(i) $\overline{(1-x)A} = (0)$ *if and only if x is a left identity of norm* 1 *for A.*

(ii) *If A is unital then $1-x$ is left invertible in A if and only if $1-x$ is right invertible in A. If A is nonunital then x is left quasi-invertible in A if and only if x is right quasi-invertible in A. These statements are also equivalent to any one, and all, of the following statements:* $\overline{(1-x)A} = A$, $(1-x)A = A$, $\overline{A(1-x)} = A$, *and* $A(1-x) = A$.

(iii) *If A is unital then the statements in* (ii) *are also equivalent to the same statements, but with A replaced throughout by the Banach subalgebra generated by* 1 *and x.*

(iv) *If A has a left identity of norm* 1 *write this element as e, else set $e = 0$. Every* 1*-regular right ideal of A is trivial if and only if the spectral radius $r(a) < \|a\|$ for all $a \in A \setminus \mathbb{C}\, e$; and if and only if* $\mathrm{Ball}(A) \setminus \{e\}$ *is composed entirely of quasi-invertible elements.*

Proof. (i) Clear.

(ii) and (iii) If A is unital then $\overline{(1-x)A}$ has a left bai (e_n), defined by $e_n = 1 - (1/n) \sum_{k=1}^{n} x^k$. Let B be the closure of the algebra generated by 1 and x. Then $(e_n) \subset (1-x)B$. Suppose that $\overline{(1-x)A} = A$. Clearly $\overline{(1-x)B} \subset B$. Conversely, if $b \in B \subset \overline{(1-x)A}$, then $e_n b \to b$, so that $b \in \overline{(1-x)B}$. Thus $\overline{(1-x)B} = B = \overline{B(1-x)}$. Therefore $1-x$ is invertible in B, and hence also in A, by the Neumann series lemma. Conversely, if $1-x$ is right invertible in A then $(1-x)A = A$. This proves the unital case of (ii) and (iii).

For a nonunital Banach algebra A, let A^1 be a unitization of A. If $\overline{(1-x)A} = A$ then $x \in \overline{(1-x)A} \subset \overline{(1-x)A^1}$. If (e_n) is a left bai for $\overline{(1-x)A^1}$ as in the last paragraph, then $e_n(1-x) \to 1-x$, so $e_n = e_n(1-x) + e_n x \to 1 - x + x = 1$. Thus $A^1 = \overline{(1-x)A^1}$, and so x is quasi-invertible. Conversely, it is clear that x right quasi-invertible in A implies that $\overline{(1-x)A} = A$.

(iv) If the condition on the spectral radius holds, then for $a \in \mathrm{Ball}(A) \setminus \mathbb{C}\, e$, we have $r(a) < 1$, so that a is quasi-invertible, or equivalently $\overline{(1-a)A} = A$. If $a \in \mathbb{C}\, e$ the corresponding ideals are clearly trivial.

Supposing every 1-regular right ideal is trivial, then $1 \notin \mathrm{Sp}_A(x)$ for all $x \in A \setminus \mathbb{C}\, e$ of norm 1. Multiplying x by unimodular scalars shows that the unit circle does not intersect $\mathrm{Sp}_A(x)$. Thus $r(x) < 1$. By scaling, $r(x) < \|x\|$ for all $x \in A \setminus \mathbb{C}\, e$.

The rest of (iv) is evident from the above. ∎

REMARK 3.5. Of course the $r(x) < \|x\|$ condition in (iv) is equivalent to $a^n \to 0$ for all $a \in \mathrm{Ball}(A) \setminus \mathbb{C}\, e$.

For operator algebras one can refine the last result further. The equivalences of (i)–(ii) below is probably a well known observation.

PROPOSITION 3.6. *Let A be a closed subalgebra of $B(H)$, and $x \in \mathrm{Ball}(A)$.*

If A is unital, then the following are equivalent (and are equivalent to the other conditions in (ii) *and* (iii) *in the last theorem);*

(i) $1 - x$ *is invertible in A.*

(ii) $\|1 + x\| < 2$.

If A is nonunital, then x is quasi-invertible in A if and only if (ii) *holds with respect to A^1.*

If A is any operator algebra then the conditions in (iv) *of the last theorem hold if and only if* $\|1+x\| < 2$ *for all* $x \in \mathrm{Ball}(A)$ *with* $x \neq e$*, and if and only if* $\nu(x) < \|x\|$ *for all* $x \in A \setminus \mathbb{C}\, e$*. Here* ν *is the numerical radius in* A^1*.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $\|I_H + x\| = 2$. If $\zeta_n \in \mathrm{Ball}(H)$ satisfies $\|\zeta_n + x\zeta_n\|^2 \to 4$, then $\|\zeta_n\|$, $\|x\zeta_n\|$, and $\mathrm{Re}(\langle x\zeta_n, \zeta_n\rangle)$, all converge to 1, and so $\|\zeta_n - x\zeta_n\|^2 \to 0$. Thus $\|\zeta_n\| \leqslant \|(1-x)^{-1}\| \|\zeta_n - x\zeta_n\| \to 0$, which is a contradiction.

(ii) $\Rightarrow$ (i) Let $b = (x+1)/2$. If $\|x+1\| < 2$, then by the Neumann series lemma, $1-b$ is invertible in any operator subalgebra containing 1 and x, and in particular in A. Hence (i) holds.

If $\|x+1\| < 2$ for all $x \in \mathrm{Ball}(A) \setminus \mathbb{C}\, e$, then the unit circle does not intersect the numerical range in A^1 of such x, since $|1 + \varphi(\mathrm{e}^{\mathrm{i}\theta} x)| < 2$ for all states φ on A. So $\nu(x) < 1$. By scaling, $\nu(x) < \|x\|$ for all $x \in A \setminus \mathbb{C}\, e$.

The rest of the assertions of the theorem are obvious. ∎

PROPOSITION 3.7. *Let A be an* nc*-discrete approximately unital operator algebra. If J is a right ideal in A then J is a regular* r*-ideal in A if and only if J is* 1*-regular and if and only if* $J = e^{\perp} A$ *for a projection* $e \in A$*. Also, the following are equivalent:*

(i) *Every* 1*-regular right ideal is trivial.*

(ii) $\Delta(A) \subset \mathbb{C}\, 1$.

(iii) *A contains no nontrivial projections.*

Also, A has no nontrivial r*-ideals if and only if $M(A)$ contains no nontrivial projections.*

Proof. If $(1-y)A \subset J$ as above, and $J = eA$ for a projection $e \in M(A)$, then $e(1-y) = 1-y$. That is $e^{\perp} = e^{\perp} y \in A$. Conversely, $e^{\perp} \in A$ implies that eA is 1-regular.

That (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i), is now clear. To see that (i) $\Rightarrow$ (ii), note that if $\Delta(A)$ contains a hermitian h then $\nu(h) = \|h\|$, so $h \in \mathbb{C}\, 1$ by the last assertion of Proposition 3.6. Thus $\Delta(A) \subset \mathbb{C}\, 1$.

The last part follows from Proposition 2.12(iii). ∎

PROPOSITION 3.8. *Let A be an operator algebra which contains nontrivial* 1*-regular ideals (or equivalently,* $\mathrm{Ball}(A) \setminus \{1\}$ *is not composed entirely of quasi-invertible elements). Then proper maximal* r*-ideals of A exist. Indeed, if* $y \in \mathrm{Ball}(A)$ *is not quasi-invertible then* $(1-y)A$ *is contained in a proper* (1*-regular) maximal* r*-ideal. The unit ball of the intersection of the* 1*-regular maximal* r*-ideals of A is composed entirely of quasi-invertible elements of A.*

Proof. We adapt the classical route. For a non-quasi-invertible $y \in \mathrm{Ball}(A)$, let (J_t) be an increasing set of proper r-ideals, each containing $(1-y)A$. Then $J = \bigcup_t J_t$ is a right ideal which does not contain y, or else there is a t with $a = ya + (1-y)a \in J_t$ for all $a \in A$. The closure $\overline{J}$ of J is an r-ideal since it equals the closure of $\bigcup_t \overline{J}_t$. Also $\overline{J}$ is proper, since the closure of a proper regular ideal is proper [32]. Thus by Zorn's lemma, $(1-y)A$ is contained in a (regular) maximal r-ideal. Let

I be the intersection of the proper 1-regular maximal r-ideals. If $y \in \mathrm{Ball}(I)$, but y is not quasi-invertible, then $y \notin \overline{(1-y)A}$. Let K be a maximal proper r-ideal containing $\overline{(1-y)A}$. Then $y \notin K$ (for if $y \in K$ then $a = (1-y)a + ya \in K$ for all $a \in A$), and K is regular. So $y \in I \subset K$, a contradiction. Hence every element of $\mathrm{Ball}(I)$ is quasi-invertible. ∎

REMARK 3.9. (i) One may replace r-ideals in the last result with l-ideals, or HSA's.

(ii) In connection with the last result we recall from algebra that the Jacobson radical is the intersection of all maximal (regular) one-sided ideals.

4. MATRICIAL OPERATOR ALGEBRAS

In this section, we shall only consider separable algebras. This is a blanket assumption, and will be taken for granted hereafter. This is only for convenience, the general case is almost identical.

If $(q_k)_{k=1}^{\infty}$ are elements in a normed algebra A then we say that $\sum_k q_k = 1$ *right strictly* if $\sum_k q_k a = a$ for all $a \in A$, with the convergence in the sense of nets (indexed by finite subsets of $\mathbb{N}$). Similarly for *left strictly*, and *strict convergence* means both left and right strict convergence. If A is a Banach algebra and the q_k are idempotents with $q_j q_k = 0$ for $j \neq k$, and if $\sum_k q_k = 1$ strictly, then $(\sum_{k=1}^{n} q_k)$ is an approximate identity for A. If A is left or right essential (that is, $A \subset B(A)$ bicontinuously via left or right multiplication) then $(\sum_{k=1}^{n} q_k)$ is a bai for A. Indeed by a simple argument involving the principle of uniform boundedness, there is a constant K such that $\left\| \sum_{k \in J} q_k \right\| \leqslant K$ for all finite sets J. (We are not saying that convergent nets are bounded, but that convergent series have bounded partial sums, which follows because the sets J are finite.) Now suppose that in addition, A is a closed subalgebra of $B(H)$. In this case we claim that, up to similarity, we may assume that the q_k are orthogonal projections and that $(\sum_{k=1}^{n} q_k)$ is a cai for A. Indeed, since by the above the finite partial sums of $\sum_k q_k$ are uniformly bounded, by basic similarity theory there exists an invertible operator S with $S^{-1} q_k S$ a projection for all k. To see this, note that the set $\{1 - 2\sum_{k \in E} q_k\}$, where E is any finite subset of $\mathbb{N}$, is a bounded abelian group of invertible operators. Hence by Lemma XV.6.1 of [18], there exists an invertible S with $S^{-1}(1 - 2\sum_{k \in E} q_k)S$ unitary for every E. This forces $e_k = S^{-1} q_k S$ to be a projection for all k. Then $B = S^{-1} A S$ is a subalgebra of $B(H)$ with a cai $(\sum_{k=1}^{n} e_k)$, for a sequence of mutually orthogonal projections e_k in B.

Because of the trick in the last paragraph, we will usually suppose in the remainder of this section that the idempotents q_k are projections. This corresponds to the "isometric case" of the theory. The "isomorphic case" of our theory below sometimes follows from the "isometric case" by the similarity trick above, however we will make rarely mention this variant of the theory.

PROPOSITION 4.1. *Let $(q_k)_{k=1}^{\infty}$ be mutually orthogonal projections in an operator algebra A. Then $\sum_k q_k = 1$ right strictly if and only if A is the closure of $\sum_k q_k A$. In this case $(\sum_{k=1}^{n} q_k)$ is a left* cai*; and it is a* cai *if and only if A is approximately unital, and if and only if A is also the closure of $\sum_k Aq_k$.*

Proof. The first part is obvious. If A is approximately unital then $(\sum_{k=1}^{n} q_k)$ must converge weak* to an identity 1 for A^{**}. The closure of $\sum_k Aq_k$ is an l-ideal with support projection $e \geqslant \sum_{k=1}^{n} q_k$, so $e = 1$ and A is the closure of $\sum_k Aq_k$. ∎

PROPOSITION 4.2. *If A is an Arens regular Banach algebra with idempotents $(q_k)_{k=1}^{\infty}$ with $\sum_k Aq_k$ or $\sum_k q_k A$ dense in A (for example, if $\sum_k q_k = 1$ left or right strictly), then A is a right ideal in A^{**} if and only if $q_k A$ is reflexive for all k. If A is topologically simple and Arens regular, then A is a right ideal in A^{**} if eA is reflexive for some idempotent $e \in A$.*

Proof. This is probably well known, and so we will just sketch the proof. We write "w-" for the weak topology. We use the fact that $T : X \to Y$ is w-compact if and only if the w-closure of $T(\mathrm{Ball}(X))$ is w-compact and if and only if $T^{**}(X^{**}) \subset Y$ [31]. Suppose that A is a right ideal in A^{**}. We have $\mathrm{Ball}(q_k A) \subset q_k \mathrm{Ball}(A)$, and the w-closure of the latter set is w-compact in A. A little argument shows that it is w-compact in $q_k A$. Since $\mathrm{Ball}(q_k A)$ w-closed in $q_k A$, it is w-compact, and so $q_k A$ is reflexive. Conversely, if $q_k A$ is reflexive then $L_{q_k} : A \to A$ is w-compact since the w-closure in A of $L_{q_k}(\mathrm{Ball}(A))$ is contained in a multiple of $\mathrm{Ball}(q_k A)$, which is w-compact in $q_k A$, hence in A. Inside $LM(A)$, A is the closure of finite sums of terms of the form aL_{q_k} or $L_{q_k}a$, for $a \in A$, which are all weakly compact. Since the weakly compact operators form an ideal, left multiplication by any element of A is weakly compact, and hence A is a right ideal in A^{**}. We only need $k = 1$ in the above argument if A is topologically simple, for then Aq_1A is dense in A. ∎

DEFINITION 4.3. We now define a class of examples which fit in the above context. We say that an operator algebra A is *matricial* if it has a full set of matrix units $\{T_{ij}\}$, whose span is dense in A. Thus $T_{ij}T_{kl} = \delta_{jk}T_{il}$, where δ_{jk} is the Kronecker delta. Define $q_k = T_{kk}$. We say that a matricial operator algebra A is *1-matricial* if $\|q_k\| = 1$ for all k, that is, if and only if the q_k are orthogonal projections. We mostly focus on 1-matricial algebras (other matricial operator algebras appear only occasionally, for example in Corollary 4.27). We will think of two 1-matricial algebras as being the same if they are completely isometrically isomorphic.

As noted at the start of the section, we are only interested in separable (or finite dimensional) algebras, and in this case we prefer the following equivalent description of 1-matricial algebras. Consider a (finite or infinite) sequence $T_1, T_2, \ldots$ of invertible operators on a Hilbert space K, with $T_1 = I$. Set $H = \ell^2 \otimes^2 K = K^{(\infty)} = K \oplus^2 K \oplus^2 \cdots$ (in the finite sequence case, $H = K^{(n)}$). Define $T_{ij} = E_{ij} \otimes T_i^{-1} T_j \in B(H)$ for $i, j \in \mathbb{N}$, and let A be the closure of the span of the T_{ij}. Then $T_{ij} T_{kl} = \delta_{jk} T_{il}$, so that these are matrix units for A. Then A is a 1-matricial algebra, and all separable or finite dimensional 1-matricial algebras are completely isometrically isomorphic to one which arises in this way (by the proof of Theorem 4.8 below). Let $q_k = T_{kk}$, then $\sum_k q_k = 1$ strictly.

A *σ-matricial* algebra is a c_0-direct sum of 1-matricial algebras. Since we only care about the separable case these will all be countable (or finite) direct sums. It would certainly be better to call these σ-1-matricial algebras, or something similar, but since we shall not really consider any other kind, we drop the "1" for brevity.

LEMMA 4.4. *Any* 1*-matricial algebra A is approximately unital, topologically simple, hence semisimple and semiprime, and is a compact modular annihilator algebra. It is an* HSA *in its bidual, so has the unique Hahn–Banach extension property in Theorem* 2.10 *of* [9]*. It also has dense socle, with the q_k algebraically minimal projections with $A = \bigoplus_k^{\mathrm{c}} q_k A = \bigoplus_k^{\mathrm{r}} A q_k$. The canonical representation of A on Aq_1 is faithful and irreducible, so that A is a primitive Banach algebra.*

Proof. Clearly $(\sum_{k=1}^n q_k)$ is a cai. Also, $q_j A q_k = \mathbb{C}\, T_{jk}$. Thus $q_j A q_j = \mathbb{C}\, q_j$, so the q_k are algebraically minimal projections with $A = \bigoplus_k^{\mathrm{c}} q_k A$ by Theorem 7.2 of [6]. Also $q_j A = \overline{\mathrm{Span}(\{T_{jk} : k \in \mathbb{N}\})}$. If J is a closed ideal in A, and $0 \neq x \in J$, then $q_j x \neq 0$, and $q_j x q_k \neq 0$, for some j, k. Hence $T_{jk} \in J$, and so $T_{pq} \in J$ for all $p, q \in \mathbb{N}$, since these are matrix units. So $A = J$. Thus A is topologically simple, hence semisimple and semiprime. Thus the $q_k A$ are minimal right ideals, so A has dense socle, and by Proposition 8.7.6 of [32] we have that A is a compact modular annihilator algebra. Note that $q_j A^{**} q_k \subset \mathbb{C}\, T_{jk} \subset A$, and so $AA^{**} q_k \subset A$ and $AA^{**} A \subset A$. Hence A is an HSA in its bidual, so has the unique Hahn–Banach extension property in Theorem 2.10 of [9]. The representation of A on Aq_1 is faithful, since $aAq_1 = (0)$ implies $aAq_1 A = (0)$, hence $aA = (0)$ and $a = 0$. It is also irreducible, since Aq_1 is a minimal left ideal. ∎

REMARK 4.5. If a Banach space X has the unique Hahn–Banach extension property in Theorem 2.10 of [9], then by [29], it is *Hahn–Banach smooth* in X^{**}, hence it is a *HB-subspace* of X^{**}, and X^* has the Radon–Nikodym property. By the work of Godefroy and collaborators, if X^* has the latter property then there is a unique contractive projection from $X^{(4)}$ onto X^{**}, and X^* is a strongly unique predual (see e.g. [38]). Thus all of the above holds if X is a σ-matricial operator algebra.

COROLLARY 4.6. *A 1-matricial algebra A is a right (respectively left, two-sided) ideal in its bidual if and only if $q_1 A$ (respectively Aq_1, $q_1 A$ and Aq_1) is reflexive.*

REMARK 4.7. (i) It is known that semisimple (and many semiprime) annihilator algebras are ideals in their bidual ([32], Corollary 8.7.14). In particular, a 1-matricial annihilator algebra is an ideal in its bidual. We conjecture that a 1-matricial algebra A is bicontinuously isomorphic to $\mathbb{K}(\ell^2)$ if and only if it is an annihilator algebra.

(ii) It is helpful to know that in any 1-matricial algebra, $(T_{1k}) = (E_{1k} \otimes T_k)$ is a monotone Schauder basis for $q_1 A$. Indeed, clearly the closure of the span of the T_{1k} equals $q_1 A$, and if $n < m$ then

$$\Big\| \sum_{k=1}^{n} \alpha_k T_{1k} \Big\|^2 = \Big\| \sum_{k=1}^{n} |\alpha_k|^2 T_k T_k^* \Big\| \leqslant \Big\| \sum_{k=1}^{m} |\alpha_k|^2 T_k T_k^* \Big\| = \Big\| \sum_{k=1}^{m} \alpha_k T_{1k} \Big\|^2 .$$

The following is a first characterization of 1-matricial algebras. Others may be derived by adding to the characterizations of c_0-sums of 1-matricial algebras below, the hypothesis of topological simplicity.

THEOREM 4.8. *If A is a topologically simple left or right essential operator algebra with a sequence of nonzero algebraically minimal idempotents (q_k) with $q_j q_k = 0$ for $j \neq k$, and $\sum_k q_k = 1$ strictly, then A is similar to a 1-matricial algebra. If further the q_k are projections, then A is unitarily isomorphic to a 1-matricial algebra.*

Proof. By the similarity trick at the start of this section, we may assume that the q_k are projections. Since A is semiprime, the $q_k A$ are minimal right ideals, and A has dense socle. Any nonzero $T \in B(q_j A, q_k A)_A$ is invertible, and if $S, T \in B(q_j A, q_k A)_A$ then $T^{-1} S \in \mathbb{C}\, q_j$. Thus $B(q_j A, q_k A)_A = \mathbb{C}\, T$. We now show that the "left multiplication map" $\theta : q_j A q_k \to CB(q_k A, q_j A)_A$ is a completely isometric isomorphism. Certainly it is completely contractive and one-to-one. Also, if $T \in CB(q_k A, q_j A)_A$ then $T(q_k) \in q_j A q_k$ with $T(q_k) q_k a = T(q_k a)$. Thus θ^{-1} is a contraction, and similarly it is completely contractive. Choose $0 \neq T_k \in q_1 A q_k$, with $q_1 = T_1$. Write T_k^{-1} for the inverse of $T_k \in B(q_j A, q_k A)_A$, an element of $q_k A q_1$ with $T_k^{-1} T_k = q_k$ and $T_k T_k^{-1} = q_1$. Then $T_{jk} = T_j^{-1} T_k \in q_j A q_k$, and so $q_j A q_k = \mathbb{C}\, T_j^{-1} T_k$. Any $a \in A$ may be approximated first by $\sum_{k=1}^{n} q_k a$, and then by $\sum_{j,k=1}^{n} q_k a q_j$. Thus A is the closure of the span of $\{T_{ij}\}$, which are a set of matrix units for A.

If $A \subset B(H_0)$ nondegenerately, set $K_k = q_k(H_0) = T_k^{-1}(H_0)$, and let $K = q_1 H_0$. Then $K_k \cong K$ via T_k and T_k^{-1}. Since these are Hilbert spaces, there is a unitary $U_k : K_k \to K$, and hence a unitary isomorphism $U : H_0 = \bigoplus_k^2 K_k \to K^{(\infty)}$. It is easy to see UAU^* is a 1-matricial algebra, so that A is unitarily equivalent to a 1-matricial algebra. ∎

LEMMA 4.9. *Let A be an infinite dimensional* 1*-matricial algebra. Then A is completely isomorphic to $\mathbb{K}(\ell^2)$ if and only if A is topologically isomorphic to a C^*-algebra (as Banach algebras), and if and only if $(\|T_k\|\|T_k^{-1}\|)$ is bounded (where T_k is as in Definition* 4.3*). If $\|T_k\|\|T_k^{-1}\| \leqslant 1$ for all k then $A = \mathbb{K}(\ell^2)$ completely isometrically.*

Proof. If $\rho : A \to \mathbb{K}(\ell^2)$ is a bounded homomorphism and $e_k = \rho(q_k)$, then e_k are finite rank idempotents. If ρ is surjective, then the idempotents e_k are rank one, since they are algebraically minimal. They are mutually orthogonal and uniformly bounded, so there is an invertible $S \in B(\ell^2)$ with $S^{-1}e_kS$ rank one projections, say $S^{-1}e_kS = |\xi_k\rangle\langle\xi_k|$, ξ_k a unit vector in ℓ^2. Since these projections are mutually orthogonal, (ξ_k) is orthonormal. Then $\rho(T_k) = e_1\rho(T_k)e_k$, so that $S^{-1}\rho(T_k)S = \lambda_k|\xi_1\rangle\langle\xi_k|$ for some scalars λ_k. There is a constant such that $\|T_k\| \leqslant C|\lambda_k|$. Similarly, $S^{-1}\rho(T_k^{-1})S = \mu_k|\xi_k\rangle\langle\xi_1|$, and $\|T_k^{-1}\| \leqslant D|\mu_k|$. Thus $\|T_k\|\|T_k^{-1}\| \leqslant CD|\lambda_k\mu_k|$. However since $q_1 = T_kT_k^{-1}$ we have

$$S^{-1}\rho(q_1)S = |\xi_1\rangle\langle\xi_1| = \lambda_k\mu_k|\xi_1\rangle\langle\xi_1|,$$

so that $\lambda_k\mu_k = 1$. Hence $\|T_k\|\|T_k^{-1}\| \leqslant CD$.

Suppose that $\|T_k\|\|T_k^{-1}\| \leqslant M^2$ for each k. By multiplying by appropriate constants, we may assume that $\|T_k\| = \|T_k^{-1}\| \leqslant M$ for all k. Let $x = \mathrm{diag}\{T_1, T_2, \ldots\}$, and $x^{-1} = \mathrm{diag}\{T_1^{-1}, T_2^{-1}, \ldots\}$. The map $\theta(a) = xax^{-1}$ is a complete isomorphism from A onto $\mathbb{K}(\ell^2)$, and θ is isometric if $M = 1$.

If A were isomorphic to a C^*-algebra B, then B is a topologically simple C^*-algebra with dense socle, so is a dual algebra in the sense of Kaplansky. Since it is topologically simple, $B = \mathbb{K}(\ell^2)$. It is a simple consequence of similarity theory that a bicontinuous isomorphism from $\mathbb{K}(\ell^2)$ is a complete isomorphism. ∎

An operator algebra will be called a *subcompact* 1*-matricial algebra*, if it is (completely isometrically isomorphic to) a 1-matricial algebra with the space K in the definition of a 1-matricial algebra (the second paragraph of 4.3) being finite dimensional.

LEMMA 4.10. *A* 1*-matricial algebra A is subcompact if and only if A is completely isometrically isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$, and if and only if its C^*-envelope is an annihilator C^*-algebra. In this case, A is an ideal in its bidual, and q_kA (respectively Aq_k) is linearly completely isomorphic to a row (respectively column) Hilbert space. Here q_k is as in Definition* 4.3*. Indeed, if a* 1*-matricial algebra A is bicontinuously (respectively isometrically) isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$, then A is bicontinuously (respectively isometrically) isomorphic to a subcompact* 1*-matricial algebra.*

Proof. That a subcompact 1-matricial algebra is a subalgebra of $\mathbb{K}(\ell^2)$ follows from the definition. We leave it as an exercise that the separable operator algebras whose C^*-envelope is an annihilator C^*-algebra, are precisely the subalgebras of $\mathbb{K}(\ell^2)$. If θ was a bicontinuous homomorphism from A onto a subalgebra of $\mathbb{K}(\ell^2)$, then $e_k = \theta(q_k)$ is a finite rank idempotent. Hence $e_k\,\mathbb{K}(\ell^2)$ is Hilbertian. Thus $q_k A$ is Hilbertian, and similarly Aq_k is Hilbertian. These are reflexive, and so A is an ideal in its bidual by Corollary 4.6. If H_0 is the closure of $\theta(A)(\ell^2)$, then the compression of θ to H_0 is a nondegenerate bicontinuous homomorphism, with range easily seen to be inside $\mathbb{K}(H_0)$. So we may assume that θ is nondegenerate from the start. As in Theorem 4.8, there is an invertible operator S on ℓ^2 with $p_k = S^{-1}e_k S$ mutually orthogonal projections. These are compact, so finite dimensional. Now appeal to Theorem 4.8 and its proof to see that A is bicontinuously isomorphic to a subcompact 1-matricial algebra. The other cases are similar. ∎

REMARK 4.11. (i) Suppose that in the last lemma, also $(\|T_k\|\|T_k^{-1}\|)$ is unbounded, so that A is not isomorphic to a C^*-algebra as Banach algebras. In this case A is not amenable (nor has the total reduction property of Gifford [21], etc.). For amenability implies the total reduction property, and the total reduction property for subalgebras of $\mathbb{K}(\ell^2)$, implies by [21] that A is similar to a C^*-algebra. It is interesting to ask if a σ-matricial algebra is amenable (or has the total reduction property, etc.) if and only if it is isomorphic to a C^*-algebra. Probably no 1-matricial algebras are amenable, biprojective, have Gifford's reduction property, etc., unless it is isomorphic to a C^*-algebra, but this needs to be checked.

(ii) We do not know if 1-matricial algebras are bicontinuously isomorphic if and only if they are completely isomorphic.

EXAMPLE 4.12. Let $K = \ell^2_2$, and $T_k = \text{diag}\{k, 1/k\}$. In this case by the above A is an ideal in its bidual, but is not topologically isomorphic to $\mathbb{K}(\ell^2)$ as Banach algebras. Here $q_1 A$ is a row Hilbert space and Aq_1 is a column Hilbert space. Note that A is not an annihilator algebra by Theorem 8.7.12 of [32], since $(q_1 A)^*$ is not isomorphic to Aq_1 via the canonical pairing.

EXAMPLE 4.13. Let $K = \ell^2$, and $T_k = E_{kk} + (1/k)I$.

Claim. $q_1 A$ is not reflexive. Indeed the Schauder basis (T_{1k}) (see Remark 4.7) fails the first part of the well known two part test for reflexivity [31], because $\sum_{k=1}^{\infty} T_k T_k^*$ converges weak* but not in norm. Or one can see that $q_1 A \cong c_0$ by Lemma 4.15 below. Here T_k^{-1} has k in all diagonal entries but one, which has a positive value $< k$. It follows that Aq_1 is a column Hilbert space. By Corollary 4.6, A is a left ideal in its bidual, but is not a right ideal in its bidual. This is interesting since any C^*-algebra which is a left ideal in its bidual is also a right ideal in its bidual [38].

Note that this is not an annihilator algebra by Theorem 8.7.12 of [32], since $(q_1 A)^*$ is not isomorphic to Aq_1. Also A is not bicontinuously isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$ by Lemma 4.10.

EXAMPLE 4.14. Let $K = \ell^2$ and $T_k = I - \sum_{i=1}^{k}(1 - \sqrt{i/k})E_{ii}$. It is easy to see that $q_k A$ is a row Hilbert space and Aq_k is a column Hilbert space, for all k. Thus A is an ideal in its bidual. Also A is not isomorphic to $\mathbb{K}(\ell^2)$ since $(\|T_k\|\|T_k^{-1}\|)$ is unbounded. Some of the authors are currently using examples such as these to test conjectures about 1-matricial algebras. For example, an argument by the second and third author in [37] gives a negative answer to the question "if $q_k A$ is a row Hilbert space and Aq_k is a column Hilbert space for all k, then is A subcompact?" Indeed, we show there that the C^*-envelope of the present example is not an annihilator C^*-algebra (so A is not subcompact by Lemma 4.10).

For a 1-matricial algebra A, if we only care about the norms on $q_1 A, Aq_1$ (as opposed to $q_2 A$ etc.) then we also may assume that $T_k \geqslant 0$ for all k, by replacing T_k by $(T_k T_k^*)^{1/2}$. This does not change the norm on $q_1 A$ and Aq_1. Note that if we are given *any* not necessarily invertible $T_k \geqslant 0$, for all $k \in \mathbb{N}$, then we can set $S_k = g_k(T_k)$, where $g_k(t) = \chi_{[0,1/2^k)}(t) + t\chi_{[1/2^k,\infty)}(t)$. Then S_k is a small perturbation of T_k which is invertible. Using this trick one can build 1-matricial algebras such that $q_1 A$ is "very bad":

LEMMA 4.15. *If $T_2, T_3, \ldots$ are arbitrary elements of norm $\geqslant 1$ in an operator space X, then there exists a 1-matricial algebra A with $q_1 A$ bicontinuously isomorphic to $\overline{\mathrm{Span}}\{E_{1k} \otimes T_k\} \subset R_\infty(X)$. (One may suppose without loss that X contains I, the identity operator on a Hilbert space on which X is represented, and set $T_1 = I$.) Also, A may be chosen such that Aq_1 is a row Hilbert space, if $\sum_{k=2}^{n} |\alpha_k|^4 \leqslant \left\| \sum_{k=2}^{n} |\alpha_k|^2 T_k^2 \right\|^2$ for all scalars α_k and $n \in \mathbb{N}$.*

Proof. Assume that $I \in X \subset B(K)$. As above, we may assume $T_k \geqslant 0$, and form S_k as described. Also, $T_k \leqslant S_k$, $(1/2^k)\, I \leqslant S_k$, and $S_k - T_k \leqslant (1/2^k)I$ and $S_k^2 - T_k^2 \leqslant (1/2^k)I$. Then

$$\sum_{k=1}^{n} |\alpha_k|^2 T_k^2 \leqslant \sum_{k=1}^{n} |\alpha_k|^2 S_k^2 \leqslant \sum_{k=1}^{n} |\alpha_k|^2 T_k^2 + \sum_{k=1}^{n} \frac{|\alpha_k|^2}{2^k} I \leqslant \sum_{k=1}^{n} |\alpha_k|^2 T_k^2 + \sup_k |\alpha_k|^2 I,$$

so that

$$\left\| \sum_{k=1}^{n} |\alpha_k|^2 T_k^2 \right\|^{1/2} \leqslant \left\| \sum_{k=1}^{n} |\alpha_k|^2 S_k^2 \right\|^{1/2} \leqslant \left\| \sum_{k=1}^{n} |\alpha_k|^2 T_k^2 \right\|^{1/2} + \sup_k |\alpha_k|.$$

If $1 \leqslant \|T_k\|$ then the right hand side is dominated by $2\left\| \sum_{k=1}^{n} |\alpha_k|^2 T_k^2 \right\|^{1/2}$, so that

$$\overline{\mathrm{Span}(\{E_{1k} \otimes S_k : k \in \mathbb{N}\})} \cong \overline{\mathrm{Span}(\{E_{1k} \otimes T_k : k \in \mathbb{N}\})}$$

bicontinuously (hence they are reflexive or nonreflexive simultaneously). Thus we obtain a 1-matricial algebra A formed from the S_k, S_k^{-1}, with $q_1 A$ bicontinuously isomorphic to the closure of the span of $E_{1k} \otimes T_k$ in $R_\infty(X)$.

Assume that $\sqrt{\sum_{k=2}^n |\alpha_k|^4} \leqslant \left\| \sum_{k=2}^n |\alpha_k|^2 T_k^2 \right\| \leqslant \left\| \sum_{k=2}^n |\alpha_k|^2 S_k^2 \right\|$. Since $\|I+T\| = 1 + \|T\|$ if $T \geqslant 0$, it is easy to see that we can replace all occurrences of the symbols "$k=2$", in the last formula, by "$k=1$". Let $R_k = S_k \oplus t_k I$. Then

$$\left\| \sum_{k=1}^n |\alpha_k|^2 R_k^2 \right\| = \max \left\{ \left\| \sum_{k=1}^n |\alpha_k|^2 S_k^2 \right\|, \sum_{k=1}^n |\alpha_k|^2 t_k^2 \right\}.$$

The last sum is dominated by $\sqrt{\sum_{k=1}^n |\alpha_k|^4}$, if $\sum_k t_k^4 \leqslant 1$, and so there is no change in the norm on $q_1 A$: $\left\| \sum_{k=1}^n |\alpha_k|^2 R_k^2 \right\| = \left\| \sum_{k=1}^n |\alpha_k|^2 S_k^2 \right\|$. Moreover, $S_k^{-1} \leqslant 2^k I$, and so $\left\| \sum_{k=1}^n |\alpha_k|^2 S_k^{-2} \right\| \leqslant \sum_{k=1}^n |\alpha_k|^2 (2^k)^2$. Therefore $\left\| \sum_{k=1}^n |\alpha_k|^2 R_k^{-2} \right\| = \sum_{k=1}^n |\alpha_k|^2 / t_k^2$ if $t_k \leqslant 1/2^k$. A similar fact holds at the matrix level. This forces Aq_1 to be a row Hilbert space, since the map $(\alpha_1, \alpha_2, \ldots) \mapsto [\alpha_1 t_1 R_1^{-1} : \alpha_2 t_2 R_2^{-1} : \cdots]$ is a complete isometry from the finitely supported elements in ℓ^2, with its row operator space structure, into Aq_1. Thus for example we may take $t_k = 1/2^k$, and obtain a 1-matricial algebra A formed from the R_k, R_k^{-1}, with Aq_1 a row Hilbert space, and $q_1 A$ bicontinuously isomorphic to the closure of the span of $E_{1k} \otimes T_k$ in $R_\infty(X)$. ∎

EXAMPLE 4.16. An example of a 1-matricial algebra A with $q_1 A$ reflexive (isomorphic to ℓ^4) but not isomorphic to a Hilbert space, and Aq_1 a row Hilbert space, is obtained from Lemma 4.15 by taking $T_2, T_3, \ldots$ to be the canonical basis for $O\ell^2$. Here $O\ell^2$ is Pisier's operator Hilbert space [35], and $T_1 = I_H$ is the identity operator on H where $O\ell^2 \subset B(H)$. Here $\left\| \sum_{k=2}^n |\alpha_k|^2 T_k^2 \right\|^{1/2} = \left(\sum_{k=2}^n |\alpha_k|^4 \right)^{1/4}$.

One may vary this example by replacing $O\ell^2$ with other "classical" operator spaces, to obtain 1-matricial algebras with other interesting features.

REMARK 4.17. (i) For a C^*-algebra A, it is well known that every minimal left or right ideal is a Hilbert space (since it is a C^*-module over $\mathbb{C} e \cong \mathbb{C}$), as is A/J for a maximal left or right ideal J (any maximal left ideal is the left kernel of a pure state φ, and then $A/J \subset A^{**}(1-p)$ where $1-p$ is a minimal projection in A^{**} (see p. 87 in [34]). For a minimal projection q in a W^*-algebra M, qMq is one dimensional since it is projectionless, so Mq is a Hilbert space as in the minimal ideal argument above).

The above gives, in contrast, very nice (semisimple, etc.) approximately unital operator algebras A with an r-ideal J (respectively K) which is maximal (respectively minimal) amongst all the right ideals, such that A/J (respectively

K) is not Hilbertian, indeed is not reflexive, or is reflexive but is not Hilbertian. For if A is one of the 1-matricial algebras in our examples, one can show that $J = \overline{\sum_{k \neq 1} q_k A}$ is maximal amongst all the right ideals, and $A/J \cong q_1 A$ and $A/(q_1 A) \cong J$. The latter is because for example $L_{q_1} : A \to q_1 A$ is a complete quotient map with kernel J.

(ii) For a 1-matricial algebra A, one may consider the associated Haagerup tensor product $Aq_1 \otimes_h q_1 A$, which is also an operator algebra. Such algebras are considered in the interesting paper [3]. Immediately several questions arise, which may be important, such as if there are some useful sufficient conditions for when this algebra is isomorphic to A (if it is, then A is completely isomorphic to $\mathbb{K}(\ell^2)$).

PROPOSITION 4.18. *Let A be a σ-matricial algebra. Then A is a Δ-dual algebra.*

Proof. Clearly A has a positive cai. Also $\Delta(\bigoplus_k^0 A_k) = \bigoplus_k^0 \Delta(A_k)$, so that we may assume that A is a 1-matricial algebra. If $x \in \Delta(A)$ then so is $q_i x q_j$ for any i, j. Note that $q_i x q_j \neq 0$ if and only if $T_i T_i^* \in \mathbb{C}\, T_j T_j^*$. If we assume, as we may without loss of generality, that $\|T_k\| = 1$ for all k, then the latter is equivalent to $T_i T_i^* = T_j T_j^*$. This gives an equivalence relation $\sim$ on $\mathbb{N}$, and we may partition into equivalence classes, E_k say, each consisting of natural numbers. Let B_k be the closure of the span of the $E_{ij} \otimes T_i^{-1} T_j$, for $i, j \in E_k$. These are 1-matricial algebras, which are selfadjoint, hence C^*-algebras. Thus $B_k \cong \mathbb{K}(H_k)$ for a Hilbert space H_k. Note that the relation $T_i T_i^* = T_j T_j^*$ above implies that $\|T_i\|$ is constant on E_k, and by taking inverses we also have $\|T_i^{-1}\|$ constant on E_k. Since $q_i x q_j = 0$ if i and j come from distinct equivalence classes, $\Delta(A)$ decomposes as a c_0-sum $\Delta(A) = \bigoplus_k^0 B_k$. Indeed, clearly $B_k \subset \Delta(A)$, and any $x \in \Delta(A)_{\rm sa}$ is approximable by a selfadjoint finitely supported matrix in $\Delta(A)_{\rm sa}$, and hence by a finite sum of elements from the B_k. Hence $\Delta(A)$ is an annihilator C^*-algebra. ∎

A pleasant feature of 1-matricial algebras is that their second duals have a simple form:

LEMMA 4.19. *If A is a 1-matricial algebra defined by a system of matrix units $\{T_{ij}\}$ in $B(K^{(\infty)})$ as in Definition 4.3, then*

$$A^{**} \cong \{T \in B(K^{(\infty)}) : q_i T q_j \in \mathbb{C}\, T_{ij} \ \ \forall\, i, j\}.$$

Thus A^{**} is the collection of infinite matrices $[\beta_{ij} T_i^{-1} T_j]$, for scalars β_{ij}, which are bounded operators on $K^{(\infty)}$.

Proof. Write N for the space on the right of the last displayed equation. This is weak* closed. Suppose that A is represented nondegenerately on a Hilbert space H in such a way that $I_H = 1_{A^{**}} \in A^{**} \subset B(H)$, the latter as a weak* closed subalgebra, with the σ-weak topology agreeing on A^{**} with the weak* topology of A^{**}. Then we have $q_i A^{**} q_j = \mathbb{C}\, T_{ij}$. That is, $A^{**} \subset N$ completely isometrically. If $x \in N$, and if $x_n = (\sum_{k=1}^n q_k) x (\sum_{k=1}^n q_k)$, then $x_n \in A$, and $x_n \to x$ WOT, hence weak*. Thus $x \in \overline{A}^{w*} = A^{**}$. ∎

LEMMA 4.20. *Let A be a σ-matricial algebra. If p is a projection in the second dual of A, then p lies in $M(A)$ and in $M(\Delta(A))$, and is thus open in the sense of* [9]. *Hence A is* nc*-discrete. Also,*

$$\Delta(A^{**}) = \Delta(A)^{**} = M(\Delta(A)) = \Delta(M(A)).$$

Proof. We may assume that A is a 1-matricial algebra. Let $x \in \Delta(A^{**})_{\mathrm{sa}}$. If $x_n = (\sum_{k=1}^n q_k) x (\sum_{k=1}^n q_k)$, then $x_n \in \Delta(A)$, and $x_n \to x$ WOT, hence weak*. Thus $x \in \Delta(A)^{\perp\perp}$. Hence

$$\Delta(A^{**}) \subset \Delta(A)^{**} = \Delta(M(A)) = M(\Delta(A)),$$

using Proposition 2.11. Therefore all of these sets are equal since $\Delta(A)$, and hence $\Delta(A)^{\perp\perp}$, are subsets of $\Delta(A^{**})$. Thus any projection $p \in A^{**}$ is in $M(A)$, and hence is open by 2.1 in [9]. ∎

REMARK 4.21. By the above, and using also the notation in the proof of Proposition 4.18, for any 1-matricial algebra A we have $\Delta(A) = \bigoplus_k^0 B_k$, where B_k are C^*-subalgebras of $\Delta(A)$ corresponding to the equivalence relation $\sim$ on $\mathbb{N}$, and $B_k \cong \mathbb{K}(H_k)$ for a Hilbert space H_k. It follows by Lemma 4.20 that

$$\Delta(A^{**}) = \Delta(A)^{**} = \bigoplus_k^\infty B_k^{**} \cong \bigoplus_k^\infty B(H_k).$$

Recall, by Lemma 4.19, we may write any element of A^{**} as a matrix $\left[\beta_{ij} T_i^{-1} T_j\right]$, for scalars β_{ij}. One may ask what this matrix looks like if $x \in \Delta(A^{**})$. In this case, $\beta_{ij} = 0$ if i and j are in different equivalence classes for the relation $\sim$ discussed in the proof of Proposition 4.18. Indeed if $x = x^*$ then it is easy to see that

$$\beta_{ij}\, T_j T_j^* = \overline{\beta}_{ji}\, T_i T_i^*, \quad \forall i, j.$$

Assume, as we may, that $\|T_k\| = 1$ for all k. Taking norms we see that $|\beta_{ij}| = |\beta_{ji}|$. It follows that $\beta_{ij} = \overline{\beta}_{ji}$; and also, if $\beta_{ij} \neq 0$ then $i \sim j$. Thus $\beta_{ij} = 0$ if i and j are in different equivalence classes.

PROPOSITION 4.22. *Let A be an operator algebra such that for every nonzero projection p in A, $pq \neq 0$ for some algebraically minimal projection $q \in A$. Then every $*$-minimal projection in A is algebraically minimal. This holds in particular for σ-matricial algebras.*

Proof. If p is $*$-minimal, $pq \neq 0$ as above, then $(1/t)pqp$ is an algebraically minimal projection for some $t > 0$, and thus equals p. Hence p is algebraically minimal. ∎

We now give some "Wedderburn type" structure theorems. See e.g. [26], [28] for some other operator algebraic "Wedderburn type" results in the literature.

THEOREM 4.23. *Let A be an approximately unital semiprime operator algebra. The following are equivalent:*

(i) *A is completely isometrically isomorphic to a σ-matricial algebra.*

(ii) *A is the closure of $\sum_k q_k A$ for mutually orthogonal algebraically minimal projections $q_k \in A$.*

(iii) *A is the closure of the joint span of minimal right ideals which are also* r*-ideals (these are the qA for algebraically minimal projections $q \in A$).*

(iv) *A is Δ-dual, and every $*$-minimal projection in A is algebraically minimal.*

(v) *A is Δ-dual, and every nonzero projection in A dominates a nonzero algebraically minimal projection in A.*

(vi) *A is* nc*-discrete, and every nonzero projection in $M(A)$ dominates a nonzero algebraically minimal projection in A.*

(vii) *A is* nc*-discrete, and every nonzero* HSA *D in A containing no nonzero projections of A except possibly an identity for D, is one-dimensional.*

Proof. Clearly (i) implies (ii), and (ii) implies (iii), and (v) implies (iv). Also fairly obvious are that (vi) implies (v) (using Corollary 2.14(iii)) and (vii).

(ii) $\Rightarrow$ (i) By Proposition 4.1 we have $\sum_k q_k = 1$ strictly. We partition the (q_k) into equivalence classes I_j, according to whether $q_k A \cong q_j A$ or not. Note that $q_k A q_j = (0)$ if j, k come from distinct classes, by the idea in the proof of Theorem 4.8 above. If j, k come from the same class then $q_k A q_j$ is one dimensional, $q_k A q_j = \mathbb{C}\, T_{kj}$ say. Let $e_j = \sum_{k \in I_j} q_k$. Then either $e_j T_{pq} = 0 = T_{pq} e_j$ (if j is in a different class to p, q) or $e_j T_{pq} = T_{pq} = T_{pq} e_j$ (if j is in the same class as p, q). So e_j is in the center of $M(A)$. Then $B = e_j A$ is an ideal in A, and for $b \in B$, $\sum_{k \in I_j} q_k b = \sum_k q_k b = b$, and similarly $b \sum_{k \in I_j} q_k = b$. As in earlier proofs B is generated by a set of matrix units T_{ij} which it contains, and hence is topologically simple. By Theorem 4.8, $B = e_j A$ is a 1-matricial algebra. The map $A \to \bigoplus^0_{j \in J} e_j A$ is a completely isometric isomorphism, since any $a \in A$ is approximable in norm by finite sums of term of the form $q_j a q_k$, each contained in some $e_j A$.

(iii) $\Rightarrow$ (v) Given (iii), the joint support of all the algebraically minimal projections is 1 (e.g. as in the proof of (v) $\Rightarrow$ (ii) below). Thus the closure of the sum of the $q\Delta(A)$ for all the algebraically minimal projections q, is $\Delta(A)$ (since the weak* closure in the second dual contains 1). So by Exercise 4.7.20(ii) of [17], $\Delta(A)$ is an annihilator C^*-algebra with support projection 1, and hence A is Δ-dual. Let e be a nonzero projection in $M(A)$. Then since the joint support of all algebraically minimal projections is 1, $eq \neq 0$ for an algebraically minimal projection q. We have $(eqe)^2 = eqeqe = teqe$ for some $t > 0$, so that $(1/t)eqe$ is an algebraically minimal projection dominated by e.

(v) $\Rightarrow$ (ii) Let (e_k) be a maximal family of mutually orthogonal algebraically minimal projections in A, and let $e = \sum_k e_k \in A^{**}$. If A is Δ-dual, and 1 is the identity in A^{**}, then 1 is also the identity of $M(\Delta(A)) = \Delta(A)^{**}$, so that $1 - e \in M(\Delta(A))$. Hence if $e \neq 1$ then $1 - e$ dominates a nonzero $*$-minimal projection in $\Delta(A)$, which in turn dominates a nonzero algebraically minimal projection in A, contradicting maximality of (e_k). So $\sum_k e_k = 1$. The closure L of $\sum_k e_k A$ is A. This is because $L^{\perp\perp}$ is the weak* closure of sums of the $e_k A^{**}$ by e.g. A.3 in [15], which contains 1 and hence equals A^{**}. So $L = A \cap L^{\perp\perp} = A$.

(iii) $\Rightarrow$ (vi) We have by the above that (iii) implies (i), which implies by Lemma 4.20 that A is nc-discrete. The proof of (iii) implies (v) also gives the other part of (vi).

(iv) $\Rightarrow$ (v) Given a projection $e \in M(A)$, we have $e \in \Delta(M(A)) = M(\Delta(A))$ by Proposition 2.11. Since $\Delta(A)$ is an annihilator C^*-algebra, e majorizes a nonzero $*$-minimal projection (since this is true for algebras of compact operators), which by (iv) is algebraically minimal.

(vii) $\Rightarrow$ (iv) As we said in Proposition 2.12, $\Delta(A)$ is an annihilator C^*-algebra. Given a nonzero projection $p \in M(A) \setminus A$, then either the HSA pAp is one-dimensional, in which case p dominates the identity of pAp, or it is not one-dimensional, in which case p dominates a nonzero projection in A by (vii). Thus A is Δ-dual by Corollary 2.14(iii). Now (iv) is clear.

That (iii) is equivalent to (i) also follows from Theorem 4.31. ∎

REMARK 4.24. If A is a one-sided ideal in A^{**}, then A is nc-discrete by Proposition 2.16. In this case, one may remove the condition "A is nc-discrete" in (vi)–(vii), and one may replace "A is Δ-dual" by "$\Delta(A)$ acts nondegenerately on A" in (iv) and (v).

The following is another characterization of σ-matricial algebras.

THEOREM 4.25. *Let A be an approximately unital semiprime operator algebra such that $\Delta(A)$ acts nondegenerately on A. Suppose also that every $*$-minimal projection $p \in A$ is also minimal among all idempotents (that is, there are no nontrivial idempotents in pAp). The following are equivalent:*

(i) *A is completely isometrically isomorphic to a σ-matricial algebra.*

(ii) *A is compact.*

(iii) *A is a modular annihilator algebra.*

(iv) *The socle of A is dense.*

(v) *A is semisimple and the spectrum of every element in A has no nonzero limit point.*

Proof. We first point out a variant of (iv) in the last theorem: if A is a Δ-dual algebra which is semiprime, and if every nonzero $*$-minimal projection p is minimal among the idempotents in A, and pAp is finite dimensional (or equivalently, p is in the socle, or is "finite rank"), then A is completely isometrically isomorphic to a σ-matricial algebra. To see this, note that by Proposition 2.5, pAp is semiprime, hence is one-dimensional, or equivalently p is algebraically minimal (for if not, then by Wedderburn's theorem pAp contains nontrivial idempotents, contradicting the hypothesis). Thus Theorem 4.23(iv) holds.

If (ii), (iii), (iv), or (v) hold, then it is known that pAp is finite dimensional for every projection $p \in A$ (some of these follow from the ideas in 8.6.4 and 8.5.4 in [32]). Suppose that p is $*$-minimal. By the last paragraph, we will be done if we can show that any one of (ii)–(v) imply that $\Delta(A)$ is an annihilator C^*-algebra. If A is compact then so is $\Delta(A)$, hence it is an annihilator C^*-algebra. By 8.7.6 in [32], (iv) implies (ii) and (iii). If A is a modular annihilator algebra then the spectrum condition in (v) holds by 8.6.4 in [32]. If the spectrum condition in (v) holds, then by the spectral permanence theorem, if $x \in \Delta(A)_{\mathrm{sa}}$ and B is a C^*-algebra generated by A, then $\mathrm{Sp}_{\Delta(A)}(x) \setminus \{0\} = \mathrm{Sp}_A(x) \setminus \{0\} = \mathrm{Sp}_B(x) \setminus \{0\}$, which has no nonzero limit point. So $\Delta(A)$ is an annihilator C^*-algebra by 4.7.20 (vii) of [17]. ∎

EXAMPLE 4.26. In the last theorems, most of the hypotheses seem fairly sharp, as one may see by considering examples such as the disk algebra, Example 4.30, or the following example. Let $B = RDR^{-1}$, where D is the diagonal copy of c_0 in $B(\ell^2)$, and $R = I + (1/2)S$ where S is the backwards shift. Indeed R could be any invertible operator such that the commutant of R^*R contains no nontrivial projections in D. This example has most of the properties in Theorem 4.25: its second dual is isometrically identifiable with $R\overline{D}^{w*}R^{-1}$ in $B(\ell^2)$, which is unital, and so B is approximately unital; B is semiprime and satisfies (ii)–(v) in Theorem 4.25, since D does. Moreover, B has no nontrivial projections. Indeed, if $q = RpR^{-1}$ is a projection then p is an idempotent in D, hence is a projection. That $q = q^*$ implies that p is in the commutant of R^*R, which forces $p = 0 = q$. On the other hand, B does not satisfy (i) of Theorem 4.25, hence has no positive cai. Thus it is not Δ-dual although it is nc-discrete, indeed it is an ideal in its bidual, and its diagonal C^*-algebra is an annihilator C^*-algebra. Thus this example illustrates the importance of the condition that $\Delta(A)$ acts nondegenerately on A in the last theorem. One may vary this example by letting $A = B \oplus \mathbb{C}$, where $B = RDR^{-1}$ as above. This has exactly one nontrivial projection. Variants of this example are also useful to illustrate hypotheses in others of our results, such as replacing D by the diagonal copy of ℓ^∞ in $B(\ell^2)$.

COROLLARY 4.27. *Let A be a semiprime left or right essential operator algebra, containing algebraically minimal idempotents $(q_k)_{k=1}^\infty$ with $q_j q_k = 0$ for $j \neq k$, and*

$\sum_k q_k = 1$ *strictly. Then A is semisimple and A is completely isomorphic to a σ-matricial algebra.*

Proof. By the similarity trick from the start of this section, we can assume that the q_k are projections, and A is the closure of $\sum_k q_k A$. We may then appeal to Theorem 4.23(ii). ∎

We now consider a class of algebras which are a commutative variant of matricial operator algebras, and are ideals in their bidual.

PROPOSITION 4.28. *Let A be a commutative operator algebra with no nonzero annihilators in A, and possessing a sequence of nonzero algebraically minimal idempotents (q_k) with $q_j q_k = 0$ for $j \neq k$, and $\overline{\sum_k A q_k} = A$. Then A is a semisimple annihilator algebra with dense socle, and A is an ideal in its bidual. If further the q_k are projections (respectively $\sum_k q_k = 1$ strictly), and if A is left essential, then $A \cong c_0$ isometrically (respectively $A \cong c_0$ isomorphically).*

Proof. If $x \in J(A)$, the Jacobson radical, then $xq_k \in J(A) \cap \mathbb{C}\, q_k = 0$ for all k, since $J(A)$ contains no nontrivial idempotents. Thus $xA = 0$ and $x = 0$, and so A is semisimple. If J is a closed ideal in A with $q_k J \neq (0)$ for all k, then $q_k \in J$ since $q_k J \subset J \cap \mathbb{C}\, q_k$. Thus $\sum_k A q_k \subset J$ and $J = A$. So A is an annihilator algebra. The $q_k A$ are minimal ideals and so A has dense socle. By Proposition 4.2, A is an ideal in its bidual. We leave the other assertions as an exercise. ∎

REMARK 4.29. If in addition to the conditions in the first sentence of Proposition 4.28, $\Delta(A)$ acts nondegenerately on A then A is Δ-dual and nc-discrete (using Proposition 2.16).

EXAMPLE 4.30. The following example illustrates the distinction between the condition $\overline{\sum_k q_k A} = \overline{\sum_k A q_k} = A$, and the condition $\sum_k q_k = 1$ strictly (the latter defining algebras isomorphic or similar to a σ-matricial algebra by Corollary 4.27). Inside $B = M_2 \oplus^\infty M_2 \oplus^\infty \cdots$, we consider idempotents $q_{2k} = 0 \oplus \cdots \oplus 0 \oplus e_k \oplus 0 \oplus \cdots$ and $q_{2k+1} = 0 \oplus \cdots \oplus 0 \oplus f_k \oplus 0 \oplus \cdots$, where e_k, f_k are idempotents in M_2 with $e_k f_k = 0, e_k + f_k = I_2$, and $\|e_k\|, \|f_k\| \to \infty$. For example, consider the rank one operators $e_k = [1 : 1] \otimes [-k : k+1]$ and $f_k = [(k+1)/k : 1] \otimes [k : -k]$ in M_2. Let A be the closure of the span of these idempotents (q_k), which has cai, and may be viewed as a subalgebra of $\mathbb{K}(\ell^2)$. The algebra A is of the type discussed in the last result, and the remark after it. Indeed it is a "dual Banach algebra" in the sense of Kaplansky. However A is not isomorphic to a σ-matricial algebra, indeed is not isomorphic to c_0, since the algebraically minimal idempotents in A are not uniformly bounded, whereas they are in c_0.

Let A be any operator algebra. If $e_1, \dots, e_n$ are algebraically minimal projections in A, set $e = e_1 e_2 \cdots e_n$. Then $e^2 = t e_1 e_2 \cdots e_n = te$, for some t with $|t| \leqslant 1$. Note that $t = 0$ if and only if e is nilpotent, whereas if $t \neq 0$ then $(1/t)e$ is an algebraically minimal idempotent. The set E of linear combinations of such products is a $*$-subalgebra of $\Delta(A)$, and so $\overline{E}$ is the C^*-subalgebra B of $\Delta(A)$ generated by the algebraically minimal projections in A. Note that the sum of all minimal right ideals of B is dense in B, so that B is an annihilator C^*-algebra. We write B as Δ-soc(A).

For an operator algebra A, define the *r-socle* r-soc(A) to be the closure of the sum of r-ideals of the form eA for algebraically minimal projections e. This is an r-ideal, with support projection f equal to the "join" of all the algebraically minimal projections. Thus r-soc$(A) = fA^{**} \cap A$. Note that $f \in M(B) = B^{**}$ where $B = \Delta$-soc(A). Similarly, ℓ-soc$(A) = A^{**}f \cap A$ is the closure of the sum of l-ideals of the form Ae for such e, and h-soc(A) is the matching HSA $fA^{**}f \cap A$. We say that the i-socle exists if r-soc$(A) = \ell$-soc(A), an approximately unital ideal, which also equals h-soc(A) in this case. Note that r-soc$(A) \cap J(A) = (0)$ by p. 671 of [32], hence h-soc$(A) \cap J(A) = (0)$.

THEOREM 4.31. *Let A be a semiprime operator algebra. Then* h-*soc*(A) *is a σ-matricial algebra.*

Proof. We use the notation above. Let $B = \Delta$-soc(A), an annihilator C^*-algebra. Let $D = $ h-soc$(A) = fA^{**}f \cap A$, an approximately unital semiprime operator algebra by Proposition 2.5. Set $J = fA^{**} \cap A$. Let $(f_k)_{k \in E}$ be a maximal family of mutually orthogonal algebraically minimal projections in D, and set $e = \sum_{k \in E} f_k \in M(B)$. Note that $e \leqslant f$. Suppose that $f \neq e$. Then $(f-e)g \neq 0$ for some algebraically minimal projection g in A (or else $(f-e)f = 0$, which is false). Then $p = t(f-e)g(f-e)$ is an algebraically minimal projection for some $t > 0$, which lies in B since $f, e \in M(B)$. Thus $p \in A \cap fAf \subset D$, contradicting the maximality of the family. So $f = \sum_{k \in E} f_k$, and $J = \bigoplus_{k \in E}^{c} f_k A$ by the argument that (v) implies (ii) in Theorem 4.23 ($f \in (\bigoplus_{k \in E}^{c} f_k A)^{\perp\perp}$ so $J^{\perp\perp} = fA^{**} = (\bigoplus_{k \in E}^{c} f_k A)^{\perp\perp}$). The partial sums of $\sum_{k \in E} f_k$ are a positive left cai for J, so they are a cai for D [9]. So $\sum_{k \in E} f_k = f$ strictly on D. By Theorem 4.23, D is a σ-matricial algebra. ∎

If A is a σ-matricial algebra, then the r-ideals, l-ideals, and HSA's of A are of a very nice form:

PROPOSITION 4.32. *If A is a σ-matricial algebra, then for every* r-*ideal (respectively* l-*ideal) J of A, there exist mutually orthogonal algebraically minimal projections $(f_k)_{k \in I}$ in A with $\sum_k f_k = 1$ strictly on A, and $J = \bigoplus_{k \in E}^{c} f_k A$ (respectively $J = \bigoplus_{k \in E}^{r} A f_k$), for some set $E \subset I$.*

Proof. By Proposition 2.16, every r-ideal J equals fA for a projection $f \in M(A)$. Then $D = fAf$ is the HSA corresponding to J, and Af is the corresponding l-ideal. We follow the proof of Theorem 4.31. Let $(f_k)_{k\in E}$ be as in that proof, then $e = \sum_{k\in E} f_k \in M(\Delta(A)) = \Delta(M(A))$. If $f \neq e$ then $f - e \in M(A)$, so $f - e$ dominates a nonzero algebraically minimal projection in D, producing a contradiction. So $f = \sum_{k\in E} f_k$, and $J = \bigoplus^{\mathrm{c}}_{k\in E} f_k A$ as before. A similar argument shows that $Af = \bigoplus^{\mathrm{r}}_{k\in E} Af_k$.

By a maximality argument (similar to the proof of (v) implies (ii) in Theorem 4.23), we can enlarge $(f_k)_{k\in E}$ to a set (f_k) in A with $A = \bigoplus^{\mathrm{c}}_{k} f_k A$. Then $\sum_k f_k = 1$ strictly on A. ∎

COROLLARY 4.33. *Every* HSA *in a* 1*-matricial algebra (respectively in a* σ*-matricial algebra) is a* 1*-matricial algebra (respectively a* σ*-matricial algebra).*

Proof. We may assume that A is a 1-matricial algebra. Continuing the proof of Proposition 4.32: as in the proof of Theorem 4.31, $\sum_{k\in E} f_k = f$ strictly on D. By Proposition 2.5, D is topologically simple. By Theorem 4.8, D is a 1-matricial algebra. ∎

5. CHARACTERIZATIONS OF C^*-ALGEBRAS OF COMPACT OPERATORS

An interesting question is whether every approximately unital operator algebra with the property that all closed right ideals have a left cai (and/or similarly for left ideals), is a C^*-algebra. The following is a partial result along these lines:

THEOREM 5.1. *Let A be a semiprime approximately unital operator algebra. The following are equivalent:*

(i) *Every minimal right ideal of A has a left* cai *(or equivalently by Lemma* 2.3*, equals pA for a projection $p \in A$).*

(ii) *Every algebraically minimal idempotent in A has range projection in A.*

If either of these hold, and if A has dense socle, then A is completely isometrically isomorphic to an annihilator C^*-algebra.

Proof. (i) $\Rightarrow$ (ii) If e is an algebraically minimal idempotent in A, then eA is a minimal right ideal, hence an r-ideal by (i). By Lemma 2.3, $eA = pA$ for a projection $p \in A$. We have $pe = e, ep = p$, which forces p to be the range projection of e.

(ii) $\Rightarrow$ (i) Every minimal right ideal equals eA for an algebraically minimal idempotent e. The range projection p of e satisfies $pe = e, ep = p$, so that $eA = pA$.

Suppose that these hold, and A has dense socle. We will argue that A satisfies Theorem 4.23(iv), hence is a σ-matricial algebra. As in the proof of Theorem 4.25, A is a semisimple modular annihilator algebra, and $\Delta(A)$ is a nonzero annihilator C^*-algebra. Let e_Δ be the identity of $\Delta(A)^{**}$, and set $J = e_\Delta A^{**} \cap A$, a closed right ideal in A. If e is an algebraically minimal idempotent not in J, and if q is its range projection, which is in A, then q is not in J, or else $e = qe$ would be in J. However $q \leqslant e_\Delta$ since $q \in \Delta(A)$, so that $q \in J$. Thus every algebraically minimal idempotent in A is in J. Hence J contains the socle, so that $J = A$ and $e_\Delta = 1$. Therefore A is Δ-dual. If p is a $*$-minimal projection in A, then pA contains a nonzero algebraically minimal idempotent $e \in A$, by Theorem 8.4.5(h) of [32]. By the hypothesis, $eA = fA$ for a projection $f \in A$. Clearly f is algebraically minimal, and $f \in eA \subset pA$, so that $pf = f = fp$. Thus $p = f$ is algebraically minimal. By Theorem 4.23(iv), A is a σ-matricial algebra.

It remains to show that every 1-matricial algebra A satisfying the given condition, is a C^*-algebra of compact operators. To this end, fix integers $i \neq j$ and let $x = q_i + q_j + T_{ij} + T_{ji} \in (q_i + q_j)A(q_i + q_j)$. Then $xA = eA$ for a projection $e \in A$, since x is a scalar multiple of an algebraically minimal idempotent. We have $(q_i + q_j)e = e = e(q_i + q_j)$. Suppose that the i-j entry of e is zero, which forces e to be q_i, q_j, or $q_i + q_j$. Let $u = T_{ij} + T_{ji}$, then $ux = x$. There exists $(a_n) \subset A$ with $xa_n \to e$, and it follows that $ue = e$, which is false. This contradiction shows that the i-j entry of e is nonzero. By the proof of Proposition 4.18, $T_i T_i^* = T_j T_j^*$. Since i, j were arbitrary, it follows as in Proposition 4.18 that A is selfadjoint, and $A \cong \mathbb{K}(H)$ for a Hilbert space H. ∎

We say that a left ideal in A is *A-complemented* if it is the range of a bounded idempotent left A-module map.

LEMMA 5.2. *Let A be an operator algebra with a bounded right approximate identity.*

(i) *Every A-complemented closed left ideal J in A has a right* bai, *and also a nonzero right annihilator. Indeed $J = Ae$ for some idempotent $e \in A^{**}$.*

(ii) *If every closed left ideal in A is A-complemented, then A is a semiprime right annihilator algebra.*

(iii) *If A has a* bai, *and if J is a two-sided ideal in A which is both right and left complemented, then $J = eA$ for an idempotent e in the center of $M(A)$. Also, J has a* bai.

(iv) *If A has a right* cai, *then every contractively A-complemented closed left ideal J in A has a right* cai, *and the e in* (i) *may be chosen to be a contractive projection in $M(A)$.*

Proof. (i) Let $P : A \to J$ be the projection, with $\|P\| \leqslant K'$. If (e_t) is the right bai, then $xP(e_t) = P(xe_t) \to P(x) = x$ for $x \in J$. Thus J has a right bai. There is a weak* convergent subnet $P(e_{t_\mu}) \to r$ weak* in A^{**}. Then $ar = \text{w*-}\lim_\mu P(ae_{t_\mu}) = P(a) \in J$ for all $a \in A$. So $r \in RM(A) \cap J^{\perp\perp}$. Also, $xr = x$ for $x \in J$. Hence $J = Ar$, and $\|r\| \leqslant KK'$, if K is a bound for the right bai. If $ArA = (0)$

then $ar^2 = ar = 0$ for all $a \in A$, so that $r = 0$. Thus if J is nontrivial then J has a nonzero right annihilator. Also, $J^{\perp\perp}$ is a weak* closed left ideal containing r so $A^{**}r = J^{\perp\perp}$.

(ii) In this case, A is a right annihilator algebra, by (i). Also A is semiprime, since if J were a two-sided ideal with $J^2 = (0)$, then $ArAr = (0)$, and $r^4 = r = 0$.

(iii) In this case, J has a right and a left bai, hence a bai [32].

(iv) This is a slight modification of the proof of (i). ∎

COROLLARY 5.3. *Let A be a semisimple approximately unital operator algebra such that every closed left ideal in A is contractively A-complemented, or equivalently equals $J = Ap$ for a projection $p \in M(A)$. Then A is completely isometrically isomorphic to an annihilator C^*-algebra.*

Proof. As we said above, A is a right annihilator algebra. If it is semisimple then it has dense socle by Proposition 8.7.2 of [32]. Now apply Theorem 5.1. ∎

REMARK 5.4. (i) This result is related to theorems of Tomiuk and Alexander (see e.g. [40], [1] concerning "complemented Banach algebras", but the proofs and conclusions are quite different).

(ii) We imagine that a semisimple approximately unital operator algebra such that every closed left ideal in A is A-complemented, is isomorphic to an annihilator C^*-algebra. Certainly in this case A is a right annihilator algebra by the last lemma. It therefore has dense socle by Proposition 8.7.2 of [32], and an argument from [27] shows that $A = \overline{\sum_i Ae_i}$ for mutually orthogonal algebraically minimal idempotents e_i in A. Also by [27], we have a family $\{S_i : i \in I\}$ of two-sided closed ideals of A, with $S_iS_j = (0)$ if $i \neq j$, and whose union is dense in A. If we assume that ideals are *uniformly* A-complemented, then there is a constant K, and idempotents f_i with $\|f_i\| \leqslant K$ and $S_i = Af_i$ for each i. For any finite $J \subset I$ we have $A(\sum_{i\in J} f_i)$ is complemented too, so that $\left\| \sum_{i\in J} f_i \right\| \leqslant K$. We may thus use a similarity as at the start of Section 4 to reduce to the case that f_i are projections. The canonical map $\bigoplus_{i\in I}^{f} S_i \to A$ is an isometric homomorphism with respect to the ∞-norm on the direct sum, so that $A \cong \bigoplus_{i\in I}^{0} S_i$. Thus if we are assuming that closed ideals are uniformly complemented then we have reduced the question to the case that A is a topologically simple annihilator algebra.

We now give a characterization amongst the C^*-algebras, of C^*-algebras consisting of compact operators. There are many such characterizations in the literature, however we have not seen the one below, in terms of the following notions introduced by Hamana. If X contains a subspace E then we say that X is an *essential extension* (respectively *rigid extension*) of E if any complete contraction with domain X (respectively from X to X) is completely isometric (respectively is the identity map) if it is completely isometric (respectively is the identity map) on E. If X is injective then it turns out that it is rigid if and only if it is essential, and in this case we say X is an *injective envelope* of E, and write $X = I(E)$. See e.g. 4.2.3 in [11], or the works of Hamana; or [33] for some related topics.

THEOREM 5.5. *If A is a C^*-algebra, the following are equivalent:*
(i) *A is an annihilator C^*-algebra.*
(ii) *A^{**} is an essential extension of A.*
(iii) *A^{**} is an injective envelope of A.*
(iv) *$I(A^{**})$ is an injective envelope of A.*
(v) *Every surjective complete isometry $T : A^{**} \to A^{**}$ maps A onto A.*
(iv) *A is nuclear and A^{**} is a rigid extension of A.*

Proof. Clearly (i) $\Rightarrow$ (iii) (from e.g. Lemma 3.1(ii) of [23]), and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) by standard diagram chasing. Since A is nuclear if and only if A^{**} is injective, we have (vi) if and only if (iii).

Item (ii) implies that the normal extension of every faithful $*$-representation of A is faithful on A^{**}. This implies that A^{**} is injective, the latter since $\pi_{\mathrm{a}}(A)'' \cong \bigoplus_i^\infty B(H_i)$ is injective (see Lemma 4.3.8 of [34]), where π_{a} is the atomic representation of A. So (ii) $\Leftrightarrow$ (iii). Moreover, in the notation above, these imply that $\pi_{\mathrm{a}}(A)''$ is an injective envelope of A, and hence $\pi_{\mathrm{a}}(A)$ contains $\bigoplus_i^0 K(H_i)$ by Lemma 3.1(iii) of [23]. Thus A has a subalgebra B with $\pi_{\mathrm{a}}(B) = \bigoplus_i^0 K(H_i)$, and $\widetilde{\pi}_{\mathrm{a}}(B^{\perp\perp}) = \pi_{\mathrm{a}}(A)''$. Hence $B^{\perp\perp} = A^{**}$, and so $A = B$. So (ii) $\Rightarrow$ (i).

From e.g. Proposition 3.3 of [38], (i) $\Rightarrow$ (v). Conversely, if p is a projection in A^{**} then $u = 1 - 2p$ is unitary, and so $(1 - 2p)A \subset A$ if (v) holds. Indeed clearly $p \in M(A)$, so that $A^{**} \subset M(A)$. Thus A is an ideal in A^{**}, which implies (i) [24]. ∎

REMARK 5.6. We are not sure if in (vi) one may drop the nuclearity condition. By standard diagram chasing, for nonselfadjoint algebras (or operator spaces), (ii) is equivalent to (iv), and to $A^{**} \subset I(A)$ unitally. Also (i) $\Rightarrow$ (ii) for nonselfadjoint algebras of compact operators, indeed if A is an operator algebra with cai, which is a left or right ideal in its bidual, then we have (ii) (since $LM(A)$ and $RM(A)$ may be viewed in $I(A)$, see e.g. Chapter 4 of [11]), and also (v) (by Proposition 3.3 of [38]). Also (ii) implies that A^{**} is a rigid extension (since $I(A)$ is), and this works for operator spaces too. It is easy to see that A^{**} being a rigid extension of an operator space A, implies that every surjective complete isometry $T : A^{**} \to A^{**}$ is weak* continuous (a property enjoyed by all C^*-algebras). To see this, let $\widetilde{T_{|A}} : A^{**} \to A^{**}$ be the weak* continuous extension of $T_{|A}$, then $T^{-1} \circ \widetilde{T_{|A}} = I_{A^{**}}$ by rigidity.

In [22], [23], Hamana defines the notion of a *regular* extension of a C^*-algebra. It is not hard to see that A^{**} is an essential extension of A if and only if it is a regular extension. This uses the fact that (ii) is equivalent to $A^{**} \subset I(A)$, and the fact that the regular monotone completion of A from [22], resides inside $I(A)$ (see [37] for more details if needed).

6. ONE-SIDED IDEALS IN TENSOR PRODUCTS OF OPERATOR ALGEBRAS

Amongst other things, in this section we extend several known results about the Haagerup tensor products of C^*-algebras (mainly from [5], [15]), to general operator algebras, and give some applications. For example, we investigate the one-sided M-ideal structure of the Haagerup tensor products of nonselfadjoint operator algebras.

We will write $M \otimes^{\sigma \mathrm{h}} N$ for the *σ-Haagerup tensor product* (see e.g. [20], [19], [12], [11]). We will repeatedly use the fact that for operator spaces X and Y, we have $(X \otimes_{\mathrm{h}} Y)^{**} \cong X^{**} \otimes^{\sigma \mathrm{h}} Y^{**}$ (see e.g. 1.6.8 in [11]). We recall from Section 3 of [12] that the Haagerup tensor product and σ-Haagerup tensor product of unital operator algebras is a unital operator space (in the sense of [12]), and also is a unital Banach algebra. We write $\mathrm{Her}(D)$ for the hermitian elements in a unital space D (recall that h is hermitian if and only if $\varphi(h) \in \mathbb{R}$ for all $\varphi \in \mathrm{Ball}(D^*)$ with $\varphi(1) = 1$).

LEMMA 6.1. *If A and B are unital operator spaces then* $\mathrm{Her}(A \otimes_{\mathrm{h}} B) = A_{\mathrm{sa}} \otimes 1 + 1 \otimes B_{\mathrm{sa}}$ *and* $\Delta(A \otimes_{\mathrm{h}} B) = \Delta(A) \otimes 1 + 1 \otimes \Delta(B)$. *Similarly, if M and N are unital dual operator algebras, then* $\mathrm{Her}(M \otimes^{\sigma \mathrm{h}} N) = M_{\mathrm{sa}} \otimes 1 + 1 \otimes N_{\mathrm{sa}}$ *and* $\Delta(M \otimes^{\sigma \mathrm{h}} N) = \Delta(M) \otimes 1 + 1 \otimes \Delta(N)$.

Proof. If A and B are unital operator spaces then $A \otimes_{\mathrm{h}} B$ is a unital operator space (see [12]), and $\mathrm{Her}(A \otimes_{\mathrm{h}} B) \subset \mathrm{Her}(C^*(A) \otimes_{\mathrm{h}} C^*(B))$. By a result in [5], it follows that if $u \in \mathrm{Her}(A \otimes_{\mathrm{h}} B)$ then there exist $h \in C^*(A)_{\mathrm{sa}}, k \in C^*(B)_{\mathrm{sa}}$ such that $u = h \otimes 1 + 1 \otimes k$. It is easy to see that this forces $h \in A, k \in B$. For example if φ is a functional in $A^{\perp}$ then $0 = (\varphi \otimes I_B)(u) = \varphi(h)1$, so that $h \in (A^{\perp})_{\perp} = A$. Conversely, it is obvious that $A_{\mathrm{sa}} \otimes 1 + 1 \otimes B_{\mathrm{sa}} \subset \mathrm{Her}(A \otimes_{\mathrm{h}} B)$. Indeed the canonical maps from A and B into $A \otimes_{\mathrm{h}} B$ must take hermitians to hermitians. This gives the first result, and taking spans gives the second.

Now let M and N be unital dual operator algebras. Again it is obvious that $M_{\mathrm{sa}} \otimes 1 + 1 \otimes N_{\mathrm{sa}} \subset \mathrm{Her}(M \otimes^{\sigma \mathrm{h}} N)$. For the other direction, we may assume that $M = N$ by the trick of letting $R = M \oplus N$. It is easy to argue that $M \otimes^{\sigma \mathrm{h}} N \subset R \otimes^{\sigma \mathrm{h}} R$, since M and N are appropriately complemented in R. If $W^*_{\max}(M)$ is the "maximal von Neumann algebra" generated by M, then by Theorem 3.1(i) of [12] we have $M \otimes^{\sigma \mathrm{h}} M \subset W^*_{\max}(M) \otimes^{\sigma \mathrm{h}} W^*_{\max}(M)$. So (again using the trick in the first paragraph of our proof) we may assume that M is a von Neumann algebra.

By a result of Effros and Kishimoto ([19], Theorem 2.5), $\mathrm{Her}(M \otimes^{\sigma \mathrm{h}} M)$ equals

$$\mathrm{Her}(CB_{M'}(B(H))) \subset \mathrm{Her}(CB(B(H))) = \{h \otimes 1 + 1 \otimes k : \ h, k \in B(H)_{\mathrm{sa}}\},$$

the latter by a result of Sinclair and Sakai (see e.g. Lemma 4.3 of [14]). By a small modification of the argument in the first paragraph of our proof it follows that $h, k \in M$. The final result again follows by taking the span. ∎

THEOREM 6.2. *Let M and N be unital dual operator algebras. If $\Delta(M)$ is not one-dimensional then $\Delta(M) \cong \mathcal{A}_{\mathrm{l}}(M \otimes^{\sigma \mathrm{h}} N)$. If $\Delta(N)$ is not one-dimensional then $\Delta(N) \cong \mathcal{A}_{\mathrm{r}}(M \otimes^{\sigma \mathrm{h}} N)$. If $\Delta(M)$ and $\Delta(N)$ are one-dimensional then*

$$\mathcal{A}_{\mathrm{l}}(M \otimes^{\sigma \mathrm{h}} N) = \mathcal{A}_{\mathrm{r}}(M \otimes^{\sigma \mathrm{h}} N) = \mathbb{C}\, I.$$

Proof. We just prove the first and the last assertions. Let M and N be unital dual operator algebras, and let $X = M \otimes^{\sigma \mathrm{h}} N$. The map $\theta : \mathcal{A}_{\mathrm{l}}(X) \to X$ defined by $\theta(T) = T(1)$ is a unital complete isometry (see the end of the notes section for 4.5 in [11]). Hence, by Corollary 1.3.8 of [11] and Lemma 6.1, it maps into $\Delta(X) = \Delta(M) \otimes 1 + 1 \otimes \Delta(N)$. The last assertion is now clear. For the first, if we can show that $\mathrm{Ran}(\theta) \subset \Delta(M) \otimes 1$, then we will be done. There is a copy of $\Delta(M)$ in $\mathcal{A}_{\mathrm{l}}(X)$ via the embedding $a \mapsto L_{a \otimes 1}$, and this is a C^*-subalgebra. Note that θ restricts to a $*$-homomorphism from this C^*-subalgebra into the free product $M * N$ discussed in [12]. Let $T \in \mathcal{A}_{\mathrm{l}}(X)_{\mathrm{sa}}$, then $\theta(T) \in X_{\mathrm{sa}}$. By Lemma 6.1, $T(1 \otimes 1) = h \otimes 1 + 1 \otimes k$, with $h \in \Delta(M)_{\mathrm{sa}}, k \in \Delta(N)_{\mathrm{sa}}$. It suffices to show that $\theta(T - L_{h \otimes 1}) = 1 \otimes k \in \Delta(M) \otimes 1$. So let $S = T - L_{h \otimes 1}$. By Proposition 1.3.11 of [11] we have for $a \in \Delta(M)_{\mathrm{sa}}$ that

$$S(a \otimes 1) = \theta(S L_{a \otimes 1}) = \theta(S) * (a \otimes 1) = (1 \otimes k) * (a \otimes 1).$$

The involution in $M * N$, applied to the last product, yields $a * k = a \otimes k \in M \otimes^{\sigma \mathrm{h}} N$. Hence

$$S(a \otimes 1) \in \Delta(M \otimes^{\sigma \mathrm{h}} N) = \Delta(M) \otimes 1 + 1 \otimes \Delta(N) \subset \Delta(M) \otimes \Delta(N).$$

Since left and right multipliers of an operator space automatically commute, we have that $\rho(\Delta(N))$ commutes with S, where $\rho : \Delta(N) \to \mathcal{A}_{\mathrm{r}}(M \otimes^{\sigma \mathrm{h}} N)$ is the canonical injective $*$-homomorphism. Thus for $b \in \Delta(N)$ we have

$$S(a \otimes b) = S(\rho(b)(a \otimes 1)) = \rho(b)(S(a \otimes 1)) = S(a \otimes 1)(1 \otimes b) \in \Delta(M) \otimes \Delta(N).$$

By linearity this is true for any $a \in \Delta(M)$ too. It follows that $\Delta(M) \otimes_{\mathrm{h}} \Delta(N)$ is a subspace of $M \otimes^{\sigma \mathrm{h}} N$ which is invariant under S. Since S is selfadjoint, it follows from Propositionn 5.2 of [15] that the restriction of S to $\Delta(M) \otimes_{\mathrm{h}} \Delta(N)$ is adjointable, and selfadjoint. Hence by Theorem 5.42 of [15] we have that there exists an $m \in \Delta(M)$ with $S(1 \otimes 1) = m \otimes 1 = 1 \otimes k$. Thus $1 \otimes k \in \Delta(M) \otimes 1$ as desired. ∎

COROLLARY 6.3. *Let A and B be approximately unital operator algebras. If $\Delta(A^{**})$ is not one dimensional then $\Delta(M(A)) \cong \mathcal{A}_{\mathrm{l}}(A \otimes_{\mathrm{h}} B)$. If $\Delta(B^{**})$ is not one* *dimensional then* $\Delta(M(B)) \cong \mathcal{A}_{\rm r}(A \otimes_{\rm h} B)$. *If* $\Delta(A^{**})$ *and* $\Delta(B^{**})$ *are one dimensional then* $\mathcal{A}_{\rm l}(A \otimes_{\rm h} B) = \mathcal{A}_{\rm r}(A \otimes_{\rm h} B) = \mathbb{C}\, I$.

Proof. We just prove the first and last relations. Let $\rho : \Delta(LM(A)) \to \mathcal{A}_{\rm l}(A \otimes_{\rm h} B)$ be the injective $*$-homomorphism given by $S \mapsto S \otimes I_B$. If $T \in \mathcal{A}_{\rm l}(A \otimes_{\rm h} B)_{\rm sa}$, then by Proposition 5.16 from [15], we have $T^{**} \in \mathcal{A}_{\rm l}(A^{**} \otimes^{\sigma {\rm h}} B^{**})_{\rm sa}$. By the last theorem, $T^{**}(a \otimes b) = T(a \otimes b) = L_{h \otimes 1}(a \otimes b)$, for some $h \in A^{**}_{\rm sa}$ and for all $a \in A$, $b \in B$. Since $T(a \otimes b)$ is in $A \otimes_{\rm h} B$, so is $L_{h\otimes 1}(a \otimes b)$ for all $a \in A$, $b \in B$. Also $L_{h \otimes 1}(a \otimes 1) = ha \otimes 1 \in A \otimes_{\rm h} B$ for all $a \in A$. So $ha \in A$ for all $a \in A$. Thus $L_h \in \Delta(LM(A)_{\rm sa})$. This shows that ρ is surjective, since selfadjoint elements span $\mathcal{A}_{\rm l}(A \otimes_{\rm h} B)$. Thus $\Delta(LM(A)) \cong \mathcal{A}_{\rm l}(A \otimes_{\rm h} B)$. By the proof of Proposition 5.1 in [9], we have $\Delta(LM(A)) = \Delta(M(A))$. This proves the first relation. If $\Delta(A^{**})$ and $\Delta(B^{**})$ are one dimensional, then so is $\Delta(M(A))$, and so is $\mathcal{A}_{\rm l}(A^{**} \otimes^{\sigma {\rm h}} B^{**})$, by the theorem. Hence the T above is in $\mathbb{C}\, I$, and this proves the last assertion. ∎

REMARK 6.4. For A, B, M, N as in the last results, it is probably true that we have $\Delta(M(A)) \cong \mathcal{A}_{\rm l}(A \otimes_{\rm h} B)$, and similarly that $\Delta(M) \cong \mathcal{A}_{\rm l}(M \otimes^{\sigma {\rm h}} N)$, if A and M are not one-dimensional, with no other restrictions. We are able to prove this if $B = N$ is a finite dimensional C^*-algebra.

The following is a complement to Theorem 5.38 of [15]:

THEOREM 6.5. *Let* A *and* B *be approximately unital operator algebras, and suppose that* $\Delta(A^{**})$ *is not one-dimensional. Then the right* M*-ideals (respectively right* M*-summands) in* $A \otimes_{\rm h} B$ *are precisely the subspaces of the form* $J \otimes_{\rm h} B$, *where* J *is a closed right ideal in* A *having a left* cai *(respectively having form* eA *for a projection* $e \in M(A)$*).*

Proof. The summand case follows immediately from Corollary 6.3. The one direction of the M-ideal case is Theorem 5.38 of [15]. For the other, suppose that I is a right M-ideal in $A \otimes_{\rm h} B$. View $(A \otimes_{\rm h} B)^{**} = A^{**} \otimes^{\sigma {\rm h}} B^{**}$. Then $I^{\perp\perp}$ is a right M-summand in $A^{**} \otimes^{\sigma {\rm h}} B^{**}$. By Theorem 6.2 we have $I^{\perp\perp} = eA^{**} \otimes^{\sigma {\rm h}} B^{**}$ for a projection $e \in A^{**}$. Let $J = eA^{**} \cap A$, a closed right ideal in A. We claim that $I = J \otimes_{\rm h} B$. Since $I = I^{\perp\perp} \cap (A \otimes_{\rm h} B)$, we need to show that $(eA^{**} \otimes^{\sigma {\rm h}} B^{**}) \cap (A \otimes_{\rm h} B) = (eA^{**} \cap A) \otimes_{\rm h} B$. By injectivity of $\otimes_{\rm h}$, it is clear that $(eA^{**} \cap A) \otimes_{\rm h} B \subset (eA^{**} \otimes^{\sigma {\rm h}} B^{**}) \cap (A \otimes_{\rm h} B)$. For the other containment, we let $u \in (eA^{**} \otimes^{\sigma {\rm h}} B^{**}) \cap (A \otimes_{\rm h} B)$, and use a slice map argument. By Corollary 4.8 of [39], we need to show that for all $\psi \in B^*$, $(1 \otimes \psi)(u) \in eA^{**} \cap A = J$. Let $\psi \in B^*$, then $\langle \widetilde{u}, 1 \otimes \psi \rangle = (1 \otimes \psi)(u) \in A$, where $\widetilde{u}$ is u regarded as an element in $A^{**} \otimes^{\sigma {\rm h}} B^{**}$. Since $u \in eA^{**} \otimes^{\sigma {\rm h}} B^{**}$, we have $\langle \widetilde{u}, 1 \otimes \psi \rangle \in eA^{**}$. So $(1 \otimes \psi)(u) \in eA^{**} \cap A = J$, and so $u \in J \otimes_{\rm h} B$ as desired.

Next we show that J has a left cai. It is clear that $J^{\perp\perp} = \overline{J}^{w^*} \subset eA^{**}$. Suppose that there is $x \in eA^{**}$ such that $x \notin J^{\perp\perp}$. Then there exists $\phi \in J^{\perp}$ such that $x(\phi) \neq 0$. Since $I = J \otimes_{\rm h} B$ and $\phi \in J^{\perp}$, we have $\phi \otimes \psi \in I^{\perp}$ for all states

ψ on B. So $I^{\perp\perp}$ annihilates $\phi \otimes \psi$, and in particular $0 = (x \otimes 1)(\phi \otimes \psi) = x(\phi)$, a contradiction. Hence $J^{\perp\perp} = eA^{**}$, and it follows from basic principles about approximate identities that J has a left cai. ∎

THEOREM 6.6. *Let M and N be unital (respectively unital dual) operator algebras, with neither M nor N equal to $\mathbb{C}$. Then the operator space centralizer algebra $Z(M \otimes_{\mathrm{h}} N)$ (respectively $Z(M \otimes^{\sigma \mathrm{h}} N)$) (see Chapter 7 of* [15]*) is one-dimensional.*

Proof. First we consider the dual case. If $\Delta(M)$ and $\Delta(N)$ are both one-dimensional then $Z(M \otimes^{\sigma \mathrm{h}} N) \subset \mathcal{A}_{\mathrm{l}}(M \otimes^{\sigma \mathrm{h}} N) = \mathbb{C}\, I$, and we are done. If $\Delta(M)$ and $\Delta(N)$ are both not one-dimensional, let P be a projection in $Z(M \otimes^{\sigma \mathrm{h}} N)$. By the theorem, $Px = ex = xf$, for all $x \in M \otimes^{\sigma \mathrm{h}} N$, for some projections $e \in M$ and $f \in N$. Then

$$e^{\perp} \otimes f = e^{\perp} \otimes ff = P(e^{\perp} \otimes f) = ee^{\perp} \otimes f = 0,$$

which implies that either $e^{\perp} = 0$ or $f = 0$. Hence $P = 0$ or $P = I$. So $Z(M \otimes^{\sigma \mathrm{h}} N)$ is a von Neumann algebra with only trivial projections, hence it is trivial.

Suppose that $\Delta(N)$ is one-dimensional, but $\Delta(M)$ is not. Again it suffices to show that any projection $P \in Z(M \otimes^{\sigma \mathrm{h}} N)$ is trivial. By Theorem 6.2, P is of the form $Px = ex$ for a projection $e \in M$. Assume that e is not 0 or 1. If $D = \mathrm{Span}\{e, 1 - e\}$, and X is the copy of $D \otimes N$ in $M \otimes^{\sigma \mathrm{h}} N$, then P leaves X invariant. Note that $X = D \otimes_{\mathrm{h}} N$, since $\otimes_{\mathrm{h}}$ is known to be completely isometrically contained in $\otimes^{\sigma h}$ (see [20]). Hence by Section 5.2 in [15] we have that the restriction of P to X is in $\mathcal{A}_{\mathrm{l}}(X) \cap \mathcal{A}_{\mathrm{r}}(X) = Z(X)$. Thus we may assume without loss of generality that $M = D = \ell^{\infty}_2$, and P is left multiplication by e_1, where $\{e_1, e_2\}$ is the canonical basis of ℓ^{∞}_2. Since P is an M-projection, $\|e_1 \otimes x + e_2 \otimes y\|_h = \max\{\|x\|, \|y\|\}$, for all $x, y \in N$. Set $x = 1_N$, and let $y \in N$ be of norm 1. Then $\|e_1 \otimes 1 + e_2 \otimes y\|_h = 1$. If we can show that $y \in \mathbb{C}\, 1_N$ then we will be done: we will have contradicted the fact that N is not one-dimensional, hence e, and therefore P, is trivial. By the injectivity of the Haagerup tensor product, we may replace N with $\mathrm{Span}\{1, y\}$. By basic facts about the Haagerup tensor product, there exist $z_1, z_2 \in \ell^{\infty}_2$ and $v, w \in N$ with $e_1 \otimes 1 + e_2 \otimes y = z_1 \otimes v + z_2 \otimes w$, and with $\|[z_1 \;\; z_2]\|^2 = \|v^* v + w^* w\| = 1$. Multiplying by $e_1 \otimes 1$ we see that $z_1(1)v + z_2(1)w = 1$, so that

$$1 \leqslant (|z_1(1)|^2 + |z_2(1)|^2)\|v^* v + w^* w\| = |z_1(1)|^2 + |z_2(1)|^2 \leqslant \|[z_1 \; z_2]\|^2 = 1.$$

From basic operator theory, if a pair of contractions have product I, then the one is the adjoint of the other. Thus v, w, and hence y, are in $\mathbb{C}\, 1$.

A similar argument works if $\Delta(M)$ is one-dimensional, but $\Delta(N)$ is not.

In the "non-dual case", use Theorem 7.4(ii) of [15] to see that $Z(M \otimes_{\mathrm{h}} N) \subset Z(M^{**} \otimes^{\sigma \mathrm{h}} N^{**}) = \mathbb{C} I$. ∎

COROLLARY 6.7. *Let A and B be approximately unital operator algebras, with neither being one-dimensional. Then $A \otimes_{\mathrm{h}} B$ contains no non-trivial complete M-ideals.*

Proof. Suppose that J is a complete M-ideal in $A \otimes_{\mathrm{h}} B$. The complete M-projection onto $J^{\perp\perp}$ is in $Z((A \otimes_{\mathrm{h}} B)^{**}) = Z(A^{**} \otimes^{\sigma \mathrm{h}} B^{**})$, and hence is trivial by Theorem 6.6. ∎

REMARK 6.8. The ideal structure of the Haagerup tensor product of C^*-algebras has been studied in [2] and elsewhere.

PROPOSITION 6.9. *Let A and B be approximately unital operator algebras with A not a reflexive Banach space, B finite dimensional and $B \neq \mathbb{C}$. If A is a right ideal in A^{**}, then $A \otimes_{\mathrm{h}} B$ is a right M-ideal in its second dual, and it is not a left M-ideal in its second dual.*

Proof. Since A is a right M-ideal in A^{**}, $A \otimes_{\mathrm{h}} B$ is a right M-ideal in $A^{**} \otimes_{\mathrm{h}} B$ by Proposition 5.38 of [15]. Since B is finite dimensional, $(A \otimes_{\mathrm{h}} B)^{**} = A^{**} \otimes_{\mathrm{h}} B$ (see e.g. 1.5.9 of [11]). Hence $A \otimes_{\mathrm{h}} B$ is a right M-ideal in its bidual. Suppose that it is also a left M-ideal. Then it is a complete M-ideal in its bidual, and therefore corresponds to a projection in $Z(A^{(4)} \otimes_{\mathrm{h}} B)$. However, the latter is trivial by Theorem 6.6. This forces $A \otimes_{\mathrm{h}} B$, and hence A, to be reflexive, which is a contradiction. So $A \otimes_{\mathrm{h}} B$ is not a left M-ideal in its bidual. ∎

REMARK 6.10. A similar argument (see [38]) shows that if Y is a non-reflexive operator space which is a right M-ideal in its bidual and if X is any finite dimensional operator space, then $Y \otimes_{\mathrm{h}} X$ is a right M-ideal in its bidual. Further, if $Z(Y^{(4)} \otimes_{\mathrm{h}} X) \cong \mathbb{C}I$ then $Y \otimes_{\mathrm{h}} X$ is not a left M-ideal in its bidual.

The last few paragraphs, and Corollary 4.6 and Example 4.13, provide natural examples of spaces which are right but not left M-ideals in their second dual. Their duals will be left but not right L-summands in their second dual, by the next result. We refer to [8], [15] for notation.

LEMMA 6.11. *If an operator space X is a right but not a left M-ideal in its second dual, then X^* is a left but not a right L-summand in its second dual.*

Proof. We first remark that a subspace J of operator space X is a complete L-summand of X if and only if it is both a left and a right L-summand. This follows e.g. from the matching statement for M-ideals ([11], Proposition 4.8.4), and the second "bullet" on p. 8 of [15]. By Proposition 2.3 of [38], X^* is a left L-summand in X^{***}, via the canonical projection $i_{X^*} \circ (i_X)^*$. Thus if X^* is both a left and a right L-summand in its second dual, then $i_{X^*} \circ (i_X)^*$ is a left L-projection by the third "bullet" on p. 8 of [15]. Hence by Proposition 2.3 of [38], X is a left M-ideal in its second dual, a contradiction. ∎

We end with some remarks complementing some other results in [38].

(i) Theorem 3.4(i) of [38] can be improved in the case that X is an approximately unital operator algebra A. Theorem 3.4(i) there, is valid for all one-sided

M-ideals, both right and left. This follows from Proposition 2.16 and Proposition 2.12.

(ii) Theorem 3.4 (iii) of [38] can also be improved in the case that X is an operator algebra A. If A is an operator algebra with right cai which is a left ideal in A^{**} (or equivalently, if A is a left M-ideal in its bidual), and if J is a right ideal in A^{**}, then $JA \subset J \cap A$. Hence if $J \cap A = (0)$ then $JA = (0)$. Thus $JA^{**} = (0)$, and hence $J = (0)$, since A^{**} has a right identity. Thus the case $J \cap A = (0)$ will not occur in the conclusion of Theorem 3.4(iii) of [38], in the case that X is an approximately unital operator algebra.

(i) One further result on L-structure: If an operator space X is a right L-summand in its bidual, then any right L-summand Y of X is a right L-summand in Y^{**}. Indeed if X is the range of a left L-projection P on X^{**}, and if Y is the range of a left L-projection Q on X, then Q^{**} and P are in the left Cunningham algebra of X^{**} ([15], p. 8–9). Note that $Q^{**}P = PQ^{**}P$ (since $\mathrm{Ran}(Q^{**}P) \subset Y \subset X$). Since we are dealing with projections in a C^*-algebra, we deduce that $PQ^{**} = Q^{**}P$. It follows that $P(Y^{\perp\perp}) \subset Y$, and so Y is a right L-subspace of X in the sense of Theorem 4.2 of [38]. By that result, Y is a right L-summand in its bidual.

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ADDED IN PROOFS. The open questions stated in Section 3 have now been solved. See e.g. "Operator algebras with contractive approximate identities" by the second author and C. J. Read, *J. Funct. Anal.* **261**(2011), 188–217; and a forthcoming paper of the same authors with the first author. We also remark that there is an obvious variant of Theorem 4.23 in terms of HSA's: a separable operator algebra A is sigma-matricial if and only if A is semiprime, a HSA in its bidual, and every HSA D in A of dimension bigger than 1, contains a nonzero projection which is not an identity for D.